SUBORDINATION PRINCIPLE AND FEYNMAN-KAC FORMULAE FOR GENERALIZED TIME-FRACTIONAL EVOLUTION EQUATIONS

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Abstract. We consider generalized time-fractional evolution equations of the form

$$u(t) = u_0 + \int_0^t k(t,s) L u(s) ds$$

with a fairly general memory kernel k and an operator L being the generator of a strongly continuous semigroup. In particular, L may be the generator L_0 of a Markov process ξ on some state space Q, or $L := L_0 + b\nabla + V$ for a suitable potential V and drift b. Moreover, L may be the generator of a subordinate semigroup or a Schrödinger type group. This class of evolution equations includes in particular time- and space- fractional heat and Schrödinger type equations. We show that a subordination principle holds for such evolution equations and obtain Feynman-Kac formulae for solutions of these equations with the use of different stochastic processes, such as subordinate Markov processes and randomly scaled Gaussian processes. In particular, we obtain some Feynman-Kac formulae with generalized grey Brownian motion and other related self-similar processes with stationary increments.

Keywords: anomalous diffusion, time-fractional evolution equations, fractional calculus, subordination principle, Feynman-Kac formulae, randomly scaled Gaussian processes, generalized grey Brownian motion, time-changed Markov processes, Marichev-Saigo-Maeda generalized fractional operators, Hille-Phillips functional calculus.

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1. Introduction

Many natural phenomena exhibit a diffusive behaviour such that the displacement distribution has a non-Gaussian form and / or its variance is not linear in time. Such phenomena are usually called *anomalous diffusion* and are observed in many complex systems, ranging from turbulence and plasma physics to soft matter and neuro-physiological systems (see, e.g., [40, 41, 52] and references therein). Many different models have been proposed for the description of such phenomena. One of the earliest approaches obtains different regimes of anomalous diffusion as proper scaling limits of continuous time random walks. Stochastic processes which arise as such scaling limits are *Markov processes time-changed by* so-called *inverse subordinators* (see, e.g., [2, 30, 31, 38, 39] and references therein). In the frame of this approach, time- and / or space-fractional evolution equations emerge as governing equations for the underlying stochastic processes. The basic evolution equation in this context is the time- and / or space-fractional heat equation which replaces the standard heat equation, the basic equation of the classical diffusion models:

$$u(t,x) = u_0(x) - \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left(-\frac{1}{2}\Delta\right)^{\gamma} u(s,x) ds, \qquad \beta \in (0,1], \quad \gamma \in (0,1]. \tag{1}$$

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Here $-\left(-\frac{1}{2}\Delta\right)^{\gamma}$, $\gamma\in(0,1)$, is the fractional Laplacian [32]. Equation (1) serves as a governing equation for the process $(Y_{\mathcal{E}^{\beta}_t})_{t\geqslant0}$ which is a symmetric 2γ-stable Lévy process $(Y_t)_{t\geqslant0}$ time-changed by an independent inverse β-stable subordinator $(\mathcal{E}^{\beta}_t)_{t\geqslant0}$.

Another direction in the theoretical description of anomalous diffusion emerges by modelling the diffusion in complex media and interprets the anomalous character of the diffusion as a consequence of a very heterogeneous character of the environment [7, 8, 15, 26, 52, 53]. Some of the models of diffusion in complex media are based on *randomly scaled Gaussian processes* (RSGP), see, e.g., [46, 47, 54] and references therein. In particular, processes of the form $(\sqrt{A}G_t)_{t\geqslant0}$, where $(G_t)_{t\geqslant0}$ is a Gaussian process and A is a nonnegative random variable, which is independend of $(G_t)_{t\geqslant0}$, are considered. The most well-known RSGP of this type is the *generalized grey Brownian motion* (GGBM) $(X^{\alpha,\beta}_t)_{t\geqslant0}$, $\alpha\in(0,2)$, $\beta\in(0,1]$. The GGBM $(X^{\alpha,\beta}_t)_{t\geqslant0}$ was introduced in works of Mainardi, Mura and their coauthors [42, 43, 44], and can be realized as

$$X^{\alpha,\beta}_t := \sqrt{A_\beta}B^{\alpha/2}_t, \tag{2}$$

where $B^{\alpha/2}_t$ is a fractional Brownian motion (FBM) with Hurst parameter $\alpha/2$ and A_β is a nonnegative random variable with $\mathbb{E}[e^{-\lambda A_\beta}]=E_\beta(-\lambda)$ (here E_β is the Mittag-Leffler function with parameter β). Due to the properties of fractional Brownian motion, the GGBM (and some further related RSGP) are *self-similar processes with stationary increments* (SSSI), which makes such processes attractive for modelling. Note however, that fractional Brownian motion (and hence GGBM) is neither a Markov process, nor a semimartingale. The governing equation of GGBM is the following time-stretched time-fractional heat equation

$$u(t,x)=u_0(x)+\frac{\alpha}{\beta\Gamma(\beta)}\int_0^t s^{\frac{\alpha}{\beta}-1}\left(t^{\frac{\alpha}{\beta}}-s^{\frac{\alpha}{\beta}}\right)^{\beta-1}\frac{1}{2}\Delta u(s,x)ds. \tag{3}$$

Equation (3) reduces to the time-fractional heat equation (i.e. equation (1) with $\gamma:=1$) if $\alpha:=\beta$. For $\gamma\in(0,1)$, the time- and space-fractional heat equation (1) is shown to be the governing equation for another SSSI RSGP (see [47]). Therefore, both classes of stochastic processes discussed above can be used to solve the same time- (and space-) fractional evolution equations. These classes of processes have however very different nature and properties (see, e.g., [10]). Let us finally mention another type of RSGP considered in the literature. It is represented by the *scaled Brownian motion with random diffusivity* (see, e.g., [12] and references therein) which can be thought of as a solution $(X_t)_{t\geqslant0}$ of a heuristic stochastic equation $X_t=\int_0^t\dot{B}_s\sqrt{A_s\tau(s)}ds$, where $(\dot{B}_t)_{t\geqslant0}$ is a white noise, $(A_t)_{t\geqslant0}$ is a suitable nonnegative stochastic process which is independent of Brownian motion $(B_t)_{t\geqslant0}$ and $\tau:[0,\infty)\to[0,\infty)$ is a deterministic function (usually a power function). The processes $(B_{At^\theta})_{t\geqslant0}$ and $(\sqrt{A}B_{t^\theta})_{t\geqslant0}$ may be considered as special cases of scaled Brownian motion with random diffusivity (cf. [12]).

In this paper, we study a general class of evolution equations of the form

$$u(t)=u_0+\int_0^t k(t,s)Lu(s)ds,\qquad t>0, \tag{4}$$

where k is a fairly general memory kernel and L is the infinitesimal generator of a strongly continuous semigroup $(T_t)_{t\geqslant0}$ acting on some Banach space X. We identify conditions on the memory kernel k which admit to write the solution operator of this equation in the form

$$\mathrm{Dom}(L)\to X,\quad u_0\mapsto\int_0^\infty(T_au_0)\,\mathcal{P}_{A(t)}(da)$$

for a family $(\mathcal{P}_{A(t)})_{t\geqslant 0}$ of probability measures on the positive real line, which depends on k only. We, thus, consider this representation as a subordination principle associated to the memory kernel k. We state the subordination principle in Section 2, and in particular discuss how to obtain stochastic representations of the solution, if the L is (a Bernstein function of) the infinitesimal generator of a Markov process (plus a potential). The most natural stochastic reprensentations of such an approach are given in terms of time-changed Markov processes. In Section 3, we explain, however, how to arrive at representations in terms of non-Markovian processes such as generalized grey Brownian motion or even in terms of stochastic differential equations driven by more general randomly scaled fractional Brownian motions. In Section 3, we discuss also evolution equations (4) in the special case of kernels k of convolution type; we present some examples of such kernels and discuss an equivalent form of equation (4) in the setting of generalized Caputo type fractional derivatives. Finally, the proofs are provided in Section 4. While the main results can be considered as generalizations of our previous results in [5] beyond the case of pseudo-differential operators L associated to Lévy processes, the proofs are completely different, relating (an approximate version of) the subordination principle to a family of Volterra equations via the Hille-Phillips functional calculus.

2. Main results

In this section, we state our main results. We start from a general abstract setting and extract some Feynman-Kac formulae as special cases afterwards.

Assumption 2.1. Let X be a Banach space with a norm $\|\cdot\|_X$. Let $(T_t)_{t\geqslant 0}$ be a strongly continuous semigroup on X with generator $(L, \mathrm{Dom}(L))$.

We consider the evolution equation (4) with operator L as in Assumption 2.1, with $u_0 \in \mathrm{Dom}(L)$, $u : [0,\infty) \to X$ and k satisfying the following Assumptions 2.2–2.3.

Assumption 2.2. We consider a Borel-measurable kernel $k : (0,\infty)\times(0,\infty) \to \mathbb{R}$ satisfying the condition: $\exists\, \alpha^* \in [0,1)$ and $\exists\, \varepsilon > 0$ such that for each $T > 0$

$$K_T := \sup_{0<t\leqslant T} t^{\alpha^* - \frac{1}{1+\varepsilon}} \|k(t,\cdot)\|_{L^{1+\varepsilon}((0,t))} < \infty.$$

In order to identify the family of probability measures $(\mathcal{P}_{A(t)})_{t\geqslant 0}$ for the subordination, we specify their Laplace transform in terms of the memory kernel k. To this end we define the function $\Phi : [0,\infty)\times\mathbb{C} \to \mathbb{C}$ via

$$\Phi(t,\lambda) := \sum_{n=0}^{\infty} c_n(t)\lambda^n, \tag{5}$$

$$c_0(t) := 1 \qquad \forall\ t \geqslant 0 \qquad \text{and}$$

$$c_n(t) := \begin{cases} \int_0^t k(t,s)c_{n-1}(s)ds, & \forall\ t > 0, \\ 0 & t = 0, \end{cases} \qquad n \in \mathbb{N}, \tag{6}$$

It has been shown in [5] that, under Assumption 2.2, the function Φ is well-defined (i.e., the integrals in the recursion formula exist) and, for fixed t, entire in λ.

