

Lieb's permanental dominance conjecture

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Abstract

We survey the impact of Lieb's influential paper "Proofs of some conjectures on permanents" [*J. Math. Mech.* **16** 1966, 127–134], which introduced the famous permanental dominance conjecture. This conjecture has defied all attacks for over half a century, although a number of related conjectures have recently been resolved.

1 Introduction

This is a survey article focusing on the legacy of Lieb's paper [28] on permanents of matrices in \mathcal{H}_n , the set of $n \times n$ positive semi-definite Hermitian matrices. Let \mathcal{S}_n denote the symmetric group on $\{1, 2, \dots, n\}$ and $\mathbf{1}_n$ denote the identity permutation in \mathcal{S}_n . The *permanent* of an $n \times n$ complex matrix $A = [a_{i,j}]$ is defined by

$$\text{per } A = \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^n a_{i,\sigma(i)}.$$

More generally, if G is a subgroup of \mathcal{S}_n and χ is any character of G then the (normalised) *generalised matrix function* f_χ is defined by

$$f_\chi(A) = \frac{1}{\chi(\mathbf{1}_n)} \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}.$$

If $A \in \mathcal{H}_n$ then $f_\chi(A)$ is a non-negative real number. If $G = \mathcal{S}_n$ and χ is irreducible then f_χ is called a (normalised) *immanant*. If χ is the principal/trivial character then f_χ is the permanent, while if χ is the alternating character then f_χ is the determinant. Taking G to be the trivial group yields the diagonal product

$$h(A) = \prod_{i=1}^n a_{i,i}.$$

These three special examples of generalised matrix functions are related by

$$0 \leq \det A \leq h(A) \leq \text{per } A \tag{1}$$

for all $A \in \mathcal{H}_n$. The second inequality was shown by Hadamard [20] and the last inequality is due to Marcus [29], [30].

Lieb's work [28] was motivated by results such as (1). He observed that “the few inequalities that are known for the permanent are suspiciously similar to certain special cases of classical inequalities for the determinants of such matrices – the only difference being that the direction of the inequality is reversed”. Applying this principle to the classical result of Schur [50], which states that $\det A \leq f_\chi(A)$ for all $A \in \mathcal{H}_n$, Lieb proposed:

Conjecture 1. *Let G be a subgroup of \mathcal{S}_n , and let χ be a character of G . Then $\text{per } A \geq f_\chi(A)$ for any $A \in \mathcal{H}_n$.*

This conjecture subsequently became known as the “permanental dominance conjecture”. However, as noted by Zhang [62], there is some confusion in the literature between this name and the “permanent on top conjecture”. We will use the latter name for the related Conjecture 4 below.

The special case of Conjecture 1 in which χ has degree 1 had earlier been the subject of an open problem in [31], which was then listed as Problem 2 in Minc's catalogue of open problems [39]. Conjecture 1 was listed as Conjecture 42 in [40]. As we shall see in §2, the permanental dominance conjecture has motivated a lot of research, whilst gaining a degree of notoriety. Despite considerable scrutiny, it remains open 56 years later. The other key contribution from Lieb [28] was the following result. An alternative proof of this theorem was subsequently given by Djoković [9].

Theorem 1. *Let*

$$A = \begin{bmatrix} B & C \\ C^* & D \end{bmatrix} \in \mathcal{H}_n \quad (2)$$

where B and D are $b \times b$ and $d \times d$ blocks, respectively. For any scalar λ , define

$$A_\lambda = \begin{bmatrix} \lambda B & C \\ C^* & D \end{bmatrix}.$$

Now consider $P(\lambda) = \text{per } A_\lambda$ as a polynomial in λ . All coefficients of this polynomial are real and nonnegative.

