

THE DUISTERMAAT-HECKMAN FORMULA AND CHERN-SCHWARTZ-MACPHERSON CLASSES

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For Victor Guillemin, my friend, advisor, and inspiration

ABSTRACT. Let M be a smooth complex projective variety, bearing a Kähler symplectic form ω and a Hamiltonian action of a torus T , with finitely many fixed points M^T . One standard form of the Duistermaat-Heckman theorem gives a formula for M 's Duistermaat-Heckman measure $DH_T(M, \omega)$ as an alternating sum of projections of cones, with overall direction determined by a Morse decomposition of M .

Using Victor Ginzburg's construction of Chern-Schwartz-MacPherson classes, we show that these individual cone terms can themselves be interpreted as Duistermaat-Heckman measures of cycles in T^*M . (This has a similar goal to the symplectic cobordism approach of Viktor Ginzburg, Guillemin, and Karshon.) Our approach also suggests extensions of the formula, including the Brianchon-Gram theorem.

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1. THE DUISTERMAAT-HECKMAN FORMULA (APRÉS [GiLerS96])

Except for a minor twist, the material in this section is by now completely classical (no pun intended), and we include it largely to fix notation. The minor twist will be the inclusion of a cycle in our manifold. For more leisurely treatments we direct the reader to [GiLerS96, HKa12].

Given

- a complex projective manifold $M \subseteq \mathbb{P}V$ where V bears a fixed Hermitian form $\langle \cdot, \cdot \rangle$, hence
- a symplectic form ω on M , restricted from the Fubini-Study form on $\mathbb{P}V$,
- an algebraic cycle $C \subseteq M$, i.e. a formal \mathbb{Z} -linear combination $\sum_i n_i C_i$ of subvarieties $C_i \subseteq M$ of a fixed dimension, and
- an action $T \curvearrowright V$ of a compact torus T preserving each of M , C and $\langle \cdot, \cdot \rangle$, hence
- a homomorphism $\rho : T \rightarrow \mathcal{U}(V)$ and a moment map $\Phi : M \rightarrow \mathfrak{t}^*$ made by composing $M \hookrightarrow \mathbb{P}V \hookrightarrow \mathfrak{u}(V)^* \xrightarrow{\rho^*} \mathfrak{t}^*$, where $\mathfrak{u}(V)^*$, \mathfrak{t}^* are the duals of the Lie algebras,

we have three ways to think about the **Duistermaat-Heckman measure** $\text{DH}_T(C \subseteq M, \omega)$:

- (1) We can use ω to define a Liouville measure on the smooth part of each C_i , push those measures forward with Φ_* to \mathfrak{t}^* , and consider the sum of those measures, weighted by the coefficients (n_i) .
- (2) We can take the Fourier transform of $\int_M \exp(\tilde{\omega})[C]$, where $\tilde{\omega} = \omega - \Phi$ is the equivariant extension of the symplectic form (as in [ABo84]). Note that if C is a smooth subvariety (formally given coefficient 1), then $\int_M \exp(\tilde{\omega})[C] = \int_C \exp(\tilde{\omega}|_C)$ so $\text{DH}_T(C \subseteq M, \omega) = \text{DH}_T(C \subseteq C, \omega|_C)$.
- (3) (Following [BrPro90] or [GiLerS96, §3.4]) We can compute the weight multiplicity diagrams of $\bigoplus_i \Gamma(C_i; \mathcal{O}(d))^{n_i}$ on T 's weight lattice T^* , divide the multiplicities by $d^{\dim C}$ and the lattice spacing by d , and consider the limit of the measure (now on \mathfrak{t}^*) as $d \rightarrow \infty$.

In [ABo84] the first two ways are related (in the case $C = M$, but our generalization is an easy change) when M^T is finite, computing the integral

$$\int_M \exp(\tilde{\omega})[C] = \sum_{f \in M^T} \frac{\exp(\tilde{\omega})|_f [C]|_f}{\prod_{\lambda \in \text{wts}(T_f M)} \lambda} = \sum_{f \in M^T} \exp(-\Phi(f)) \frac{[C]|_f}{\prod_{\lambda \in \text{wts}(T_f M)} \lambda}$$

where $\text{wts}(T_f M)$ denotes the set of weights (with multiplicity) in the isotropy action of T on the tangent space $T_f M$, and $\alpha|_f$ denotes the pullback of a class α along the (T -equivariant) inclusion of the point f .

For general C of complex dimension d , we can (nonuniquely) write the restriction $[C]|_f$ of its equivariant cohomology class $[C] \in H_T^*(M)$ to the point f as

$$[C]|_f = \sum_{S \in \binom{\text{wts}(T_f M)}{d}} n_{f,S} \prod_{\lambda \in \text{wts}(T_f M) \setminus S} \lambda, \quad \text{for some } n_{f,S} \in \mathbb{Z}$$

where S runs over sub-multisets of $\text{wts}(T_f M)$ of size d . (Proof: pass to the formal neighborhood of f in M , then apply [KnM05, Lemma D], which is stated there over polynomial rings but applies without change to power series rings.)

