ON UPPER BOUNDS FOR THE FIRST ℓ^2-BETTI NUMBER

CARSTEN FELDKAMP AND STEFFEN KIONKE

ABSTRACT. This article presents a method for proving upper bounds for the first ℓ^2-Betti number of groups using only the geometry of the Cayley graph. As an application we prove that Burnside groups of large prime exponent have vanishing first ℓ^2-Betti number.

Our approach extends to generalizations of ℓ^2-Betti numbers, that are defined using characters. We illustrate this flexibility by generalizing results of Thom-Peterson on q-normal subgroups to this setting.

Over the last 30 years the ℓ^2-Betti numbers have become a major tool in the investigation of infinite groups. The purpose of this article is to explore the first ℓ^2-Betti number of groups using only the geometry of the Cayley graph. Our method is based on Pichot's observation [11, Propositon 2] that the first ℓ^2-Betti number can be expressed with the *rate of relations* in the Cayley graph. It follows from an elementary identity (see Lemma 1.1) that explicit cycles in the Cayley graph give rise to upper bounds for the first ℓ^2-Betti number. Surprisingly, these elementary bounds can be used to prove new results.

Theorem 0.1. *Let p be a prime and let G be a torsion group of exponent p. Then $b_1^{(2)}(G) \leq 2p-2$.*

Using a theorem of Gaboriau this implies a vanishing result for the first ℓ^2-Betti number of Burnside groups $B(m,p)$ of exponent p.

Corollary 0.2. *Let p be a prime number. If p is sufficiently large, then $b_1^{(2)}(B(m,p)) = 0$.*

On the other hand, suppose that $b_1^{(2)}(B(m,p)) \neq 0$ for some prime p. Then Theorem 0.1 offers a simple solution of the restricted Burnside problem for m-generated p-groups using the multiplication formula for ℓ^2-Betti numbers of finite index subgroups.

Our method can neatly be adapted to character-theoretic generalizations of the first ℓ^2-Betti number. We recall that every character ψ (see [8, Def. 2.5]) of the group G, gives rise to a ψ-Betti number $b_1^{\psi}(G)$; see [8] or Section 1. The ordinary Betti numbers and the ℓ^2-Betti numbers are special cases of this construction. However, it is difficult to calculate or bound ψ-Betti numbers under general assumptions of ψ.

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We extend Pichot's observation to the general setting and we use our method to generalize a result of Thom-Peterson [10, Theorem 5.6] to ψ-Betti numbers; see Corollary 3.3. Even for ℓ^2-Betti numbers our argument contains a new proof of their result. This provides a convenient way to bound (and sometimes calculate) ψ-Betti numbers in some generality. We illustrate this by proving a vanishing result for certain ψ-Betti numbers of right-angled groups; see Theorem 3.6.

In Section 1 we discuss basic results on ψ-Betti numbers and we introduce our main method. In Section 2 we apply it in the case of p-torsion groups. Section 3 is concerned with q-normality and presents applications to ψ-Betti numbers.

1. Betti numbers and the Cayley graph

The following simple result is essential for our approach.

Lemma 1.1. *Let $\mathcal{H}$ be a Hilbert space and let $W \subseteq \mathcal{H}$ be a subspace. Let $P\colon \mathcal{H} \to \overline{W}$ denote the orthogonal projection onto the closure of W. Then for all $v \in \mathcal{H}$*

$$\langle Pv, v\rangle = \sup_{w\in W} \frac{|\langle w, v\rangle|^2}{\langle w, w\rangle}$$

where the supremum is taken over all non-zero elements of W (and is defined to be 0 if $W = 0$).

Proof. For $v = 0$ the assertion is obvious. We may assume that $\|v\| = 1$. For all $w \in W$, we note that

$$|\langle w, v\rangle|^2 = |\langle Pw, v\rangle|^2 = |\langle w, Pv\rangle|^2 \overset{C.S.}{\leq} \|w\|^2\|Pv\|^2 = \langle Pv, v\rangle\|w\|^2.$$

If $w \neq 0$ we obtain

$$\frac{|\langle w, v\rangle|^2}{\langle w, w\rangle} \leq \langle Pv, v\rangle.$$

In particular, the proof is complete if $Pv = 0$.