Assumption 2.3. Let the function Φ be constructed from the kernel k via formulas (5), (6). We assume that the restriction of the function $\Phi(t,-\cdot)$ on $(0,\infty)$ is completely monotone for all $t \geqslant 0$, i.e., for each $t \geqslant 0$, there exists a nonnegative random variable $A(t)$ whose distribution $\mathcal{P}_{A(t)}$ has the Laplace transform given by $\Phi(t,-\cdot)$:

$$\int_0^\infty e^{-\lambda a}\mathcal{P}_{A(t)}(da) = \Phi(t,-\lambda), \qquad \forall\, \lambda \in \mathbb{C}, \quad \mathrm{Re}\,\lambda \geqslant 0. \tag{7}$$

Note that $\mathcal{P}_{A(0)} = \delta_0$ and $A(0) = 0$ a.s. since $\Phi(0,-\lambda) \equiv 1$.

Typical examples of kernels k satisfying Assumptions 2.2–2.3 are kernels of convolution type and homogeneous kernels related to operators of generalized fractional calculus (see Section 3 below, cf. [5]). Recall that a kernel k is *homogeneous of degree* $\theta - 1$ for some $\theta > 0$ if

$$k(t, ts) = t^{\theta-1} k(1, s), \qquad \forall\ t \in (0, \infty), \quad s \in (0,1). \tag{8}$$

Theorem 2.1. *Let Assumption 2.1 hold. Let k satisfy Assumption 2.2 and assume that the corresponding function Φ satisfies Assumption 2.3. Then:*
(i) For each $t \geqslant 0$, the operator $\Phi(t, L)$ given by the Bochner integral

$$\Phi(t, L)\varphi := \int_0^\infty T_a \varphi\, \mathcal{P}_{A(t)}(da), \qquad \varphi \in X, \tag{9}$$

is well defined and it is a bounded linear operator on X.
(ii) For each $t > 0$ and each $u_0 \in \operatorname{Dom}(L)$, the function

$$u(t) := \Phi(t, L) u_0 \tag{10}$$

solves equation (4) *and it holds* $\lim_{t\searrow 0} u(t) = u_0$.
(iii) Suppose additionally that k is homogeneous of order $\theta - 1$ for some $\theta > 0$. Then one can choose $A(t) := At^\theta$ in (9)*, where A is a nonnegative random variable such that*

$$\int_0^\infty e^{-\lambda a} \mathcal{P}_A(da) = \Phi(1, -\lambda) \qquad \forall \lambda \in \mathbb{C}, \quad \operatorname{Re}\lambda \geqslant 0. \tag{11}$$

Remark 2.1. Theorem 2.1 provides a subordination principle for evolution equations of the form (4): Solution (10) of equation (4) is obtained from the solution $T_t u_0$ of the corresponding standard evolution equation $\frac{\partial u}{\partial t} = Lu$, $u(0) = u_0$, via a "subordination" with respect to the "subordinator" $(A(t))_{t\geqslant 0}$.

If the semigroup $(T_t)_{t\geqslant 0}$ has a stochastic representation, then, due to subordination formula (9), the family $(\Phi(t, L))_{t\geqslant 0}$ has a stochastic representation as well. We present some examples of such stochastic representations below.

Corollary 2.1. *(i) Let Q be a Polish*[1] *space endowed with a Borel σ-field $\mathscr{B}(Q)$ and $(\Omega, \mathcal{F}, \mathbb{P}^x, (\xi_t)_{t\geqslant 0})_{x\in Q}$ be a (universal) Markov process with state space $(Q, \mathscr{B}(Q))$. Assume that the corresponding transition semigroup $(T^0_t)_{t\geqslant 0}$, $T^0_t u_0(x) := \mathbb{E}^x\left[u_0(\xi_t)\right]$, is a strongly continuous semigroup on some Banach space $X \subset B_b(Q)$ (where $B_b(Q)$ is the space of all bounded Borel measurable functions on Q). Let $(L_0, \operatorname{Dom}(L_0))$ be the generator of $(T^0_t)_{t\geqslant 0}$. Let Assumption 2.2 and Assumption 2.3 hold. Let further $(A(t))_{t\geqslant 0}$ be taken to be independent from $(\xi_t)_{t\geqslant 0}$. Then, for each $u_0 \in \operatorname{Dom}(L_0)$, the function*

$$u(t, x) := \mathbb{E}^x\left[u_0(\xi_{A(t)})\right], \qquad t \geqslant 0, \quad x \in Q, \tag{12}$$

solves the evolution equation

$$\begin{aligned} u(t,x) &= u_0(x) + \int_0^t k(t,s) L_0 u(s,x)\,ds, \qquad t > 0, \quad x \in Q, \\ \lim_{t\searrow 0} u(t,x) &= u_0(x), \qquad x \in Q. \end{aligned} \tag{13}$$

(ii) Let additionally $V : Q \to \mathbb{R}$ be a Borel measurable function with $\sup_{x\in Q} V(x) \leqslant c$ for some $c \in \mathbb{R}$ such that the (closure of the) operator $(L_0{+}V, \operatorname{Dom}(L_0{+}V))$ generates

[1] A Polish space is a separable completely metrizable topological space.

a strongly continuous semigroup $(T_t)_{t\geqslant 0}$ on X with stochastic representation[2]

$$T_t u_0(x) := \mathbb{E}^x\left[u_0(\xi_t)\exp\left(\int_0^t V(\xi_s)ds\right)\right], \qquad t\geqslant 0, \quad x\in Q, \quad u_0\in X.$$

Then, for each initial condition $u_0 \in \mathrm{Dom}(L_0+V)$, the following Feynman-Kac formula

$$u(t,x) := \mathbb{E}^x\left[u_0(\xi_{A(t)})\exp\left(\int_0^{A(t)} V(\xi_s)ds\right)\right], \qquad t\geqslant 0, \quad x\in Q, \tag{14}$$

provides a solution to the evolution equation

$$u(t,x) = u_0(x) + \int_0^t k(t,s)\big(L_0 u(s,x) + V(x)u(s,x)\big)ds, \qquad t>0, \quad x\in Q, \tag{15}$$
$$\lim_{t\searrow 0} u(t,x) = u_0(x), \qquad x\in Q.$$

(iii) Suppose additionally that k is homogeneous of order $\theta-1$ for some $\theta>0$, then one can choose $A(t) := At^\theta$ in (12) *and in* (14)*, where A is a nonnegative random variable which is independent from $(\xi_t)_{t\geqslant 0}$ and satisfies* (11).

Remark 2.2. If we choose $(\xi_t)_{t\geqslant 0}$ to be an $\mathbb{R}^d$-valued Lévy process, $\xi_t := x + Y_t$ under $\mathbb{P}^x$, and $X := C_\infty(\mathbb{R}^d)$, then Corollary 2.1 (i) implies Theorem 1 (ii) in [5], where the initial condition u_0 is even required to be a member of the Schwartz space of rapdily decreasing smooth functions. If we take now a bounded continuous potential $V : \mathbb{R}^d \to \mathbb{R}$, all assumptions of Corollary 2.1 (ii) are fullfilled[3] and hence the Feynman-Kac formula (14) holds. Note that, if $V\equiv 0$, only one-dimensional marginal distributions of the process $(\xi_{A(t)})_{t\geqslant 0}$ are relevant for the Feynman-Kac formula (14) and the process $(\xi_{A(t)})_{t\geqslant 0}$ can be replaced by any other process with the same one-dimensional marginal distributions. If V is a nonzero constant, some particular changings of the process $(\xi_{A(t)})_{t\geqslant 0}$ are possible. For example, if $(\xi_t)_{t\geqslant 0}$ is a δ-stable Lévy process, one may replace $\xi_{A(t)}$ by $(A(t))^{1/\delta}\zeta$, where a random variable ζ has the same distribution as ξ_1 and is independent from $(A(t))_{t\geqslant 0}$. In the case of nonconstant potential V it is not possible to change the structure of the process $(\xi_{A(t)})_{t\geqslant 0}$ since the whole process $(\xi_s)_{s\geqslant 0}$ is needed in (14).

We next wish to apply the semigroup $(T_t)_{t\geqslant 0}$ associated to an infinitesimal generator L in order to represent the solution of the evolution equation with memory kernel k and the (space-)fractional operator $-(-L)^\gamma$. In order to cover this and related situations, we use subordination in the sense of Bochner [6, 50]. Recall that subordination in the sense of Bochner is a random time change of a given process $(\xi_t)_{t\geqslant 0}$ by an independent 1-dimensional increasing Lévy process (subordinator) $(\eta_t^f)_{t\geqslant 0}$. Any subordinator can be characterized in terms of its Laplace exponent f: $\mathbb{E}\left[e^{-\lambda\eta_t^f}\right] = e^{-tf(\lambda)}$; any such f is a Bernstein function and is determined uniquely by its Lévy-Khintchine representation

$$f(\lambda) = a + b\lambda + \int_{(0,\infty)}\left(1-e^{-\lambda s}\right)\nu(ds),$$

where a, $b\geqslant 0$ and ν is a measure on $(0,\infty)$ satisfying $\int_{(0,\infty)}\min(s,1)\nu(ds)<\infty$. Let $(T_t)_{t\geqslant 0}$ be a strongly continuous contraction semigroup on some Banach space

[2] This is the classical Feynman-Kac formula which holds under very mild assumptions on processes and potentials, cf. [9] Chapter 3.3.2, [11, 28]. For example, this Feynman-Kac formula holds in the case $X := C_\infty(\mathbb{R}^d)$ = the space of continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$ vanishing at infinity, $(\xi_t)_{t\geqslant 0}$ is a Feller diffusion on $\mathbb{R}^d$, $V:\mathbb{R}^d\to\mathbb{R}$ is a bounded continuous function.