It becomes now very tempting to Fourier transform the sum term by term. As is well-understood, defining the transform of such singular terms requires some choices in regularization, which we recapitulate in a moment. For now, we choose $\vec{v} \in \mathfrak{t}$ not perpendicular to any of the (finitely

many) weights in $\{T_f M : f \in M^T\}$, use it to define $\lambda_+ := \text{sign}(\langle \vec{v}, \lambda \rangle) \lambda$, and flip the signs on some of our $(n_{f,S})$ so that

$$\frac{[C]|_f}{\prod_{\lambda \in \text{wts}(T_f M)} \lambda} = \sum_{S \in (\text{wts}(T_f M)_d)} n_{f,S} \frac{1}{\prod_{\lambda \in S} \lambda_+}$$

and having now rewritten the “equivariant multiplicity” of $[C]$ like so, we have

$$\int_M \exp(\tilde{\omega}) [C] = \sum_{f \in M^T, S \in (\text{wts}(T_f M)_d)} n_{f,S} \frac{\exp(-\Phi(f))}{\prod_{\lambda \in S} \lambda_+}$$

1.1. Cone terms. Define a **cone term** associated to a weight $\mu \in T^*$ and a multiset $P = \{\lambda\}$ of weights in T^* as any multiple of

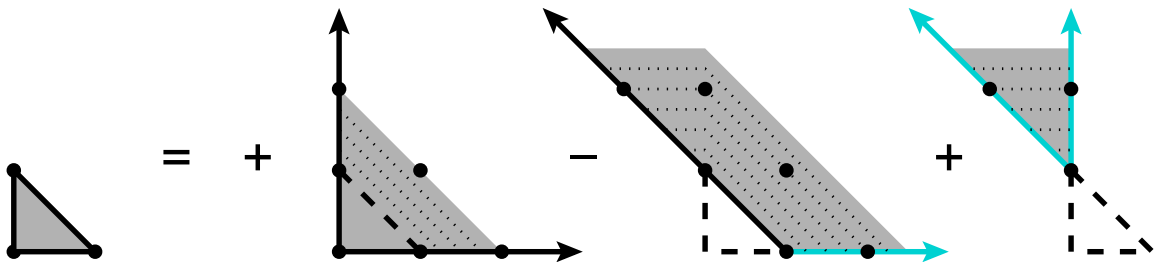
$$\text{cone}(\mu, \{\lambda\}) := \pi_*(\text{Lebesgue measure on } \mathbb{R}_{\geq 0}^P),$$

where

$$\begin{aligned} \pi : \mathbb{R}_{\geq 0}^P &\rightarrow \mathfrak{t}^* \\ (x_\lambda)_{\lambda \in P} &\mapsto \mu + \sum_{\lambda \in P} x_\lambda \lambda \end{aligned}$$

This measure is only locally finite on \mathfrak{t}^* if π is proper, or equivalently, if the vectors $\lambda \in P$ live in an open half-space in T^* . This, we will arrange by requiring $\langle \vec{v}, \lambda \rangle > 0$. At this point we *define* the Fourier transform of $\exp(-\Phi(f))/\prod_{\lambda \in S} \lambda_+$ to be $\text{cone}(\Phi(f), S)$. Summing these cone terms, we have arrived at **Heckman’s formula** (as it is called in [GiLerS96, §3.3]) for the Duistermaat-Heckman measure.

Example: $\mathbb{C}P^2$. Let $T^2 \hookrightarrow \mathbb{C}^3$ with weights $P = \{(0, 0), (1, 0), (0, 1)\}$, and let $\vec{v} = (1, 2)$. Let M and C be the projectivization $\mathbb{C}P^2$. Then $\text{DH}_T(\mathbb{C}P^2, \omega)$ is Lebesgue measure on the triangle with vertices P . The formula above computes it as pictured, where cyan edges indicate flipped edges, i.e. those for which $\lambda_+ = -\lambda$. (The dotted lines, which indicate a certain 3-dimensionality of the picture, will be explained later.)



1.2. Fourier equivalent measures. There is a clear subtlety – if we change \vec{v} , it changes the individual terms, so what precisely is guaranteeing that the final sum of measures is independent of \vec{v} ? (Other than the obvious: the measure was originally defined as a pushforward not depending on this choice.)

In what is to come the only measures we need consider on \mathfrak{t}^* will be linear combinations of cone terms. Call two cone terms $C \cdot \text{cone}(\mu, \lambda)$ and $D \cdot \text{cone}(\mu', \lambda')$ **Fourier equivalent** if $\mu = \mu'$, the two multisets $\{\lambda\}, \{\lambda'\}$ agree up to negating k many weights, and $C = (-1)^k D$. Call two measures Fourier equivalent if one can be obtained from the other by replacing terms in one with Fourier equivalent terms to obtain the other. We thank Terry Tao for pointing out the following lemma:

Lemma 1. *If f, g are Fourier equivalent linear combinations of cone terms, and there is a pointy cone P such that f, g are both supported with P , then $f = g$.*

Proof. Let ∂_λ denote the differencing operation $(\partial_\lambda f)(x) = f(x) - f(x + \lambda)$. If $\partial_\nu f = \partial_\nu g$ with $\nu \neq \vec{0}$, then f and g differ by a function h invariant under translation by ν . This h will also be supported inside P , hence $h = 0$. So we can simplify the equality to be checked by applying such operators, without losing information.

If we take any single cone term $\text{cone}(\mu, \{\lambda\})$ and apply $\prod_\lambda \partial_\lambda$, the result is the projection of a parallelepiped. At this point the two measures we are comparing are compactly supported, and are thus determined by their Fourier transforms. \square

Heckman's formula produces a measure supported in a translate of the cone spanned by $\{\lambda_+ : \lambda \text{ a weight in some } T_f M\}$, which is a proper cone (i.e. pointy) by the assumption on $\vec{\nu}$. Since the manifold M is compact its DH measure is compactly supported, hence is supported in some translate of any proper cone. At this point we apply the lemma.