For the converse we assume $Pv \neq 0$. Let $\varepsilon \in (0, 1)$. Since W is dense in $\overline{W}$, there is $w \in W$ with $\|Pv - w\| < \varepsilon\|Pv\|^2$ and we deduce

$$|\langle w, v\rangle| \geq \langle Pv, v\rangle - |\langle w - Pv, v\rangle| \overset{C.S.}{\geq} \langle Pv, v\rangle - \|w - Pv\| \geq (1 - \varepsilon)\langle Pv, v\rangle.$$

In addition, we note that $\|w\| = \|w - Pv + Pv\| \leq (1 + \varepsilon)\|Pv\|$ and so

$$\frac{|\langle w, v\rangle|^2}{\langle w, w\rangle} \geq \frac{(1 - \varepsilon)^2}{(1 + \varepsilon)^2}\langle Pv, v\rangle$$

The assertion follows as ε can be arbitrarily close to 0. □

Let G be a group. A *character* of G is a function $\psi\colon G \to \mathbb{C}$ of positive type, which is constant on conjugacy classes of G and satisfies $\psi(1_G) = 1$; see [8, Def. 2.5]. Let $\mathrm{Ch}(G)$ denote the space of all characters of G. Every character $\psi \in \mathrm{Ch}(G)$ gives rise to a semi-definite G-invariant inner product $\langle g, h\rangle_\psi = \psi(h^{-1}g)$ on the group ring $\mathbb{C}[G]$. Passing to the completion provides us with a tracial Hilbert G-bimodule $\ell^\psi(G)$; see [8, Def. 2.1]. Using the GNS construction, this provides a tracial von Neumann algebra and a notion of dimension, which can be used to define the ψ-Betti numbers $b_k^\psi(G)$

of G, provided that G satisfies suitable finiteness properties. Specifically b_0^ψ is defined for all groups and b_1^ψ is defined for all finitely generated groups.

For the regular character δ with $\delta(g) = 0$ for all $g \neq 1_G$, one has $\ell^\delta(G) = \ell^2(G)$ and one obtains the famous ℓ^2-Betti numbers $b_k^{(2)}(G)$. The constant character (i.e., $\psi(g) = 1$ for all g) gives rise to the ordinary rational Betti numbers of G since $\ell^\psi(G) \cong \mathbb{C}$.

Definition 1.2. Let G be a group and let $\psi \in \mathrm{Ch}(G)$. A subgroup $K \leq G$ is *ψ-regular*, if $\psi|_K$ is the regular character on K, i.e. $\psi(k) = 0$ for all $k \in K\backslash\{1\}$.

Here we are mainly interested in the first Betti numbers $b_1^\psi(G)$. It will however be useful and instructive to initially consider the 0-th Betti number. Let J_G denote the augmentation ideal in $\mathbb{C}[G]$, i.e. the set of elements $w = \sum_{g\in G} w_g g$ which satisfy $\sum_{g\in G} w_g = 0$.

Lemma 1.3. *Let G be a group and let $\psi \in \mathrm{Ch}(G)$ be a character.*

(a) $b_0^\psi(G) = 1 - \sup_{w\in J_G} \frac{\langle w,1\rangle_\psi^2}{\langle w,w\rangle_\psi}$ where the supremum is taken over all non-zero elements of J_G.
(b) If $G = \bigcup_{i\in I} G_i$ is a directed union of subgroups G_i, then $\lim_{i\in I} b_0^\psi(G_i) = b_0^\psi(G)$.
(c) $b_0^\psi(G) \leq \frac{1}{|K|}$ for every ψ-regular subgroup $K \leq G$.

Remark 1.4. It is well-known that $b_0^{(2)}(G) = \frac{1}{|G|}$; see [9, Thm. 1.35 (8)].

Proof. Let S be a generating set for G. We consider the initial segment of the associated free resolution of $\mathbb{C}$:

$$\mathbb{C}[G]^S \xrightarrow{\partial_1} \mathbb{C}[G] \longrightarrow \mathbb{C}.$$

The image of ∂_1 is the augmentation ideal. We take the tensor product with $\ell^\psi(G)$ and deduce that

$$b_0^\psi(G) = 1 - \dim_\psi(\overline{J_G}).$$

where $\overline{J_G}$ denotes the closure of the image of the augmentation ideal in $\ell^\psi(G)$. Let $P\colon \ell^\psi(G) \to \overline{J_G}$ denote the orthogonal projection. By definition

$$\dim_\psi(\overline{J_G}) = \langle P(1), 1\rangle_\psi$$

and assertion (a) follows from Lemma 1.1. Let $G = \bigcup G_i$ be a direct union of subgroups, then $J_G = \bigcup J_{G_i}$ and (b) follows immediately from (a).

