Existence results for a class of quasilinear Schrödinger equations with singular or vanishing potentials

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Abstract

Given two continuous functions $V(r) \geq 0$ and $K(r) > 0$ ($r > 0$), which may be singular or vanishing at zero as well as at infinity, we study the quasilinear elliptic equation

$$-\Delta w + V(|x|)\, w - w\left(\Delta w^2\right) = K(|x|)g(w) \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$. To study this problem we apply a change of variables $w = f(u)$, already used by several authors, and find existence results for nonnegative solutions by the application of variational methods. The main features of our results are that they do not require any compatibility between how the potentials V and K behave at the origin and at infinity, and that they essentially rely on power type estimates of the relative growth of V and K, not of the potentials separately. Our solutions satisfy a weak formulations of the above equation, but we are able to prove that they are in fact classical solutions in $\mathbb{R}^N \backslash \{0\}$. To apply variational methods, we have to study the compactness of the embedding of a suitable function space into the sum of Lebesgue spaces $L_K^{q_1} + L_K^{q_2}$, and thus into L_K^q $(= L_K^q + L_K^q)$ as a particular case. The nonlinearity g has a double-power behavior, whose standard example is $g(t) = \min\{t^{q_1 - 1}, t^{q_2 - 1}\}$, recovering the usual case of a single-power behavior when $q_1 = q_2$.

Keywords. Quasilinear elliptic PDEs, unbounded or decaying potentials, Orlicz-Sobolev spaces, compact embeddings

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1 Introduction

In the present paper, we study the following quasilinear elliptic equation

$$-\Delta w + V(|x|)\, w - w\left(\Delta w^2\right) = K(|x|)g(w) \quad \text{in } \mathbb{R}^N \tag{1.1}$$

where $N \geq 3$, $V \geq 0$ and $K > 0$ are given potentials, and $g : \mathbb{R} \to \mathbb{R}$ is a continuous nonlinearity such that $g(0) = 0$. Searching for standing waves solutions, this equation derives from an evolution Schrödinger equation which has been used to study several physical phenomena (see [21, 24, 29] and the references therein), such as laser beams in matter [9] and quasi-solitons in superfluids films [20].

It is not easy to apply variational methods to study (1.1), because the (formally) associated functional presents unusual integral terms, like $\int_{\mathbb{R}^N} w^2 |\nabla w|^2 dx$. In recent times, a great amount of work has been made on equation (1.1) and several techniques have been introduced to overcome these difficulties (see [1,12–18,21,22,25–28,30–33,38,39] and the references therein). In this paper, following an idea introduced

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in [24], we exploit a change of variable $w = f(u)$ where f satisfies a suitable ordinary differential equation (see Section 2). The problem in the new unknown u can be faced with usual variational methods, working in an Orlicz-Sobolev space. This idea has been used in [14, 21, 31, 32], among others.

In almost all the papers dealing with (1.1), the potential V (be it radial or nonradial) is supposed to be positive and bounded away from zero at infinity. At the best of our knowledge, the only papers dealing with a potential V allowed to vanish at infinity are [1, 21, 22, 31] (see also [32] for equation (1.1) in presence of a parameter). In [1] and [21], the authors respectively prove existence and nonexistence results assuming that V is bounded. In [22], existence of solutions is obtained for possibily singular V's but bounded K's. In [31], which is the paper that inspired our work, both V and K can be singular or vanishing at zero or at infinity, and the authors prove existence of solution assuming that the potentials are radial and essentially behave as powers of $|x|$ as $|x| \to 0$ and $|x| \to \infty$ (see the paper introduction for more precise assumptions).

Here we study equation (1.1) via the change of variable $w = f(u)$ in the case in which both V and K are radial potentials that may be singular or vanishing at zero as well as at infinity. This implies that, even in the new variational setting brought in by the variable change, the usual embeddings theorems for Sobolev spaces are not available, and new embedding theorems need to be proved. We observe that, for semilinear and p-laplacian elliptic equations, this has been done in several papers: see e.g. the references in [6, 7, 19] for a bibliography concerning the usual Laplace equation, [3, 5, 8, 11, 34–36, 40, 41] for equations involving the p-laplacian, and [10, 37] for problems with a potential A on the derivatives (see also [4] for biharmonic equations).

The main novelty in our approach (with respect to the previous literature, and especially to [31]) is two-folded. First, we look for embeddings of a suitable function space not into a single (weighted) Lebesgue space L^q_K but into a sum of Lebesgue spaces $L^{q_1}_K + L^{q_2}_K$. This allows to study separately the behaviour of the potentials V and K at 0 and ∞, assuming independent sets of hypotheses about these behaviours. Second, we assume hypotheses not on V and K separately but on their ratio, so admitting asymptotic behaviors of general kind for the two potentials, not only power-like (cf. Section 8).

As a conclusion, our approach shows that, in order to have solutions, the potentials V and K can have independent behaviours at zero and at infinity, no needing to satisfy compatibility conditions between such behaviours. Moreover, what does really count are not the growths of V and K separately, but only how they grow (or decay) relatively to one another.

The paper is organized as follows. In Section 2 we introduce our hypotheses on V and K, the change of variables $w = f(u)$ and the main function spaces X and E we will work in. In Section 3 we state a general result concerning the embedding properties of E into $L^{q_1}_K + L^{q_2}_K$ (Theorem 3.1) and some explicit conditions ensuring that the embedding is compact (Theorems 3.2 and 3.3). The general result is proved in Section 4, the explicit conditions in Section 5. In Section 6 we introduce our hypotheses on the nonlinearity g, we study the main properties of the functional I associated to the dual problem and of its critical points, which give rise to solutions to (1.1). In Section 7 we apply our embedding results to get existence of non negative solutions for equation (1.1), stating and proving our main existence result, which is Theorem 7.1. Section 8 is devoted to concrete examples of potentials V and K satisfying our hypotheses, though escaping the previous literature.

Notations. We end this introductory section by collecting some notations used in the paper.

• $\mathbb{R}_+ = (0, +\infty) = \{x \in \mathbb{R} : x > 0\}$.

• For every $R > 0$, we set $B_R = \{x \in \mathbb{R}^N : |x| < R\}$.

• For any subset $A \subseteq \mathbb{R}^N$, we denote $A^c := \mathbb{R}^N \setminus A$. If A is Lebesgue measurable, $|A|$ stands for its measure.

• $\hookrightarrow$ denotes *continuous* embeddings.

• If Y is a Banach space, Y' is its dual.

• $C_c^\infty(\Omega)$ is the space of the infinitely differentiable real functions with compact support in the open set $\Omega \subseteq \mathbb{R}^N$. If Ω has radial symmetry, $C_{c,r}^\infty(\Omega)$ is the subspace of $C_c^\infty(\Omega)$ made of radial functions.

• For any measurable set $A \subseteq \mathbb{R}^N$, $L^q(A)$ and $L^q_{\text{loc}}(A)$ are the usual real Lebesgue spaces. If $\rho : A \to \mathbb{R}_+$ is a measurable function, then $L^p(A, \rho(z)\, dz)$ is the real Lebesgue space with respect to the measure $\rho(z)\, dz$ (dz stands for the Lebesgue measure on $\mathbb{R}^N$). In particular, if $K : \mathbb{R}_+ \to \mathbb{R}_+$ is measurable, we denote $L^q_K(A) := L^q(A, K(|x|)\, dx)$.

• For $N \geq 3$, $2^* = \frac{2N}{N-2}$ is the critical exponent of Sobolev embeddings.

2 Hypotheses and preliminary results

Throughout this paper, we assume $N \geq 3$ and the following hypothesis (**H**) on V, K:

(**H**) $V : \mathbb{R}_+ \to [0, +\infty)$ and $K : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, and there is $C > 0$ such that for all $r \in (0, 1)$ one has

$$V(r) \leq \frac{C}{r^2}.$$

We begin by introducing the function f we need to define the Orlicz-Sobolev space in which we will work. This function is defined as the solution of the following Cauchy problem:

$$\begin{cases} f'(t) = \frac{1}{\sqrt{1+2f(t)^2}} & \text{in } \mathbb{R} \\ f(0) = 0 \end{cases} \tag{2.1}$$

The following lemma gives the main properties of the solution of (2.1). For the proofs see [14, 31].