In what is to come we will need noncompact extensions of (1-3), as to be found in [PraWu94, Th92]. Instead of compactness of M , one reduces to the case M connected, then asks that M^\top be compact, and finally that some component of M^\top be **attractive**, meaning that all of the isotropy weights in its normal bundle lie in an open half-space of T^* . The moment map is then proper, so (1) makes sense, and its image lies inside a proper cone, as was needed in the argument above. One uses the AB/BV localization theorem to make sense of (2), and although $\Gamma(C_i; \mathcal{O}(d))$ is infinite-dimensional its weight spaces are finite-dimensional, making sense of (3).

Finally, we will need to relax the nondegeneracy of the symplectic form, to a closed 2-form. This changes (1) in that the moment map is no longer determined up to translation (unless the form is generically nondegenerate) but is explicit extra data. It does not affect definition (2). For definition (3) one still needs $[\omega]$ to be the first Chern class of a holomorphic line bundle $\mathcal{O}(1)$, and the weight multiplicity diagram is now made using the entire Euler characteristic of the sheaf cohomology of $\mathcal{O}(d)$, rather than just H^0 the space of sections. In the rest of the paper we work primarily with definition (2), based off the AB/BV localization formula.

2. CHERN-SCHWARTZ-MACPHERSON CLASSES

We follow [Gi86] for our treatment of CSM classes, as derived from \mathcal{D} -modules.

2.1. An exact sequence of \mathcal{D} -modules. Let $\beta : B \hookrightarrow A$ be the closed inclusion of one smooth complex manifold into another, in codimension 1, defined by the vanishing of a function f . Then there is a short exact sequence of \mathcal{D}_A -modules

$$0 \rightarrow \mathcal{O}_A \xrightarrow{\cdot f} \mathcal{O}_{A \setminus B} \rightarrow \beta_*(\mathcal{O}_B) \rightarrow 0$$

where \mathcal{D}_A is the sheaf of differential operators on A , and β_* is the pushforward of \mathcal{D} -modules.

Our running example is very simple: $\{0\} \hookrightarrow \mathbb{C}$. Since \mathbb{C} is affine, instead of working with sheaves we can take global sections $\Gamma(\mathcal{D}_{\mathbb{C}}) \cong \mathbb{C}[\hat{x}, \frac{d}{dx}]$. Then the short exact sequence of $\mathbb{C}[\hat{x}, \frac{d}{dx}]$ -modules

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbb{C}[\hat{x}, \frac{d}{dx}]/\langle \frac{d}{dx} \rangle & \hookrightarrow & \mathbb{C}[\hat{x}, \frac{d}{dx}]/\langle \frac{d}{dx}\hat{x} \rangle & \twoheadrightarrow & \mathbb{C}[\hat{x}, \frac{d}{dx}]/\langle \hat{x} \rangle \rightarrow 0 \\ & & & & 1 & \mapsto & \hat{x} & & 1 & \mapsto & 1 \end{array}$$

define ODEs $\frac{d}{dx}f = 0$ $(\frac{d}{dx}\hat{x})f = 0$ $\hat{x}f = 0$

with solutions 1 x^{-1} δ ($\delta = \text{Dirac delta}$)

and those solutions can be identified with generators of our $\mathcal{D}_{\mathbb{C}}$ -modules:

$$0 \rightarrow \mathbb{C}[x] \hookrightarrow x^{-1}\mathbb{C}[x^{\pm}] \twoheadrightarrow \delta\mathbb{C}[\delta] \rightarrow 0$$

2.2. Characteristic cycles. The \mathcal{O}_A -algebra \mathcal{D}_A is generated by vector fields, sections of TA , which are used to build directional derivatives and therefore define (noncommuting) operators on \mathcal{O}_A . If we instead interpret sections of TA as fiberwise linear functions *on* T^*A , then they generate a different, commutative, algebra: the sheaf of (polynomial, not just linear) functions on T^*A . One can make the relation more precise: the degree of differential operators induces a filtration on \mathcal{D}_A , whose associated graded algebra $\text{gr } \mathcal{D}_A$ is \mathcal{O}_{T^*A} .

Given a \mathcal{D}_A -module \mathcal{F} , one could hope to filter it as well, compatibly with the \mathcal{D}_A filtration. At that point $\mathcal{O}_{T^*A} \circlearrowleft \text{gr } \mathcal{F}$, and we can consider its support cycle $\text{supp}(\text{gr } \mathcal{F}) \subseteq \mathcal{O}_{T^*A}$ (which will typically have multiplicities). It turns out (see e.g. [Bj13, definition 1.8.5]) that such “good filtrations” exist, not uniquely enough to canonically define the sheaf $\text{gr } \mathcal{F}$, but uniquely enough to well-define its support cycle. In the running example from §2.1, we get the following short exact sequence of modules over $\text{gr } \mathcal{D}_{\mathbb{C}} \cong \mathbb{C}[x, y]$:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbb{C}[x, y]/\langle y \rangle & \hookrightarrow & \mathbb{C}[x, y]/\langle xy \rangle & \twoheadrightarrow & \mathbb{C}[x, y]/\langle y \rangle \rightarrow 0 \\ & & & & 1 & \mapsto & x & & 1 & \mapsto & 1 \end{array}$$

The gradedness of $\text{gr } \mathcal{F}$ can be interpreted as its bearing a circle action. This has the specific consequence that $\text{supp}(\text{gr } \mathcal{F})$ is a **conical** cycle inside T^*A , meaning, invariant under the dilation action $\mathbb{C}^{\times} \circlearrowleft T^*A$ that scales the cotangent vectors. Consequently, we get a well-defined class

$$[\text{supp}(\text{gr } \mathcal{F})] \in H_{\mathbb{C}^{\times}}^*(T^*A) \cong H_{\mathbb{C}^{\times}}^*(A) \cong H^*(A) \otimes H_{\mathbb{C}^{\times}}^*(\text{pt}) \cong H^*(A)[\hbar]$$

in the dilation-equivariant cohomology of T^*A , taking \hbar as the generator of $H_{\mathbb{C}^{\times}}^*(\text{pt})$.