Let $K \leq G$ be a ψ-regular subgroup. Let $T \subseteq K \setminus \{1\}$ be a finite subset. Then

$$w = |T| \cdot 1_G - \sum_{k\in T} k \in J_G.$$

Since K is ψ-regular, the elements of K are orthonormal and we deduce

$$\frac{|\langle w, 1\rangle_\psi|^2}{\langle w, w\rangle_\psi} = \frac{|T|^2}{|T|^2 + |T|} = \frac{|T|}{|T| + 1}.$$

Now (a) implies $b_0^\psi(K) \leq 1 - \frac{|T|}{|T|+1} = \frac{1}{|T|+1}$. Statement (c) follows by taking $T = K \setminus \{1\}$ if K is finite respectively letting $|T|$ tend to ∞ otherwise. $\square$

We would like to apply the same ideas to the first ψ-Betti number $b_1^\psi(G)$. However, up to now we only have a definition of $b_1^\psi(G)$ for all finitely generated groups G. We also require a definition for groups which are not finitely generated. This could be done using Lück's generalized dimension function (discussed in [9, §6.1, 6.2]), but this is not convenient for our purposes and for simplicity we work with the following variation.

Definition 1.5. Let G be a group and let $\psi \in \mathrm{Ch}(G)$. Then

$$\bar{b}_1^\psi(G) := \liminf_{H \leq G} b_1^\psi(H)$$

where the limit is taken over the directed system of all finitely generated subgroups $H \leq G$.

Remark 1.6. For a finitely generated group $\bar{b}_1^\psi(G) = b_1^\psi(G)$. In general however, $\bar{b}_1^\psi(G)$ can be strictly larger than the properly defined value of the first ψ-Betti number. It is easy to see this for the ordinary Betti numbers. For instance, it follows from the methods developed in [5] that $\langle (x_i)_{i\in\mathbb{Z}} \mid x_i x_{i+1} x_i^{-1} = x_{i+1}^2 \rangle$, is a perfect and locally indicable group, i.e., the ordinary rational Betti number of every finitely generated subgroup is ≥ 1.

For the classical ℓ^2-Betti number the inequality $b_1^{(2)}(G) \leq \bar{b}_1^{(2)}(G)$ follows from the argument given in the proof of [9, Theorem 7.2 (3)].

For later reference we state the following observation.

Lemma 1.7. *Let G be a group and let $\psi \in \mathrm{Ch}(G)$. If $G = \bigcup_{i\in I} G_i$ is a directed union of subgroups G_i, then*

$$\bar{b}_1^\psi(G) \leq \liminf_{i\in I} \bar{b}_1^\psi(G_i)$$

Proof. Let $\varepsilon > 0$. There is a finitely generated subgroup $H_0 \leq G$ such that $b_1^\psi(H) \geq \bar{b}_1^\psi(G) - \varepsilon$ for all finitely generated subgroups H that contain H_0. Since H_0 is finitely generated, there is $i \in I$ such that $H_0 \subseteq G_i$. Thus for all $j \geq i$ we have $\bar{b}_1^\psi(G_j) \geq \bar{b}_1^\psi(G) - \varepsilon$. □