Lemma 2.1. *There is a unique solution $f \in C^\infty(\mathbb{R}, \mathbb{R})$ of (2.1). Such a solution is odd, strictly increasing, and surjective (hence invertible). Moreover, it satisifes the following properties:*

(1) $|f'(t)| \leq 1$ *for all* $t \in \mathbb{R}$*;*

(2) $|f(t)| \leq |t|$ *for all* $t \in \mathbb{R}$*;*

(3) $f(t)/t \to 1$ *as* $t \to 0$*;*

(4) $f(t)/\sqrt{t} \to 2^{1/4}$ *as* $t \to +\infty$*;*

(5) $f(t)/2 \leq t f'(t) \leq f(t)$ *for all* $t \geq 0$*;*

(6) $|f(t)| \le 2^{1/4}\sqrt{|t|}$ *for all* $t \in \mathbb{R}$*;*

(7) There is a constant $C_1 > 0$ *such that*

$$|f(t)| \ge C_1|t| \quad \text{if } |t| \le 1; \qquad |f(t)| \ge C_1\sqrt{|t|} \quad \text{if } |t| \ge 1;$$

(8) There are two positive constants c_1, c_2 *such that* $|t| \le c_1|f(t)| + c_2 f(t)^2$ *for all* $t \in \mathbb{R}$*;*

(9) $|f(t)f'(t)| \le \frac{1}{\sqrt{2}}$ *for all* $t \in \mathbb{R}$*;*

(10) The function $f(t)^2$ *is strictly convex;*

(11) There is a constant $C > 0$ *such that* $f(2t)^2 \le Cf(t)^2$ *for all* $t \in \mathbb{R}$*.*

We now use the function f to define a change of unknown: we call w the solution of (1.1) that we are looking for and we set $w = f(u)$, where u is the new unknown, living in a suitable space that we are going to define. In this way, to get solutions w to (1.1) we will look for solutions u to the equation

$$-\Delta u + V(|x|)\, f(u)f'(u) = K(|x|)g(f(u))f'(u) \quad \text{in } \mathbb{R}^N, \tag{2.2}$$

which will be obtained as critical points of the following functional:

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(|x|)f(u)^2 dx - \int_{\mathbb{R}^N} K(|x|)G(f(u))\, dx \tag{2.3}$$

The critical points of I and their relations with solutions of (1.1) will be studied in Section 6, my means of the following hypotheses on the nonlinearity: $g : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$(\mathbf{g_1})$ $\exists\theta > 2$ such that $0 \le 2\theta G(t) \le g(t)\, t$ for all $t \in \mathbb{R}$;

$(\mathbf{g_2})$ $\exists t_0 > 0$ such that $G(t_0) > 0$, where $G(t) = \int_0^t g(s)ds$;

$(\mathbf{g}_{q_1,q_2})$ there exits a constant $C > 0$ such that $|g(t)| \le C \min\left\{|t|^{q_1-1}, |t|^{q_2-1}\right\}$ for all $t \in \mathbb{R}$.

We notice that these hypotheses imply $q_1, q_2 \ge 2\theta$. We also observe that, if $q_1 \ne q_2$, the double-power growth condition $(\mathbf{g}_{q_1,q_2})$ is more stringent than the more usual single-power one, since it implies $|g(t)| \le C|t|^{q-1}$ for $q = q_1$, $q = q_2$ and every q in between. On the other hand, we will never require $q_1 \ne q_2$ in $(\mathbf{g}_{q_1,q_2})$, so that our results will also concern single-power nonlinearities as long as we can take $q_1 = q_2$.

In this section and in the following ones, we introduce the function space E in which we will obtain critical points of I and we study the relevant compactness results for E.

First, we introduce the space $D_r^{1,2}(\mathbb{R}^N)$, which is the closure of $C_{c,r}^\infty(\mathbb{R}^N)$ with respect to the norm $||u||_{1,2} := \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{1/2}$. It is well known that $D_r^{1,2}(\mathbb{R}^N)$is a Hilbert space. Then we define a second Hilbert space

$$X := \left\{ u \in D_r^{1,2}(\mathbb{R}^N) \;\Big|\; \int_{\mathbb{R}^N} V(|x|)|u|^2 dx < +\infty \right\}$$

endowed with the norm $||u|| := \left(||u||_{1,2}^2 + ||u||^2_{L^2(\mathbb{R}^N, V(|x|)dx)} \right)^{1/2}$. Finally we introduce the main function space that we will use, which is

$$E := \left\{ u \in D_r^{1,2}(\mathbb{R}^N) \;\Big|\; \int_{\mathbb{R}^N} V(|x|)f(u)^2 dx < +\infty \right\}.$$

In E, we first define the norm

$$||u||_o := \inf_{k>0} \frac{1}{k}\left[1 + \int_{\mathbb{R}^N} V(|x|) f(ku)^2 dx\right],$$

which is an Orlicz norm. Then we introduce the norm

$$||u|| := ||u||_{1,2} + ||u||_o.$$

The space E, endowed with the norm $||.||$, is an Orlicz-Sobolev space. In the results, we recall its main properties.

Theorem 2.2. *$(E, ||.||)$ is a Banach space and the following continuous embedding holds:*

$$E \hookrightarrow D_r^{1,2}\left(\mathbb{R}^N\right).$$

Proof. The fact that E is a Banach space derives from the general theory of Orlicz spaces, together with the properties of the function f stated in the above lemma, in particular (10) and (11) (see [31]). The embedding is obvious from the definitions of E and its norm. □

Corollary 2.3. *There are constants $S_N, C_N > 0$ (only depending on N) such that for all $u \in E$ it holds:*

$$\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{1/2^*} \leq S_N\, ||u||, \qquad |u(x)| \leq C_N \frac{||u||}{|x|^{\frac{N-2}{2}}} \quad a.e.\, x \in \mathbb{R}^N.$$

Proof. These are well known properties of any $u \in D_r^{1,2}\left(\mathbb{R}^N\right)$. □

Lemma 2.4. *(1) There exists $C > 0$ such that for all $u \in E$ one has*

$$\frac{\int_{\mathbb{R}^N} V(|x|) f(u)^2 dx}{1 + \left(\int_{\mathbb{R}^N} V(|x|) f(u)^2 dx\right)^{1/2}} \leq C\, ||u||.$$

(2) If $u_n \to u$ in E, then

$$\int_{\mathbb{R}^N} V(|x|) \left|f(u_n)^2 - f(u)^2\right| \, dx \to 0 \quad \textit{and} \quad \int_{\mathbb{R}^N} V(|x|) \left|f(u_n) - f(u)\right|^2 \, dx \to 0.$$

(3) If $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ and

$$\int_{\mathbb{R}^N} V(|x|) f(u_n)^2 \, dx \to \int_{\mathbb{R}^N} V(|x|) f(u)^2 \, dx$$

then $||u_n - u||_o \to 0$.