Notice that $\text{per } A = P(1)$ is the sum of the coefficients of $P(\lambda)$, which therefore dominates any individual coefficient. In particular, it dominates the coefficient of λ^b , which gives:

Corollary 1. *For A in (2) we have $\text{per } A \geq (\text{per } B)(\text{per } D)$. Equality holds if and only if A has a zero row or C is the zero matrix.*

Also, if $b = d$ then $(\text{per } C)(\text{per } C^*)$ is the constant term in $P(\lambda)$, which yields:

Corollary 2. *If $b = d$ in (2) then $\text{per } A = P(1) \geq (\text{per } B)(\text{per } D) + (\text{per } C)(\text{per } C^*)$. Equality holds if and only if A has a zero row or C is the zero matrix.*

In his MathSciNet review of [28], Marcus described the proofs of these results as “extremely ingenious and intricate arguments”. Lieb also stated a determinantal analogue of Theorem 1.

The following notation will be used throughout this paper. The direct sum of matrices A and B will be denoted $A \oplus B$, while their tensor/Kronecker product will be denoted $A \otimes B$ and their Hadamard (elementwise) product will be denoted $A \circ B$. For any matrix A the submatrix obtained by deleting row i and column j from A will be denoted $A(i|j)$. The Hermitian adjoint (conjugate transpose) of A will be denoted A^* , while \overline{A} will denote the conjugate of A . We use \mathcal{C}_n to denote the subset of all matrices $[a_{i,j}] \in \mathcal{H}_n$ that satisfy $a_{i,i} = 1$ for all $1 \leq i \leq n$. The matrices in \mathcal{C}_n are called *correlation matrices*.

2 Related conjectures

In this section we examine the relationships between a raft of conjectures that are related to Lieb's permanental dominance conjecture. To assist the reader to keep track of all these conjectures, we summarise their relationships by the implications pictured in Figure 1. We show only the implications which have historically been demonstrated in the literature prior to the point at which a conjecture has been proved or refuted. Of course, once a conjecture has been resolved there are many new (uninteresting) implications that could be added. On the topic of resolution, the current status of each conjecture is indicated in Figure 1 by a subscript on the conjecture's number. A \checkmark indicates the conjecture has been proved, a \times shows that it has been disproved and an $?$ indicates that it remains open. These details will gradually unfold in the historical account below.

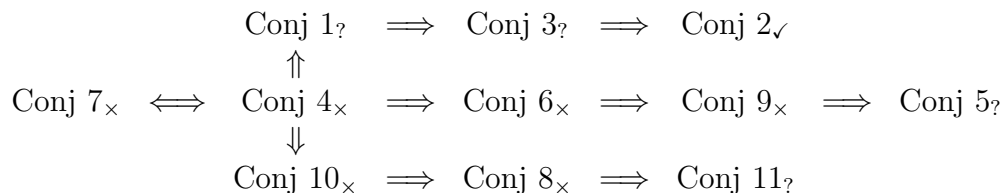


Figure 1: Relationship between various conjectures

We start by noting that Lieb's work was motivated by two other conjectures. Firstly, Corollary 1 solved the following conjecture by Marcus and Newman, first published as Conjecture 8 in [31]. It is a permanental analogue of a classical result by Fischer [14], who showed that $\det A \leq (\det B)(\det D)$ for A partitioned as in (2).

Conjecture 2. *For A in (2) we have $\text{per } A \geq (\text{per } B)(\text{per } D)$.*

Secondly, we have the following conjecture, which was credited to Marcus as Conjecture 9 in [31].

Conjecture 3. *Let $A \in \mathcal{H}_{mk}$ be partitioned into $k \times k$ blocks $A_{i,j}$, $i, j = 1, 2, \dots, m$. Let G be the $m \times m$ matrix whose (i, j) entry is $\text{per}(A_{i,j})$ for $i, j = 1, 2, \dots, m$. Then*

$$\text{per } A \geq \text{per } G. \tag{3}$$

If the $A_{i,i}$ are positive definite, then equality holds in (3) if and only if

$$A = A_{11} \oplus A_{22} \oplus \dots \oplus A_{mm}.$$

It was noted in [31] that Conjecture 3 implies Conjecture 2. Also, Lieb [28] showed that Conjecture 1 implies Conjecture 3, and his Corollary 2 proves the $m = 2$ case. Pate [42] proved that Conjecture 3 holds when A is real and m is large enough relative to k .

