

# Position of the centroid of a planar convex body

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**Abstract.** It is well known that any planar convex body  $A$  permits to inscribe an affine-regular hexagon  $H_A$ . We prove that the centroid of  $A$  belongs to the homothetic image of  $H_A$  with ratio  $\frac{4}{21}$  and the center in the center of  $H_A$ . This ratio cannot be decreased.

**Keywords:** convex body, centroid, affine-regular hexagon

**MSC:** Primary: 52A10

## 1 Introduction

This paper concerns the position of the centroid of a planar convex body, i.e., a closed bounded convex set. Recall that the notion of centroid is discussed by, among others, Bonnesen and Fenchel [2], Grünbaum [3], Hammer [4] and Neumann [5].

As usual, by an *affine-regular hexagon* we understand a non-degenerated affine image of the regular hexagon. Besicovitch [1] proved that for every planar convex body  $A$  there exists an affine-regular hexagon  $H_A$  inscribed in  $A$ . Our aim is to prove that the centroid of  $A$  belongs to the homothetic image  $\frac{4}{21}H_A$  of  $H_A$  with ratio  $\frac{4}{21}$  and the center in the center of  $H_A$ . In general, this ratio cannot be lessened, which is explained at the end of the paper.

For a compact set  $C$  of the Euclidean plane  $E^2$  denote by  $\text{cen}_x(C)$  and  $\text{cen}_y(C)$  the first and the second coordinates of the centroid of  $C$ . Let compact sets  $B_1, \dots, B_n \subset E^2$  with non-empty interiors have disjoint interiors and  $B = \bigcup_{j=1}^n B_j$ . It is well known that

$$\text{cen}_x(B) = \frac{\sum_{j=1}^n \text{cen}_x(B_j) \cdot \text{area}(B_j)}{\sum_{j=1}^n \text{area}(B_j)}, \quad \text{cen}_y(B) = \frac{\sum_{j=1}^n \text{cen}_y(B_j) \cdot \text{area}(B_j)}{\sum_{j=1}^n \text{area}(B_j)}. \quad (1)$$

## 2 The position of the centroid of a convex body with respect to an inscribed affine-regular hexagon

Let  $D \subset E^2$  and  $\ell$  be a straight line. Imagine  $D$  as the union of segments (including one-point segments) being intersections of  $D$  by straight lines perpendicular to  $\ell$ . Shift every such a segment perpendicularly to  $\ell$  in order to obtain its image centered at  $\ell$ . Denote the union of

all these obtained segments by  $\text{sym}_\ell D$ . It is the result of the Steiner symmetrization of  $D$ . The proof of the following lemma is given in a number of books. For instance in Section 40 of [2].

**Lemma.** *If  $D \subset E^2$  is convex, then  $\text{sym}_\ell D$  is convex.*

**Theorem.** *Let  $A \subset E^2$  be a convex body and  $H_A$  be an affine-regular hexagon inscribed in  $A$ . Then the centroid of  $A$  belongs to the homothetic image of  $H_A$  with ratio  $\frac{4}{21}$  and center in the center of  $H_A$ .*

*Proof.* For better clarity, we divide the proof into a preliminary text mostly on notations, and then Parts 1–8 with considerations.

We do not lose the generality assuming that the successive vertices  $a_1, \dots, a_6$  of  $H_A$  are  $(1, 1)$ ,  $(-1, 1)$ ,  $(-2, 0)$ ,  $(-1, -1)$ ,  $(1, -1)$ ,  $(2, 0)$ , see Figure. Denote by  $o$  the center  $(0, 0)$  and by  $a$  the midpoint of  $a_1 a_2$ . Since we deal with  $\text{cen}_y(A)$ , by Lemma we may assume that  $x = 0$  is an axis of symmetry of  $A$ .

In order to prove the assertion, let us show that for any side of  $\frac{4}{21}H_A$  the centroid of  $A$  is on the same side of the straight line containing this side which contains  $o$ . Observe that it is enough to show this for one side of the hexagon  $\frac{4}{21}H_A$ . Let us provide this task for the side connecting  $\frac{4}{21}a_1$  and  $\frac{4}{21}a_2$ .