[3] In this case, V is a bounded perturbation of L; the semigroup generated by $L+V$ exists, is again strongly continuous and has the required Feynman-Kac representation.

$(X, \|\cdot\|_X)$ with generator $(L, \mathrm{Dom}(L))$. The family of operators $(T_t^f)_{t\geqslant 0}$ defined by the Bochner integral

$$T_t^f \varphi := \int_0^\infty T_s\varphi \mathcal{P}_{\eta_t^f}(ds), \qquad \varphi \in X,$$

is said to be subordinate to $(T_t)_{t\geqslant 0}$ with respect to the convolution semigroup of measures $\left(\mathcal{P}_{\eta_t^f}\right)_{t\geqslant 0}$, where $\mathcal{P}_{\eta_t^f}$ is the distribution of η_t^f. The family $(T_t^f)_{t\geqslant 0}$ is again a strongly continuous contraction semigroup on the space X whose generator $(L^f, \mathrm{Dom}(L^f))$ is the closure of the operator $(-f(-L), \mathrm{Dom}(L))$, where

$$-f(-L)\varphi := -a\varphi + bL\varphi + \int_{(0,\infty)} (T_s\varphi - \varphi)\, \nu(ds), \qquad \varphi \in \mathrm{Dom}(L).$$

If $(T_t)_{t\geqslant 0}$ is the transition semigroup of a Feller process $(\xi_t)_{t\geqslant 0}$ and $(\eta_t^f)_{t\geqslant 0}$ is an independent subordinator, then $(T_t^f)_{t\geqslant 0}$ is the transition semigroup of the (again Feller) process $(\xi_{\eta_t^f})_{t\geqslant 0}$. Further information on subordination in the sense of Bochner and all related objects can be found e.g. in [51].

Consider now the function $\Phi^f(t, -\cdot) := \Phi(t, -f(\cdot))$. If the function $\Phi(t, -\cdot)$ is completely monotone, so is[4] the function $\Phi^f(t, -\cdot)$. Hence there exists a family of nonnegative random variables whose Laplace transform is given by $\Phi^f(t, -\cdot)$, $t \geqslant 0$. Using distributions of these random variables and a strongly continuous contraction semigroup $(T_t)_{t\geqslant 0}$ with generator $(L, \mathrm{Dom}(L))$, one can define the operator $\Phi^f(t, L)$ analogously to (9).

Corollary 2.2. *Let Assumption 2.1 hold. Let k satisfy Assumption 2.2 and the corresponding function Φ satisfy Assumption 2.3. Let $(A(t))_{t\geqslant 0}$ be a family of nonnegative random variables satisfying* (7)*. Let $(\eta_t^f)_{t\geqslant 0}$ be a subordinator corresponding to a Bernstein function f which is independent from $(A(t))_{t\geqslant 0}$.*

(i) It holds:

$$\Phi^f(t, L)\varphi = \int_0^\infty T_s\varphi \mathcal{P}_{\eta^f_{A(t)}}(ds) = \Phi(t, L^f)\varphi, \qquad \varphi \in X.$$

Moreover, for each $t > 0$ and each $u_0 \in \mathrm{Dom}(L^f)$, the function $u(t) := \Phi^f(t, L)u_0$ solves the evolution equation

$$u(t) = u_0 + \int_0^t k(t,s) L^f u(s) ds, \qquad t > 0 \tag{16}$$
$$\lim_{t\searrow 0} u(t) = u_0.$$

(ii) Let all assumptions of Corollary 2.1 be fulfilled and L in part (i) above be given by $L := L_0 + V$, where L_0 and V are as in Corollary 2.1. Let additionally $V \leqslant 0$. Let $(\xi_t)_{t\geqslant 0}$ be a Markov process with generator L_0 which is independent from $(\eta_t^f)_{t\geqslant 0}$ and $(A(t))_{t\geqslant 0}$. Then for $u_0 \in \mathrm{Dom}((L_0 + V)^f)$ the function

$$u(t,x) := \mathbb{E}^x \left[u_0\left(\xi_{\eta^f_{A(t)}}\right) e^{\int_0^{\eta^f_{A(t)}} V(\xi_s) ds} \right] \tag{17}$$

solves the evolution equation

$$u(t,x) = u_0(x) + \int_0^t k(t,s)\, (L_0 + V)^f\, u(s,x) ds. \tag{18}$$

(iii) Suppose additionally that k is homogeneous of order $\theta - 1$ for some $\theta > 0$. Let A be a nonnegative random variable which satisfies (11) *and is independent from $(\eta_t^f)_{t\geqslant 0}$ and $(\xi_t)_{t\geqslant 0}$. Then we can take $A(t) := At^\theta$ in* (17)*.*

[4] As a composition of a Bernstein function f and a completely monotone function $\Phi(t, -\cdot)$.

Remark 2.3. When L is a pseudo-differential operator associated to a Lévy process and $V \equiv 0$, then we obtain Theorem 1 (iii) in [5] as a special case.

Remark 2.4. Theorem 2.1 can be applied also to generalized time-fractional Schrödinger type equations. Note, that different types of fractional analogues of the standard Schrödinger equation have been discussed in the literature, see, e.g., [3, 16, 20, 33, 45]. Such equations seem to be physically relevant; in particular, some of them arise from the standard quantum dynamics under special geometric constraints [25, 49]. So, let $X := L^2(\mathbb{R}^d)$ be the Hilbert space of complex-valued square integrable functions; X plays the role of the state space of a quantum system. Let $(\mathcal{H}, \mathrm{Dom}(\mathcal{H}))$ be a (bounded from below) self-adjoint operator in X playing the role of the Hamiltonian (energy operator) of this quantum system. Then $(L, \mathrm{Dom}(L)) := (-i\mathcal{H}, \mathrm{Dom}(\mathcal{H}))$ does generate a strongly continuous contraction semigroup $(T_t^{\mathcal{H}})_{t\geqslant 0}$ on X by the Stone theorem. Let k, Φ, $(A(t))_{t\geqslant 0}$ be as in Theorem 2.1. Then, by Theorem 2.1,

$$u(t,x) := \mathbb{E}\left[T^{\mathcal{H}}_{A(t)} u_0(x)\right] \tag{19}$$

solves[5] the generalized time-fractional Schrödinger type equation

$$u(t,x) = u_0(x) - i\int_0^t k(t,s)\mathcal{H}u(s,x)ds, \tag{20}$$

where the equality above is understood as the equality of two elements of the space X. For a few particular choices of the Hamiltonian, some stochastic representations of the corresponding semigroup $(T_t^{\mathcal{H}})_{t\geqslant 0}$ are known in the literature (see, e.g., [24, 14, 13]). Inserting these stochastic representations into (19), one obtains Feynman-Kac formulae (which may be local in the space variables) for the corresponding generalized time-fractional Schrödinger type equation (20).

Remark 2.5. Under assumptions of Theorem 2.1 let $u(t)$ be the solution of equation (4) given by formula (10). Consider now the following class of time-stretchings $\mathcal{G} := \big\{g : [0,\infty) \to [0,\infty)$ such that $g(\tau) \nearrow \infty$ as $\tau \nearrow \infty$, $g(\tau) = \int_0^\tau \dot{g}(\sigma)d\sigma$ for some $\dot{g} \in L^1_{loc}([0,\infty))$, $g(\tau) > 0$ and $\dot{g}(\tau) > 0$ for all $\tau > 0\big\}$. The change of variables $t = g(\tau)$, $g \in \mathcal{G}$, shows that

$$v(\tau) := u(g(\tau)) = \Phi(g(\tau), L)u_0 = \int_0^\infty T_a u_0 \, \mathcal{P}_{A(g(\tau))}(da)$$

solves the time-stretched equation

$$v(\tau) = u_0 + \int_0^\tau \kappa_g(\tau,\sigma)Lv(\sigma)d\sigma, \qquad \tau > 0, \tag{21}$$

where the kernel κ_g is defined via

$$\kappa_g(\tau,\sigma) := k(g(\tau), g(\sigma))\dot{g}(\sigma).$$

In particular, any stochastic representation of a solution $u(t)$ of equation (4) induces the corresponding stochastic representation for a solution $v(t)$ of the time-stretched equation(21) for the whole class $\mathcal{G}$ of time-stretchings.

3. Feynman-Kac formulae based on randomely scaled Gaussian processes and further remarks

Due to Corollary 2.1 and Corollary 2.2, the most natural stochastic reprensentations for evolution equations of the form (4) with L being (a Bernstein function of) the generator of a Markov process (plus a potential term) are given in terms of time-changed Markov processes. In the special case when the memory kernel k is homogeneous, one may sometimes use randomly scaled Gaussian processes (or

[5]Similar results can be found in [29] for the equation (20) with a particular memory kernel k.

other randomly scaled processes which are self-similar and have stationary increments) in the obtained stochastic representations. Let us discuss this case. For this recall first a class of memory kernels k from [5] which satisfy Assumptions 2.2-2.3 and are homogeneous.