For any $A \in \mathcal{H}_n$ we define the *Schur power matrix* $\pi(A)$ to be the $n! \times n!$ matrix whose (σ, τ) entry is $\prod_{t=1}^n a_{\sigma(t), \tau(t)}$, where σ and τ run over all permutations in \mathcal{S}_n . The following conjecture was introduced by Soules [53] and included as Conjecture 31 in [39].

Conjecture 4. *Let $A = [a_{i,j}] \in \mathcal{H}_n$. Then $\text{per } A$ is the maximum eigenvalue of $\pi(A)$.*

If $A \in \mathcal{H}_n$ then $\pi(A)$ is a principal submatrix of the n -fold Kronecker product $\otimes^n A$, and hence $\pi(A) \in \mathcal{H}_{n!}$. It is easy to see that the row and column sums of $\pi(A)$ equal $\text{per } A$ and hence $\text{per } A$ is an eigenvalue. Schur [50] (cf. [3]) implicitly showed that $\det A$ is also an eigenvalue, and indeed it is the lowest eigenvalue of $\pi(A)$. So Conjecture 4 is equivalent to the assertion that all eigenvalues of $\pi(A)$ lie in the interval $[\det A, \text{per } A]$. It can be shown (see Lemma 1 in [44]) that $f_\chi(A)$ is an eigenvalue of $\pi(A)$ for any character χ of a subgroup of \mathcal{S}_n , from which it follows that Conjecture 4 implies Conjecture 1.

Pate [43] proved a special case of Conjecture 4 in order to show that Conjecture 1 holds for immanants associated with two part partitions. Soules [54] showed that if Conjecture 4 is false for real matrices then for the smallest order for which it fails there must be a counterexample which is singular, has zero row sums and has several other properties. Interestingly, as we will see shortly, Drury [11] did eventually show that Conjecture 4 fails for real matrices.

Showing similar intuition to that which motivated Lieb [28], Chollet [8] proposed the following permanental analogue of Oppenheim's inequality for determinants:

Conjecture 5. *If $A, B \in \mathcal{H}_n$ then*

$$\text{per}(A \circ B) \leq (\text{per } A)(\text{per } B). \quad (4)$$

Chollet himself showed that it suffices to consider the case when $B = \overline{A}$, when (4) reduces to $\text{per}(A \circ \overline{A}) \leq (\text{per } A)^2$. By a standard scaling argument, we also lose no generality by assuming that $A \in \mathcal{C}_n$. The argument goes like this. If $A = [a_{i,j}]$ has a zero row then both sides of (4) are zero and there is nothing to prove. So we may assume that $a_{i,i} > 0$ for each i . Now define a diagonal matrix $D = [d_{i,i}]$ by $d_{i,i} = a_{i,i}^{-1/2}$. Replace A by DAD and note that $DAD \in \mathcal{C}_n$. Also both $\text{per}(A \circ \overline{A})$ and $(\text{per } A)^2$ have been scaled by the same factor, namely $h(A)^{-2}$, which justifies the claim.

Let $V = V(A) = \pi(A)\pi(A)^* = \pi(A)^2$. Note that for any k ,

$$\begin{aligned} V_{k,k} &= \sum_{j=1}^{n!} \pi(A)_{k,j} \pi(A)_{j,k}^* = \sum_{j=1}^{n!} |\pi(A)_{k,j}|^2 = \sum_{\tau \in \mathcal{S}_n} \left| \prod_{i=1}^n a_{i,\tau(i)} \right|^2 = \sum_{\tau \in \mathcal{S}_n} \prod_{i=1}^n |a_{i,\tau(i)}|^2 \\ &= \text{per}([|a_{i,j}|^2]) = \text{per}(A \circ \overline{A}). \end{aligned}$$