Proof. For the proof see [27], [14], [31]. We just point out that the proofs also work in our hypotheses, which are a little different from theirs. □

Corollary 2.5. *Assuming $(\mathbf{H})$, we have the continuous embedding $X \hookrightarrow E$.*

Proof. This is Corollary 2.1 of [14]. Their proof can be repeated in our case. □

Lemma 2.6. *$C_{c,r}^{\infty}(\mathbb{R}^N)$ is dense in E.*

Proof. The proof is contained in the Master's degree thesis [23], which is unpublished and in Italian, so we give it here. Let $u \in E$ and assume first that $\operatorname{supp} u$ is bounded. By standard results, there is a sequence $\{\varphi_n\}_n \subseteq C^\infty_{c,r}(\mathbb{R}^N)$ such that $\varphi_n \to u$ in $D^{1,2}_r(\mathbb{R}^N)$ and $\varphi_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$. We have to prove that $||\varphi_n - u||_o \to 0$. Thanks to Lemma 2.4, it is enough to prove the following claim:

$$\int_{\mathbb{R}^N} V(|x|)f(\varphi_n)^2\,dx \to \int_{\mathbb{R}^N} V(|x|)f(u)^2\,dx.$$

By hypothesis, there is $M > 0$ such that $\operatorname{supp} u \subseteq B_M$. As the sequence $\{\varphi_n\}_n$ is obtained by convolution of u with a family of mollifiers, we can assume $\operatorname{supp}\varphi_n \subseteq B_{M+1}$ for all n. Hence we have

$$\int_{\mathbb{R}^N} V(|x|)f(\varphi_n)^2\,dx = \int_{\{|x|\le 1\}} V(|x|)f(\varphi_n)^2\,dx + \int_{\{1\le|x|\le M+1\}} V(|x|)f(\varphi_n)^2\,dx.$$

We have $V(|x|)f(\varphi_n(x))^2 \to V(|x|)f(u(x))^2$ a.e., and we will apply Dominated Convergence Theorem. If $|x| \le 1$ and $x \neq 0$, using $(\mathbf{H})$ and Lemma 2.1 we have

$$V(|x|)f(\varphi_n(x))^2 \le C\frac{f(\varphi_n(x))^2}{|x|^2} \le C\frac{\varphi_n(x)^2}{|x|^2}.$$

As $\varphi_n \to u$ in $D^{1,2}_r(\mathbb{R}^N)$, from Hardy's inequality we get $\frac{\varphi_n^2}{|x|^2} \to \frac{u^2}{|x|^2}$ in $L^1(\mathbb{R}^N)$, whence, up to a subsequence, there exists a function $h \in L^1(\mathbb{R}^N)$ such that $\frac{\varphi_n^2}{|x|^2} \le h$. This implies

$$V(|x|)f(\varphi_n(x))^2 \le Ch$$

for a.e. $x \in B_1$. By Dominated Convergence Theorem we get

$$\int_{\{|x|\le 1\}} V(|x|)f(\varphi_n)^2\,dx \to \int_{\{|x|\le 1\}} V(|x|)f(u)^2\,dx.$$

If $1 \le |x| \le M+1$ we get $V(|x|)f(\varphi_n)^2 \le C\varphi_n^2$. As $\varphi_n \to u$ in $L^{2^*}(\mathbb{R}^N)$, we have $\varphi_n \to u$ in $L^2(B_{M+1})$, whence there exists $h_1 \in L^1(B_{M+1})$ such that $\varphi_n(x)^2 \le h_1(x)$ for a.e. $x \in B_{M+1}$. Hence $V(|x|)f(\varphi_n)^2 \le Ch_1$ and by Dominated Convergence

$$\int_{\{1\le|x|\le M+1\}} V(|x|)f(\varphi_n)^2\,dx \to \int_{\{1\le|x|\le M+1\}} V(|x|)f(u)^2\,dx.$$

This concludes the proof if $\operatorname{supp} u$ is bounded. In the general case, we choose a sequence of standard truncation functions $\{\zeta_n\}_n$. It is easy to show that $\zeta_n u \to u$ in E for every $u \in E$, and combining this result with the previous one we get the thesis. □

Lemma 2.7. *For any r, R such that $0 < r < R$, the embedding*

$$E \hookrightarrow L^2(B_R \setminus \overline{B}_r)$$

is continuous and compact.

Proof. The embedding result is easily proved for the space $D^{1,2}_r(\mathbb{R}^N)$, so the thesis derives from the continuous embedding $E \hookrightarrow D^{1,2}_r(\mathbb{R}^N)$. □

3 Compactness results for the space E

Let $N \geq 3$ and let V and K be as in $(\mathbf{H})$. In this section we state the main compactness results of this paper, concerning the space E, defined as above. The compactness results that we state here will be proved in Sections 4 and 5. They concern the embedding properties of E into the sum space

$$L_K^{q_1} + L_K^{q_2} := \left\{ u_1 + u_2 : u_1 \in L_K^{q_1}\left(\mathbb{R}^N\right),\, u_2 \in L_K^{q_2}\left(\mathbb{R}^N\right) \right\}, \quad 1 < q_i < \infty.$$

We recall from [8] that such a space can be characterized as the set of measurable mappings $u : \mathbb{R}^N \to \mathbb{R}$ for which there exists a measurable set $A \subseteq \mathbb{R}^N$ such that $u \in L_K^{q_1}(A) \cap L_K^{q_2}(A^c)$. It is a Banach space with respect to the norm

$$\|u\|_{L_K^{q_1}+L_K^{q_2}} := \inf_{u_1+u_2=u} \max\left\{ \|u_1\|_{L_K^{q_1}(\mathbb{R}^N)}, \|u_2\|_{L_K^{q_2}(\mathbb{R}^N)} \right\}$$

and the continuous embedding $L_K^q \hookrightarrow L_K^{q_1} + L_K^{q_2}$ holds for all $q \in [\min\{q_1, q_2\}, \max\{q_1, q_2\}]$. Our general embedding result is Theorem 3.1 below. The assumptions of this result are quite general but not so easy to check, so more handy conditions ensuring these general assumptions will be provided by the next results.

To state our results we introduce the following functions of $R > 0$ and $q > 1$:

$$\mathcal{S}_0(q, R) := \sup_{u \in E,\, \|u\|=1} \int_{B_R} K(|x|)\, |u|^q \, dx, \tag{3.1}$$

$$\mathcal{S}_\infty(q, R) := \sup_{u \in E,\, \|u\|=1} \int_{\mathbb{R}^N \setminus B_R} K(|x|)\, |u|^q \, dx. \tag{3.2}$$

Clearly $\mathcal{S}_0(q, \cdot)$ is nondecreasing, $\mathcal{S}_\infty(q, \cdot)$ is nonincreasing and both of them can be infinite at some R.

Theorem 3.1. *Let $q_1, q_2 > 1$.*

(i) If

$$\mathcal{S}_0(q_1, R_1) < \infty \quad \textit{and} \quad \mathcal{S}_\infty(q_2, R_2) < \infty \quad \textit{for some } R_1, R_2 > 0, \tag{$\mathcal{S}'_{q_1,q_2}$}$$

then E is continuously embedded into $L_K^{q_1}(\mathbb{R}^N) + L_K^{q_2}(\mathbb{R}^N)$.

(ii) If

$$\lim_{R \to 0^+} \mathcal{S}_0(q_1, R) = \lim_{R \to +\infty} \mathcal{S}_\infty(q_2, R) = 0, \tag{$\mathcal{S}''_{q_1,q_2}$}$$

then E is compactly embedded into $L_K^{q_1}(\mathbb{R}^N) + L_K^{q_2}(\mathbb{R}^N)$.

It is obvious that $(\mathcal{S}''_{q_1,q_2})$ implies $(\mathcal{S}'_{q_1,q_2})$. Moreover, these assumptions can hold with $q_1 = q_2 = q$ and therefore Theorem 3.1 also concerns the embedding properties of X into L_K^q, $1 < q < \infty$.

We now look for explicit conditions on V and K implying $(\mathcal{S}''_{q_1,q_2})$ for some q_1 and q_2. More precisely, in Theorem 3.2 we will find a range of exponents q_1 such that $\lim_{R \to 0^+} \mathcal{S}_0(q_1, R) = 0$, while in Theorem 3.3 we will do the same for exponents q_2 such that $\lim_{R \to +\infty} \mathcal{S}_\infty(q_2, R) = 0$.

For $\alpha \in \mathbb{R}$, $\beta \in [0,1]$, we define three functions $\alpha^*(\beta)$, $q_0^*(\alpha,\beta)$, $q_\infty^*(\alpha,\beta)$ by setting

$$\alpha^*(\beta) := \max\left\{\beta\frac{N+2}{2} - 1 - \frac{N}{2}, \frac{\beta}{2}(3N-2) - N\right\},$$

$$q_0^*(\alpha,\beta) := \frac{2\alpha + 2N - \beta(N+2)}{N-2}, \qquad q_\infty^*(\alpha,\beta) := 2\,\frac{\alpha + N - 2\beta}{N-2}.$$

Notice that $\alpha^*(\beta) = \beta\frac{N+2}{2} - 1 - \frac{N}{2} = -\frac{N+2}{2}(1-\beta)$ when $0 \le \beta \le \frac{1}{2}$, and $\alpha^*(\beta) = \frac{\beta}{2}(3N-2) - N$ when $\frac{1}{2} \le \beta \le 1$.