Perhaps the simplest example is $\mathcal{F} = \mathcal{O}_A$. Then $\text{supp}(\text{gr } \mathcal{F})$ is the zero section $A \subseteq T^*A$, and its associated class is a $(-\hbar)$ -homogenized version of the total Chern class of the tangent bundle of A .

In this paper the only (complexes of) D_M -modules we will need consider are of the form $R\iota_*(\mathcal{O}_A)$ for $\iota: A \hookrightarrow M$ the inclusion of a locally closed submanifold. Hereafter we write

$$\text{cc}(A \subseteq M) := \sum_i (-1)^i \text{supp}(\text{gr } R^i\iota_*(\mathcal{O}_A))$$

to denote the resulting “characteristic cycle”, a conical Lagrangian cycle inside T^*M defining an element $[\text{cc}(A \subseteq M)] \in H_{\mathbb{C}^{\times}}^*(T^*M) \cong H^*(M)[\hbar]$. In the running example the derived pushforward $R^i\iota_*$ vanishes for $i > 0$, but in other examples such as $\mathbb{C}^2 \setminus 0 \hookrightarrow \mathbb{C}^2$ (described in more detail later) one must include some such higher pushforwards.

In general $\text{cc}(A \subseteq M)$ is very complicated, with many components with various multiplicities. One component is the closure of the conormal bundle to A , and each other component is the closure of the conormal bundle to some locally closed submanifold $B \subseteq \overline{A} \setminus A$.

2.3. Chern-Schwartz-MacPherson classes and their additivity. Recall that a **constructible function** f on M is a finite linear combination (with \mathbb{Z} -coefficients, say) of characteristic functions of closed subvarieties. By splitting a subvariety into its regular and singular locus, we can instead think of f as a linear combination $\sum_{A \in \mathcal{A}} n_A 1_A$ of characteristic functions of locally closed algebraic submanifolds $A \subseteq M$. This expansion is not unique, however, so we need to treat it with care.

For example, write $1_{\mathbb{C}} = 1_{\mathbb{C}^\times} + 1_0$. To these three subsets we can associate $\mathcal{D}_{\mathbb{C}}$ -modules as computed in our running example, and the characteristic cycles $\{y = 0\}$, $\{xy = 0\}$, $\{x = 0\}$ respectively. The exactness of the sequence from §2.1 leads to a vanishing of its Euler characteristic, as an equation on cycles:

$$\text{cc}(\mathbb{C} \subseteq \mathbb{C}) \quad - \quad \text{cc}(\mathbb{C}^\times \subseteq \mathbb{C}) \quad + \quad \text{cc}(\{0\} \subseteq \mathbb{C}) \quad = \quad 0$$

Pictorially: $-$ minus $+$ plus | = 0

This alternating-sum statement doesn't quite match $1_{\mathbb{C}} - 1_{\mathbb{C}^\times} - 1_0 = 0$. To fix this mismatch, for $\iota_* : A \subseteq M$ a locally closed submanifold, we define its **Chern-Schwartz-MacPherson class** $\text{csm}(1_A)$ as

$$\text{csm}(1_A) := (-1)^{\text{codim}_M A} [\text{cc}(A \subseteq M)] \in H_{\mathbb{C}^\times}^*(T^*M).$$

With the signs integrated into the definition, it is then a theorem that this definition on $\{1_A\}$ extends in a well-defined way to constructible functions, at which point it is additive. (While it wasn't important in the running example, in bigger examples this additivity relies on cc having been defined using the derived pushforward.)

The traditional definition is slightly off from this – it lives in homology rather than cohomology, and is dehomogenized by setting $-\hbar$ to 1. We take this opportunity to rant about the horrific unnaturalness of considering inhomogeneous elements of cohomology, insofar as cohomology should so very often be understood as the associated graded to K -theory. In an associated graded space, only homogeneous elements can properly be asked to possess lifts. While this concludes the rant, we will retain the powers of \hbar through the rest of this paper.

The Deligne-Grothendieck conjecture, proven by MacPherson [Ma74], characterized these CSM classes by a recurrence relation (a functoriality under proper pushforward) and a base case ($A = M$ smooth and proper). One philosophical reason to prefer the description from [Gi86] recapitulated here is its individual definition for each $A \subseteq M$, rather than reliance on a recurrence relation.

2.4. Weber's divisibility property. Taking $A \subseteq M$ as before, and a torus T acting on M preserving A , then we can use the same definition to associate an **equivariant CSM class** $\text{csm}(1_A) \in H_{T \times \mathbb{C}^\times}^*(T^*M)$ (a slightly different approach appears in [Oh06]).

Lemma 2. [We12, theorem 20] *If $p \in M^T$ is isolated, and $p \notin A$, then $\text{csm}(1_A)|_p \equiv 0 \pmod{\hbar}$.*

That requires a straightforward bit of translation from [We12], as CSM classes there are inhomogeneous.