Also the sum of the elements in row k of V is

$$\sum_{j=1}^{n!} V_{k,j} = \sum_{j=1}^{n!} \sum_{i=1}^{n!} \pi(A)_{k,i} \pi(A)_{i,j} = \sum_{i=1}^{n!} \pi(A)_{k,i} \sum_{j=1}^{n!} \pi(A)_{i,j} = \sum_{i=1}^{n!} \pi(A)_{k,i} \text{per } A = (\text{per } A)^2.$$

So to prove Conjecture 5 it suffices to show that $\pi(A)^2$ has row sums that exceed its diagonal entries. Interestingly, if $A \in \mathcal{C}_n$ then $\pi(A)$ itself has the targetted property, since its row sums are $\text{per } A$, which exceeds $h(A) = 1 = a_{k,k}$ for each k .

Zhang [61] initiated a new line of attack on Conjecture 5 by proving several properties of a ‘‘maximising matrix’’, namely a matrix $X \in \mathcal{C}_n$ that maximises $\text{per}(A \circ X)$ for a given $A \in \mathcal{C}_n$.

Soon after Chollet's conjecture, Bapat and Sunder [2] proposed the following conjecture, which is stronger, given (1). Both conjectures are implied by Conjecture 4 (see, for example, [36]).

Conjecture 6. *If $A, B \in \mathcal{H}_n$ then $\text{per}(A \circ B) \leq (\text{per } A)h(B)$.*

Using a similar scaling argument to the one we gave for Conjecture 5, Zhang [60] noted that Conjecture 6 is true if and only if it is true for all correlation matrices A and B . Zhang also showed that it is true if $A, B \in \mathcal{C}_n$ and every off-diagonal entry of B is equal to some fixed t in the interval $[0, 1]$.

Bapat and Sunder [3] proved Conjecture 4 for $n \leq 3$. Marcus and Sandy [32] note that the $n = 3$ case of Conjecture 5 follows immediately (see also Gregorac and Hentzel [17]). In the same paper, Bapat and Sunder [3] gave a reformulation of Conjecture 4 which then appeared as Conjecture 32 in [40] as follows:

Conjecture 7. *Let c be a complex valued function on \mathcal{S}_n satisfying*

$$\sum_{\sigma, \tau \in \mathcal{S}_n} \overline{x(\tau)} c(\sigma\tau^{-1}) x(\sigma) \geq 0$$

for all complex valued functions x on \mathcal{S}_n . Then

$$c(\mathbf{1}_n) \text{ per } A \geq \sum_{\sigma \in \mathcal{S}_n} c(\sigma) \prod_{i=1}^n a_{i, \sigma(i)}$$

for any $A = [a_{i,j}] \in \mathcal{H}_n$.

Bapat and Sunder's paper [3] also contained this conjecture:

Conjecture 8. *If A is positive definite, then $\text{per } A$ is the largest eigenvalue of the matrix $[a_{i,j} \text{ per } A(i|j)]$.*

Much later, Beasley [4] made the following conjecture, which is stronger than Conjecture 5 and weaker than Conjecture 6:

Conjecture 9. *If $A, B \in \mathcal{H}_n$ then*

$$\text{per}(A \circ B) \leq \max \{ (\text{per } A)h(B), (\text{per } B)h(A) \}. \quad (5)$$

He proved this conjecture holds for $n \leq 3$ as well as noting that (5) is true if and only if it holds for all correlation matrices (using a similar scaling argument to that seen above).