Theorem 3.2. *Assume that there exists $R_1 > 0$ such that*

$$\sup_{r\in(0,R_1)} \frac{K(r)}{r^{\alpha_0} V(r)^{\beta_0}} < +\infty \quad \textit{for some } 0 \le \beta_0 \le 1 \textit{ and } \alpha_0 > \alpha^*(\beta_0). \tag{3.3}$$

Then $\lim_{R\to 0^+} \mathcal{S}_0(q_1, R) = 0$ *for every* $q_1 \in \mathbb{R}$ *such that*

$$\max\{1, 2\beta_0\} < q_1 < q_0^*(\alpha_0, \beta_0). \tag{3.4}$$

Notice that, as $\beta \le 1$, it holds $\alpha^*(\beta) \ge -N(1-\beta)$. Also notice that the inequality $\max\{1, 2\beta_0\} < q_0^*(\alpha_0,\beta_0)$ is equivalent to $\alpha_0 > \alpha^*(\beta_0)$, so that such inequality is automatically true in (3.4) and does not ask for further conditions on α_0 and β_0.

Theorem 3.3. *Assume that there exists $R_2 > 0$ such that*

$$\sup_{r>R_2} \frac{K(r)}{r^{\alpha_\infty} V(r)^{\beta_\infty}} < +\infty \quad \textit{for some } 0 \le \beta_\infty \le 1 \textit{ and } \alpha_\infty \in \mathbb{R}. \tag{3.5}$$

Then $\lim_{R\to +\infty} \mathcal{S}_\infty(q_2, R) = 0$ *for every* $q_2 \in \mathbb{R}$ *such that*

$$q_2 > \max\{1, 2\beta_\infty, q_\infty^*(\alpha_\infty, \beta_\infty)\}. \tag{3.6}$$

Remark 3.4.

1. We mean $V(r)^0 = 1$ for every r (even if $V(r) = 0$). In particular, if $V(r) = 0$ for $r > R_2$, then Theorem 3.3 can be applied with $\beta_\infty = 0$ and assumption (3.5) means

$$\operatorname*{ess\,sup}_{r>R_2} \frac{K(r)}{r^{\alpha_\infty}} < +\infty \quad \text{for some } \alpha_\infty \in \mathbb{R}.$$

   Similarly for Theorem 3.2 and assumption (3.3), if $V(r) = 0$ for $r \in (0, R_1)$.

2. The assumptions of Theorems 3.2 and 3.3 may hold for different pairs (α_0, β_0), $(\alpha_\infty, \beta_\infty)$. In this case, of course, one chooses them in order to get the ranges for q_1, q_2 as large as possible. For example, assume that V is bounded in a neighbourhood of 0. If condition (3.3) holds true for a pair (α_0, β_0), then (3.3) also holds for all pairs (α_0', β_0') such that $\alpha_0' < \alpha_0$ and $\beta_0' < \beta_0$. Therefore, since $\max\{1, 2\beta\}$ is nondecreasing in β and $q_0^*(\alpha,\beta)$ is increasing in α and decreasing in β, it is convenient to choose $\beta_0 = 0$ and the best interval where one can take q_1 is $1 < q_1 < q_0^*(\overline{\alpha}, 0)$ with $\overline{\alpha} := \sup\left\{\alpha_0 : \operatorname{ess\,sup}_{r\in(0,R_1)} K(r)/r^{\alpha_0} < +\infty\right\}$ (here we mean $q_0^*(+\infty, 0) = +\infty$).

4 Proof of Theorem 3.1

In this section we assume, as usual, $N \geq 3$ and hypothesis $(\mathbf{H})$.

Lemma 4.1. *Let $R > r > 0$ and $1 < q < \infty$. Then there exist $\tilde{C} = \tilde{C}(N, r, R, q) > 0$ and $l = l(q) > 0$ such that $q - 2l > 0$ and $\forall u \in E$ one has*

$$\int_{B_R \setminus B_r} K(|x|)\, |u|^q \, dx \leq \tilde{C}\, \|K\|_{L^\infty(B_R \setminus B_r)} \|u\|^{q-2l} \left(\int_{B_R \setminus B_r} |u|^2 \, dx \right)^l . \tag{4.1}$$

Proof. Let $u \in E$ and fix $t > 1$ such that $t'q > 2$ (where $t' = t/(t-1)$). Then, by Hölder inequality and the pointwise estimates of Corollary 2.3, we have

$$\begin{aligned}
\int_{B_R \setminus B_r} K(|x|)\, |u|^q \, dx &\leq \left(\int_{B_R \setminus B_r} K(|x|)^t \, dx \right)^{\frac{1}{t}} \left(\int_{B_R \setminus B_r} |u|^{t'q} \, dx \right)^{\frac{1}{t'}} \\
&\leq |B_R \setminus B_r|^{\frac{1}{t}} \|K\|_{L^\infty(B_R \setminus B_r)} \left(\int_{B_R \setminus B_r} |u|^{t'q-2} |u|^2 \, dx \right)^{\frac{1}{t'}} \\
&\leq |B_R \setminus B_r|^{\frac{1}{t}} \|K\|_{L^\infty(B_R \setminus B_r)} \left(\frac{C_N \|u\|}{r^{\frac{N-2}{2}}} \right)^{q-2/t'} \left(\int_{B_R \setminus B_r} |u|^2 \, dx \right)^{\frac{1}{t'}} .
\end{aligned}$$

This proves (4.1), setting $l = 1/t'$ and $\tilde{C} = |B_R \setminus B_r|^{\frac{1}{t}} \left(C_N r^{-(N-2)/2} \right)^{q-2/t'}$. ∎

We now prove Theorem 3.1. Recall the definitions (3.1)-(3.2) of the functions $\mathcal{S}_0$ and $\mathcal{S}_\infty$, and the following result from [8] concerning convergence in the sum of Lebesgue spaces.

Proposition 4.2 ([8, Proposition 2.7])**.** *Let $\{u_n\} \subseteq L_K^{p_1} + L_K^{p_2}$ be a sequence such that $\forall \varepsilon > 0$ there exist $n_\varepsilon > 0$ and a sequence of measurable sets $E_{\varepsilon,n} \subseteq \mathbb{R}^N$ satisfying*

$$\forall n > n_\varepsilon, \quad \int_{E_{\varepsilon,n}} K(|x|)\, |u_n|^{p_1} \, dx + \int_{E_{\varepsilon,n}^c} K(|x|)\, |u_n|^{p_2} \, dx < \varepsilon. \tag{4.2}$$

Then $u_n \to 0$ in $L_K^{p_1} + L_K^{p_2}$.

Proof of Theorem 3.1. We prove each part of the theorem separately.

(i) By the monotonicity of $\mathcal{S}_0$ and $\mathcal{S}_\infty$, it is not restrictive to assume $R_1 < R_2$ in hypothesis $\left(\mathcal{S}'_{q_1,q_2}\right)$. In order to prove the continuous embedding, let $u \in E$, $u \neq 0$. Then we have

$$\int_{B_{R_1}} K(|x|)\, |u|^{q_1} \, dx = \|u\|^{q_1} \int_{B_{R_1}} K(|x|) \frac{|u|^{q_1}}{\|u\|^{q_1}} dx \leq \|u\|^{q_1} \mathcal{S}_0(q_1, R_1) \tag{4.3}$$

and, similarly,

$$\int_{B_{R_2}^c} K(|x|)\, |u|^{q_2} \, dx \leq \|u\|^{q_2} \mathcal{S}_\infty(q_2, R_2) . \tag{4.4}$$

We now use (4.1) of Lemma 4.1 and Lemma 2.7 to deduce that there exists a constant $\tilde{C}_1 > 0$, independent from u, such that

$$\int_{B_{R_2} \setminus B_{R_1}} K(|x|)\, |u|^{q_1} \, dx \leq \tilde{C}_1 \|u\|^{q_1} . \tag{4.5}$$

Hence $u \in L_K^{q_1}(B_{R_2}) \cap L_K^{q_2}(B_{R_2}^c)$ and thus $u \in L_K^{q_1} + L_K^{q_2}$. Moreover, if $u_n \to 0$ in E, then, using (4.3), (4.4) and (4.5), we get

$$\int_{B_{R_2}} K(|x|)\,|u_n|^{q_1}\,dx + \int_{B_{R_2}^c} K(|x|)\,|u_n|^{q_2}\,dx = o(1)_{n\to\infty},$$

which means $u_n \to 0$ in $L_K^{q_1} + L_K^{q_2}$ by Proposition 4.2.

