Double Inequalities for Complete Monotonicity Degrees of Remainders of Asymptotic Expansions of the Gamma and Digamma Functions

Mohamed Bouali

Abstract

Motivated by several conjectures posed in the paper " Completely monotonic degrees for a difference between the logarithmic and psi functions",we confirm in this work some conjectures on completely monotonic degrees of remainders of the asymptotic expansion of the logarithm of the gamma function and the digamma function and we give two bounded for this degrees.

1 Introduction

Completely monotonic functions have attracted the attention of many authors. Mathematicians have proved many interesting results on this topic. For example, Koumandos [8] obtained upper and lower polynomial bounds for the function $x/(e^x-1)$, $x>0$, with coefficients of the Bernoulli numbers B_k. This enabled him to give simpler proofs of some results of H. Alzer and F. Qi et al., concerning complete monotonicity of certain functions involving the functions $\Gamma(x)$, $\psi(x)$ and the polygamma functions $\psi^{(n)}(x)$, $n=1,2,....$, [5].

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I which alternate successively in sign, that is, $(-1)^n f^{(n)}(x) \geq 0$ for all $x \in I$ and all $n \in \mathbb{N}$. See for example [[25], Chap VIII], [[26], Chap I], and [[27], Chap IV].

A notion of completely monotonic degree was invented first in reference [7] and reviewed in the recent paper [20]. It can be used to measure and differentiate complete monotonicity more accurately, and it is also introduced in [7, 9, 10, 11, 12, 13, 16, 17, 18, 19] and closely related references.

Let $f(x)$ be a completely monotonic function on $(0,+\infty)$ and denote $f(+\infty) = \lim_{x\to+\infty} f(x) \geq 0$. When the function $x^r[f(x)-f(+\infty)]$ is completely monotonic on $(0,+\infty)$ if and only if $0 \leq r \leq \alpha$, the number α, denoted by $\deg_{cm}^{x}[f]$, is called the completely monotonic degree of $f(x)$ with respect to $x \in (0,+\infty)$. For more studies on complete monotonicity, the reader is also referred to [7, 9, 10, 13, 17, 18].

For $x > 0$, the classical gamma function $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ first introduced by L. Euler, is one of the most important functions in mathematical analysis. It often appears in asymptotic series, hypergeometric series, Riemann zeta function, number theory, and so on.

In [[2], Theorem 8], [[14], Theorem 2], and [28], the functions

$$R_n(x) = (-1)^n[\log\Gamma(x)-(x-\frac{1}{2})\log(x)+x+\frac{1}{2}\log(2\pi)-\sum_{k=1}^{n}\frac{B_{2k}}{2k(2k-1)}\frac{1}{x^{2k-1}},$$

for $n \geq 0$ were proved to be completely monotonic on $(0, +\infty)$, where an empty sum is understood to be 0 and the Bernoulli numbers B_n are define by the following series [21, 23, 24] by

$$\frac{z}{e^z-1} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty}\frac{b_{2n}}{(2n)!}z^{2n}, \qquad |z| < 2\pi.$$

Which implies that the functions $(-1)^m R_n^{(m)}$ for $m, n \geq 0$ are completely monotonic on $(0, +\infty)$. By the way, we call the function $(-1)^n R_n(x)$ for $n \geq 0$ the remainders of asymptotic formula of $\log\Gamma(x)$. See [[1], p. 257, 6.1.40] and [[15], p. 140, 5.11.1]. The completely monotonic degree of the function $R_n(x)$ for $n \geq 0$ with respect to x, $(0, +\infty)$ was proved in [[12], Theorem 2.1] to be at least n.