Example 3.1. Let $b > 0$, $a \geqslant b$, $\mu \geqslant \frac{b}{a} - 1$, $\nu > \max\{a-b, -a\mu\}$. Consider the Marichev-Saigo-Maeda kernel (cf. Sec. 4 in [5])

(22)
$$k(t,s) := \frac{a}{\Gamma(b/a)}(t^a - s^a)^{\frac{b}{a}-1} t^{a-\nu} s^{\nu-1} F_3\left(\frac{\nu}{a} - 1, \frac{b}{a}, 1, \mu, \frac{b}{a}, 1 - \left(\frac{s}{t}\right)^a, 1 - \left(\frac{t}{s}\right)^a\right),$$

where $0 < s < t$ and F_3 is Appell's third generalization of the Gauss hypergeometric function: $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$, $\gamma \notin -\mathbb{N}$,

$$F_3(\alpha, \alpha', \beta, \beta', \gamma, x, y) := \sum_{m,n\geqslant 0} \frac{(\alpha)_m (\beta)_m (\alpha')_n (\beta')_n}{(\gamma)_{m+n} n! m!} x^m y^n,$$

$$(\delta)_\nu := \begin{cases} 1, & \nu = 0, \quad \delta \in \mathbb{C} \\ \delta(\delta-1)\cdot \ldots \cdot (\delta+n-1), & \nu = n \in \mathbb{N}, \quad \delta \in \mathbb{C}. \end{cases}$$

The kernel k is homogeneous of degree $b-1$ and satisfies Assumption 2.2 (cf. Theorem 4 in [5]). The corresponding function Φ has the following form:

$$\Phi(t,\lambda) = \Gamma(q_2) E^{q_3}_{q_1,q_2}(\lambda t^b), \tag{23}$$

where

$$q_1 := \frac{b}{a}, \quad q_2 := \frac{\nu}{a} + \mu, \quad q_3 := 1 + \frac{\nu - a}{b}, \tag{24}$$

and $E^{q_3}_{q_1,q_2}$ is the three parameter Mittag-Leffler (or Prabhakar) function[6] $E^{q_3}_{q_1,q_2}(\lambda) := \sum_{n=0}^\infty \frac{(q_3)_n}{\Gamma(q_1 n + q_2) n!} \lambda^n$. Under our assumptions on the parameters, the function $\Phi(t, -\cdot)$ is completely monotone and hence Assumption 2.3 is fulfilled. As corresponding random variables $(A(t))_{t\geqslant 0}$ one may take $A(t) := A_{b,a,\mu,\nu} t^b$, where $A_{b,a,\mu,\nu}$ is a non-negative random variable with Laplace transform $\Gamma(q_2) E^{q_3}_{q_1,q_2}(-\lambda)$.

Let now $\alpha \in (0,2)$, $\beta \in (0,1]$. In the special case $b := \alpha$, $a := \frac{\alpha}{\beta}$, $\nu := a$, $\mu := 0$, the Marichev-Saigo-Maeda kernel (22) reduces to the kernel which appears in the governing equation for generalized grey Brownian motion:

$$k(t,s) := \frac{\alpha}{\beta\Gamma(\beta)} s^{\frac{\alpha}{\beta}-1}\left(t^{\frac{\alpha}{\beta}} - s^{\frac{\alpha}{\beta}}\right)^{\beta-1}, \qquad \beta \in (0,1], \quad \alpha \in (0,2). \tag{25}$$

The corresponding function Φ in (23) reduces to the classical Mittag-Leffler function: $\Phi(t,\lambda) = E_\beta(\lambda t^\alpha)$. And, as the corresponding random variables $(A(t))_{t\geqslant 0}$, one may take $A(t) := A_\beta t^\alpha$, where A_β is a non-negative random variable with Laplace transform $E_\beta(-\cdot)$. For $\beta \in (0,1)$, such random variable A_β has probability density function $M_\beta \mathbb{1}_{[0,\infty)}$, where $M_\beta(z) := \sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma(-\beta n + (1-\beta))}$ is the Mainardi-Wright function. Generally, probability density function of $A_{b,a,\mu,\nu}$ (when exists) is given in terms of Fox H-functions (cf. Remark 10 of [5]).

Let us now present some Feynman-Kac formulae for evolution equations of type (4) with homogeneous kernel k on the base of randomely scaled Gaussian processes.

Example 3.2. (i) Under the assumptions of Corollary 2.2 consider the Bernstein function $f(\lambda) := \lambda^\gamma$, $\gamma \in (0,1]$. Then the operator L^f is the fractional power of the operator L, i.e. $L^f = -(-L)^\gamma$ (cf. [55, 51]), and $(\eta^f_t)_{t\geqslant 0}$ is a γ-stable subordinator. Let k be homogeneous of degree $\theta - 1$ for some $\theta > 0$ and take $A(t) = At^\theta$ according

[6]The function $E^{q_3}_{q_1,q_2}$ is well-defined on the whole $\mathbb{C}$ for $\operatorname{Re} q_1 > 0$ and is an entire function.

to Corollary 2.2 (iii). Then the random variable $\eta^f_{A(t)}$ has the same distribution as $A^{1/\gamma}\eta^f_1 t^{\theta/\gamma}$. We may replace the "subordinator" $\left(\eta^f_{A(t)}\right)_{t\geqslant 0}$ in (17) by a new "subordinator" $\left(\mathcal{A}t^{\theta/\gamma}\right)_{t\geqslant 0}$ with

$$\mathcal{A} := A^{1/\gamma}\eta^f_1. \tag{26}$$

This allows to split randomness and time-dependence in the random time-change. Thus, we obtain the following Feynman-Kac formula

$$\begin{aligned} u(t,x) :&= \mathbb{E}^x\left[u_0\left(\xi_{\mathcal{A}t^{\theta/\gamma}}\right)\exp\left(\int_0^{\mathcal{A}t^{\theta/\gamma}} V(\xi_s)ds\right)\right] \\ &= \mathbb{E}^x\left[u_0\left(\xi_{\mathcal{A}t^{\theta/\gamma}}\right)\exp\left(\mathcal{A}\frac{\theta}{\gamma}\int_0^t s^{\frac{\theta}{\gamma}-1}V(\xi_{\mathcal{A}s^{\theta/\gamma}})ds\right)\right] \end{aligned} \tag{27}$$

for the evolution equation

$$u(t,x) = u_0(x) - \int_0^t k(t,s)\left(-L_0 - V\right)^\gamma u(s,x)ds.$$

(ii) Let k, A, $(\eta^f_t)_{t\geqslant 0}$ and $\mathcal{A}$ be as in part (i) of this example. Let $V := c$ for some $c \leqslant 0$, $\xi_t := x + B_t + wt$ under $\mathbb{P}^x$, where $(B_t)_{t\geqslant 0}$ is a standard d-dimensional Brownian motion, which is independent from A and $(\eta^f_t)_{t\geqslant 0}$, and $w \in \mathbb{R}^d$ is some fixed vector. Let $X^{\mathcal{A},\gamma,\theta}_t := B_{\mathcal{A}t^{\theta/\gamma}}$ or $X^{\mathcal{A},\gamma,\theta}_t := \sqrt{\mathcal{A}}B_{t^{\theta/\gamma}}$, or, if $H := \frac{\theta}{2\gamma} \in (0,1)$, $X^{\mathcal{A},\gamma,\theta}_t := \sqrt{\mathcal{A}}B^H_t$, where $\left(B^H_t\right)_{t\geqslant 0}$ is a d-dimensional fractional Brownian motion[7] with Hurst parameter H which is independent from A and $(\eta^f_t)_{t\geqslant 0}$. Note that all three options of the process $(X^{\mathcal{A},\gamma,\theta}_t)_{t\geqslant 0}$ have the same one-dimensional marginal distributions. Then, due to Feynman-Kac formula (27),

$$u(t,x) = \mathbb{E}\left[u_0\left(x + X^{\mathcal{A},\gamma,\theta}_t + \mathcal{A}wt^{\theta/\gamma}\right)e^{c\mathcal{A}t^{\theta/\gamma}}\right], \tag{28}$$

solves the evolution equation

$$u(t,x) = u_0(x) - \int_0^t k(t,s)\left(-\frac{1}{2}\Delta - w\nabla - c\right)^\gamma u(s,x)ds. \tag{29}$$

Therefore, we have obtained a Feynman-Kac formula (28) for the evolution equation (29) in terms of two different classes of randomly scaled Gaussian processes: randomly scaled slowed-down / speeded-up Brownian motion $\left(\sqrt{\mathcal{A}}B_{t^{\theta/\gamma}}\right)_{t\geqslant 0}$ and (if $H := \frac{\theta}{2\gamma} \in (0,1)$) randomly scaled fractional Brownian motion $\left(\sqrt{\mathcal{A}}B^H_t\right)_{t\geqslant 0}$. If k is a Marichev-Saigo-Maeda kernel (22) then $\theta = b$, $A = A_{b,a,\mu,\nu}$ in distribution. In the special case of the GGBM-kernel (25), we have $\theta = \alpha$, $A = A_\beta$ in distribution, and hence we may use generalized grey Brownian motion in formula (28) as the process $(X^{\mathcal{A},\gamma,\theta}_t)_{t\geqslant 0}$.

The result of Example 3.2 (ii) can be generalized beyond the case of a constant diffusion coefficient, as detailed in the case of dimension $d = 1$ in space in the following proposition. As can be seen from the proof, this generalization requires to move from a Brownian motion to a stochastic differential driven by a Brownian motion in the Stratonovich sense in order to apply Corollary 2.2.

Proposition 3.1. *Let $\gamma \in (0,1]$ and suppose the kernel k is homogeneous of order $\theta - 1$ for some $\theta > 0$ and Assumption 2.2, Assumption 2.3 are satisfied. Let $\mathcal{A}$ be a non-negative random variable constructed by* (26) *in Example 3.2 (i). Assume*

[7]Recall that a 1-dimensional fractional Brownian motion $(B^H_t)_{t\geqslant 0}$ with Hurst parameter $H \in (0,1)$ is a centered Gaussian process with covariance structure $\mathbb{E}[B^H_t B^H_s] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$. A d-dimensional fractional Brownian motion with Hurst parameter H is a vector of d independent 1-dimensional ones.