(ii) Assume hypothesis $\left(\mathcal{S}''_{q_1,q_2}\right)$. Let $\varepsilon > 0$ and let $u_n \rightharpoonup 0$ in E. Then $\{\|u_n\|\}_n$ is bounded and, arguing as for (4.3) and (4.4), we can take $r_\varepsilon > 0$ and $R_\varepsilon > r_\varepsilon$ such that for all n one has

$$\int_{B_{r_\varepsilon}} K(|x|)\,|u_n|^{q_1}\,dx \leq \|u_n\|^{q_1}\ \mathcal{S}_0(q_1, r_\varepsilon) \leq \left(\sup_n \|u_n\|^{q_1}\right)\mathcal{S}_0(q_1, r_\varepsilon) < \frac{\varepsilon}{3}$$

and

$$\int_{B_{R_\varepsilon}^c} K(|x|)\,|u_n|^{q_2}\,dx \leq \left(\sup_n \|u_n\|^{q_2}\right)\mathcal{S}_\infty(q_2, R_\varepsilon) < \frac{\varepsilon}{3}.$$

Using (4.1) of Lemma 4.1 and the boundedness of $\{\|u_n\|\}$ again, we infer that there exist two constants $\tilde{C}_2, l > 0$, independent from n, such that

$$\int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} K(|x|)\,|u_n|^{q_1}\,dx \leq \tilde{C}_2 \left(\int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} |u_n|^2\,dx\right)^l,$$

where

$$\int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} |u_n|^2\,dx \to 0 \quad \text{as } n \to \infty \quad (\varepsilon \text{ fixed})$$

thanks to Lemma 2.7. Therefore we obtain

$$\int_{B_{R_\varepsilon}} K(|x|)\,|u_n|^{q_1}\,dx + \int_{B_{R_\varepsilon}^c} K(|x|)\,|u_n|^{q_2}\,dx < \varepsilon$$

for all n sufficiently large, which means $u_n \to 0$ in $L_K^{q_1} + L_K^{q_2}$ (Proposition 4.2). This concludes the proof of part (ii). $\square$

5 Proof of Theorems 3.2 and 3.3

Assume as usual $N \geq 3$ and hypothesis $(\mathbf{H})$.

Lemma 5.1. *Let $R_0 > 0$ and assume*

$$\Lambda := \sup_{x\in B_{R_0}} \frac{K(|x|)}{|x|^\alpha V(|x|)^\beta} < +\infty \quad \textit{for some } 0 \leq \beta \leq 1 \textit{ and } \alpha \in \mathbb{R}.$$

Let $u \in E$ and assume that there exist $\nu \in \mathbb{R}$ and $m > 0$ such that

$$|u(x)| \leq \frac{m}{|x|^\nu} \quad \textit{almost everywhere in } B_{R_0}.$$

Then there exists a constant $C = C(N, R_0, \alpha, \beta) > 0$ such that $\forall R \in (0, R_0)$ and $\forall q > \max\{1, 2\beta\}$, one has

$$
\int_{B_R} K(|x|)\,|u|^q\,dx
$$

$$
\leq \begin{cases}
\Lambda C m^{q-1}\left(\int_{B_R}|x|^{\frac{\alpha-\nu(q-1)}{N+2}2N}dx\right)^{\frac{N+2}{2N}}\;||u|| & \text{if } \beta=0,\\
\Lambda C\left[m^{q-1}\left(\int_{B_R}|x|^{\frac{\alpha-\nu(q-1)}{N+2-\beta(N+2)}2N}\,dx\right)^{\frac{N+2-\beta(N+2)}{2N}}||u||^{1-\beta}+R^{N(1-\beta)+\alpha}\right]\left(\int_{B_R}V(|x|)f(u)^2dx\right)^{\beta} & \text{if } 0<\beta<\frac{1}{2},\\
\Lambda C\left[m^{q-\beta}\left(\int_{B_R}|x|^{\frac{\alpha-\nu(q-\beta)}{1-\beta}}\,dx\right)^{1-\beta}+R^{N(1-\beta)+\alpha}\right]\left(\int_{B_R}V(|x|)f(u)^2dx\right)^{\beta}, & \\
 & \text{if } \frac{1}{2}\leq\beta<1,\\
\Lambda C\left[m^{q-1}\left(\int_{B_R}|x|^{2\alpha-2\nu(q-1)}V(|x|)\,f(u)^2dx\right)^{\frac{1}{2}}\left(\int_{B_R}V(|x|)f(u)^2dx\right)^{\frac{1}{2}}+R^{\alpha}\int_{B_R}V(|x|)f(u)^2dx\right] & \text{if } \beta=1.
\end{cases}
$$

Proof. Let us take $R\in(0,R_0)$ and define

$$
B_R^1=B_R\cap\{x\in\mathbb{R}^N\mid|u(x)|\geq1\},\quad B_R^2=B_R\cap\{x\in\mathbb{R}^N\mid|u(x)|<1\}.
$$

Recall that, by Lemma 2.1, there is $C_1>0$ such that $|f(t)|\geq C_1|t|$ when $|t|\leq1$ and $|f(t)|\geq C_1|t|^{1/2}$ when $|t|\geq1$. This implies $|f(u(x))|\geq C_1|u(x)|^{1/2}$ when $x\in B_R^1$ and $|f(u(x))|\geq C_1|u(x)|$ when $x\in B_R^2$, whence

$$
\int_{B_R^1}V(|x|)f(u)^2dx\geq C_1^2\int_{B_R^1}V(|x|)|u|dx,\quad \int_{B_R^2}V(|x|)f(u)^2dx\geq C_1^2\int_{B_R^2}V(|x|)|u|^2dx \tag{5.1}
$$

We distinguish several cases, where we will use Hölder inequality many times.

Case $\beta=0$. We apply Hölder inequality with exponents $2^*=\frac{2N}{N-2}$ and $\frac{2N}{N+2}$, and standard Sobolev inequality (Corollary 2.3), in order to get

$$
\begin{aligned}
\frac{1}{\Lambda}\int_{B_R}K(|x|)\,|u|^q\,dx &\leq \int_{B_R}|x|^{\alpha}\,|u|^{q-1}\,|u|\,dx\\
&\leq\left(\int_{B_R}\left(|x|^{\alpha}\,|u|^{q-1}\right)^{\frac{2N}{N+2}}dx\right)^{\frac{N+2}{2N}}\left(\int_{B_R}|u|^{2^*}\,dx\right)^{\frac{1}{2^*}}\\
&\leq m^{q-1}S_N\left(\int_{B_R}|x|^{\frac{\alpha-\nu(q-1)}{N+2}2N}\,dx\right)^{\frac{N+2}{2N}}\|u\|.
\end{aligned}
$$

Case $0<\beta<1/2$. We write

$$
\frac{1}{\Lambda}\int_{B_R}K(|x|)\,|u|^q\,dx=\frac{1}{\Lambda}\int_{B_R^1}K(|x|)\,|u|^q\,dx+\frac{1}{\Lambda}\int_{B_R^2}K(|x|)\,|u|^q\,dx.
$$