Let $A \in \mathcal{H}_n$ and $1 \leq k \leq n$. We define $\mathcal{C}_k(A)$ to be an $\binom{n}{k} \times \binom{n}{k}$ matrix whose α, β entry is $\text{per } A[\alpha|\beta] \times \text{per } A(\alpha|\beta)$. Here, the entries are indexed by $\alpha, \beta \subset \{1, 2, \dots, n\}$ with $|\alpha| = |\beta| = k$ (we can impose lexicographic order on these row and column indices, just for concreteness, but the order is not important for our purposes). Also $A[\alpha|\beta]$ is the submatrix induced by rows α and columns β , whereas $A(\alpha|\beta)$ is the submatrix formed by the removal of those rows and columns. Pate [49] used Corollary 1 to deduce that $\text{per } A$ dominates the diagonal entries of $\mathcal{C}_k(A)$. He also showed that $\text{per } A$ is an eigenvalue of $\mathcal{C}_k(A)$, and made the following conjecture.

Conjecture 10. *Let $A \in \mathcal{H}_n$ and $1 \leq k \leq n$. Then the largest eigenvalue of $\mathcal{C}_k(A)$ is $\text{per } A$.*

The $k = 1$ case of Conjecture 10 is precisely Conjecture 8. Pate showed that Conjecture 8 holds for a "large subcone" of \mathcal{H}_n and also for all nonnegative real matrices. Pate also raised the challenge of proving the following consequence of Conjecture 8:

Conjecture 11. *For $A \in \mathcal{H}_n$,*

$$a_{11} \text{ per } A(1|1) - a_{12} \text{ per } A(1|2) - a_{21} \text{ per } A(2|1) + a_{22} \text{ per } A(2|2) \leq 2 \text{ per } A.$$

Conjecture 4 had achieved quite some prominence, in part because of the number of consequences a proof would have (see Figure 1). However, in the last five years or so a cascade of counterexamples have demolished most of those conjectures. Shchesnovich [51] was the first to disprove Conjecture 4, using a numerical search to find a rank 2 complex counterexample $A = W^*W$, where

$$W = \begin{bmatrix} 4 - 2i & 2 + 3i & -4 + 4i & -3 - 4i & 1 \\ 2 + 4i & 3i & 2 + 4i & 3i & -5 + 7i \end{bmatrix}.$$

He found that $\text{per } A = 814016640$ is smaller than the largest eigenvalue of $\pi(A)$, which is $320(2185775 + \sqrt{160600333345})$.

Drury [10] found a rank 2 complex $A \in \mathcal{C}_7$ such that A and $B = A^*$ provide a counterexample to Conjecture 6. His $A = X^*X$ where

$$X = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & \sqrt{2} & 1 & \omega & \omega^2 & \omega^3 & \omega^4 \end{bmatrix}$$

and $\omega = e^{2\pi i/5}$ is a primitive fifth root of unity. Note that $\text{per}(A \circ B) = 6185/128 > 45 = \text{per } A$. Drury then followed that in [11] with a geometrically inspired 16×16 real counterexample to the same conjecture. He took a compound of a regular dodecahedron and a regular icosahedron inscribed together in the unit sphere. He then chose one representative from each of the 16 pairs of antipodal vertices and wrote its coordinates as one row in a 16×3 matrix Y . The real correlation matrix $A = YY^*$ provided a counterexample to Conjecture 6 (with $B = A$). Drury also noted that A gives a real counterexample to Conjecture 4. He did not observe, but it is easy to see, that both his counterexamples to Conjecture 6 are also counterexamples to Conjecture 9. Finally, Drury [12] gave a rank 2 complex counterexample to Conjecture 8. Let $A = Z^*Z$, where

$$Z = \begin{bmatrix} -7 + 4i & 9 - 3i & -6 + 2i & 3 + 4i & 7 + 6i & 4 - 4i & i & 5 - 8i \\ 4 - 5i & 1 + 4i & -8 - 2i & -7 + 4i & 1 - 4i & 1 - 8i & 8 - 6i & 1 - 3i \end{bmatrix}.$$

Then $\text{per } A = 2977257622144118400$ and the largest eigenvalue of $[a_{i,j} \text{ per } A(i|j)]$ is approximately 1.7% larger at $\approx 3.028 \times 10^{18}$.