3. THE MAIN THEOREM: THE GEOMETRY OF CONE TERMS

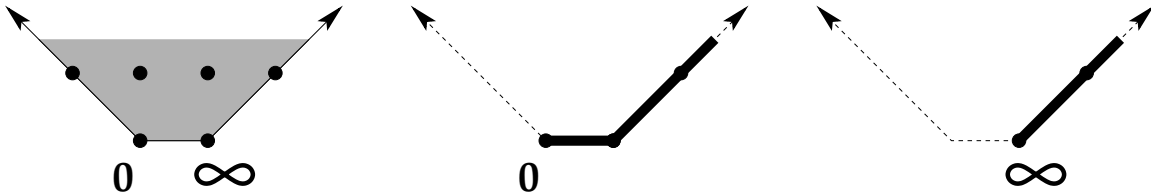
We are now ready to give an algebro-geometric interpretation of the individual cone terms in Heckman’s formula. This is in similar spirit to the approach of [GuOKaGi02], where these cone terms are interpreted as components of the (other end of the) boundary of a noncompact symplectic cobordism. It would be interesting to connect the two approaches, perhaps through the algebraic cobordism of [LevP09].

Our input is a complex projective symplectic manifold (M, ω) with an algebraic action of a torus T , and a Białynicki-Birula decomposition defined using a circle $S \hookrightarrow T$ (or equivalently, a Morse decomposition defined using a component of T ’s moment map). We assume M^T finite, and S generic enough that $M^S = M^T$, then write the decomposition into attracting sets as $M = \coprod_{p \in M^T} M_p^\circ$. Each inclusion $\iota^p : M_p^\circ \hookrightarrow M$ defines a Lagrangian cycle $cc(M_p^\circ \subseteq M)$ in T^*M , as in §2.

We give T^*M the degenerate 2-form $\omega_+ := \pi^*(\omega)$ where $M \xhookrightarrow{\iota} T^*M \xrightarrow{\pi} M$ are the inclusion and projection. This choice is dictated by wanting ω_+ to be *invariant*, not just a weight vector, under the dilation action $\mathbb{C}^\times \curvearrowright T^*M$ on the fibers, and wanting $\iota^*(\omega_+)$ to be ω . (There is a familiar nondegenerate 2-form “ $d\alpha$ ” available on T^*M , which one might be tempted to add to ω_+ . As that form is exact it wouldn’t affect our cohomology-based calculations, but it *would* spoil the dilation-invariance.) Now we compute:

$$\begin{aligned}
 & \text{DH}_{\mathbb{C}^\times \times T}(M \subseteq M, \omega) \\
 = & \text{DH}_{\mathbb{C}^\times \times T}(M \subseteq T^*M, \omega_+) && \text{since } \iota^*(\omega_+) = \omega \\
 = & \text{F.T.} \int_{T^*M} [M] \exp(\tilde{\omega}_+) = \text{F.T.} \int_{T^*M} \text{csm}(1_M) \exp(\tilde{\omega}_+) \\
 = & \text{F.T.} \int_{T^*M} \text{csm} \left(\sum_p 1_{M_p^\circ} \right) \exp(\tilde{\omega}_+) \\
 = & \sum_p \text{F.T.} \int_{T^*M} \text{csm} (1_{M_p^\circ}) \exp(\tilde{\omega}_+) && \text{by additivity of CSM classes} \\
 = & \sum_p \text{F.T.} \int_{T^*M} (-1)^{\text{codim}_M M_p^\circ} [cc(M_p^\circ \subseteq M)] \exp(\tilde{\omega}_+) && \text{as in §2.3} \\
 = & \sum_p (-1)^{\text{codim}_M M_p^\circ} \text{F.T.} \int_{T^*M} [cc(M_p^\circ \subseteq M)] \exp(\tilde{\omega}_+) \\
 = & \sum_p (-1)^{\text{codim}_M M_p^\circ} \text{DH}_{\mathbb{C}^\times \times T}(cc(M_p^\circ \subseteq M) \subseteq T^*M, \omega_+)
 \end{aligned}$$

Before comparing (not equating!) our terms $\text{DH}_{\mathbb{C}^\times \times T}(cc(M_p^\circ \subseteq M) \subseteq T^*M, \omega_+)$ to the cone terms in the Heckman formula, we consider the basic example $M = \mathbb{C}\mathbb{P}^1 = \mathbb{C} \coprod \{\infty\}$. Pictured below are the moment images of T^*M , $cc(\mathbb{C} \subseteq \mathbb{C}\mathbb{P}^1)$, $cc(\{\infty\} \subseteq \mathbb{C}\mathbb{P}^1)$ all with respect to the two-torus $\mathbb{C}^\times \times T$, where T is the maximal torus of $\text{PGL}_2(\mathbb{C})$ and \mathbb{C}^\times acts by dilation on the cotangent fibers.



Note that in this tiny example, T^*M is toric w.r.t. our augmented torus $\mathbb{C}^\times \times T$, but this will never happen in larger examples.

Even without the pictures, there is an obvious difference between the terms in this alternating sum vs. the ones in the Heckman formula: in *this* sum, the terms involve an extra \mathbb{C}^\times action,

dilating the fibers of the cotangent bundle. (Note too that the moment image of $cc(\mathbb{C} \subseteq \mathbb{CP}^1)$ is not a polytope, which can be blamed on the characteristic cycle being reducible.)

To drop that action, consider the inclusion $T \hookrightarrow S^1 \times T$ inducing $T^* \times \mathbb{Z} \rightarrow T^*$, which

- on the cohomology algebra level, amounts to setting $\hbar \rightarrow 0$, and
- on the moment polytope level, amounts to pushing forward the measure along the projection $\mathfrak{t}^* \times \mathbb{R} \rightarrow \mathfrak{t}^*$.