Applying Hölder inequality first with conjugate exponents $\frac{1}{\beta}$ and $\frac{1}{1-\beta}$, then with 2^* and $\frac{2N}{N+2}$, we get

$$
\begin{aligned}
\frac{1}{\Lambda}\int_{B_R^1}K(|x|)\,|u|^q\,dx&\leq\int_{B_R^1}|x|^{\alpha}\,V(|x|)^{\beta}\,|u|^q\,dx=\int_{B_R^1}|x|^{\alpha}\,V(|x|)^{\beta}\,|u|^{q-\beta}\,|u|^{\beta}\,dx\\
&\leq\left(\int_{B_R^1}\left(|x|^{\alpha}\,|u|^{q-1}\,|u|^{1-\beta}\right)^{\frac{1}{1-\beta}}dx\right)^{1-\beta}\left(\int_{B_R^1}V(|x|)\,|u|\,dx\right)^{\beta}\\
&\leq\frac{1}{C_1^{2\beta}}\left(\int_{B_R^1}|x|^{\frac{\alpha}{1-\beta}}\,|u|^{\frac{q-1}{1-\beta}}\,|u|\,dx\right)^{1-\beta}\left(\int_{B_R^1}V(|x|)\,f(u)^2dx\right)^{\beta}
\end{aligned}
$$

$$\leq \frac{1}{C_1^{2\beta}} \left(\int_{B_R^1} \left(|x|^{\frac{\alpha}{1-\beta}} |u|^{\frac{q-1}{1-\beta}} \right)^{\frac{2N}{N+2}} dx \right)^{(1-\beta)\frac{N+2}{2N}} \left(\int_{B_R^1} |u|^{2^*} dx \right)^{\frac{1-\beta}{2^*}} \left(\int_{B_R^1} V(|x|) f(u)^2 dx \right)^{\beta}$$

$$\leq \frac{S_N^{1-\beta}}{C_1^{2\beta}} m^{q-1} \left(\int_{B_R} |x|^{\frac{\alpha-\nu(q-1)}{1-\beta}\frac{2N}{N+2}} dx \right)^{(1-\beta)\frac{N+2}{2N}} ||u||^{1-\beta} \left(\int_{B_R} V(|x|) f(u)^2 dx \right)^{\beta}.$$

On the other hand, as $q > 1 > 2\beta$, $\alpha > -N(1-\beta)$ and $|u(x)| < 1$ in B_R^2, we get

$$\frac{1}{\Lambda} \int_{B_R^2} K(|x|) |u|^q dx \leq \int_{B_R^2} |x|^{\alpha} V(|x|)^{\beta} |u|^{q-2\beta} |u|^{2\beta} dx \leq \left(\int_{B_R^2} \left(|x|^{\alpha} |u|^{q-2\beta} \right)^{\frac{1}{1-\beta}} dx \right)^{1-\beta} \left(\int_{B_R^2} V\ |u|^2 dx \right)^{\beta}$$

$$\leq \frac{1}{C_1^{2\beta}} \left(\int_{B_R^2} |x|^{\frac{\alpha}{1-\beta}} dx \right)^{1-\beta} \left(\int_{B_R^2} V f(u)^2 dx \right)^{\beta} \leq C(N,\alpha,\beta) R^{N(1-\beta)+\alpha} \left(\int_{B_R} V f(u)^2 dx \right)^{\beta}.$$

The thesis follows by summing the two inequalities we have obtained.

Case $\beta = \frac{1}{2}$. We have

$$\frac{1}{\Lambda} \int_{B_R^1} K(|x|) |u|^q dx \leq \int_{B_R^1} |x|^{\alpha} V(|x|)^{1/2} |u|^q dx = \int_{B_R^1} |x|^{\alpha} |u|^{q-\frac{1}{2}} V(|x|)^{\frac{1}{2}} |u|^{\frac{1}{2}} dx$$

$$\leq \left(\int_{B_R^1} |x|^{2\alpha} |u|^{2q-1} dx \right)^{\frac{1}{2}} \left(\int_{B_R^1} V(|x|) |u| dx \right)^{\frac{1}{2}}$$

$$\leq \frac{m^{q-1/2}}{C_1} \left(\int_{B_R} |x|^{2\alpha-\nu(2q-1)} dx \right)^{\frac{1}{2}} \left(\int_{B_R} V(|x|) f(u)^2 dx \right)^{\frac{1}{2}},$$

while

$$\frac{1}{\Lambda} \int_{B_R^2} K(|x|) |u|^q dx \leq \int_{B_R^2} |x|^{\alpha} V(|x|)^{1/2} |u| |u|^{q-1} dx$$

$$\leq \left(\int_{B_R^2} |x|^{2\alpha} |u|^{2(q-1)} dx \right)^{1/2} \left(\int_{B_R^2} V(|x|) |u|^2 dx \right)^{1/2} \leq \frac{1}{C_1} \left(\int_{B_R} |x|^{2\alpha} dx \right)^{1/2} \left(\int_{B_R} V(|x|) f(u)^2 dx \right)^{1/2}$$

$$= C(N,\alpha,\beta) R^{\alpha+N/2} \left(\int_{B_R} V(|x|) f(u)^2 dx \right)^{1/2}.$$

As before, the thesis follows from the two inequalities we have obtained.

Case $1/2 < \beta < 1$. We will apply Hölder inequality with conjugate exponents $p = p' = \frac{1}{2}$, or $p = \frac{1}{2\beta-1} > 1$ and $p' = \frac{1}{2-2\beta}$. As above, we will estimate separately the two integrals $\int_{B_R^1} K(|x|) |u|^q dx$ and $\int_{B_R^2} K(|x|) |u|^q dx$. We have

$$\frac{1}{\Lambda} \int_{B_R^1} K(|x|) |u|^q dx \leq \int_{B_R^1} |x|^{\alpha} V(|x|)^{\beta} |u|^q dx = \int_{B_R^1} |x|^{\alpha} V(|x|)^{\frac{2\beta-1}{2}} |u|^{q-\frac{1}{2}} V(|x|)^{\frac{1}{2}} |u|^{\frac{1}{2}} dx$$

$$\leq \left(\int_{B_R^1} |x|^{2\alpha} V(|x|)^{2\beta-1} |u|^{2q-1} dx \right)^{\frac{1}{2}} \left(\int_{B_R^1} V(|x|) |u| dx \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{C_1} \left(\int_{B_R^1} |x|^{2\alpha} |u|^{2q-2\beta} V(|x|)^{2\beta-1} |u|^{2\beta-1} dx \right)^{\frac{1}{2}} \left(\int_{B_R^1} V(|x|) f(u)^2 dx \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{C_1} \left(\int_{B_R^1} |x|^{\frac{\alpha}{1-\beta}} |u|^{\frac{q-\beta}{1-\beta}} dx \right)^{1-\beta} \left(\int_{B_R^1} V(|x|) |u| dx \right)^{\frac{2\beta-1}{2}} \left(\int_{B_R^1} V(|x|) f(u)^2 dx \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{C_1^{2\beta}}\, m^{q-\beta}\left(\int_{B_R}|x|^{\frac{\alpha-\nu(q-\beta)}{1-\beta}}\,dx\right)^{1-\beta}\left(\int_{B_R}V\left(|x|\right)f(u)^2dx\right)^{\beta}.$$

On the other hand

$$\frac{1}{\Lambda}\int_{B_R^2}K\left(|x|\right)|u|^q\,dx\leq\int_{B_R^2}|x|^{\alpha}\,V\left(|x|\right)^{\beta}|u|^q\,dx=\int_{B_R^2}|x|^{\alpha}\,V\left(|x|\right)^{\beta}|u|^{2\beta}\,|u|^{q-2\beta}\,dx$$

$$\leq\left(\int_{B_R^2}|x|^{\frac{\alpha}{1-\beta}}\,|u|^{\frac{q-2\beta}{1-\beta}}dx\right)^{1-\beta}\left(\int_{B_R^2}V\left(|x|\right)|u|^2\,dx\right)^{\beta}\leq\frac{1}{C_1^{2\beta}}\left(\int_{B_R}|x|^{\frac{\alpha}{1-\beta}}\,dx\right)^{1-\beta}\left(\int_{B_R}V\left(|x|\right)f(u)^2dx\right)^{\beta}$$

$$=C(N,\alpha,\beta)R^{N(1-\beta)+\alpha}\left(\int_{B_R}V\left(|x|\right)f(u)^2dx\right)^{\beta}.$$

As before, the thesis follows from the two inequalities that we have obtained.