Tran [59] showed that Conjecture 10 is implied by Conjecture 4, before developing the following (human checkable) counterexample to both conjectures. Consider the rank 2 matrix

$$H = \begin{bmatrix} 3 & 1 - 2i & -1 & 1 + 2i & 1 \\ 1 + 2i & 3 & 1 - 2i & -1 & 1 \\ -1 & 1 + 2i & 3 & 1 - 2i & 1 \\ 1 - 2i & -1 & 1 + 2i & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \in \mathcal{H}_5.$$

Then $\text{per } H = 504$ and the spectrum of $\pi(H)$ is $512^5, 504^1, 448^5, 384^4, 320^4, 240^4, 160^4, 0^{93}$. In particular, $\text{per } H$ is not the largest eigenvalue of $\pi(H)$, so H is a counterexample to Conjecture 4. Tran also showed that H is a counterexample to Conjecture 10. Of course, on the basis of the implications in Figure 1, [10], [11] and [12] all provided counterexamples to Conjecture 4. Interestingly, none of the counterexamples mentioned so far appear to disprove Conjecture 1, Conjecture 5 or Conjecture 11.

Both Drury [12] and Zhang [62] pose questions regarding the smallest orders for which the disproved conjectures fail. Conjecture 4 is known to hold for $n \leq 3$ and fail for $n \geq 5$. The $n = 4$

case remains open. Shchesnovich [51] searched for counterexamples for order 4 but was unable to find any. Similarly, there is the question of the order of the smallest real counterexample (currently the smallest known has order 16). Another challenge is to search for new bounds on the eigenvalues of $\pi(A)$ now that we know they can exceed $\text{per } A$.

It is well-known that $\pi(A)$ is unitarily similar to a block diagonal matrix where the blocks are indexed by the irreducible representations of \mathcal{S}_n . Hence each eigenvalue of $\pi(A)$ can be associated with an irreducible character of \mathcal{S}_n , and hence with a partition of n . In the same way, any counterexamples to Conjecture 4 can also be associated with a particular partition of n . Drury [12] calculated that the partitions associated with the counterexamples in [51] and [12] are $(3, 2)$ and $(7, 1)$ respectively. He then conjectured that these are the only problematic cases in the following sense:

Conjecture 12. *Suppose that the Ferrers diagram for a partition Π of n does not contain the Ferrers diagram for $(3, 2)$ nor $(7, 1)$. Then for all $A \in \mathcal{H}_n$, the largest eigenvalue of the block of $\pi(A)$ associated with Π is at most $\text{per } A$.*

To clarify what “contains” means in this last conjecture, the Ferrers diagram for Π contains $(3, 2)$ if the first part of Π is at least 3 and the second part is at least 2. Drury [12] also conjectures that Conjecture 8 holds for real matrices, and that Conjecture 6 holds for real matrices of rank 2.

3 Lieb’s legacy

In the previous section we examined many conjectures which had been inspired either directly or indirectly by Lieb’s paper [28]. Those conjectures have undoubtedly prompted much research of permanents of Hermitian matrices, even though many of the conjectures are now known to be incorrect. In this section we examine other aspects of the legacy of [28].

We begin by briefly reviewing some of the highlights of progress on Conjecture 1. Any reader who is interested in further details is encouraged to seek out the expositions by Merris [36], [37], James [23], [24], Pate [44], [46], [47], Cheon and Wanless [7], Bapat [1] and Zhang [62], among others.

Merris [35] gave the following upper bound on generalised matrix functions, in the spirit of Conjecture 1, but known to be weaker [36].

Theorem 2. *Let G be a subgroup of \mathcal{S}_n , and let χ be a character of G . Then*

$$(h(A^n))^{1/n} \geq f_\chi(A)$$

for any $A \in \mathcal{H}_n$.