In the 2-dimensional pictures above, that amounts to projecting the measures to the horizontal line, from which to obtain the usual formula for $DH_T(\mathbb{CP}^1, \omega)$:

One might consider those 2-dimensional pictures above as “bent” versions of the half-lines that we really want, but that we only obtain after the dilation action is suppressed. Looking back at the \mathbb{CP}^2 example in §1, this extra action can be pictured by seeing the triangle as flat in the page, and the other regions as coming out of the page. The dotted lines indicate level sets in those 3-dimensional pictures.

We have now arrived at the main theorem, giving geometric interpretation to the individual terms in the Heckman formula: it is almost correct (and conjecturally correct) to say they are *themselves* DH measures, not of M but of the characteristic cycles $cc(M_p^\circ \subseteq M) \subseteq T^*M$.

Theorem 1. *Let $S \hookrightarrow T \curvearrowright (M, \omega)$, $M = \coprod_{p \in M^T} M_p^\circ$ be as described at the beginning of §3. Then $DH_{\mathbb{C}^\times \times T}(cc(M_p^\circ \subseteq M) \subseteq T^*M, \omega_+)$ is Fourier equivalent (and conjecturally equal) to a measure whose projection to \mathfrak{t}^* is proper, and that projection is p 's cone term from the Heckman formula.*

There are two subtleties in the theorem's statement. What we are really after could reasonably be called $DH_T(cc(M_p^\circ \subseteq M) \subseteq T^*M, \omega_+)$. One problem is that the T -moment map on T^*M isn't proper for $\dim M > 0$. We *believe* that its restriction to $cc(M_p^\circ \subseteq M)$ is proper, but (a) this has been frustratingly elusive and (b) that characteristic cycle is typically singular so we prefer to keep our integration definition on the manifold T^*M . (In §4.1 we study a slightly different situation where the T -moment map is *not* proper on the characteristic cycle.)

Proof. Let C denote the cycle $cc(M_p^\circ \subseteq M)$. We recall that it consists of the closure of the conormal bundle to M_p° union various other conormal varieties living over $\overline{M_p^\circ} \setminus M_p^\circ$. In particular

$$[C]_p = \prod_{\lambda \in T_p M} \begin{cases} \lambda & \text{if } \lambda \text{ defines a positive } S\text{-weight} \\ \hbar - \lambda & \text{if } \lambda \text{ defines a negative } S\text{-weight} \end{cases}$$

(no λ will define the S -weight 0, by our choice of S).

The Fourier transform of C 's DH measure on $\mathfrak{t}^* \times \mathbb{R}$ is $\sum_{f \in M^T} \exp(-\Phi(f)) \frac{[C]_f}{\prod_{\lambda \in \text{wts}(T_f M)} \lambda(\hbar - \lambda)}$. To associate a measure to it (which might only be Fourier equivalent to the actual DH measure), as in §1 we need to flip some weights in denominators. We make that choice using the generator of S 's Lie algebra. It is easy to see that the resulting measure has proper projection along the composite $\mathfrak{t}^* \times \mathbb{R} \rightarrow \mathfrak{t}^* \rightarrow \mathfrak{s}^*$, hence has proper projection to \mathfrak{t}^* .

To compute the projection to \mathfrak{t}^* , on the Fourier transform side, amounts to setting $\hbar \rightarrow 0$. Now we use lemma 2 to note that $[C]|_f \equiv 0 \pmod{\hbar}$ for $f \neq p$. Hence our sum reduces to a single term

$$\left(\exp(-\Phi(p)) \frac{[C]|_p}{\prod_{\lambda \in \text{wts}(T_p M)} \lambda(\hbar - \lambda)} \right) \Big|_{\hbar \rightarrow 0} = (-1)^{\text{codim}_M M_p^\circ} \exp(-\Phi(p)) \Big/ \prod_{\lambda \in \text{wts}(T_p M)} \lambda_+$$

which is exactly the term in the localization formula. \square

It is worth spelling out the interconnectedness of the different points of view in the case of the flag manifold, as in Heckman's thesis, which gives the asymptotic version of Kostant's multiplicity formula (see [GiLerS96, §3] for the connection). Our derivation is based on the $\mathcal{D}_{G/B}$ -modules associated to Bruhat cells; the global sections of these are the Verma modules. The exact sequence given in §2.1 for a single divisor, when extended to the full Bruhat decomposition, gives the BGG resolution involving those Verma modules (see [Ke77]). The complexity we meet here, with the "bending" of the individual cone terms, is closely related to the complexity (the non-simplicity) of Verma modules. (It is not *quite* the same complexity, as even a simple \mathcal{D} -module can have reducible characteristic cycle, a well-known example being that of Kashiwara-Saito.)

4. THE BRIANCHON-GRAM THEOREM AND OTHER EXTENSIONS

Let M be a smooth projective toric variety, with a moment polytope $P \subset \mathfrak{t}^*$. Instead of using a Morse decomposition, we consider the full decomposition $M = \bigcup_{F \subset P} M_F^\circ$ into T^C -orbits, one for each face of P . Then as in §2 we obtain

$$(1) \quad \text{DH}_{\mathbb{C}^\times \times T}(M \subseteq M, \omega) = \sum_F (-1)^{\text{codim}_P F} \text{DH}_{\mathbb{C}^\times \times T}(\text{cc}(M_F^\circ \subseteq M) \subseteq T^*M, \omega_+)$$

(though the codim in the exponent is now the *real* codimension). Then as in theorem 1, we project the measure from $\mathbb{R} \times \mathfrak{t}^*$ to \mathfrak{t}^* .