Case $\beta=1$. Recall that $\beta=1$ implies $q>2$ and $\alpha>0$. We have

$$\frac{1}{\Lambda}\int_{B_R^1}K\left(|x|\right)|u|^q\,dx\leq\int_{B_R^1}|x|^{\alpha}V\left(|x|\right)^{1/2}|u|^{q-1/2}V\left(|x|\right)^{1/2}|u|^{1/2}dx$$

$$\leq\left(\int_{B_R^1}|x|^{2\alpha}V\left(|x|\right)|u|^{2q-1}dx\right)^{1/2}\left(\int_{B_R^1}V\left(|x|\right)|u|dx\right)^{1/2}$$

$$\leq\frac{1}{C_1}\left(\int_{B_R^1}|x|^{2\alpha}|u|^{2q-2}V\left(|x|\right)|u|dx\right)^{1/2}\left(\int_{B_R^1}V\left(|x|\right)f(u)^2dx\right)^{1/2}$$

$$\leq\frac{m^{q-1}}{C_1^2}\left(\int_{B_R}|x|^{2\alpha-2\nu(q-1)}V\left(|x|\right)f(u)^2dx\right)^{1/2}\left(\int_{B_R}V\left(|x|\right)f(u)^2dx\right)^{1/2}.$$

On the other hand

$$\frac{1}{\Lambda}\int_{B_R^2}K\left(|x|\right)|u|^q\,dx\leq\int_{B_R^2}|x|^{\alpha}|u|^{q-2}V\left(|x|\right)|u|^2dx\leq\frac{1}{C_1^2}\,R^{\alpha}\int_{B_R}V\left(|x|\right)f(u)^2dx.$$

The thesis easily follows. □

The following lemma is analogous to the previous one, dealing with B_R^c instead of B_R.

Lemma 5.2. *Let* $R_0>0$ *and assume that*

$$\Lambda:=\sup_{x\in B_{R_0}^c}\frac{K\left(|x|\right)}{|x|^{\alpha}\,V\left(|x|\right)^{\beta}}<+\infty\quad\textit{for some } 0\leq\beta\leq1 \textit{ and } \alpha\in\mathbb{R}.$$

Let $u\in E$ *and assume that there exist* $\nu,m>0$ *such that*

$$|u\left(x\right)|\leq\frac{m}{|x|^{\nu}}\quad\textit{almost everywhere in } B_{R_0^c}.$$

Set $\gamma(m):=m/R_0^{\nu}+1$. *Then there exists a constant* $C=C(N,R_0,\beta)>0$ *such that* $\forall R>R_0$ *and* $\forall q>\max\{1,2\beta\}$, *one has*

$$\int_{B_R^c}K\left(|x|\right)|u|^{q-1}\,|h|\,dx$$

$$\leq\begin{cases}\Lambda C\gamma(m)^{2\beta}m^{q-1}\left(\int_{B_R^c}|x|^{\frac{\alpha-\nu(q-1)}{N+2(1-2\beta)}2N}\,dx\right)^{\frac{N+2(1-2\beta)}{2N}}\|u\|^{1-2\beta}\left(\int_{B_R^c}Vf(u)^2dx\right)^{\beta} & \textit{if } 0\leq\beta\leq\frac{1}{2}\\ \Lambda C\gamma(m)^{2\beta}m^{q-2\beta}\left(\int_{B_R^c}|x|^{\frac{\alpha-\nu(q-2\beta)}{1-\beta}}\,dx\right)^{1-\beta}\left(\int_{B_R^c}Vf(u)^2dx\right)^{\beta} & \textit{if } \frac{1}{2}<\beta<1\\ \Lambda C\gamma(m)^{2}m^{q-2}\left(\int_{B_R^c}|x|^{2(\alpha-\nu(q-2))}\,V\left(|x|\right)f(u)^2dx\right)^{\frac{1}{2}}\left(\int_{B_R^c}Vf(u)^2dx\right)^{\frac{1}{2}} & \textit{if } \beta=1.\end{cases}$$

Proof. We start by noticing that, thanks to the hypotheses, we have

$$|u(x)| \leq \frac{m}{|x|^{\nu}} \leq \frac{m}{R_0^{\nu}} \quad \text{for all } |x| \geq R_0.$$

Since $\gamma(m) = m/R_0^{\nu} + 1$, we have $|u(x)| \leq \gamma(m)$ in $B_{R_0}^c$ and $\gamma(m) \geq 1$. Recalling that $|f(t)| \geq C_1|t|$ when $|t| \leq 1$, and that $f(t)^2$ is even and increasing on $\mathbb{R}_+$, for all $R \geq R_0$ we have

$$\int_{B_R^c} V(|x|)|u|^2 dx = \gamma(m)^2 \int_{B_R^c} V(|x|) \left|\frac{u}{\gamma(m)}\right|^2 dx \leq \left(\frac{\gamma(m)}{C_1}\right)^2 \int_{B_R^c} V(|x|) f\left(\left|\frac{u}{\gamma(m)}\right|\right)^2 dx$$

$$\leq \left(\frac{\gamma(m)}{C_1}\right)^2 \int_{B_R^c} V(|x|) f(u)^2 dx.$$

Case $\beta = 0$. Here the argument is exactly the same as in the case $\beta = 0$ of the previous lemma, so we do not repeat it. We apply Hölder inequality with exponents $2^* = \frac{2N}{N-2}$ and $\frac{2N}{N+2}$, together with the standard Sobolev inequality, to get

$$\frac{1}{\Lambda} \int_{B_R^c} K(|x|)|u|^q dx \leq m^{q-1} C \left(\int_{B_R^c} |x|^{\frac{\alpha-\nu(q-1)}{N+2}2N} dx\right)^{\frac{N+2}{2N}} \|u\|.$$

Case $0 < \beta < \frac{1}{2}$. Thanks to Hölder inequalities with pairs of conjugate exponents $\frac{1}{\beta}$ and $\frac{1}{1-\beta}$, and $\frac{2^*(1-\beta)}{1-2\beta}$ and $\frac{2N(1-\beta)}{N+2(1-2\beta)}$, we obtain

$$\frac{1}{\Lambda} \int_{B_R^c} K(|x|)|u|^q dx \leq \int_{B_R^c} |x|^{\alpha}|u|^{q-2\beta} V(|x|)^{\beta} |u|^{2\beta} dx$$

$$\leq \left(\int_{B_R^c} \left(|x|^{\alpha}|u|^{q-2\beta}\right)^{\frac{1}{1-\beta}} dx\right)^{1-\beta} \left(\int_{B_R^c} V(|x|)|u|^2 dx\right)^{\beta}$$

$$\leq \left(\frac{\gamma(m)}{C_1}\right)^{2\beta} \left(\int_{B_R^c} \left(|x|^{\alpha}|u|^{q-1}|u|^{1-2\beta}\right)^{\frac{1}{1-\beta}} dx\right)^{1-\beta} \left(\int_{B_R^c} V f(u)^2 dx\right)^{\beta}$$

$$\leq \left(\frac{\gamma(m)}{C_1}\right)^{2\beta} \left(\int_{B_R^c} \left(|x|^{\frac{\alpha}{1-\beta}}|u|^{\frac{q-1}{1-\beta}}\right)^{\frac{2N(1-\beta)}{N+2(1-2\beta)}} dx\right)^{\frac{N+2(1-2\beta)}{2N}} \left(\int_{B_R^c} |u|^{2^*} dx\right)^{\frac{1-2\beta}{2^*}} \left(\int_{B_R^c} V f(u)^2 dx\right)^{\beta}$$

$$\leq \left(\frac{\gamma(m)}{C_1}\right)^{2\beta} C_N^{1-\beta} m^{q-1} \left(\int_{B_R^c} |x|^{\frac{\alpha-\nu(q-1)}{N+2(1-2\beta)}2N} dx\right)^{\frac{N+2(1-2\beta)}{2N}} ||u||^{1-2\beta} \left(\int_{B_R^c} V f(u)^2 dx\right)^{\beta}.$$

The result follows with $C = C_N^{1-\beta}/C_1^{2\beta}$.