$w \in \mathbb{R}$, $c \leqslant 0$, and $\sigma \in C^2(\mathbb{R})$ is a bounded function with bounded first and second derivatives. Consider the linear operator $(L_{(\sigma,w)}, \mathrm{Dom}(L_{(\sigma,w)}))$ in $C_\infty(\mathbb{R})$ which is defined by

$$L_{(\sigma,w)}\varphi(x) := \frac{\sigma^2(x)}{2}\frac{d^2}{dx^2}\varphi(x) + \left(w + \frac{1}{2}\sigma'(x)\right)\sigma(x)\frac{d}{dx}\varphi(x), \qquad \varphi \in \mathrm{Dom}(L_{(\sigma,w)}),$$
$$\mathrm{Dom}(L_{(\sigma,w)}) := \left\{\varphi \in C^2(\mathbb{R}) \ :\ \varphi,\ L_{(\sigma,w)}\varphi \in C_\infty(\mathbb{R})\right\}.$$

Let $u_0 \in \mathrm{Dom}(L_{(\sigma,w)})$ and denote by $g_\sigma : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ the solution to the parametrized family of ODEs

$$\frac{\partial}{\partial y} g_\sigma(y,x) = \sigma(g_\sigma(y,x)), \qquad g_\sigma(0,x) = x. \tag{30}$$

Let $(B_t)_{t\geqslant 0}$ be a standard Brownian motion independent from $\mathcal{A}$. Let $X_t^{\mathcal{A},\gamma,\theta} := B_{\mathcal{A}t^{\theta/\gamma}}$ or $X_t^{\mathcal{A},\gamma,\theta} := \sqrt{\mathcal{A}}B_{t^{\theta/\gamma}}$, or, if $H := \frac{\theta}{2\gamma} \in (0,1)$, $X_t^{\mathcal{A},\gamma,\theta} := \sqrt{\mathcal{A}}B_t^H$, where $\left(B_t^H\right)_{t\geqslant 0}$ is a 1-dimensional fractional Brownian motion with Hurst parameter H which is independent from $\mathcal{A}$. Then

$$u(t,x) = \mathbb{E}\left[u_0\left(g_\sigma\left(X_t^{\mathcal{A},\gamma,\theta} + w\mathcal{A}t^{\theta/\gamma}, x\right)\right)e^{c\mathcal{A}t^{\theta/\gamma}}\right] \tag{31}$$

$$= \mathbb{E}\left[u_0\left(g_\sigma\left(X_t^{\mathcal{A},\gamma,\theta}, x\right)\right)e^{\mathcal{A}t^{\theta/\gamma}\left(c-\frac{w^2}{2}\right)+wX_t^{\mathcal{A},\gamma,\theta}}\right]. \tag{32}$$

solves the evolution equation

$$u(t,x) = u_0(x) - \int_0^t k(t,s)\left(-L_{(\sigma,w)} - c\right)^\gamma u(s,x)ds. \tag{33}$$

The proof of Proposition 3.1 will be given in Section 4.

Remark 3.1. Let $H := \frac{\theta}{2\gamma} \in (0,1)$ and $X_t^{\mathcal{A},\gamma,\theta} := \sqrt{\mathcal{A}}B_t^H$, where $\left(B_t^H\right)_{t\geqslant 0}$ is a 1-dimensional fractional Brownian motion with Hurst parameter H as in Proposition 3.1. We now interpret the Feynman-Kac formula (32) from the point of view of stochastic differential equations with respect to $\left(X_t^{\mathcal{A},\gamma,\theta}\right)_{t\geqslant 0}$ in the rough path sense. Assume $H > 1/3$. Then almost every path of $\left(X_t^{\mathcal{A},\gamma,\theta}\right)_{t\geqslant 0}$ is Hölder continuous with some index larger than 1/3. Let $\mathbb{X}_{t,s} := \frac{1}{2}\left(X_t^{\mathcal{A},\gamma,\theta} - X_s^{\mathcal{A},\gamma,\theta}\right)^2$. Then $\mathcal{X} := (X^{\mathcal{A},\gamma,\theta}, \mathbb{X})$ is a lift to a geometric rough path (see [18]). Consider $Z_t^x := g_\sigma\left(X_t^{\mathcal{A},\gamma,\theta}, x\right)$. Then, by the Itô formula for geometric rough paths, see again [18],

$$Z_t^x = x + \int_0^t \sigma(Z_s^x)dX_s^{\mathcal{A},\gamma,\theta}, \tag{34}$$

since $g_\sigma \in C^3(\mathbb{R})$. Hence, the stochastic representation for the solution of (33) in (32) can be rewritten in the form

$$u(t,x) = \mathbb{E}\left[u_0\left(Z_t^x\right)e^{\mathcal{A}t^{\theta/\gamma}\left(c-\frac{w^2}{2}\right)+wX_t^{\mathcal{A},\gamma,\theta}}\right]$$

This form resembles the classical Feynman-Kac formula for parabolic Cauchy problems in terms of stochastic differential equations driven by a Brownian motion. However, the stochastic differential equation (34) is driven by a randomly scaled fractional Brownian motion, which is neither a semimartingale nor a Markov process (unless $H = 1/2$), to account for the memory kernel and the space fractionality in (33), while maintaining the stationary increment property of the driving process.

Let us now discuss evolution equations of the form (4) in the special case, when the kernel k is of convolution type.

Remark 3.2. Suppose the kernel k has the form $k(t,s) := \mathfrak{K}(t-s)$, where $\mathfrak{K} : (0,\infty) \to \mathbb{R}$ is continuous and satisfies $|\mathfrak{K}(t)| \leqslant Mt^{\beta-1}e^{\gamma t}, \quad t>0$, for some constants $M,\gamma \geqslant 0$ and $\beta \in (0,1]$. Let $(\mathcal{L}\mathfrak{K})(\cdot)$ be the Laplace transform of $\mathfrak{K}$. By Theorem 3 in [5], if there exists a nonnegative stochastic process $(A(t))_{t\geqslant 0}$ such that almost all its paths are right-continuous with left limits and such that

$$\int_0^\infty e^{-\sigma t}\mathbb{E}\left[e^{-\lambda A(t)}\right]dt = \frac{1}{\sigma}\frac{1}{1+\lambda(\mathcal{L}\mathfrak{K})(\sigma)} \tag{35}$$

for every $\lambda \geqslant 0$ and sufficiently large $\sigma \geqslant \sigma_0(\lambda)$, then the function $\Phi(t,-\cdot)$ constructed from $k(t,s) = \mathfrak{K}(t-s)$ by (5), (6) is completely monotone for every $t \geqslant 0$ and the above process $(A(t))_{t\geqslant 0}$ satisfies (7). In particular, consider the case when $\mathcal{L}\mathfrak{K} = 1/h$ for some Bernstein function h. Then h is the Laplace exponent of some Lévy subordinator $(\eta^h_t)_{t\geqslant 0}$. The corresponding inverse subordinator $(\mathcal{E}^h_t)_{t\geqslant 0}$ is defined via $\mathcal{E}^h_t := \inf\left\{s>0 : \eta^h_s > t\right\}$. It has been shown in [39] (formula (3.14)) that (in the case when the Lévy measure ν of $(\eta^h_t)_{t\geqslant 0}$ satisfies $\nu(0,\infty)=\infty$) the double Laplace transform of the distribution $\mathcal{P}_{\mathcal{E}^h_t}(da)$ with respect to both time and space variables is equal to

$$\int_0^\infty e^{-\sigma t}\mathbb{E}\left[e^{-\lambda \mathcal{E}^h_t}\right]dt = \frac{h(\sigma)}{\sigma(h(\sigma)+\lambda)} = \frac{1}{\sigma}\frac{1}{1+\lambda(\mathcal{L}\mathfrak{K})(\sigma)}.$$

Hence, one may take $A(t) := \mathcal{E}^h_t$, $t \geqslant 0$, in this case. Note that the assumption $\nu(0,\infty)=\infty$ guarantees that the sample paths of $(\eta^h_t)_{t\geqslant 0}$ are a.s. strictly increasing, i.e. almost all paths $t \mapsto \mathcal{E}^h_t$ are continuous (cf. [36]).

Example 3.3. Let us mention the following functions $\mathfrak{K}_1$ and $\mathfrak{K}_2$ providing admissible kernels k of convolution type and having Laplace transform $1/h$ for some Bernstein function h (cf. [4]): for $1 \geqslant \beta > \beta_1 > \ldots > \beta_m > 0$, $b_j > 0$, $j = 1,\ldots,m$

$$\mathfrak{K}_1(t) := \frac{t^{\beta-1}}{\Gamma(\beta)} + \sum_{j=1}^m b_j \frac{t^{\beta_j-1}}{\Gamma(\beta_j)}$$

with the corresponding Bernstein function $h_1(\sigma) := \left(\sigma^{-\beta} + \sum_{j=1}^m b_j\sigma^{-\beta_j}\right)^{-1}$ and

$$\mathfrak{K}_2(t) := t^{\beta-1}E_{(\beta-\beta_1,\ldots,\beta-\beta_m),\beta}\left(-b_1t^{\beta-\beta_1},\ldots,-b_mt^{\beta-\beta_m}\right)$$

with the multinomial Mittag-Leffler function [19, 23] (for $z_j \in \mathbb{C}$, $\beta \in \mathbb{R}$, $\alpha_j > 0$, $j = 1,\ldots,m$)

$$E_{(\alpha_1,\ldots,\alpha_m),\beta}(z_1,\ldots,z_m) := \sum_{n=0}^\infty \sum_{\substack{n_1+\ldots+n_m=n \\ n_1\in\mathbb{N}_0,\ldots,n_m\in\mathbb{N}_0}} \frac{n!}{n_1!\cdots n_m!}\frac{\prod_{j=1}^m z_j^{n_j}}{\Gamma\left(\beta+\sum_{j=1}^m \alpha_j n_j\right)}.$$

The kernel $\mathfrak{K}_2$ corresponds to the Bernstein function $h_2(\sigma) := \sigma^\beta + \sum_{j=1}^m b_j\sigma^{\beta_j}$. The corresponding functions $\Phi_1(t,-\lambda)$ and $\Phi_2(t,-\lambda)$ are found in [4] in terms of the multinomial Mittag-Leffler function:

$$\Phi_1(t,-\lambda) := E_{(\beta,\beta_1,\ldots,\beta_m),1}\left(-\lambda t^\beta, -\lambda t^{\beta_1},\ldots,-\lambda t^{\beta_m}\right),$$
$$\Phi_2(t,-\lambda) := 1-\lambda t^\beta E_{(\beta,\beta-\beta_1,\ldots,\beta-\beta_m),\beta+1}\left(-\lambda t^\beta, -\lambda t^{\beta_1},\ldots,-\lambda t^{\beta_m}\right).$$