Denote by  $\bar{a}_i$  the intersection of the straight lines containing  $a_i a_{i+1}$  and  $a_{i-1} a_{i-2}$  for  $i = 1, \dots, 6 \pmod{6}$ , see Figure. We define the star  $S(H_A)$  over  $H_A$  as the union of  $H_A$  and six triangles  $T_i(H_A) = a_{i-1} \bar{a}_i a_i$ , where  $i = 1, \dots, 6$  and where  $a_0$  means  $a_6$ . From the convexity of  $A$  we conclude that  $A \subset S(H_A)$ .

We do not make our considerations narrower assuming that the centroid of  $A$  is over or on the axis  $y = 0$ . Since our aim is to show that  $\text{cen}_y(A) \leq \frac{4}{21}$  for every convex body  $A$ , it is sufficient to consider only such convex bodies  $A$  which are disjoint with the interiors of  $T_4(H_A)$ ,  $T_5(H_A)$  and  $T_6(H_A)$ . Still the closure of  $A \setminus \bigcup_{i=4}^6 T_i(H_A)$  is a convex body with  $H$  inscribed and the centroid at the same or higher level.

Provide any supporting straight line  $L_1$  of  $A$  at  $a_1$  and the symmetric (with respect to  $x = 0$ ) supporting line  $L_2$  of  $A$  at  $a_2$ . Denote by  $u = (0, w)$  the intersection point of  $L_1$  (and thus of  $L_2$ ) with the axis  $x = 0$ . Since the second coordinates of  $a$  and  $\bar{a}_2$  are equal to 1 and 2, respectively, we have  $w \in [1, 2]$ .

Since  $L_1$  passes through  $u = (0, w)$  and  $a_1 = (1, 1)$ , it has the equation  $y - 1 = (-w + 1)(x - 1)$ . Its point of intersection with the segment  $a_6 \bar{a}_1$  (being a subset of the straight line  $y = x - 2$ ) is  $m_1 = (\frac{2+w}{w}, \frac{2-w}{w})$ . Similarly, we get the symmetric point  $m_2$  being the intersection of  $L_2$  with the segment  $a_3 \bar{a}_3$ .