Theorem 2. *Let (M, ω) is a symplectic toric manifold with moment polytope $P \subseteq \mathfrak{t}^*$, let $M_F^\circ \subseteq M$ be the T -orbit corresponding to a face $F \subseteq P$, and let v be any vertex of F . Let $T_\mu F$ denote the (real) tangent space to an interior point μ of F , and $W \subset T^*$ denote the primitive integer vectors along the edges from v out of F .*

Then the pushforward to \mathfrak{t}^ of the measure $\text{DH}_{\mathbb{C}^\times \times T}(\text{cc}(M_F^\circ \subseteq M) \subseteq T^*M, \omega_+)$ is*

$$\pi_*(\text{Lebesgue measure on } T_\mu F \times \mathbb{R}_{\geq 0}^W), \quad \text{where} \quad \pi: T_\mu F \times \mathbb{R}_{\geq 0}^W \rightarrow \mathfrak{t}^*$$

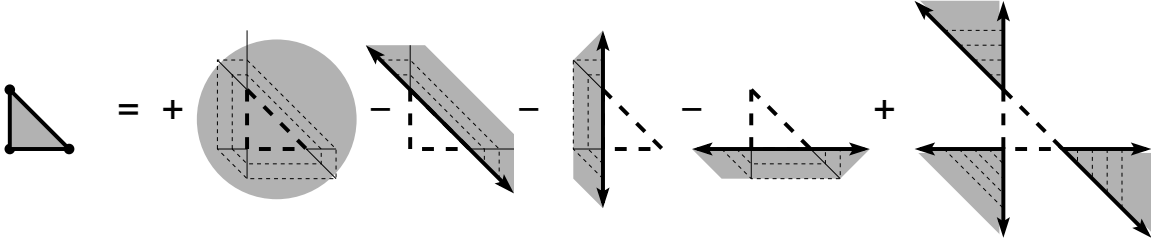
$$(\vec{v}, (x_\lambda)_{\lambda \in W}) \mapsto \mu + (\vec{v} - \sum_{\lambda \in W} x_\lambda \lambda)$$

where we normalize the measure on $T_\mu F \leq \mathfrak{t}^$ using its intersection with the lattice T^* .*

If we pushforward the LHS of equation (1) to \mathfrak{t}^ , we get Lebesgue measure on P , and the pushforward of the RHS gives the "Brianchon-Gram formula": an alternating sum over all F of the cone centered at F , with lineality space $T_\mu F$, and generators W as defined above.*

The Brianchon-Gram formula was given a Heckman-like derivation also in [HKa12], through a somewhat technical construction of a function with one critical point in the interior of each face.

Revisiting $\mathbb{C}P^2$. Before getting into the proof of theorem 2, we look again at the $\mathbb{C}P^2$ example from §1. There are seven orbits, where the open orbit gives the entire plane, the 1-dimensional orbits give half-planes, and the fixed points give sectors. (As before, we have attempted to indicate the \hbar direction out of the plane, using dashed level sets. The first characteristic cycle has seven components and the next three each have three.) We exhort the reader to check that the choices of vertices $v \in F$ are immaterial.



Proof of theorem 2. For this we use the decomposition of M into its $T^{\mathbb{C}}$ -orbits. For each such orbit $E^{\circ} \subset M$, with closure we call E , observe that $1_{E^{\circ}} = \sum_{F \subseteq E} (-1)^{\text{codim}_E F} 1_F$ where the sum is over smaller $T^{\mathbb{C}}$ -orbit closures. Then

$$\text{csm}(1_{E^{\circ}}) = \sum_{F \subseteq E} (-1)^{\text{codim}_E F} \text{csm}(1_F)$$

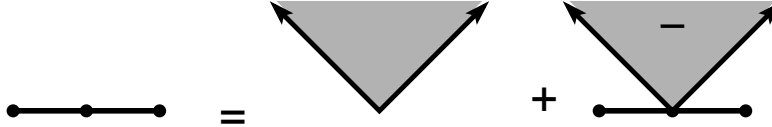
and since F is smooth, and closed in M , its characteristic cycle is just its conormal bundle $C_M F$. Consequently $[\text{cc}(1_{E^{\circ}})] = \sum_{F \subseteq E} [C_M F]$ (where the sign we had from inclusion-exclusion cancels with the one in Ginzburg’s formula for CSM classes).

To understand the DH measure associated to this sum, consider the (non-central) hyperplane arrangement defining the polytope P , and many other regions in \mathfrak{t}^* . Not every region touches P (unless P is a product of simplices – for a first cautionary example, consider a trapezoid), so we work in a small open neighborhood P_+ of P to avoid consideration of those other regions. Each hyperplane divides space into an “inside” (where P is) and an “outside”. The moment polytopes of the individual $C_M F$ in the sum, intersected with P_+ , are the exactly the regions that touch F and are on the outside of each of the hyperplanes through F . When we add them, we get the Brianchon-Gram term associated to F . \square

There is another theorem also called Brianchon-Gram, in which the cones point inward rather than outward, but the total is $(-1)^{\dim P}$ times Lebesgue measure on P . One can obtain that from this by scaling the symplectic form on M by -1 , which turns P inside out.

The additivity of CSM classes suggests that we wildly generalize to *any* T -invariant decomposition of M into locally closed submanifolds. We give an example now to demonstrate the dangers.