Most progress on the permanental dominance conjecture has been made on its specialisation to immanants. For example, as a culmination of a series of papers by Pate and others, we know [46] that Conjecture 1 is true for all immanants when $n \leq 13$. In comparison, the general conjecture has only been shown for $n \leq 3$. Tabata [57], [58] examined the $n = 3$ case in detail and showed strict inequality holds when χ is not the principal character. Interestingly, James [24] discovered that the following matrix in \mathcal{H}_4

$$\begin{bmatrix} \sqrt{3} & i & i & -i \\ -i & \sqrt{3} & i & i \\ -i & -i & \sqrt{3} & -i \\ i & -i & i & \sqrt{3} \end{bmatrix}$$

achieves equality in Conjecture 1 when G is the alternating group A_4 and χ is the character of that group that satisfies $\chi((12)(34)) = 1$ and $\chi((234)) = e^{2\pi i/3}$. It follows that for any $n \geq 4$ there is at least one non-principal character which achieves equality in Conjecture 1 for some $A \in \mathcal{H}_n$.

To describe the results which have been proved on immanants we define a partial order \preceq on the set of partitions of an integer n . Let λ and μ be two such partitions and let χ and χ' be the characters associated with λ and μ respectively by the well known bijection between partitions of n and irreducible characters of \mathcal{S}_n . By $\lambda \preceq \mu$ we will mean that $f_\chi(H) \leq f_{\chi'}(H)$ for all $H \in \mathcal{H}_n$. Let $J_n \in \mathcal{H}_n$ be the matrix in which every entry is 1. Note that, $f_{(3,1)}(J_2 \oplus J_2) < f_{(2,2)}(J_2 \oplus J_2)$ so it is not true that $(2, 2) \preceq (3, 1)$. Similarly, $f_{(3,1)}(J_3 \oplus J_1) > f_{(2,2)}(J_3 \oplus J_1)$ so that $(3, 1) \preceq (2, 2)$ also fails. Hence \preceq is not a total order. James [24] notes that using similar tests involving direct products of blocks J_i for various values of i , it can be shown that for two partitions λ, μ of n to satisfy $\lambda \preceq \mu$ it is necessary but not sufficient that μ majorises λ .

The result of Schur [50] implies that $(1^n) \preceq \lambda$ for all partitions λ of n . The specialisation of the permanent dominance conjecture to immanants asserts that for all λ ,

$$\lambda \preceq (n). \quad (6)$$

The following beautiful theorem of Heyfron [22] shows that the ‘‘single-hook immanants’’ are neatly ordered between the determinant and permanent.

Theorem 3.

$$(1^n) \preceq (2, 1^{n-2}) \preceq (3, 1^{n-3}) \preceq \cdots \preceq (n-1, 1) \preceq (n). \quad (7)$$

Heyfron’s theorem confirmed a conjecture originally made by Merris [34] (note that Merris [36] attributes the conjecture to himself and Pierce). A number of partial results in this direction had been obtained by Merris and Watkins [38] and Johnson and Pierce [26], [27], for example.

Stembridge [55], [56] notes that the analogue of the permanent dominance conjecture is trivially true for totally positive matrices (real matrices with non-negative minors). He considers to what extent analogues of Theorem 3 hold for this class of matrices. He also considers inequalities for immanants of so called Jacobi-Trudi matrices, which are closely connected with the combinatorics of symmetric functions. Haiman [21] develops these ideas and makes connections with Kazhdan-Lusztig theory and characters of Hecke algebras.

Chan and Lam [5], [6] sharpened the inequalities in (7) in the case of matrices which are the Laplacians of trees. More recently, Nagar and Sivasubramanian [41] generalised Chan and Lam’s work to q-Laplacians.