Stimulated by the above results and related ones, Qi conjectured in [31] that: the completely monotonic degrees of $R_n(x)$ for $n \geq 0$ with respect to x, $(0, +\infty)$ satisfy

$$\deg_{cm}^{x}(R_0) = 0, \qquad \deg_{cm}^{x}(R_1) = 1, \tag{1.1}$$

and

$$\deg_{cm}^{x}(R_n) = 2(n-1), \quad n \geq 2. \tag{1.2}$$

The completely monotonic degrees of $-R_n'(x)$ for $n \geq 0$ with respect to x,

$$\deg_{cm}^{x}(-R_0') = 1, \qquad \deg_{cm}^{x}(-R_1') = 2. \tag{1.3}$$

and

$$\deg_{cm}^{x}(-R_n') = 2n-1, \quad n \geq 2. \tag{1.4}$$

The completely monotonic degrees of $(-1)^m R_n^{(m)}(x)$ for $m \geq 2$ and $n \geq 0$ with respect to x,

$$\deg_{cm}^{x}((-1)^m R_0^{(m)}) = m-1, \quad \deg_{cm}^{x}((-1)^m R_1^{(m)}) = m, \tag{1.5}$$

and

$$\deg_{cm}^{x}((-1)^m R_n^{(m)}) = m + 2(n-1), \quad n \geq 2. \tag{1.6}$$

Proposition 1.1 *For all $n \geq 2$, the completely monotonic degree of the function $R_n(x)$ with respect to $x > 0$ satisfies:*

$$2(n-1) \leq \deg_{cm}^{x}(R_n) < 2n-1.$$

Proof. In [Qi and Mansour], it is proved that

$$-R'_{n+1}(x) = \int_0^\infty f_n(t)e^{-xt}dt,$$

where

$$f_n(t) = (-1)^n(\frac{1}{t} - \frac{1}{2}\coth\frac{t}{2} + \sum_{k=1}^{n+1}\frac{B_{2k}}{(2k)!}t^{2k-1}).$$

Moreover, R_{n+1} is a completely monotonic of degree at lest $n+1$, hence, $\lim_{x\to\infty} R_{n+1}(x) = 0$. Hence for all $x > 0$, We have,

$$R_{n+1}(x) = \int_0^\infty g_n(t)e^{-xt}dt,$$

where

$$g_n(t) = (-1)^n(\frac{1}{t^2} - \frac{1}{2t}\coth\frac{t}{2} + \sum_{k=1}^{n+1}\frac{B_{2k}}{(2k)!}t^{2k-2}).$$

Integrate by part yields

$$x^{2n}R_{n+1}(x) = \int_0^\infty (g_n(t))^{(2n)}e^{-xt}dt,$$

$$g_n^{(2n)}(t) = (-1)^n((\frac{1}{t^2} - \frac{1}{2t}\coth\frac{t}{2})^{(2n)} + \frac{B_{2n+2}}{(2n+2)(2n+1)}). \tag{1.7}$$

We use the Legendre integral formula. See for instance [4] (page 265) and [6] (page 92). For all $t > 0$,

$$\int_0^\infty \frac{\sin(xt)}{e^x - 1}dx = \frac{\pi}{2}\coth(\pi t) - \frac{1}{2t}.$$

Integrate by part yields,

$$\int_0^\infty \frac{\sin(xt)}{e^x-1}dx = [\frac{1}{2}\sin(xt)\log(1+e^{-2x}-2e^{-x})]_0^\infty - \frac{t}{2}\int_0^\infty \cos(xt)\log(1+e^{-2x}-2e^{-x})dx.$$

Hence,

$$\frac{1}{(2\pi)^2}\int_0^\infty \cos(\frac{xt}{2\pi})\log(1+e^{-2x}-2e^{-x})dx = \frac{1}{t^2} - \frac{1}{2t}\coth(\frac{t}{2}).$$

Applying the theorem of derivation under the integral sign, it follows that

$$g_n^{(2n)}(t) = \frac{1}{(2\pi)^{2n+2}}\int_0^\infty x^{2n}\cos(\frac{xt}{2\pi})\log(1+e^{-2x}-2e^{-x})dx + \frac{(-1)^n B_{2n+2}}{(2n+2)(2n+1)}.$$

Since,

$$\frac{1}{t^2}-\frac{1}{2t}\coth(\frac{t}{2})=-\sum_{k=0}^{\infty}\frac{B_{2k+2}}{(2k+2)!}t^{2k},\qquad |t|<2\pi.$$

We deduce that,

$$\frac{1}{(2\pi)^{2n+2}}\int_0^\infty x^{2n}\log(1+e^{-2x}-2e^{-x})dx=(-1)^{n+1}\frac{2^{2n}B_{2n+2}}{(2n+2)(2n+1)}.$$