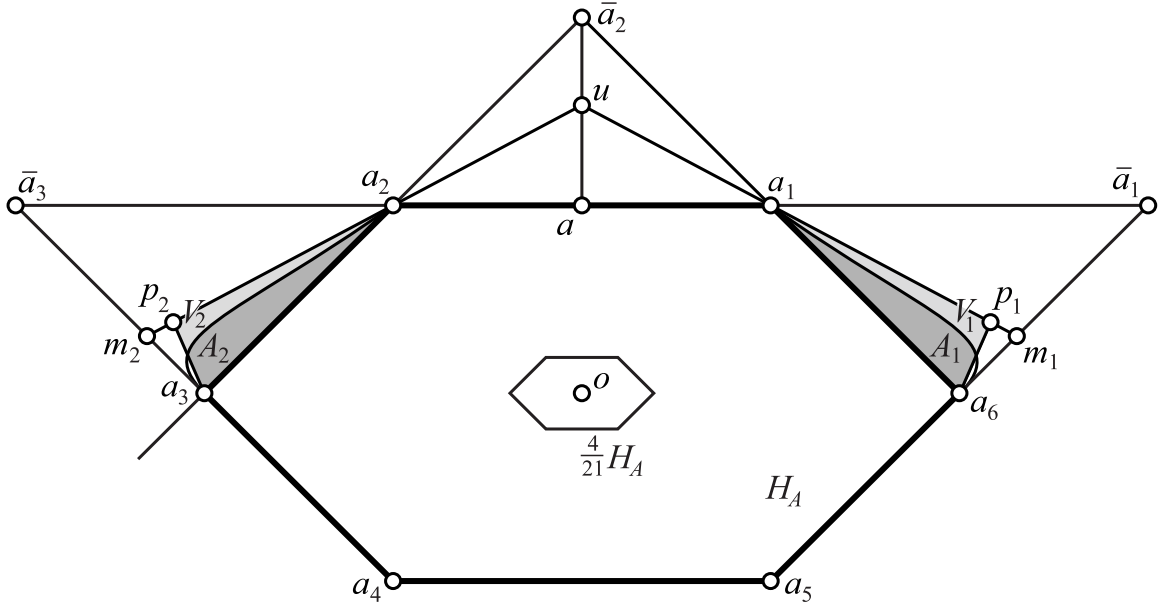


Figure. Illustration to the proof of Theorem

Later we explain the geometric meaning of the following number

$$w_0 = \frac{1}{3}(\sqrt[3]{44 - 3\sqrt{177}} + \sqrt[3]{44 + 3\sqrt{177}} - 1) = 1.6589670\dots$$

Parts 3–7 lead to the proof of our theorem for  $w \in [w_0, 2]$  and Part 8 for  $w \in [1, w_0]$ .

**Part 1** where we introduce a heptagon and find its  $\text{cen}_y$ .

Let  $z \in [0, 1]$ . Since  $a_1 = (1, 1)$  and  $m_1 = (\frac{2+w}{w}, \frac{2-w}{w})$ , every point  $p_1(z)$ , or shortly  $p_1$ , of  $a_1m_1$  has the form  $(1-z)a_1 + zm_1$ . So  $p_1 = ((1-z) + z\frac{2+w}{w}, (1-z) + z\frac{2-w}{w}) = (z\frac{2}{w} + 1, z\frac{2-2w}{w} + 1)$ . The symmetric point with respect to  $x = 0$  is denoted by  $p_2$ . The second coordinates of them are  $z\frac{2-2w}{w} + 1$ .

Consider the heptagon  $G = up_2a_3a_4a_5a_6p_1$ . The area of each of the two symmetric wings  $W_1 = a_1a_6p_1$  and  $W_2 = a_2a_3p_2$  of  $G$  is  $z\frac{2-w}{w}$  and  $\text{cen}_y$  of each wing of this heptagon is  $\frac{1+z\frac{2-2w}{w}+1}{3} = \frac{2+z\frac{2-2w}{w}}{3}$ . The area of the triangle  $a_1ua_2$  is  $\frac{1}{2} \cdot 2 \cdot (w-1) = w-1$  and its  $\text{cen}_y$  is  $\frac{2+w}{3}$ . Moreover, the area of  $H_A$  is 6 and its  $\text{cen}_y$  is 0. Taking all this into account and having in mind that  $G = H_A \cup a_1a_6p_1 \cup a_2a_3p_2 \cup a_1ua_2$ , by the right part of (1) we conclude that

$$\text{cen}_y(G) = \frac{0 + \frac{2}{3}(2 + z\frac{2-2w}{w})z\frac{2-w}{w} + \frac{2+w}{3}(w-1)}{6 + 2z\frac{2-w}{w} + w - 1}$$

which, after a simplification, equals to

$$\frac{2(2 + z\frac{2-2w}{w})z\frac{2-w}{w} + w^2 + w - 2}{6z\frac{2-w}{w} + 3w + 15}. \quad (2)$$

**Part 2** whose aim is to show the following statement

Denote by  $\nu$  the numerator and by  $\delta$  the denominator of  $\text{cen}_y(G)$  as in (2) (so  $\text{cen}_y(G) = \frac{\nu}{\delta}$ ). Consider a truncation of the wings  $W_i$  of  $G$  to symmetric convex subsets  $A_i = W_i \cap A$  for  $i = 1, 2$ . Put  $V_i = W_i \setminus A_i$  for  $i = 1, 2$  and  $V = V_1 \cup V_2$ . We have

$$\frac{\nu - \text{area}(V)\text{cen}_y(V)}{\delta - \text{area}(V)} \leq \frac{\nu}{\delta} \quad \text{iff} \quad \text{cen}_y(V) \geq \frac{\nu}{\delta}. \quad (3)$$