Assume that G is finitely generated and that S is a finite generating set. The Cayley graph $\mathrm{Cay}(G,S)$ is the directed graph with vertex set G and edges

$$E_{G,S} = \{(g, gs) \mid g \in G, s \in S\}.$$

The edge $(1_G, s)$ will be denoted by $\bar{s}$. The Cayley graph is equipped with a left action of G. Let $\mathbb{C}[E_{G,S}]$ be the vector space with basis $E_{G,S}$ and let $\partial\colon \mathbb{C}[E_{G,S}] \to \mathbb{C}[G]$ denote the boundary map. A *finite cycle* in $\mathrm{Cay}(G,S)$ is an element $z \in \mathbb{C}[E_{G,S}]$ with $\partial(z) = 0$. Let $Z_{G,S}$ denote the space of finite cycles. If $\psi \in \mathrm{Ch}(G)$ is a character, then the semi-definite inner product $\langle \cdot, \cdot \rangle_\psi$ extends to a G-invariant semi-definite inner product on $\mathbb{C}[E_{G,S}]$ such that the edges $\{\bar{s} \mid s \in S\}$ are orthonormal; this means

$$\langle (g, gs), (h, ht) \rangle_\psi = \begin{cases} 0 & \text{if } s \neq t \\ \psi(h^{-1}g) & \text{if } s = t \end{cases}$$

The following extends [11, Prop. 2] and is our main tool.

Lemma 1.8. *Let G be a group and let S be a finite generating set. Let $\psi \in \mathrm{Ch}(G)$ be a character. Then*

$$b_1^{\psi}(G) = |S| - 1 + b_0^{\psi}(G) - \sum_{s\in S} \sup_{z\in Z_{G,S}} \frac{|\langle z, \bar{s}\rangle_\psi|^2}{\langle z, z\rangle_\psi}$$

where the suprema are taken over all non-zero elements of $Z_{G,S}$.

Proof. The finite generating set S provides us with a presentation $G \cong F/R$ of G where F is the free group over S and R is the subgroup of relations. We consider the initial segment of the associated free resolution of $\mathbb{C}$:

$$\mathbb{C}[G]^R \xrightarrow{\partial_2} \mathbb{C}[G]^S \xrightarrow{\partial_1} \mathbb{C}[G] \longrightarrow \mathbb{C}.$$

Tensoring with $\ell^{\psi}(G)$ gives

$$\ell^{\psi}(G)^R \xrightarrow{\partial_2} \ell^{\psi}(G)^S \xrightarrow{\partial_1} \ell^{\psi}(G).$$

The middle term is naturally isomorphic to the completion of $\mathbb{C}[E_{G,S}]$ with respect to $\langle\cdot,\cdot\rangle_\psi$. The image of ∂_2 is the closure of $Z_{G,S}$. The ψ-dimension of the closure of the image of ∂_1 is $1 - b_0^{\psi}(G)$. We deduce that

$$b_1^{\psi}(G) = |S| - (1 - b_0^{\psi}(G)) - \dim_\psi(\overline{Z_{G,S}}).$$

Let $P\colon \ell^{\psi}(G)^S \to \overline{Z_{G,S}}$ denote the orthogonal projection. By definition

$$\dim_\psi(\overline{Z_{G,S}}) = \sum_{s\in S} \langle P\bar{s}, \bar{s}\rangle_\psi$$

and the result follows from Lemma 1.1 □

Remark 1.9. It seems surprising that the value on the right hand side is independent from the chosen set of generators. This is a consequence of the homotopy invariance of the ψ-Betti numbers, which can be proven using the standard argument; e.g. [7, Thm. 3.18] or [9].

2. Torsion groups

In view of Remark 1.6, the following result implies Theorem 0.1.

Theorem 2.1. *Let p be a prime. Let G be a torsion group of exponent p. Then $\bar{b}_1^{(2)}(G) \leq 2p - 2$.*

Proof. We may assume that G is infinite (and $b_0^{(2)} = 0$), otherwise $b_1^{(2)}(G) = 0$ and there is nothing to show. By the definition of $\bar{b}_1^{(2)}(G)$ (see Remark 1.6), we may assume that G is finitely generated.

We choose a minimal generating set S of G and denote the number of elements by $N = |S|$. Since all elements of G have prime order, all pairwise distinct elements $a, b, c \in S$ satisfy

$$\langle ac\rangle \cap \langle ab\rangle = \{1\} \tag{1}$$

Suppose for a contradiction that there are three distinct elements $a, b, c \in S$ with $\langle ac\rangle \cap \langle ab\rangle \neq \{1\}$, then these cyclic groups of prime order coincide and

$$ac = (ab)^k$$

for some $k \in \mathbb{N}$, i.e., $c \in \langle a, b\rangle$ which contradicts the minimality of S.