Thus,

$$g_n^{(2n)}(t)=\frac{1}{(2\pi)^{2n+2}}\int_0^\infty x^{2n}\Big(\cos(\frac{xt}{2\pi})-1\Big)\Big(\log(1+e^{-2x}-2e^{-x})\Big)dx.$$

Let $\theta(x)=\log(1+e^{-2x}-2e^{-x})$. Then, $\theta'(x)=2(e^{-x}-e^{-2x})\geq 0$ and $\lim_{x\to\infty}\theta(x)=0$. then, $g_n^{(2n)}(t)>0$ for all $t>0$. Which implies that $\deg_{\text{cm}}^t(R_{n+1})\geq 2n$.

Assume $t^\alpha R_n(t)$ is completely monotonic. Then $\alpha\leq -\frac{tR_n'(t)}{R_n(t)}$ for all $t>0$. Since,

$$\frac{tR_n'(t)}{R_n(t)}=\frac{t(\psi(t)-\log t)-\frac{1}{2}+\sum_{k=1}^{n+1}\frac{B_{2k}}{2k}\frac{1}{t^{2k-1}}}{\log\Gamma(t)-(t-\frac{1}{2})\log(t)+t+\frac{1}{2}\log(2\pi)-\sum_{k=1}^{n+1}\frac{B_{2k}}{2k(2k-1)}\frac{1}{t^{2k-1}}},$$

Hence,

$$\lim_{t\to 0}\frac{-tR_n'(t)}{R_n(t)}=2n+1.$$

Furthermore,

$$x^{2n+1}R_{n+1}(x)=\int_0^\infty g_n^{(2n+1)}(t)e^{-xt}dt,$$

and

$$g_n^{(2n+1)}(t)=\frac{-1}{(2\pi)^{2n+3}}\int_0^\infty x^{2n+1}\sin(\frac{xt}{2\pi})\log(1+e^{-2x}-2e^{-x}))dx.$$

Hence, $x^{2n+1}R_{n+1}(x)$ is not completely monotonic. Then, $\deg_{\text{cm}}^t(R_{n+1})\in[2n,2n+1)$ for all n.

Proposition 1.2 *There is $m_0\in\mathbb{N}$ such that for $m\geq m_0$, the function $(-1)^m x^{m-1}R_0^{(m)}(x)$ is not completely monotonic.*

Proof. We have

$$R_0(x)=\log\Gamma(x)-(x-\frac{1}{2})\log x+x-\frac{1}{2}\log(2\pi).$$

then, for all $x>0$ and all $m\geq 2$,

$$(-1)^m R_0^{(m)}(x)=(-1)^m\psi^{(m-1)}(x)-\frac{(m-1)!}{2x^m}-\frac{(m-2)!}{x^{m-1}}.$$

$$(-1)^m R_0^{(m)}(x) = \int_0^\infty (\frac{t^{m-1}}{1-e^{-t}} - \frac{t^{m-1}}{2} - t^{m-2})e^{-xt}dt,$$

then, for all $m \geq 2$,

$$(-1)^m x^{m-1} R_0^{(m)}(x) = \int_0^\infty ((\frac{t^{m-1}}{1-e^{-t}})^{(m-1)} - \frac{(m-1)!}{2})e^{-xt}dt.$$

Set $g_m(t) = \left(\frac{t^{m-1}}{1-e^{-t}}\right)^{(m-1)} - \frac{(m-1)!}{2}$. Assume that $x^{m-1}\varphi_m$ is completely monotonic for all m, then, $g_m(t) \geq 0$ and $f_m(t) \geq \frac{(m-1)!}{2}$. Furthermore, it is proved by Alzer that for all $t > 0$, $\lim_{m\to\infty} \frac{1}{(m-1)!} f_m(t/(m-1)) = s(t)$, where,

$$s(t) = \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^\infty \frac{1}{k}\sin(\frac{t}{2k\pi}).$$

Which gives, $s(t) \geq \frac{1}{2}$ for all $t > 0$. From Theorem 2.1 (see ([3] p. 105)), we can derive that there is $a > 0$ such that $s(a) < 0$, Which gives a contradiction. The Bernstein-Widder theorem [26] implies $x^{m-1}\varphi_m(x)$ is not completely monotonic on $(0, \infty)$ for all $m \in \mathbb{N}$. This completes the proof.