Let us confirm this. We have  $\frac{\nu - \text{area}(V)\text{cen}_y(V)}{\delta - \text{area}(V)} \leq \frac{\nu}{\delta}$  iff  $\delta(\nu - \text{area}(V)\text{cen}_y(V)) \leq \nu(\delta - \text{area}(V))$  iff  $\nu \cdot \text{area}(V) \leq \delta \cdot \text{area}(V)\text{cen}_y(V)$  iff  $\nu \leq \delta \cdot \text{cen}_y(V)$  iff  $\text{cen}_y(V) \geq \frac{\nu}{\delta}$ .

Observe that  $\frac{\nu - \text{area}(V)\text{cen}_y(V)}{\delta - \text{area}(V)}$  is nothing else but  $\text{cen}_y(A')$ , where  $A' = G \setminus V$ .

**Part 3** where we start considerations for  $w \in [w_0, 2]$ .

For every  $w \in [w_0, 2]$  we are looking for the positions of  $p_1$  and thus of  $p_2$  such that  $\text{cen}_y(G)$  is the largest. For this reason let us find the derivative of the function (2) with respect to  $z$ :

$$\frac{2(w-2)[4z^2(-w^2+3w-2)+4z(w^2+4w-5)+(w^4-w^3-12w^2)]}{3w(w^2-2wz+5w+4z)^2}. \quad (4)$$

The discriminant of the quadratic function in the square bracket is  $16w^2(2w^4+4w^3-w^2-6w+1)$ . Hence (4) equals 0 for  $z = \frac{w(w^2+4w-5 \pm \sqrt{2w^4+4w^3-w^2-6w+1})}{2(w^2-3w+2)}$ . Take into account only the root

$$z_w = \frac{w(w^2+4w-5 - \sqrt{2w^4+4w^3-w^2-6w+1})}{2(w^2-3w+2)} \quad (5)$$

which is positive for every  $w \in [w_0, 2]$  (the other one is always negative here). Moreover, put  $z_2 = \lim_{w \rightarrow 2^-} z_w$ . This is  $z_2 = \frac{5}{7}$ .

We see that for any fixed  $w \in [w_0, 2]$  the global maximum of (2) as a function of  $z$  from the interval  $[0, 1]$  can be only for  $z = 0$ ,  $z = z_w$  or  $z = 1$ . Substituting these three  $z$  into (2) we see that the global maximum of (2) in the interval  $[0, 1]$  is at  $z = z_w$  for every fixed  $w \in [w_0, 2]$ .

**Part 4** where our aim is to show that for each  $w \in [w_0, 2]$  the value of (2) for  $z = z_w$  is at most  $\frac{4}{21}$ .

This task with substituting  $z = z_w$  into (2) seems to be very complicated to perform. We can get it around by performing the more general task to show that for every  $w \in [w_0, 2]$  and  $z \in [\frac{5}{7}, 1]$  we have

$$\frac{2(2 + z\frac{2-2w}{w})z\frac{2-w}{w} + w^2 + w - 2}{6z\frac{2-w}{w} + 3w + 15} \leq \frac{4}{21}. \quad (6)$$

This task is more general since  $z_w$  belongs to  $[\frac{5}{7}, 1]$  for every  $w \in [w_0, 2]$ . Really, the inequality  $z_w \leq 1$  is equivalent to  $w^6 - 7w^4 - 2w^2 + 16w^2 + 8w - 16 \geq 0$  and thus to  $(w-1)(w-2)(w+3)(w^3+w^2-2w-4) \geq 0$ , which means that it holds true in  $[1, 2]$  if and only if  $w \in [w_0, 2]$  (still  $w_0$  is the only real root of this polynomial). Moreover, the inequality  $\frac{5}{7} \leq z_w$  is equivalent to  $-7(w-2)(7w^2+22w+20) \geq 0$ , which means that it holds true in the whole interval  $[1, 2]$ , so in particular for every  $w \in [w_0, 2]$ .