Remark 3.3. Note that, in the case $k(t,s) = \mathfrak{K}(t-s)$ with $\mathcal{L}\mathfrak{K} = 1/h$ for some Bernstein function h, evolution equation (4) is equivalent (what can be shown by applying the Laplace transform w.r.t. time-variable to both equations) to the Cauchy problem

$$\mathcal{D}_t^h u(t,x) = Lu(t,x), \qquad u(0,x) = u_0(x), \qquad x \in \mathbb{R}^d, \quad t > 0, \tag{36}$$

where $\mathcal{D}_t^h$ is a generalized time-fractional derivative of Caputo type, which is defined (for sufficiently good functions $v : (0,\infty) \to \mathbb{R}$ of time variable t) via the Laplace transform (cf. [1]) by

$$\left(\mathcal{L}\left[\mathcal{D}_t^h v\right]\right)(\sigma) = h(\sigma)(\mathcal{L}v)(\sigma) - \frac{h(\sigma)}{\sigma} v(+0).$$

Therefore, the results of Theorem 2.1 and Corollaries 2.1, 2.2 provide solutions for evolution equations of the form (36) with generalized time-fractional derivatives of Caputo type $\mathcal{D}_t^h$. In the case $h(\sigma) := \sigma^\beta$, $\beta \in (0,1)$, the generalized time-fractional derivative $\mathcal{D}_t^h$ coincides with the Caputo derivative of order β. The kernel $\mathfrak{K}_1$ corresponds to a mixture of Caputo time-fractional derivatives of orders β, $\beta_1, \ldots, \beta_m$. In the case of Bernstein function $h(\sigma) := \int_0^1 \sigma^\beta \mu(d\beta)$ with a finite Borel measure μ concentrated on the interval $(0,1)$, the corresponding derivative $\mathcal{D}_t^h$ is known as *distributed order fractional derivative*.

4. Proofs

Proof of Theorem 2.1. (i) Let Assumptions 2.1, 2.2 and 2.3 hold. Since the function $\Phi(t,\cdot)$, $t \geqslant 0$, is entire by Theorem 1 in [5], the function $\Phi(t, i(\cdot))$ is also entire and is the characteristic function of the distribution $\mathcal{P}_{A(t)}$, which is concentrated on $[0,\infty)$. Therefore, we have by the Raikov theorem (cf. Theorem 3.2.1 in [34])

$$\int_{\mathbb{R}} e^{r|a|} \mathcal{P}_{A(t)}(da) = \int_0^\infty e^{ra} \mathcal{P}_{A(t)}(da) < \infty \qquad \forall\, r > 0. \tag{37}$$

Further, for any strongly continuous semigroup $(T_t)_{t \geqslant 0}$ there exist constants $M \geqslant 1$, $c \geqslant 0$ such that $\|T_t\| \leqslant M e^{ct}$, $\forall\, t \geqslant 0$, and the mapping $t \mapsto T_t \varphi$ is continuous for any $\varphi \in X$. Thus, we have $\int_0^\infty \|T_a \varphi\|_X \mathcal{P}_{A(t)}(da) < \infty$ and the Bochner integral in the r.h.s. of (9) is well defined for any $\varphi \in X$. Moreover, the operator $\Phi(t,L)$ defined by (9) is a bounded linear operator on X and $\Phi(0,L) = \mathrm{Id}$.

(ii) Recall that the following statement was proved in [5] (cf. Corollary 1 of [5]):

Lemma 4.1. *Let Assumption 2.2 hold. Then, for each $\lambda \in \mathbb{C}$, there exists a unique solution $\Phi(\cdot, -\lambda) \in B_b([0,T], \mathbb{C})$, $\forall\, T > 0$, of the following Volterra equation of the second kind*

$$\Phi(t, -\lambda) = 1 - \lambda \int_0^t k(t,s) \Phi(s, -\lambda) ds, \qquad t > 0. \tag{38}$$

Moreover, $\lim_{t \searrow 0} \Phi(t, -\lambda) = 1$ *locally uniformly with respect to $\lambda \in \mathbb{C}$, $\Phi(t,\cdot)$ is an entire function for all $t \geqslant 0$ and equalities* (5) *and* (6) *hold.*

Our aim is to lift the equality (38) to the level of operators $\Phi(t,L)$. To this aim we use the so-called Hille-Phillips functional calculus. Let us recall the main facts about this functional calculus (cf. [21, 22]).

Let $(T_t)_{t \geqslant 0}$ be as in Assumption 2.1. Consider first the case when $(T_t)_{t \geqslant 0}$ is uniformly bounded (i.e. $\|T_t\| \leqslant M$ for some $M \geqslant 1$ and all $t \geqslant 0$). Denote by $LS(\mathbb{C}_+)$ the space of functions that are Laplace transforms of complex measures on $([0,\infty), \mathscr{B}([0,\infty)))$. Let $g \in LS(\mathbb{C}_+)$ and m_g be the (unique) complex measure whose Laplace transform $\mathscr{L}[m_g]$ is given by g. One defines the operator $g(-L)$ as follows:

$$g(-L)\varphi := \int_0^\infty T_a \varphi\, m_g(da), \qquad \varphi \in X. \tag{39}$$

The right hand side of (39) is a well-defined Bochner integral and $g(-L)$ is a bounded linear operator on X, i.e. $g(-L) \in \mathcal{L}(X)$. The mapping $\mathscr{C}_T : LS(\mathbb{C}_+) \to \mathcal{L}(X)$, $g \mapsto g(-L)$, is called the *Hille-Phillips calculus* for $-L$. Note that $\mathscr{C}_T$ is an algebra homomorphism and hence $\mathscr{C}_T(g_1 g_2) = g_1(-L) \circ g_2(-L) = g_2(-L) \circ g_1(-L)$ and $\mathscr{C}_T(ag_1 + bg_2) = ag_1(-L) + bg_2(-L)$ for any $g_1, g_2 \in LS(\mathbb{C}_+)$, $a, b \in \mathbb{R}$.

Consider now the case when $(T_t)_{t\geqslant 0}$ is of type $c \geqslant 0$ (i.e., $\|T_t\| \leqslant Me^{ct}$ for some $M \geqslant 1$, $c \geqslant 0$ and all $t \geqslant 0$). Then the rescaled semigroup $(T_t^c)_{t\geqslant 0}$, $T_t^c := T_t e^{-ct}$, is uniformly bounded, strongly continuous and has generator $(L - c, \mathrm{Dom}(L))$. Then one may use the Hille-Phillips calculus $\mathscr{C}_{T^c}$ for $-(L-c)$. Consider now the space $LS(\mathbb{C}_+ - c) := \{g : g(\cdot - c) \in LS(\mathbb{C}_+)\}$. Let $g \in LS(\mathbb{C}_+ - c)$ and m_g^c be the (unique) complex measure with $\mathscr{L}[m_g^c] = g(\cdot - c)$. One defines the operator $g(-L)$ as follows:

$$g(-L)\varphi := \mathscr{C}_{T^c}\big(g(\cdot - c)\big)\varphi \equiv \int_0^\infty T_a^c \varphi \, m_g^c(da), \qquad \varphi \in X.$$

Let now m be a complex measure such that $e^{ca}m(da)$ is again a complex measure. Let $g^* := \mathscr{L}[m]$. Then it holds

$$\mathscr{L}[e^{ca}m(da)](\lambda) = \int_0^\infty e^{-\lambda a} e^{ca} m(da) = g^*(\lambda - c).$$

Hence $g^* \in LS(\mathbb{C}_+ - c)$ and $m_{g^*}^c(da) = e^{ca}m(da)$. Therefore,

$$g^*(-L)\varphi = \int_0^\infty T_a^c \varphi \, m_{g^*}^c(da) = \int_0^\infty T_a \varphi \, m(da), \qquad \varphi \in X.$$

Thus, the operator $\Phi(t, L)$ defined in (9) can be interpreted in terms of Hille-Phillips calculus as $\mathscr{C}_{T^c}\big(\Phi(t, -(\cdot - c))\big)$ due to (37).

Now we are ready to transfer equality (38) to the level of operators by means of Hille-Phillips calculus. Let $(T_t)_{t\geqslant 0}$ be of type $c \geqslant 0$ and $\rho(L)$ be the resolvent set of operator L, i.e. the resolvent operator $R_\lambda(L) := (\lambda - L)^{-1}$ is a well defined bounded operator on X for each $\lambda \in \rho(L)$. Let $\gamma > c$. Hence $\gamma \in \rho(L)$. And equality (38) implies the equality

(40)
$$\gamma \cdot \frac{\Phi(t, -\lambda) - 1}{\gamma + \lambda} = -\lambda \cdot \frac{\gamma}{\gamma + \lambda} \cdot \int_0^t k(t,s)\Phi(s, -\lambda)ds, \quad \forall t > 0, \ \forall \lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \geqslant -c.$$

Let us present each component of (40) as the Laplace transform of some complex measure on $([0,\infty), \mathscr{B}([0,\infty)))$. As we have already discussed

$$\Phi(t, -\lambda) = \mathscr{L}(\mathcal{P}_{A(t)})(\lambda) \overset{\mathscr{C}_{T^c}}{\longleftrightarrow} \Phi(t, L) = \int_0^\infty T_a \, \mathcal{P}_{A(t)}(da).$$