Remark 1.3 *By the proposition above, one deduces that* $x^m(-1)^m R_0^{(m)}(x)$ *is not completely monotonic for all* $m \in \mathbb{N}$*.*

One shows that, $m \geq 80$, $f'_m\left(\sqrt{\frac{252}{(m+4)(m+3)}}\right) < 0$.

Proposition 1.4 *For* $m \geq 3$*, the completely monotonic degree of the function* $(-1)^m R_0^{(m)}(x)$ *with respect to* $x > 0$ *is not less that* $m-2$ *and less than* $m-1$*,*

$$m-2 \leq \deg_{cm}^x((-1)^m R_0^{(m)}(x)) < m-1$$

Proof. As above, we have

$$x^{m-2}\varphi_m(x) = \int_0^\infty ((\frac{t^{m-1}}{1-e^{-t}})^{(m-2)} - \frac{(m-1)!}{2}t - (m-2)!)e^{-xt}dt.$$

Moreover for $|x| < 2\pi$,

$$\frac{x^{m-1}}{1-e^{-x}} = x^{m-2} + \frac{x^{m-1}}{2} + \sum_{k=2}^\infty \frac{B_k}{k!}x^{k+m-2},$$

hence, the $m-2$ derivative of the function $\frac{x^{m-1}}{1-e^{-x}}$ at 0 is equal to $(m-2)!$. This implies,

$$\left(\frac{t^{m-1}}{1-e^{-t}}\right)^{(m-2)} - \frac{(m-1)!}{2}t - (m-2)! = \int_0^t (f_m(t) - \frac{(m-1)!}{2})dt. \quad (1.8)$$

To show that $x^{m-2}\varphi_m(x)$ is completely monotonic, it sufficient to prove that for all $t>0$,

$$\Big(\frac{t^{m-1}}{1-e^{-t}}\Big)^{(m-2)}-\frac{(m-1)!}{2}t-(m-2)!\geq 0.$$

We saw that

$$f_m(t)=(m-1)!+(m-1)!\sum_{k=1}^{\infty}e^{-kt}L_{m-1}(kt),$$

and by the fact that, $|L_m(x)|\leq e^{\frac{x}{2}}$ for all $x>0$. It then follows that

$$|\sum_{k=1}^{\infty}e^{-kt}L_{m-1}(kt)|\leq\frac{e^{-\frac{t}{2}}}{1-e^{-\frac{t}{2}}},$$

hence,

$$(m-1)!(1-\frac{e^{-\frac{t}{2}}}{1-e^{-\frac{t}{2}}})\leq f_m(t),$$

It is easy seeing that $1-\frac{e^{-\frac{t}{2}}}{1-e^{-\frac{t}{2}}}\geq\frac{1}{2}$ if and only if $t\geq 2\log 3$. Then, for all $t\geq 2\log 3\simeq 2.19$

$$f_m(t)-\frac{(m-1)!}{2}\geq 0.$$

Moreover, H. Alzer et al. [3] (p.113) showed that for all $t\in(-2\pi,2\pi)$,

$$f_m(t)=\int_0^{\infty}s(tu)u^{m-1}e^{-u}du,$$

where,

$$s(u)=\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin(\frac{u}{2k\pi}).$$

Let $t\in[0,2\pi)$, then,

$$\int_0^t(f_m(t)-\frac{(m-1)!}{2})dt=\int_0^t(\int_0^{\infty}(s(tu)-\frac{1}{2})u^{m-1}e^{-u}du)dt.$$

then,

$$\begin{aligned}\int_0^t(f_m(t)-\frac{(m-1)!}{2})dt&=\int_0^{\infty}\Big(\int_0^t\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin(\frac{tu}{2k\pi})\Big)dt)u^{m-1}e^{-u}du\\&=4\int_0^{\infty}\Big(\sum_{k=1}^{\infty}\sin^2(\frac{tu}{4k\pi})\Big)u^{m-2}e^{-u}du\geq 0.\end{aligned}$$

This complete the proof.