Equivalently to (6), it is sufficient to show that

$$28z^2(w^2 - 3w + 2) + 20zw(2 - w) + w^2(7w^2 + 3w - 34) \quad (7)$$

is at most 0 for every point  $(w, z)$  of the rectangle  $[w_0, 2] \times [\frac{5}{7}, 1]$ .

In order to simplify evaluations consider this task in the larger rectangle  $[1, 2] \times [\frac{5}{7}, 1]$ .

Let us apply the following method of finding the global maximum of a continuous function  $f(w, z)$  in a polygon  $R \subset E^2$ . Namely, first we find the points being the solutions of the system of two equations when partial derivatives of our function  $f(w, z)$  are 0 in the interior of  $R$ . Next we write the equations of the sides in the forms  $z = g(w)$  or  $w = g(z)$ . We find the critical points in the relative interiors of each side, where the derivative of the respective equation is 0. Finally, we check the values of  $f(w, z)$  at the vertices of  $R$ . The largest value at all the found points gives the maximum value of  $f(w, z)$  in  $R$ .

In our particular case our function  $f(w, z) = 28z^2(w^2 - 3w + 2) + 20zw(2 - w) + w^2(7w^2 + 3w - 34)$  is given by (7). Moreover,  $R = [1, 2] \times [\frac{5}{7}, 1]$ . According to the recalled method we find the partial derivatives  $f'_w(w, z) = 28w^3 + 9w^2 + 56wz^2 - 40wz - 68w - 84z^2 + 40z$  and  $f'_z(w, z) = 56w^2z - 20w^2 - 168wz + 40w + 112z$ . Consider the system of equations when both are 0. Finding  $z$  from the second and substituting to the first we get three solutions:  $w \approx -1.8$ ,  $w = 0$  and  $w \approx 1.544$ . None of them is in the interval  $[w_0, 2]$ . Hence the system of equations has no solution in our  $R$ , and thus in its interior.

Let us find the critical points in the relative interiors of the sides. After substituting  $z = \frac{5}{7}$  to  $f(w, z)$  we get  $\frac{1}{7}(49w^4 + 21w^3 - 238w^2 - 100w + 200)$ . Its derivative  $\frac{1}{7}(196w^3 + 63w^2 - 476w - 100)$  is 0 only at  $w_1 = 1.5103\dots$ . Placing  $z = 1$  to our  $f(w, z)$  we get  $7w^4 + 3w^3 - 26w^2 - 44w + 56$ . Its derivative  $28w^3 + 9w^2 - 52w - 44$  equals 0 in  $[1, 2]$  only at  $w_2 = 1.5427\dots$ . Substituting  $w = 1$  to  $f(w, z)$  we get  $20z - 24$ , which is negative for every  $z \in [\frac{5}{7}, 1]$ . Placing  $w = 2$  to  $f(w, z)$  we get 0 for every  $z \in [\frac{5}{7}, 1]$ .

We have  $f(w_1, \frac{5}{7}) \approx -23.803$ ,  $f(w_2, 1) \approx -23.094$ ,  $f(1, 1) = -4$ ,  $f(1, \frac{5}{7}) = -9.714$ ,  $f(2, \frac{5}{7}) = 0$ , and  $f(2, 1) = 0$ . Thus the global maximum of  $f(w, z)$  in  $R$  is 0. Hence (7) is at most 0 and thus (6) holds true in  $R$ . We conclude that (2) for  $z = z_w$  is at most  $\frac{4}{21}$  for every  $w \in [1, 2]$  and so for every  $w \in [w_0, 2]$ .

**Part 5** where we show that  $\text{cen}_y(V) \geq \frac{\nu}{\delta}$  for any  $w \in [w_0, 2]$  and  $z_w$  in place of  $z$ .