For all $a \in S$ we have $N-1$ relations

$$(ab)^p$$

of length $2p$ for all $b \neq a$ in S. By condition (1), the only common edge in the Cayley graph is the first edge $\bar{a}$ from 1 to a. Summing up these cycles, we obtain a cycle z_a in $\mathrm{Cay}(G,S)$ with

$$\frac{\langle z_a, \bar{a}\rangle^2}{\langle z_a, z_a\rangle} = \frac{(N-1)^2}{(N-1)^2 + (N-1)(2p-1)} = \frac{1}{1+\frac{2p-1}{N-1}}.$$

We deduce from Lemma 1.8 that

$$\begin{aligned} b_1^{(2)}(G) &\leq N - 1 - \sum_{a\in S} \frac{\langle z_a, \bar{a}\rangle^2}{\langle z_a, z_a\rangle} = N - 1 - \frac{N}{1+\frac{2p-1}{N-1}} \\ &= \frac{2p-2}{1+\frac{2p-1}{N-1}} \leqslant 2p-2. \end{aligned}$$

□

Theorem 2.2. *Let p be a prime number and let G be a countable torsion group of exponent p. If G has an infinite normal subgroup N of infinite index, then*

$$b_1^{(2)}(G) = 0.$$

Proof. By Theorem 2.1 and Remark 1.6 we have $b_1^{(2)}(N) \leq \bar{b}_1^{(2)}(N) \leq 2p-2$. By Gaboriau's Theorem [4, Thm. 6.8], this implies $b_1^{(2)}(G) = 0$. □

Proof of Corollary 0.2. Recall that $B(m,p)$ denotes the Burnside group of exponent p and rank m. Since $B(1,p)$ is finite, we have $b_1^{(2)}(B(1,p)) = 0$.

Assume $m \geq 2$. For sufficiently large p, the main result of [6] implies that $B(m,p)$ contains a Q-subgroup H which is isomorphic to $B(\infty,p)$. A Q-subgroup has the property that the normal closure $\langle K\rangle^{B(m,p)}$ in $B(m,p)$ of any normal subgroup $K \trianglelefteq H$ intersects H exactly in K.

Take a projection from $B(\infty,p)$ onto $B(\infty,p)$ with an infinite kernel K. Then the normal closure $\langle K\rangle^{B(m,p)}$ is an infinite normal subgroup of $B(m,p)$ of infinite index. Now Theorem 2.2 implies the result. □

Remark 2.3.
(1) Ivanov [6] quantifies sufficiently large as $p > 10^{78}$.
(2) One can also deduce $b_1^{(2)}(B(m,p)) = 0$ for $m \geq 3$ under the assumption that $B(2,p)$ is infinite[1] using the normal subgroup $N = \ker(B(m,p) \to B(m-1,p))$.

    Indeed, let $x_1, x_2, \ldots, x_m$ be a free generating set of $B(m,p)$ such that N is the normal closure of x_1 in $B(m,p)$. Since $\langle x_1, x_2\rangle \subseteq N\langle x_2\rangle$ and $\langle x_1, x_2\rangle \cong B(p,2)$ is infinite, we deduce that N is infinite. Moreover, N has infinite index, since $B(m,p)/N \cong B(m-1,p)$.
(3) We expect that $b_1^{(2)}(B(m,p)) = 0$ for all p, m. On the other hand, if $b_1^{(2)}(B(m,p)) > 0$ holds for some m and p, then this offers a simple solution to the restricted Burnside problem for m-generated p-groups.

[1] According to Adian [2] the Burnside groups $B(2,p)$ are infinite for all $p > 100$.

More precisely, every finite index normal subgroup $N \trianglelefteq B(m,p)$ satisfies

$$|B(m,p) : N| \cdot b_1^{(2)}(B(m,p)) = b_1^{(2)}(N) \leq 2p - 2$$

by Theorem 2.1 and this inequality imposes an upper bound on the index of N.

3. q-NORMALITY AND APPLICATIONS

Lemma 3.1. *Let $G = \langle H, a \rangle$ be a group and let $\psi \in \mathrm{Ch}(G)$ be a character. Assume that $aHa^{-1} \cap H$ contains a ψ-regular subgroup of order $n \in \mathbb{N} \cup \{\infty\}$. Then*

$$\bar{b}_1^{\psi}(G) - b_0^{\psi}(G) \leqslant \bar{b}_1^{\psi}(H) + \frac{3 + 2\mathrm{Re}(\psi(a))}{n + 2 + 2\mathrm{Re}(\psi(a))}$$

In particular

$$\bar{b}_1^{\psi}(G) \leqslant \bar{b}_1^{\psi}(H).$$

for $n = \infty$.