Proposition 1.5 *For $m \geq 1$, the completely monotonic degree of the function $(-1)^m R_1^{(m)}(x)$ with respect to $x > 0$ is not less that m and less than $m + 1$,*

$$m \leq \deg_{cm}^x((-1)^m R_1^{(m)}(x)) < m + 1$$

Proof. We saw that

$$-R_1(x) = \log\Gamma(x) - (x - \frac{1}{2})\log(x) + x + \frac{1}{2}\log(2\pi) - \frac{1}{12x},$$

So, for all $m \geq 1$,

$$(-1)^m R_1^{(m)}(x) = (-1)^{m+1}\psi^{(m-1)}(x) + \frac{(m-2)!}{x^{m-1}} + \frac{(m-1)!}{2x^m} + \frac{m!}{12x^{m+1}},$$

Hence,

$$(-1)^m R_1^{(m)}(x) = \int_0^\infty \Big(\frac{t^m}{12} + \frac{t^{m-1}}{2} + t^{m-2} - \frac{t^{m-1}}{1-e^{-t}}\Big)e^{-xt}dt.$$

It follows that

$$(-1)^m x^m R_1^{(m)}(x) = \int_0^\infty (\frac{m!}{12} - f'_m(t))e^{-xt}dt,$$

where $f_m(t) = \Big(\dfrac{t^{m-1}}{1-e^{-t}}\Big)^{(m-1)}$. It is known that for $|t| < 2\pi$,

$$f_m(t) = \int_0^\infty s(tu)u^{m-1}e^{-u}du,$$

then

$$f'_m(t) = \int_0^\infty s'(tu)u^m e^{-u}du.$$

Since, for all $x \in \mathbb{R}$, $s'(x) = \dfrac{1}{2\pi^2}\displaystyle\sum_{k=1}^\infty \frac{1}{k^2}\cos(\frac{x}{2k\pi})$. Then, $|s'(x)| \leq \frac{1}{12}$. Thus,

$$\frac{m!}{12} - f'_m(t) = \int_0^\infty (\frac{1}{12} - s'(tu))t^m e^{-t}dt \geq 0, \qquad \text{for all } |t| < 2\pi. \qquad (1.9)$$

On the other hand,

$$f_m(t) = (m-1)!\sum_{k=0}^\infty e^{-kt}L_m(kt),$$

then,

$$f'_m(t) = (m-1)!\sum_{k=0}^\infty ke^{-kt}(L'_m(kt) - L_m(kt)).$$

Using the relation, $tL'_m(t) = mL_m(t) - mL_{m-1}(t)$, then,

$$tf'_m(t) = (m-1)!\sum_{k=0}^{\infty} e^{-kt}(mL_m(kt) - mL_{m-1}(kt) - ktL_m(kt)).$$

By an inequality due to Szegö (see [14, p. 168]) we have $|L_m(t)| \leq e^{\frac{t}{2}}$ for $t \geq 0$, so that we obtain for $t > 0$,

$$tf'_m(t) \leq 2\,m!\frac{e^{-t/2}}{1-e^{-t/2}} - 2t\,(m-1)!\frac{d}{dt}(\frac{e^{-t/2}}{1-e^{-t/2}}),$$

This yields the following inequality

$$tf'_m(t) \leq 2\,m!\frac{e^{-t/2}}{1-e^{-t/2}} + \,(m-1)!\frac{t}{4\sinh^2(t/4)}, \tag{1.10}$$

which gives,

$$\frac{m!}{12} - f'_m(t) \geq (m-1)!(\frac{m}{12} - \frac{2me^{-t/2}}{t(1-e^{-t/2})} - \frac{1}{4\sinh^2(t/4)}),$$

Let $K(t,m) = \frac{m}{12} - \frac{2me^{-t/2}}{t(1-e^{-t/2})} - \frac{1}{4\sinh^2(t/4)}$. It easy to see that the function $K(t,m)$ increases on the variable m if and only if $\frac{1}{12} - \frac{2e^{-t/2}}{t(1-e^{-t/2})} \geq 0$, which is true for $t \geq 4$. Moreover,

$$K'(t,1) = \frac{-2+e^{t/2}(2+t)}{(-1+e^{t/2})^2t^2} + \frac{1}{8}\frac{\coth(t/4)}{\sinh^2(t/4)} \geq 0.$$