Recall from Part 2 that  $\frac{\nu}{\delta} = \text{cen}_y(G)$ . Looking at the second coordinates of  $a_1, a_6$  and  $p_1$  we get  $\text{cen}_y(a_1 a_6 p_1) = (z_w \frac{2-2w}{w} + 1)/2$ . Hence  $\text{cen}_y(V_1) \geq (z_w \frac{2-2w}{w} + 1)/2$  and so  $\text{cen}_y(V) \geq (z_w \frac{2-2w}{w} + 1)/2$ .

We see that in order to confirm the promise of Part 5 it is sufficient to show that

$$\frac{z_w \frac{2-2w}{w} + 1}{2} \leq \frac{2(2 + z_w \frac{2-2w}{w})z_w \frac{2-w}{w} + w^2 + w - 2}{6z_w \frac{2-w}{w} + 3w + 15}, \quad (8)$$

for every  $w \in [w_0, 2]$ , where the right side is taken from (2). Instead, let us show the inequality

$$\frac{z \frac{2-2w}{w} + 1}{2} \leq \frac{2(2 + z \frac{2-2w}{w})z \frac{2-w}{w} + w^2 + w - 2}{6z \frac{2-w}{w} + 3w + 15},$$

or equivalently, let us show that

$$8z^2 - 12z^2w - 22zw^2 + 26zw + 4z^2w^2 - 6zw^3 - 2w^4 + w^3 + 19w^2 \quad (9)$$

is at most 0 for every point  $(w, z)$  of the piece of the curve  $z = z_w$  when  $w \in [w_0, 2]$ .

Instead, let us find the global maximum of (9) in a triangle containing it. Namely, in the triangle  $T$  between the straight lines  $w = 2$ ,  $z = -\frac{5}{7}w + \frac{15}{7}$ , and  $z = 1$ . Its vertices are  $(2, 1)$ ,  $(\frac{8}{5}, 1)$  and  $(2, \frac{13}{21})$ .

First let us show that the piece of the curve  $z = z_w$  for  $w \in [w_0, 2]$  is a subset of  $T$ . The reason is that  $-\frac{5}{7}w + \frac{15}{7} \leq z_w \leq 1$  for every  $w \in [w_0, 2]$ . The left inequality is equivalent to the inequality  $(2839w^4 - 10571w^3 + 18960w^2 - 284088w + 22472)(w - 1)(2 - w) \geq 0$  which holds true for every  $w \in (-\infty, \infty)$ . Thus in  $[1, 2]$  and so for every  $w \in [\frac{8}{5}, 2]$ . The right inequality  $z_w \leq 1$  is shown just after (6).

Next let us find the global maximum of (9) in  $T$  by the method described in Part 4.

Consider the system of equations  $-8w^3 - 18w^2z + 3w^2 + 8wz^2 - 44wz + 38w - 12z^2 + 26z = 0$  and  $-6w^3 + 8w^2z - 22w^2 - 24wz + 26w + 16z = 0$  (where the left sides are the partial derivatives of (9)). Finding  $z = \frac{3w^3 + 11w^2 - 13w}{4w^2 - 12w + 8}$  from the second and substituting it into the first we get the equation  $w(68w^6 - 141w^5 - 262w^4 + 359w^3 + 356w^2 - 447w + 68) = 0$  whose solutions are  $w = 0, w \approx 0.183, w \approx 0.951, w \approx 1.037$  and  $w \approx 2, 614$ . All these  $w$  are out of the interval  $[\frac{8}{5}, 2]$  which implies that all the obtained points  $(w, z)$  are out of  $T$ . Thus the system of equations has no solution in the interior of  $T$ .

Look for critical points in the relative interiors of the sides. Substituting  $z = -\frac{5}{7}w + \frac{15}{7}$  into (9) we get  $\frac{1}{49}(212w^3 - 511w^2 - 789w + 1830)$ . Its derivative  $\frac{1}{49}(848w^3 - 1533w^2 - 1578w + 1830)$  is never 0 in  $[\frac{8}{5}, 2]$ . Putting  $z = 1$  into (9) we get  $-2w^4 - 5w^3 + w^2 + 22w$ . Its derivative  $-8w^3 - 15w^2 + 2w + 22$  is never 0 in  $[\frac{8}{5}, 2]$ . Putting  $w = 2$  into (9) we get  $8z^2 - 84z + 52$ . Its derivative  $16z - 84$  is never 0 in  $[\frac{5}{7}, 1]$ .