Proof. Without loss of generality we assume that $a \notin H$. We denote the ψ-regular subgroup of order n in $H \cap aHa^{-1}$ by K. For a finite subset $S \subseteq H$, we define $H_S = \langle S \rangle$. Since K is ψ-regular, we obtain $b_0^{\psi}(H) \leqslant b_0^{\psi}(G) \leqslant \frac{1}{n}$ and $b_0^{\psi}(H_S) \leq |K \cap H_S|^{-1}$ by Remark 1.9.

For $n = \infty$, let $S \subseteq H$ be any finite subset and denote by $h_1, h_2, \ldots, h_k$ the pairwise distinct elements of $S \cap aSa^{-1} \cap K$. If $n < \infty$, we choose $S \subseteq H$ such that

$$S \cap aSa^{-1} \;=\; K \setminus \{1\} \;=\; \{h_1, h_2, \ldots, h_k\}.$$

In both situations we define $S' = S \cup \{a\}$ and $G' = \langle S' \rangle$. Lemma 1.8 implies

$$\begin{aligned} b_1^{\psi}(G') - b_0^{\psi}(G') &= |S'| - 1 - \sum_{s \in S'} \sup_{z \in Z_{G',S'}} \frac{|\langle z, \bar{s} \rangle_{\psi}|^2}{\langle z, z \rangle_{\psi}} \\ &\leq |S| + 1 - 1 - \sum_{s \in S} \sup_{z \in Z_{H_S,S}} \frac{|\langle z, \bar{s} \rangle_{\psi}|^2}{\langle z, z \rangle_{\psi}} - \sup_{z \in Z_{G',S'}} \frac{|\langle z, \bar{a} \rangle_{\psi}|^2}{\langle z, z \rangle_{\psi}} \\ &\leq b_1^{(2)}(H_S) + 1 - \sup_{z \in Z_{G,S'}} \frac{|\langle z, \bar{a} \rangle_{\psi}|^2}{\langle z, z \rangle_{\psi}}. \end{aligned} \tag{2}$$

To obtain a lower bound for $\sup_{z \in Z_{G,S'}} \frac{|\langle z, \bar{a} \rangle_{\psi}|^2}{\langle z, z \rangle_{\psi}}$, we consider the Cayley graph $\mathrm{Cay}(G', S')$ of G' and exhibit a suitable cycle z. Each relation $ah_ia^{-1}(ah_ia^{-1})^{-1}$ provides a cycle z_i of length 4 in $\mathrm{Cay}(G', S')$, i.e.,

$$z_i = \underbrace{(1, a)}_{=\bar{a}} + (a, ah_i) - (ah_ia^{-1}, ah_i) - (1, ah_ia^{-1}).$$

Note that the cycles z_i touch exactly four vertices, since $h_i \neq 1$ and $a \notin H$. In additon, the cycles z_i have no common edges, except for $\bar{a}$. We define $z = \sum_{i=1}^{k} z_i$ to be the sum of these cycles. Since $\psi(ah_ia^{-1}) = \psi(h_i) = 0$ holds for all $i \leq k$, we deduce

$$\langle z, \bar{a} \rangle_{\psi} = k \langle \bar{a}, \bar{a} \rangle_{\psi} - \sum_{i=1}^{k} \langle (ah_ia^{-1}, ah_i), \bar{a} \rangle_{\psi} = k - \sum_{i=1}^{k} \psi(ah_ia^{-1}) = k$$