Then, for $t \geq 4$ and $m \geq 1$,

$$K(t,m) \geq K(t,1).$$

Furthermore, for $t \geq 6$ we have $K(t,1) \geq K(6,1) > 0.1$. Which implies that

$$K(t,m) \geq 0 \qquad \text{for all } t \geq 6, \text{ and } m \geq 1. \tag{1.11}$$

By equations (1.9) and (1.11), we get for all $t > 0$,

$$\frac{m!}{12} - f'_m(t) \geq 0.$$

Thus $(-1)^m x^m R_1^{(m)}(x)$ is completely monotonic.

Let $m \geq 1$, we have seen that $f'_m(t) \leq m!/12$ for all $t \geq 0$ and $f'_m(0) = m!/12$. Hence, $\lim_{t\to+\infty}(m!/2 - f'_m(t))e^{-xt} = \lim_{t\to 0}(m!/2 - f'_m(t))e^{-xt} = 0$. Integrate by part yields

$$(-1)^m x^{m+1} R_1^{(m)}(x) = -\int_0^{\infty} f''_m(t)e^{-xt}dt,$$

If for all $m \geq 1$ and all $t > 0$ $f''_m(t) \leq 0$. Then, by using equation (1.10), we have, $\lim_{t\to\infty} f'_m(t) = 0$, and $f'_m(t) \geq 0$ for all $t > 0$. Therefore

$$f_m(t) \geq f_m(0) = \frac{(m-1)!}{2}.$$

Using the fact that $\lim_{m\to\infty} \frac{1}{(m-1)!} f_m(t/(m-1)) = s(t)$ for all $t \in \mathbb{R}$. It follows that, $s(t) \geq 1/2$, and $H(t) \geq 0$ for all $t > 0$. Which contradicts the result of Alzer et al [3]. Which states that $H(x_{j_k}) < -C(\log\log x_{k_j})^{1/2}$, $C > 0$, for some positive sequence x_{j_k} going to infinity as $k \to +\infty$.

The Bernstein-Widder theorem [36] implies $x^{m+1}(-1)^m R_1^{(m)}(x)$ is not completely monotonic on $(0,\infty)$ for all $m \in \mathbb{N}$. This completes the proof.

References

[1] M. Abramowitz and I. A. Stegun (Eds), Handbook of Mathematical Functions with Formu- las, Graphs, and Mathematical Tables, National Bureau of Standards, Applied Mathematics Series 55, 10th printing, Dover Publications, New York and Washington, 1972.

[2] H. Alzer, On some inequalities for the gamma and psi functions, Math. Comp. 66 (1997), no. 217, 373-389; available online at https://doi.org/10.1090/S0025-5718-97-00807-7.

[3] Alzer, H.; Berg, C.; Koumandos, S. On a conjecture of Clark and Ismail. J. Approx. Theory 2005, 134, 102-113.

[4] Bierens de Haan, D., Nouvelles tables d'intégrales définies, Amsterdam, 1867. (Reprint) G. E. Stechert & Co., New York, 1939.

[5] Clark, W.E.; Ismail, M.E.H. Inequalities involving gamma and psi functions. Anal. Appl. 2003, 1, 129-140.

[6] Erdélyi, A. et al., Tables of Integral Transforms, vols. I and II. McGraw Hill, New York, 1954.

[7] Guo, B.-N., Qi, F.: A completely monotonic function involving the trigamma function and with degree one. Appl. Math. Comput. 218, 9890-9897 (2012). https://doi.org/10.1016/j.amc.2012.03.075.

[8] Koumandos, S. Remarks on some completely monotonic functions. J. Math. Anal. Appl. 2006, 324, 1458-1461.