The value of (9) at  $(2, 1)$  is  $-44$ , at  $(\frac{8}{5}, 1)$  is  $-11.827\dots$ , and at  $(2, \frac{13}{21})$  is  $-4.598\dots$ . So the global maximum of the function (9) in  $T$  is  $-11.827\dots$ . Hence (9) is always negative in  $T$ .

Consequently, we have shown that (9) is at most 0 in  $T$  and thus that (8) is true for every  $w \in [w_0, 2]$ . Therefore  $\text{cen}_y(V) \geq \frac{\nu}{\delta}$  for  $G$  with  $z_w$  in the part of  $z$ .

**Part 6** where we show that  $\text{cen}_y(A') \leq \frac{4}{21}$  for  $w \in [w_0, 2]$ .

Recall that  $\text{cen}_y(G) = \frac{\nu}{\delta}$ . By Part 5 and by (3) we have  $\frac{\nu - \text{area}(V)\text{cen}_y(V)}{\delta - \text{area}(V)} \leq \frac{\nu}{\delta}$ . The left side is  $\text{cen}_y(A')$  and the right one is  $\text{cen}_y(G)$  with  $G$  is taken for  $z = z_w$ . By (6) it is at most  $\frac{4}{21}$ . So  $\text{cen}_y(A') \leq \frac{4}{21}$ .

**Part 7** on enlarging  $A'$  up to  $A$  which leads to the proof of our theorem for  $w \in [w_0, 2]$ .

Put  $A'_1 = A \cap p_1a_6m_1$ ,  $A''_2 = A \cap p_2a_3m_2$  and  $A'' = A'_1 \cup A''_2$ . Clearly  $A = A' \cup A''$ . We intend to show that adding  $A''$  to  $A'$  does not increase  $\text{cen}_y$ , so that  $\text{cen}_y(A) \leq \text{cen}_y(A')$ .

First let us show that if the triangles  $p_1a_6m_1$  and  $p_2a_3m_2$  are added to  $A'$ , then  $\text{cen}_y$  does not increase. Applying the easy to show implication: “if  $\text{int}(X) \cap \text{int}(Y) = \emptyset$ ,  $\text{cen}_y(X) \leq \mu$  and  $\text{cen}_y(Y) \leq \mu$ , then  $\text{cen}_y(X \cup Y) \leq \mu$  as well” and having in mind that  $\text{cen}_y(A') \leq \frac{4}{21}$  (see Part 6), it is sufficient to show that  $\text{cen}_y(p_1a_6m_1) \leq \frac{4}{21}$  (then also  $\text{cen}_y(p_2a_3m_2) \leq \frac{4}{21}$ ). Let us show this. Since  $\text{cen}_y(p_1a_6m_1) = (z_w \cdot \frac{2-2w}{w} + 1 + \frac{2-w}{w})/3$ , we have to show that this is at most  $\frac{4}{21}$ . This task is equivalent to  $7z_w(2 - 2w) \leq 4w - 14$ . After substituting  $z_w$  and providing some simplifications, this inequality is equivalent to  $h(w) \geq 0$ , where  $h(w) = 49w^6 - 196w^5 + 105w^4 + 1946w^2 + 1800w - 400$ . We have  $h''(w) = 14(105w^4 - 280w^3 + 90w^2 + 278)$ . From the fact that  $h''(w)$  always positive we conclude that  $h'(w)$  is an increasing function. Thus from  $h(1) = 3304$  we see that  $h(w) \geq 0$  for  $w \geq 1$ , and thus for every  $w \in [w_0, 2]$ . Hence  $\text{cen}_y(p_1a_6m_1) \leq \frac{4}{21}$ .