We note further (using again that K is ψ-regular) that for given $i, j \leq k$ the edges $\neq \bar{a}$ in z_i, z_j are orthogonal unless $h_i = ah_ja^{-1}$ or $ah_ia^{-1} = h_j$. Each of these cases occurs at most once for every i and then $\langle (a, ah_i), (1, ah_ja^{-1})\rangle_\psi = \psi(a)$ and $\langle (1, ah_ia^{-1}), (a, ah_j)\rangle_\psi = \overline{\psi(a)}$ respectively. We deduce

$$\langle z, z\rangle_\psi = \sum_{i,j} \langle z_i, z_j\rangle_\psi \leq \underbrace{4k}_{i=j} + \underbrace{k^2 - k + k(\psi(a) + \overline{\psi(a)})}_{i\neq j}$$

$$\leq k^2 + (3 + 2\mathrm{Re}(\psi(a)))k \tag{3}$$

and conclude

$$\frac{\langle z, \bar{a}\rangle_\psi^2}{\langle z, z\rangle_\psi} \geq \frac{1}{1 + \frac{3+2\mathrm{Re}(\psi(a))}{k}}.$$

Finally, we use this cycle in combination with inequality (2) to obtain

$$\begin{aligned} b_1^\psi(G') - b_0^\psi(G') &\leqslant b_1^{(2)}(H_S) + 1 - \sup_{z \in Z_{G,S'}} \frac{|\langle z, \bar{a}\rangle_\psi|^2}{\langle z, z\rangle_\psi} \\ &\leqslant b_1^{(2)}(H_S) + 1 - \frac{1}{1 + \frac{3+2\mathrm{Re}(\psi(a))}{k}} \\ &= b_1^{(2)}(H_S) + \frac{3 + 2\mathrm{Re}(\psi(a))}{k + 3 + 2\mathrm{Re}(\psi(a))} \quad \underset{k\to\infty}{\longrightarrow} \quad b_1^{(2)}(H_S) \end{aligned}$$

For $n = \infty$ we can make k arbitrary large. In the case $n < \infty$ we have $k = n - 1$ by construction. We note that every finitely generated subgroup of G is contained in a group of the form G'. The result follows from Lemma 1.7 and Lemma 1.3 (b).

If $n = \infty$, then G contains an infinite ψ-regular subgroup and $b_0^\psi(G) = 0$ by Lemma 1.3 (c). □

In the spirit of Popa [12] and Thom-Peterson [10] we introduce the following notion.

Definition 3.2. Let G be a group and let $\psi \in \mathrm{Ch}(G)$. A subgroup $H \leq G$ is *q-ψ-normal*, if there is a set $A \subseteq G$ such that $G = \langle H \cup A\rangle$ and $H \cap aHa^{-1}$ contains an infinite ψ-regular subgroup for all $a \in A$.

A subgroup $H \leq G$ is *weakly* q-ψ-normal, if there is an ordinal number α and an increasing chain of subgroups $H_0 = H$ to $H_\alpha = G$ such that $\bigcup_{\beta<\gamma} H_\beta$ is q-ψ-normal in H_γ for all $\gamma \leq \alpha$.

Based on Lemma 3.1 we obtain the following analog of [10, Theorem 5.6].

Corollary 3.3. *Let G be a group and let $\psi \in \mathrm{Ch}(G)$. If $H \leq G$ is a weakly q-ψ-normal subgroup, then*

$$\bar{b}_1^\psi(G) \leq \bar{b}_1^\psi(H).$$

Proof. Assume that H is q-ψ-normal. Then $G = \langle H \cup A\rangle$ and $H \cap aHa^{-1}$ contains an infinite ψ-regular subgroup for all $a \in A$. If A is finite, then the assertion follows inductively from Lemma 3.1. Assume that A is infinite. For every finite subset $B \subseteq A$, we define $G_B = \langle H \cup B\rangle$. Then $G = \bigcup_{B\subseteq A} G_B$ and Lemma 1.7 implies

$$\bar{b}_1^\psi(G) \leq \liminf_{B\subseteq A} \bar{b}_1^\psi(G_B) \leq \bar{b}_1^\psi(H).$$

The result for weakly q-ψ-normal subgroups follows by transfinite induction using Lemma 1.7. □

Corollary 3.4. *Let G be a group and let $\psi \in \mathrm{Ch}(G)$.*

(1) *If G is an HNN-extension of H with associated subgroups A, B and A contains an infinite ψ-regular subgroup, then we have $\bar{b}_1^{\psi}(G) \leq \bar{b}_1^{\psi}(H)$.*
(2) *If $G = A *_C B$ is an amalgamated product such that C is q-ψ-normal in B. Then we have $\bar{b}_1^{\psi}(G) \leq \bar{b}_1^{\psi}(A)$.*
(3) *If G contains an infinite normal amenable ψ-regular subgroup, then $b_1^{\psi}(G) = 0$.*

Proof. (1): follows immediately from Lemma 3.1.