[9] Koumandos, S.: Monotonicity of some functions involving the gamma and psi functions. Math. Compet. 77, 2261-2275 (2008). https://doi.org/10.1090/s0025-5718-08-02140-6

[10] Koumandos, S., Lamprecht, M.: Some completely monotonic functions of positive order. Math. Compet. 79, 1697-1707 (2010). https://doi.org/10.1090/s0025-5718-09-02313-8

[11] Koumandos, S., Lamprecht, M.: Complete monotonicity and related properties of some special functions. Math. Compet. 82, 1097-1120 (2013). https://doi.org/10.1090/s0025-5718-2012-02629-9

[12] Koumandos, S., Pedersen, H.L.: Completely monotonic functions of positive order and asymptotic expansions of the logarithm of Barnes double gamma function and Euler's gamma function. J. Math. Anal. Appl. 355, 33-40 (2009). https://doi.org/10.1016/j.jmaa.2009.01.042

[13] Koumandos, S., Pedersen, H.L.: Absolutely monotonic functions related to Euler's gamma function and Barnes' double and triple gamma function. Monatshefte Math. 163, 51-69 (2011). https://doi.org/10.1007/s00605-010-0197-9.

[14] S. Koumandos, Remarks on some completely monotonic functions, J. Math. Anal. Appl. 324 (2006), no. 2, 1458-1461; available online at http://dx.doi.org/10.1016/j.jmaa.2005.12. 017.

[15] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, New York, 2010; available online at http://dlmf.nist.gov/.

[16] Qi, F., Guo, B.-N.: Lévy-Khintchine representation of Toader-Qi mean. Math. Inequal. Appl. 21, 421-431 (2018). https://doi.org/10.7153/mia-2018-21-29

[17] Qi, F., Guo, B.-N.: The reciprocal of the weighted geometric mean of many positive numbers is a Stieltjes function. Quaest. Math. 41, 653-664 (2018). https://doi.org/10.13140/RG.2.2.23822.36163

[18] Qi, F., Li, W.-H.: Integral representations and properties of some functions involving the logarithmic function. Filomat 30, 1659-1674 (2016). https://doi.org/10.2298/FIL1607659Q

[19] Qi, F., Lim, D.: Integral representations of bivariate complex geometric mean and their applications. J. Comput. Appl. Math. 330, 41-58 (2018). https://doi.org/10.1016/j.cam.2017.11.047

[20] Qi, F., Liu, A.-Q.: Completely monotonic degrees for a difference between the logarithmic and psi functions. J. Comput. Appl. Math. 361, 366-371 (2019). https://doi.org/10.1016/j.cam.2019.05.001.

[21] F. Qi, A double inequality for the ratio of two non-zero neighbouring Bernoulli numbers, J. Comput. Appl. Math. 351 (2019), 1-5; available online at https://doi.org/10.1016/j.cam. 2018.10.049.

[22] F. Qi, Completely monotonic degree of a function involving the tri- and tetra-gamma func- tions, arXiv preprint (2013), available online at http://arxiv.org/abs/1301.0154

[23] F. Qi, Notes on a double inequality for ratios of any two neighbouring non-zero Bernoulli numbers, Turkish J. Anal. Number Theory 6 (2018), no. 5, 129-131; available online at https://doi.org/10.12691/tjant-6-5-1.

[24] F. Qi and R. J. Chapman, Two closed forms for the Bernoulli polynomials, J. Number Theory 159 (2016), 89-100; available online at https://doi.org/10.1016/j.jnt.2015.07.021.

[25] Mitrinovíc, D.S., Pećarić, J.E., Fink, A.M.: Classical and New Inequalities in Analysis. Kluwer Academic, Dordrecht (1993).

[26] Schilling, R.L., Song, R., Vondra¡cek, Z.: Bernstein Functions-Theory and Applications, 2nd edn. de Gruyter Studies in Mathematics, vol. 37. de Gruyter, Berlin (2012).

[27] Widder, D.V.: The Laplace Transform. Princeton University Press, Princeton (1946).

[28] Y. Xu and X. Han, Complete monotonicity properties for the gamma function and Barnes G-function, Sci. Magna 5 (2009), no. 4, 47-51.

Address: Institu préparatoire aux études d'ingénieurs de Tunis.
Campus Universitaire El-Manar, 2092 El Manar Tunis.
Email: bouali25@laposte.net