A general scheme for obtaining inequalities involving immanants is described by Pate in [45] and many such inequalities can be found throughout his papers. For example, in [47] he showed the following results for positive integers n, p and k . If $k \geq 2$ and $n \geq p+k-2$ then $(n+p-i, n^k, i) \preceq (n+p, n^k)$ for $1 \leq i \leq p$. On the other hand if $p \geq n+k-1$ then $(n+p-i, n^k, i) \preceq (n+p, n^k)$ whenever $p/2 \leq i \leq n$. In the same paper he obtained the following asymptotic result. For positive integers k and s there exists an integer $N_{k,s}$ such that for all $n \geq N_{k,s}$,

$$(n+s, n^k) \preceq (2n+s, n^{k-1}) \preceq (3n+s, n^{k-2}) \preceq \cdots \preceq (kn+n+s).$$

Corollary 1 has proven useful when deriving special cases of the permanent dominance conjecture (see e.g. Merris and Watkins [38]). For example, it immediately proves that Conjecture 1 holds when G is a Young subgroup of \mathcal{S}_n (that is, a direct product of symmetric groups) and χ is the trivial character on G .

Next we turn our attention to other applications of Theorem 1 and its corollaries. Strengthening (1), Marcus and Soules [33] found upper and lower bounds on $\text{per } A - h(A)$ and $h(A) - \det A$ for $A \in \mathcal{H}_n$. They also proved this stronger form of Corollary 1:

Theorem 4. *For A in (2) we have $\text{per } A \geq (\text{per } B)(\text{per } D) + \mu^{n-2}\|C\|^2$, where μ is the smallest eigenvalue of A and $\|\cdot\|$ denotes the Euclidean norm.*

Grone and Pierce [19] used Corollary 1 to prove the following bound for permanents of correlation matrices.

Theorem 5. *Let $A = [a_{i,j}] \in \mathcal{C}_n$. Then $\text{per } A \geq \frac{1}{n}\|A\|$, with equality if and only if $n = 2$ or A is an identity matrix or A is unitarily similar to*

$$Y_3 = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}.$$

From Theorem 5 they also deduced the following three corollaries, each of which had been conjectured by Grone and Merris [18].

Corollary 3. *If $A \in \mathcal{C}_n$ then $\text{per } A \geq (h(A^2))^{1/n}$.*

Corollary 4. *If $A \in \mathcal{H}_n$ then $(n-1)\text{per } A + \det A \geq n h(A)$.*

Corollary 5. *If $A \in \mathcal{C}_n$ is singular then $\text{per } A \geq n/(n-1)$.*

The following result of Pate [48] was motivated by Corollary 5 (cf. (1)).

Theorem 6. *If $A \in \mathcal{H}_n$ is real, and has rank at most 2, then*

$$\text{per } A \geq \frac{n!}{2^{n-1}}h(A).$$

Frenkel [15] derives analogues of Theorem 1 for Pfaffians and Hafnians, then applies his results to something called the real linear polarisation constant problem. He also remarks that in order to derive Theorem 1 it is not necessary to assume that A is positive semi-definite; it suffices for both the diagonal blocks B and D to be positive semi-definite.

In a follow-up paper, Frenkel [16] obtained an analogue of Corollary 1 for α -permanents. The α -permanent is defined by

$$\text{per}_\alpha(A) = \sum_{\sigma \in \mathcal{S}_n} \alpha^{\nu(\sigma)} \prod_{i=1}^n a_{i,\sigma(i)}, \quad (8)$$

where $\nu(\sigma)$ is the number of disjoint cycles of the permutation σ . Note that the 1-permanent is just the permanent. Also, the (-1) -permanent is equal to the determinant, up to a factor of $(-1)^n$. Frenkel's Theorem is this:

Theorem 7. *Suppose that α is either a nonnegative integer or a real number with $\alpha \geq n-1$. Then, for $A \in \mathcal{H}_n$ as in (2) we have $\text{per}_\alpha A \geq (\text{per}_\alpha B)(\text{per}_\alpha D)$ and $0 \leq (-1)^n \text{per}_{-\alpha} A \leq (-1)^n (\text{per}_{-\alpha} B)(\text{per}_{-\alpha} D)$.*

Another application of Corollary 1 to α -permanents is given by Eisenbaum [13] in her study of the stochastic comparisons between α -permanental point processes. Shirai [52] studies a function closely related to (8) and considers for what values of α it is positive on \mathcal{H}_n .

For a short paper, [28] has proved an inspiration for investigations in many different directions.

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