4.1. A problematic decomposition. Let T be one-dimensional this time, acting on V with weights $0, 1, 2$, and decompose $\mathbb{P}V$ into the projective point $[0, *, 0]$ and the open complement A . Then $\text{DH}_{T \times \mathbb{C}^{\times}}(\mathbb{C}P^2, \omega_+)$ is a piecewise-linear function times Lebesgue measure on the interval connecting $(0, 0)$ and $(2, 0)$. The inclusion $\iota : A \hookrightarrow \mathbb{C}P^2$ is perhaps already worrisome in that $R^1 \iota_* \neq 0$. Forging ahead, we calculate $\text{DH}_{T \times \mathbb{C}^{\times}}(\text{cc}(S \subseteq M) \subseteq T^*M, \omega_+)$ for $S \in \{[0, *, 0], A\}$ and obtain the following picture:



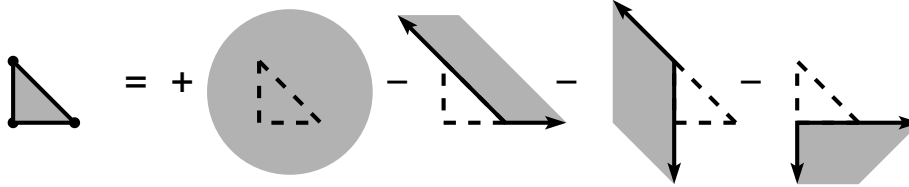
In the shaded regions we have $\pm \frac{1}{2}$ Lebesgue measure (where the “minus” comes from a contribution from the derived pushforward). There is now a serious impropriety if we try to forget the \mathbb{C}^\times action, projecting out the vertical direction.

This could be fixed by replacing these measures with Fourier equivalent ones pointing rightward. Our conjecture within theorem 1 is that in the case of BB decompositions (which does not include this example), that replacement is unnecessary.

A non-Morse decomposition. Consider the decomposition of $\mathbb{C}\mathbb{P}^2 = \{[x, y, z]\}$ into

$$\{xyz \neq 0\} \coprod \{x = 0, y \neq 0\} \coprod \{y = 0, z \neq 0\} \coprod \{z = 0, x \neq 0\}.$$

If we follow the proof of theorem 2, but use this decomposition, we get the following equality of measures



which of course one could obtain by partial cancelation of the Brianchon-Gram formula we drew after theorem 2.

We didn't here discuss nonabelian versions of our results, which hopefully would allow for a similar geometric interpretation of [Pa99]. It is worth pointing out that when $T \hookrightarrow G \curvearrowright (M, \omega)$, it is frequently possible that G 's moment map is proper even though T 's isn't (e.g. $T^1 \hookrightarrow \text{SU}(2) \curvearrowright \mathbb{C}^2$), so one can't obviously derive the nonabelian from the abelian.

We end with a question, another conjecture, and an example.

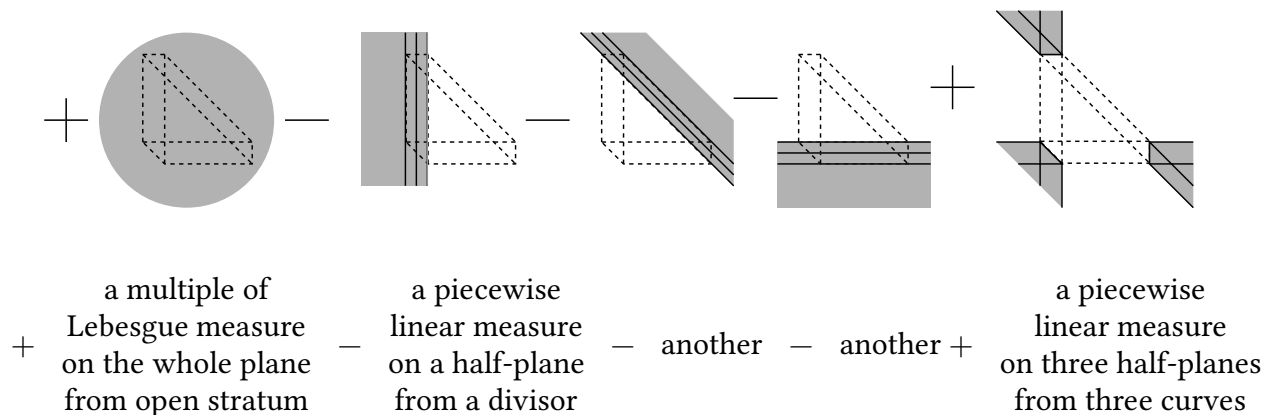
Q. Let M be a smooth complex projective variety M with a T -action, and $A \subseteq M$ a locally closed T -invariant smooth subvariety. What condition on A guarantees that the projection of $\text{supp}(\text{DH}_{T \times \mathbb{C}^\times}(\text{cc}(A) \subseteq T^*M, \omega_+))$ to \mathfrak{t}^* is proper?

There are two issues to be wary of – higher cohomology involved in defining $\text{cc}(A)$, and im-properness of A itself. The following is an attempt to deal with each of those:

Conjecture. Assume $\overline{A} \setminus A$ (the points added in the closure) supports an ample Cartier divisor in \overline{A} . Assume that there exists an open T -invariant subset $U \subseteq CA$ of the conormal bundle to A such that U/T is a proper scheme. Then the projection in the question above is proper.

We describe an example that would be covered by this conjecture. Consider the flag manifold $\text{GL}(3)/B$, its divisor $X^{231} := \overline{\text{Br}_1 \text{r}_2 B}/B$, and its rotations cX^{231}, c^2X^{231} where $c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

When we intersect those divisors pairwise, we get three \mathbb{P}^1 s. Each stratum (the open stratum, three $\mathbb{P}^1 \times \mathbb{G}_m$, three \mathbb{P}^1) has trivial normal bundle. The analogue of theorem 2 for this example, computing the usual piecewise-linear measure on the hexagon, looks as follows:



This gives a manifestly S_3 -invariant formula for the measure.

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