Furthermore, we have with Dirac delta-measure δ_0 and with exponential distribution $Exp(\gamma)$:

$$1 = \mathscr{L}(\delta_0)(\lambda) \overset{\mathscr{C}_{T^c}}{\longleftrightarrow} \mathscr{L}(\delta_0)(-L) := \int_0^\infty T_a^c \delta_0(da) = Id,$$

$$\frac{\gamma}{\gamma + \lambda} = \mathscr{L}(Exp(\gamma))(\lambda) \overset{\mathscr{C}_{T^c}}{\longleftrightarrow} \mathscr{L}(Exp(\gamma))(-L) := \int_0^\infty T_a^c \gamma e^{-\gamma a} e^{ca} da$$
$$= \int_0^\infty T_a \gamma e^{-\gamma a} da = \gamma \cdot (\gamma - L)^{-1} = \gamma \cdot R_\gamma(L).$$

Note that $R_\gamma(L)$ is a bounded operator since $\gamma \in (c, \infty) \subset \rho(L)$ (cf. [17, 48]) and $\|\gamma R_\gamma(L)\varphi - \varphi\|_X \to 0$ as $\gamma \to \infty$ for any $\varphi \in X$. Further,

$$\frac{-\lambda\gamma}{\gamma + \lambda} = -\gamma \cdot 1 + \gamma \cdot \frac{\gamma}{\gamma + \lambda} = -\gamma \cdot \mathscr{L}(\delta_0)(\lambda) + \gamma \cdot \mathscr{L}(Exp(\gamma))(\lambda) \overset{\mathscr{C}_{T^c}}{\longleftrightarrow}$$
$$-\gamma \cdot \mathscr{L}(\delta_0)(-L) + \gamma \cdot \mathscr{L}(Exp(\gamma))(-L) = -\gamma \cdot Id + \gamma^2 R_\gamma(L) =: L_\gamma.$$

Note that L_γ is the so-called *Yosida-Approximation* of L (cf. [17, 48]); L_γ is a bounded operator and $\|L\varphi - L_\gamma\varphi\|_X \to 0$ as $\gamma \to +\infty$ for each $\varphi \in \mathrm{Dom}(L)$.

Without loss of generality we now assume $k(t,s) \geq 0$ (else divide into negative and nonnegative part) and define a family of measures on $([0,\infty),\mathcal{B}([0,\infty)))$ via

$$\nu_t(B) := \int_0^t k(t,s)\mathcal{P}_{A(s)}(B)ds, \qquad B \in \mathcal{B}([0,\infty)).$$

The right hand side in the above formula is well-defined since the mapping $s \mapsto \mathcal{P}_{A(s)}(B)$ is a bounded Borel-measurable function on $[0,\infty)$ for any $B \in \mathcal{B}([0,\infty))$. Indeed, the mapping $s \mapsto \Phi(s,-\lambda)$ is Borel measurable for any $\lambda \in \mathbb{C}$ due to Assumption 2.2 and representation formulas (5), (6). And for any $s, x \geqslant 0$ holds (cf. Lemma 1.1 and the proof of Prop. 1.2 in [51])

$$\mathcal{P}_{A(s)}([0,x]) = \lim_{\lambda\to\infty} \sum_{k\leqslant \lambda x} (-1)^k \frac{\partial^k \Phi(s,-\lambda)}{\partial \lambda^k} \frac{\lambda^k}{k!}.$$

Further, it holds for measurable $g : [0,\infty) \to [0,\infty)$

$$\int_0^\infty g(a)\nu_t(da) = \int_0^t k(t,s) \int_0^\infty g(a)\mathcal{P}_{A(s)}(da)ds, \tag{41}$$

which can be seen via approximation of g by step functions from below and the use of Beppo Levi's Theorem. By choosing $g(a) := e^{-\lambda a}$ we see that

$$\int_0^\infty e^{-\lambda a}\nu_t(da) = \int_0^t k(t,s) \int_0^\infty e^{-\lambda a}\mathcal{P}_{A(s)}(da)ds = \int_0^t k(t,s)\Phi(s,-\lambda)ds =: \Psi(t,-\lambda).$$

Thereby $\Psi(t,-\lambda)$ is the Laplace transform of an appropriate measure and we get the correspondence

$$\Psi(t,-\lambda) \overset{\mathscr{C}_{T^c}}{\longleftrightarrow} \Psi(t,L) := \int_0^\infty T_a^c \nu_t^c(da)$$

where $\nu_t^c(da) := e^{ca}\nu_t(da)$. Note that ν_t^c is a bounded measure on $([0,\infty),\mathscr{B}([0,\infty)))$ due to (37). Furthermore, similar to property (41), it holds for any Bochner-integrable function $g : [0,\infty) \to X$

$$\int_0^\infty g(a)\nu_t^c(da) = \int_0^t k(t,s) \int_0^\infty g(a)e^{ca}\mathcal{P}_{A(s)}(da)ds,$$

and therefore, for any $\varphi \in X$,

$$\begin{aligned}\Psi(t,L)\varphi = \int_0^\infty T_a^c\varphi\, \nu_t^c(da) &= \int_0^t k(t,s) \int_0^\infty T_a^c \varphi\, e^{ca}\mathcal{P}_{A(s)}(da)ds \\ &= \int_0^t k(t,s)\Phi(s,L)\varphi ds.\end{aligned}$$

Thereby, all components of (40) have been transferred. Taking everything together and using that for $u_0 \in \mathrm{Dom}(L)$ holds (cf. [22])

$$L_\gamma \Psi(t,L)u_0 = \gamma L R_\gamma(L)\Psi(t,L)u_0 = \Psi(t,L)\gamma L R_\gamma(L)u_0 = \Psi(t,L)L_\gamma u_0,$$

we get

$$\gamma R_\gamma(L)\left(\Phi(t,L) - Id\right)u_0 = \Psi(t,L)L_\gamma u_0 \qquad \forall\, u_0 \in \mathrm{Dom}(L). \tag{42}$$

Taking the limit $\gamma \to +\infty$ we obtain (with $\Phi(s,L)Lu_0 = L\Phi(s,L)u_0$ for all $u_0 \in \mathrm{Dom}(L)$)

$$\begin{aligned}&(\Phi(t,L) - Id)u_0 = \Psi(t,L)Lu_0 = \int_0^t k(t,s)\Phi(s,L)Lu_0 ds = \int_0^t k(t,s)L\Phi(s,L)u_0 ds \\ &\Leftrightarrow\ \Phi(t,L)u_0 = u_0 + \int_0^t k(t,s)L\Phi(s,L)u_0 ds.\end{aligned}$$

Therefore, the function $u(t) := \Phi(t,L)u_0$ solves evolution equation (4) for any $u_0 \in \mathrm{Dom}(L)$.

For continuity at zero we evaluate equality (7) at $\lambda = -c - i\rho$, $\rho \in \mathbb{R}$, resulting in

$$\int_0^\infty e^{i\rho a} e^{ca} \mathcal{P}_{A(t)}(da) = \Phi(t, i\rho + c) \qquad \forall (t,\rho) \in [0,\infty) \times \mathbb{R}.$$

According to Lemma 4.1

$$\lim_{t \searrow 0} \int_0^\infty e^{i\rho a} e^{ca} \mathcal{P}_{A(t)}(da) = \lim_{t \searrow 0} \Phi(t, i\rho + c) = 1 \qquad \forall \rho \in \mathbb{R},$$

and by Lévy's Continuity Theorem it follows that

$$e^{ca} \mathcal{P}_{A(t)}(da) \xrightarrow{\text{weakly}} \delta_0(da), \qquad t \searrow 0.$$

We now write

$$\begin{aligned}\|u(t) - u_0\|_X = \left\| \int_0^\infty (T_a u_0 - u_0)\, \mathcal{P}_{A(t)}(da) \right\|_X &\le \int_0^\infty \|T_a u_0 - u_0\|_X e^{-ca} e^{ca} \mathcal{P}_{A(t)}(da) \\ &= \int_{\mathbb{R}} f(a) e^{ca} \mathcal{P}_{A(t)}(da),\end{aligned}$$

where

$$f : \mathbb{R} \to \mathbb{R}, \qquad a \mapsto \begin{cases} 0, & a < 0 \\ \|T_a u_0 - u_0\|_X e^{-ca}, & a \ge 0 \end{cases}$$

is a bounded and continuous function. Now

$$\lim_{t \searrow 0} \|u(t) - u_0\|_X \leqslant \lim_{t \searrow 0} \int_{\mathbb{R}} f(a) e^{ca} \mathcal{P}_{A(t)}(da) = f(0) = 0$$

by weak convergence and thus continuity at zero is shown.

(iii) Let k be homogeneous of order $\theta - 1$ for some $\theta > 0$. By the recursion formula (6) for all $t > 0$, $n \in \mathbb{N}$

$$c_n(t) = t^\theta \int_0^1 k(1,s) c_{n-1}(ts) ds = t^{n\theta} \int_0^1 k(1,s) c_{n-1}(s) ds = t^{n\theta} c_n(1).$$

And, thus, we have for all $t \geqslant 0$, $\lambda \in \mathbb{C}$ (cf. Theorem 2 in [5]):

$$\Phi(1, -t^\theta \lambda) = \sum_{n=0}^\infty c_n(1) \left(-t^\theta \lambda\right)^n = \sum_{n=0}^\infty t^{-n\theta} c_n(t) (-t^\theta \lambda)^n = \sum_{n=0}^\infty c_n(t) (-\lambda)^n = \Phi(t, -\lambda).$$

Let $A(t) \coloneqq At^\theta$, where A is a nonnegative random variable satisfying (11). Then

$$\mathscr{L}(\mathcal{P}_{A(t)})(\lambda) = \mathbb{E}\left[e^{-\lambda A t^\theta}\right] = \mathscr{L}(\mathcal{P}_A)(\lambda t^\theta) = \Phi(1, -\lambda t^\theta) = \Phi(t, -\lambda).$$

Therefore, $A(t) \coloneqq At^\theta$ has the required distribution. Theorem 2.1 is proved. □

Proof of Corollary 2.1. (i) By construction $(T_t^0)_{t \geqslant 0}$ is a strongly continuous semigroup of type 0 on X. It follows from Theorem 2.1 ii) and Fubini's theorem that

$$u(t,x) = \Phi(t, L_0) u_0(x) = \int_0^\infty \mathbb{E}^x \left[u_0(\xi_a)\right] \mathcal{P}_{A(t)}(da) = \mathbb{E}^x \left[u_0(\xi_{A(t)})\right]$$

solves the evolution equation (13).