(2): The assumptions imply that A is q-ψ-normal in $G = A *_C B$ and the assertion follows from Corollary 3.3.

(3): The infinite normal amenable subgroup N is q-ψ-normal in G and $b_1^{\psi}(N) = b_1^{(2)}(N) = 0$ by [3, Thm. 0.2]. □

We illustrate the helpfulness of q-normality with an application to right-angled groups. This notion was put forward in [1, Definition 1].

Definition 3.5 (right-angled groups)**.** A group G is *right-angled*, if it is the quotient of a right-angled Artin group A_Γ with a finite connected graph $\Gamma = (\mathcal{I}, \mathcal{E})$ such that the image of every generator σ_i $(i \in \mathcal{I})$ has infinite order in G.

The image of the generating set of A_Γ will be called a *right-angled set of generators*.

Theorem 3.6. *Let G be a right-angled group and let $\mathcal{S} = \{s_i \mid i \in \mathcal{I}\}$ be a right-angled set of generators. If $\psi \in \mathrm{Ch}(G)$ is such that the cyclic subgroup $\langle s_i \rangle$ is ψ-regular for every $i \in \mathcal{I}$. Then we have $b_1^{\psi}(G) = 0$.*

Proof. Our proof will be by induction over the number $n = |\mathcal{I}| \in \mathbb{N}$ of generators. For the base of induction we note that $b_1^{(2)}(\mathbb{Z}) = 0$. We assume for the induction step w. l. o. g. $\mathcal{S} = \{s_1, s_2, \ldots, s_n, s_{n+1}\}$ such that s_n commutes with s_{n+1} and such that $G' = \langle s_1 \ldots, s_n \rangle$ is a right-angled group with $b_1^{(2)}(G') = 0$. We claim that G' is q-ψ-normal. Indeed, set $H = \langle s_n \rangle$ and $a = s_{n+1}$, $a^{-1}G'a \cap G' \supseteq H$ and H is ψ-regular by assumption. Now the result follows from Lemma 3.1. □

Using this calculation and the approximation methods from [8] one can control the growth of Betti numbers in right-angled Artin groups with respect to normal chains with non-trivial intersection.

Corollary 3.7. *Let A_Γ be a right-angled Artin group for a finite connected graph Γ with generating set $\{\sigma_i \mid i \in \mathcal{I}\}$. Let $N_1 \trianglerighteq N_2 \trianglerighteq \ldots$ be a descending chain of finite index normal subgroups in A_Γ. If the order $\mathrm{ord}_{A_\Gamma/N_n}(\sigma_i)$ in the finite factors A_Γ/N_n is unbounded for each generator σ_i, then*

$$\lim_{n \to \infty} \frac{b_1(N_n)}{|A_\Gamma : N_n|} = 0.$$

Proof. Let ψ_n be the character of the permutation action of A_Γ on A_Γ/N_n. Since the sequence of normal subgroups is descending, the sequence ψ_n converges in $\mathrm{Ch}(A_\Gamma)$ to a character ψ. Since each character ψ_n is sofic and A_Γ is finitely presented, it follows from [8, Theorem 3.5] that

$$b_1^{\psi}(A_\Gamma) = \lim_{n\to\infty} b_1^{\psi_n}(A_\Gamma) = \lim_{n\to\infty} \frac{b_1(N_n)}{|A_\Gamma : N_n|}$$

If the order $\mathrm{ord}_{A_\Gamma/N_n}(\sigma_i)$ tends to infinity, $\psi_n(\sigma_i^k)$ vanishes for all $k \neq 0$ and all large n, i.e., $\langle\sigma_i\rangle$ is a ψ-regular subgroup. Theorem 3.6 implies that $b_1^{\psi}(A_\Gamma) = 0$ and this completes the proof. □

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Email address: `carsten.feldkamp@hhu.de`

Fakultät für Mathematik und Informatik, FernUniversität in Hagen, 58084 Hagen, Germany

Email address: `steffen.kionke@fernuni-hagen.de`