Also for adding only  $A''$  to  $A'$ , the value of  $\text{cen}_y$  does not increase. The reason is that  $\text{cen}_y(A'_1) \leq \text{cen}_y(p_1a_6m_1)$  and  $\text{cen}_y(A''_2) \leq \text{cen}_y(p_2a_3m_2)$ . The first follows from the convexity of  $A'_1$  and from the observation that every segment jointing  $a_6$  with a point of  $p_1m_1$  has in common with  $A'_1$  only a segment which is lower. Analogously, we confirm the second inequality.

We conclude that  $\text{cen}_y(A) \leq \text{cen}_y(A')$ . This and  $\text{cen}_y(A') \leq \frac{4}{21}$  (see Part 6) imply  $\text{cen}_y(A) \leq \frac{4}{21}$ .

**Part 8** where we prove our theorem for  $w \in [1, w_0]$ .

Consider the pentagon  $P = m_1um_2a_4a_5$ . It is the special case of  $G$  for  $z = 1$ . Thus substituting  $z = 1$  to (2) we see that  $\text{cen}_y(P)$  equals to  $\frac{w^4 + w^3 - 2w^2 - 4w + 8}{3w(w^2 + 3w + 4)}$ . In order to show that this is at most  $\frac{4}{21}$  for every  $w \in [1, w_0]$  take into account the equivalent inequality  $(w - 2)(7w^3 + 17w^2 + 8w - 28) \leq 0$ . Its left side equals 0 only for  $w = 2$  and  $w = w_3 \approx 0.934$ .

Consequently, this inequality and thus also the preceding one hold true in  $[w_3, 2]$ . Hence also in  $[1, w_0]$ . Resuming,  $\text{cen}_y(P) \leq \frac{4}{21}$ .

From  $z = 1$  we see that  $V_1 = a_1 a_6 m_1$  for our  $P = G$ . The second coordinates of  $a_1, a_6$  and  $m_1$  give  $\text{cen}_y(a_1 a_6 m_1) = \frac{4-2w}{3w}$ . Hence  $\text{cen}_y(V_1) \leq \frac{4-2w}{3w}$ . Thus by  $\text{cen}_y(V) = \text{cen}_y(V_1)$  we get  $\text{cen}_y(V) \leq \frac{4-2w}{3w}$ .

In order to show that the right side of (3) is now true we have to show that  $\text{cen}_y(V) \geq \text{cen}_y(P)$ , where, as  $P$  takes the role of  $G$ , the right side is denoted by  $\frac{v}{\delta}$  in (3). Hence we have to show that  $\frac{4-2w}{3w} \geq \frac{w^4+w^3-2w^2-4w+8}{3w(w^2+3w+4)}$ . This is equivalent to the inequality  $w^4 + 3w^3 - 8w - 8 \leq 0$ . A simple evaluation confirms that it is true in  $[1, w_0]$  (by the way, we have here the equality just for  $w = w_0$ ). Thus  $\text{cen}_y(V) \geq \text{cen}_y(P)$ .

The shown inequality means that the right side of (3) is fulfilled. Hence the left side of (3), i.e.,  $\text{cen}_y(A') \leq \frac{v}{\delta}$  holds true. From  $A = A'$  for our  $P = G$  we obtain  $\text{cen}_y(A) \leq \frac{v}{\delta}$ . Consequently, from  $\frac{v}{\delta} = \text{cen}_y(P) \leq \frac{4}{21}$  we conclude that  $\text{cen}_y(A) \leq \frac{4}{21}$ .

Thanks to results of Parts 7 and 8 the thesis of our theorem holds true.  $\square$

The ratio  $\frac{4}{21}$  in Theorem cannot be lessened as it follows from the example of the pentagon  $\bar{a}_2 a_3 a_4 a_5 a_6$  in the part of  $A$ , and the hexagon  $a_1 \dots a_6$  as  $H_A$ . The author expects that there are no more such examples besides the affine images of the above presented one.

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