(ii) $(T_t)_{t \geqslant 0}$ is a strongly continuous semigroup of type $\max\{c, 0\}$ on X. It follows from Theorem 2.1 ii) and Fubini's theorem that

$$\begin{aligned}u(t,x) = \Phi(t, L_0 + V) u_0(x) &= \int_0^\infty \mathbb{E}^x \left[u_0(\xi_a) exp\left(\int_0^a V(\xi_s) ds\right)\right] \mathcal{P}_{A(t)}(da) \\ &= \mathbb{E}^x \left[u_0(\xi_{A(t)}) exp\left(\int_0^{A(t)} V(\xi_s) ds\right)\right]\end{aligned}$$

solves the evolution equation (15).

(iii) Follows immediately from Theorem 2.1 (iii). □

Proof of Corollary 2.2. (i) $(T_t^f)_{t\geqslant 0}$ is a strongly continuous contraction semigroup on the Banach space X. Therefore, $(T_t^f)_{t\geqslant 0}$, k and Φ fulfill all assumptions of Theorem 2.1 and thus

$$\Phi(t,L^f)\varphi := \int_0^\infty T_s^f\varphi\mathcal{P}_{A(t)}(ds), \qquad \varphi\in X,$$

is well-defined. Let now $(A(t))_{t\geqslant 0}$ and $(\eta_t^f)_{t\geqslant 0}$ be as in the statement of Corollary 2.2. Consider the family of random variables $\left(\eta^f_{A(t)}\right)_{t\geqslant 0}$. Then

$$\mathbb{E}\left[e^{-\lambda\eta^f_{A(t)}}\right] = \int_0^\infty \mathbb{E}\left[e^{-\lambda\eta^f_a}\right]\mathcal{P}_{A(t)}(da) = \int_0^\infty e^{-af(\lambda)}\mathcal{P}_{A(t)}(da) = \Phi(t,-f(\lambda)).$$

Starting with the strongly continuous contraction semigroup $(T_t)_{t\geqslant 0}$ and the completely monotone function $\Phi^f(t,-\cdot) := \Phi(t,-f(\cdot))$, one may define

$$\Phi^f(t,L)\varphi := \int_0^\infty T_s\varphi\,\mathcal{P}_{\eta^f_{A(t)}}(ds), \qquad \varphi\in X.$$

Due to Fubini's theorem

$$\begin{aligned}\Phi(t,L^f)\varphi = \int_0^\infty T_s^f\varphi\,\mathcal{P}_{A(t)}(ds) &= \int_0^\infty\int_0^\infty T_a\varphi\,\mathcal{P}_{\eta_s^f}(da)\mathcal{P}_{A(t)}(ds)\\ &= \int_0^\infty T_a\varphi\,\mathcal{P}_{\eta^f_{A(t)}}(da) = \Phi^f(t,L)\varphi, \qquad \varphi\in X.\end{aligned}$$

Therefore, for $u_0\in \mathrm{Dom}(L^f)$, equation (16) is solved by $\Phi(t,L^f)u_0 = \Phi^f(t,L)u_0$ according to Theorem 2.1 (ii).
(ii) Since $V\leq 0$, $(T_t)_{t\geqslant 0}$ is a strongly continuous contraction semigroup and so is $(T_t^f)_{t\geqslant 0}$. It follows from Theorem 2.1 (ii) that $u(t,x) := \Phi(t,(L+V)^f)u_0$ solves evolution equation (18) and due to Fubini's theorem

$$\begin{aligned}\Phi(t,(L+V)^f)u_0 &= \int_0^\infty T_a^f u_0\mathcal{P}_{A(t)}(da)\\ &= \int_0^\infty\int_0^\infty \mathbb{E}^x\left[u_0(\xi_s)exp\left(\int_0^s V(\xi_v)dv\right)\right]\mathcal{P}_{\eta_a^f}(ds)\mathcal{P}_{A(t)}(da)\\ &= \int_0^\infty \mathbb{E}^x\left[u_0(\xi_{\eta_a^f})exp\left(\int_0^{\eta_a^f} V(\xi_v)dv\right)\right]\mathcal{P}_{A(t)}(da)\\ &= \mathbb{E}^x\left[u_0(\xi_{\eta^f_{A(t)}})exp\left(\int_0^{\eta^f_{A(t)}} V(\xi_s)ds\right)\right].\end{aligned}$$

(iii) Follows immediately from Theorem 2.1 (iii).

□

Proof of Proposition 3.1. First, note that, under our assumptions on σ, the operator $(L_{(\sigma,w)},\mathrm{Dom}(L_{(\sigma,w)}))$ does generate a strongly continuous semigroup on $C_\infty(\mathbb{R})$ (cf. [35], Sec. 3.1.2). Second, consider the pair $((\xi_t)_{t\geqslant 0},(\mathbb{P}^x)_{x\in\mathbb{R}})$ where $(\xi_t)_{t\geqslant 0}$ solves the Stratonovich SDE with respect to a standard 1-dimensional Brownian motion $(B_t)_{t\geqslant 0}$

$$d\xi_t = \sigma(\xi_s)\circ dB_t + w\sigma(\xi_t)dt$$

with $\xi_0 = x$ under $\mathbb{P}^x$. By Remark 5.2.22 in [27], the pair $((\xi_t)_{t\geqslant 0},(\mathbb{P}^x)_{x\in\mathbb{R}})$ is a Markov process with generator $L_{(\sigma,w)}$. We apply the Doss-Sussmann technique to find an explicit expression for $(\xi_t)_{t\geqslant 0}$. So, let $(B_t)_{t\geqslant 0}$ be a standard 1-dimensional Brownian motion with respect to some some probability measure $\mathbb{P}$. We write $\mathbb{E}[\cdot]$ for the expectation under $\mathbb{P}$ and $\mathbb{E}^x[\cdot]$ for the one under $\mathbb{P}^x$. Let g_σ be as in the statement of Proposition 3.1. Then, by the Itô formula for the Stratonovich integral

$$g_\sigma(B_t+wt,x) = x + \int_0^t \sigma\big(g_\sigma(B_s+ws,x)\big)\circ dB_s + \int_0^t w\sigma\big(g_\sigma(B_s+ws,x)\big)ds.$$

Hence, for every $x \in \mathbb{R}$,

$$\mathrm{Law}\big((g_\sigma(B_t + wt, x))_{t\geqslant 0}, \mathbb{P}\big) = \mathrm{Law}\big((\xi_t)_{t\geqslant 0}, \mathbb{P}^x\big).$$

In view of Corollary 2.2 and Example 3.2, there is a nonnegative random variable $\mathcal{A}$ (constructed from k and γ as in (26)) which is independent of $(B_t)_{t\geqslant 0}$ and such that

$$u(t,x) = \mathbb{E}\left[u_0\left(g_\sigma\left(B_{\mathcal{A}t^{\theta/\gamma}} + w\mathcal{A}t^{\theta/\gamma}, x\right)\right)e^{c\mathcal{A}t^{\theta/\gamma}}\right]$$

solves the evolution equation (33). Note that $\left(B_{\mathcal{A}t^{\theta/\gamma}} + w\mathcal{A}t^{\theta/\gamma}\right)_{t\geqslant 0}$, conditionally on $\mathcal{A}$, is a Gaussian process with mean $w\mathcal{A}t^{\theta/\gamma}$ and variance $\mathcal{A}t^{\theta/\gamma}$. The process $\left(\sqrt{\mathcal{A}}B_{t^{\theta/\gamma}} + w\mathcal{A}t^{\theta/\gamma}\right)_{t\geqslant 0}$ and, if $H := \frac{\theta}{2\gamma} \in (0,1)$, the process $\left(\sqrt{\mathcal{A}}B^H_t + w\mathcal{A}t^{\theta/\gamma}\right)_{t\geqslant 0}$ have the same conditional law, where $(B^H_t)_{t\geqslant 0}$ is a 1-dimensional fractional Brownian motion independent of $\mathcal{A}$. Hence Feynman-Kac formula (31) is shown. Further, we have

$$\begin{aligned}
&u(t,x) = \mathbb{E}\left[u_0\left(g_\sigma\left(\sqrt{\mathcal{A}}B_{t^{\theta/\gamma}} + w\mathcal{A}t^{\theta/\gamma}, x\right)\right)e^{c\mathcal{A}t^{\theta/\gamma}}\right]\\
&= \int_0^\infty \int_{\mathbb{R}} u_0\big(g_\sigma(\sqrt{a}z + wat^{\theta/\gamma}, x)\big)e^{cat^{\theta/\gamma}}(2\pi t^{\theta/\gamma})^{-1/2}\exp\left(-\frac{z^2}{2t^{\theta/\gamma}}\right)dz\,\mathcal{P}_{\mathcal{A}}(da)\\
&= \int_0^\infty \int_{\mathbb{R}} u_0\big(g_\sigma(\sqrt{a}y, x)\big)e^{at^{\theta/\gamma}(c-w^2/2)+w\sqrt{a}y}(2\pi t^{\theta/\gamma})^{-1/2}\exp\left(-\frac{y^2}{2t^{\theta/\gamma}}\right)dy\,\mathcal{P}_{\mathcal{A}}(da)\\
&= \mathbb{E}\left[u_0\left(g_\sigma\left(\sqrt{\mathcal{A}}B_{t^{\theta/\gamma}}, x\right)\right)e^{\mathcal{A}t^{\theta/\gamma}\left(c-\frac{w^2}{2}\right)+w\sqrt{\mathcal{A}}B_{t^{\theta/\gamma}}}\right].
\end{aligned}$$

Hence Feynman-Kac formula (32) is shown.

□

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