Case $\beta = \frac{1}{2}$. We have

$$\frac{1}{\Lambda} \int_{B_R^c} K(|x|)|u|^q dx \leq \int_{B_R^c} |x|^{\alpha}|u|^{q-1} V(|x|)^{1/2} |u| dx$$

$$\leq \left(\int_{B_R^c} |x|^{2\alpha}|u|^{2(q-1)} dx\right)^{1/2} \left(\int_{B_R^c} V|u|^2 dx\right)^{1/2}$$

$$\leq \frac{\gamma(m)}{C_1} m^{q-1} \left(\int_{B_R^c} |x|^{2\alpha-2\nu(q-1)} dx\right)^{1/2} \left(\int_{B_R^c} V f(u)^2 dx\right)^{1/2}.$$

Case $\frac{1}{2} < \beta < 1$. We use Hölder inequality with exponents $p = p' = \frac{1}{2}$ first, and then with $p = \frac{1}{2\beta - 1}$ and $p' = \frac{1}{2-2\beta}$. We get

$$\begin{aligned}
\frac{1}{\Lambda} \int_{B_R^c} K(|x|)\, |u|^q \, dx &\leq \int_{B_R^c} |x|^\alpha |u|^q V(|x|)^\beta \, dx = \int_{B_R^c} |x|^\alpha V(|x|)^{\beta - \frac{1}{2}} |u|^{q-1} V(|x|)^{\frac{1}{2}} |u| dx \\
&\leq \left(\int_{B_R^c} |x|^{2\alpha} V(|x|)^{2\beta-1} |u|^{2q-2} dx \right)^{1/2} \left(\int_{B_R^c} V|u|^2 dx \right)^{1/2} \\
&\leq \frac{\gamma(m)}{C_1} \left(\int_{B_R^c} |x|^{2\alpha} |u|^{2(q-2\beta)} V(|x|)^{2\beta-1} |u|^{2(2\beta-1)} dx \right)^{1/2} \left(\int_{B_R^c} V f(u)^2 dx \right)^{1/2} \\
&\leq \frac{\gamma(m)}{C_1} \left(\int_{B_R^c} |x|^{\frac{\alpha}{1-\beta}} |u|^{\frac{q-2\beta}{1-\beta}} dx \right)^{1-\beta} \left(\int_{B_R^c} V(|x|)\, |u|^2 dx \right)^{\frac{2\beta-1}{2}} \left(\int_{B_R^c} V f(u)^2 dx \right)^{1/2} \\
&\leq \left(\frac{\gamma(m)}{C_1} \right)^{2\beta} m^{q-2\beta} \left(\int_{B_R^c} |x|^{\frac{\alpha - \nu(q-2\beta)}{1-\beta}} dx \right)^{1-\beta} \left(\int_{B_R^c} V f(u)^2 dx \right)^{\beta}.
\end{aligned}$$

Case $\beta = 1$. In this case, hypothesis $q > \max\{1, 2\beta\}$ implies $q > 2$. Hence

$$\begin{aligned}
\frac{1}{\Lambda} \int_{B_R^c} K(|x|)\, |u|^q \, dx &\leq \int_{B_R^c} |x|^\alpha |u|^q V(|x|) \, dx = \int_{B_R^c} |x|^\alpha V(|x|)^{\frac{1}{2}} |u|^{q-1} V(|x|)^{\frac{1}{2}} |u| dx \\
&\leq \left(\int_{B_R^c} |x|^{2\alpha} V(|x|)\, |u|^{2q-2} dx \right)^{1/2} \left(\int_{B_R^c} V(|x|)\, |u|^2 dx \right)^{1/2} \\
&\leq \frac{\gamma(m)}{C_1} \left(\int_{B_R^c} |x|^{2\alpha} |u|^{2(q-2)} V(|x|)\, |u|^2 dx \right)^{1/2} \left(\int_{B_R^c} V f(u)^2 dx \right)^{1/2} \\
&\leq \left(\frac{\gamma(m)}{C_1} \right)^2 m^{q-2} \left(\int_{B_R^c} |x|^{2\alpha - 2\nu(q-2)} V f(u)^2 dx \right)^{1/2} \left(\int_{B_R^c} V f(u)^2 dx \right)^{1/2}.
\end{aligned}$$

□

We can now prove Theorems 3.2 and 3.3.

Proof of Theorem 3.2. Assume the hypotheses of the theorem and let $u \in E$ be such that $\|u\| = 1$. Let $0 < R < R_1$. We will denote by C any positive constant which does not depend on u and R. Recalling the pointwise estimates of Corollary 2.3 and the fact that

$$\sup_{x \in B_R} \frac{K(|x|)}{|x|^{\alpha_0} V(|x|)^{\beta_0}} \leq \sup_{r \in (0, R_1)} \frac{K(r)}{r^{\alpha_0} V(r)^{\beta_0}} < +\infty,$$

we can apply Lemma 5.1 with $R_0 = R_1$, $\alpha = \alpha_0$, $\beta = \beta_0$, $m = M\|u\| = M$ and $\nu = \frac{N-2}{2}$. The argument will proceed as follows: we will distinguish several cases, as in Lemma 5.1, and we will prove that in any case we get

$$\int_{B_R} K(|x|)\, |u|^{q_1} \, dx \leq C R^\delta \quad \text{for any } 0 < R < R_1, \tag{5.2}$$

with $\delta > 0$ and $C > 0$ independent from R and u. This clearly implies $\mathcal{S}_0(q_1, R) \leq CR^\delta$, and hence $\lim_{R \to 0^+} \mathcal{S}_0(q_1, R) = 0$. Recall also that if $||u|| = 1$ then $\int_{\mathbb{R}^N} V(|x|) f(u)^2 dx \leq C$, for a suitable $C > 0$ independent from u.

If $\beta_0 = 0$, we get

$$\int_{B_R} K(|x|)\,|u|^{q_1}\,dx \le C\left(\int_{B_R} |x|^{\frac{\alpha_0-\nu(q_1-1)}{N+2}2N}dx\right)^{\frac{N+2}{2N}} \le C\left(\int_0^R \rho^{\frac{\alpha_0-\frac{N-2}{2}(q_1-1)}{N+2}2N+N-1}d\rho\right)^{\frac{N+2}{2N}},$$

where

$$\frac{\alpha_0-\frac{N-2}{2}(q_1-1)}{N+2}2N+N = \frac{N}{N+2}\left[\,2\alpha_0-(N-2)(q_1-1)+N+2\,\right]=$$
$$=\frac{N(N-2)}{N+2}\left[\frac{2\alpha_0+2N}{N-2}-q_1\right]=\frac{N(N-2)}{N+2}\left[\,q_0^*(\alpha_0,0)-q_1\,\right]>0,$$

thanks to the hypotheses. Hence, by integration and simple computations, we deduce

$$\int_{B_R} K(|x|)\,|u|^{q_1}\,dx \le CR^{\frac{N-2}{2}[q_0^*(\alpha_0,0)-q_1]} = CR^{\delta}.$$

If $0<\beta_0<1/2$, we have

$$\int_{B_R} K(|x|)\,|u|^{q_1}\,dx \le C\left[\left(\int_{B_R} |x|^{\frac{\alpha_0-\frac{N-2}{2}(q_1-1)}{(N+2)(1-\beta_0)}2N}dx\right)^{\frac{(N+2)(1-\beta_0)}{2N}} + R^{\alpha_0+N(1-\beta_0)}\right],$$

where

$$\int_{B_R} |x|^{\frac{\alpha_0-\frac{N-2}{2}(q_1-1)}{(N+2)(1-\beta_0)}2N}dx = C\int_0^R \rho^{\frac{\alpha_0-\frac{N-2}{2}(q_1-1)}{(N+2)(1-\beta_0)}2N+N-1}d\rho.$$

Now observe that

$$\frac{\alpha_0-\frac{N-2}{2}(q_1-1)}{(N+2)(1-\beta_0)}2N+N=\frac{N}{(N+2)(1-\beta_0)}\left(2\alpha_0+2N-\beta_0(N+2)-(N-2)q_1\right)$$

$$=\frac{N(N-2)}{(N+2)(1-\beta_0)}\left(\frac{2\alpha_0+2N-\beta_0(N+2)}{N-2}-q_1\right)=\frac{N(N-2)}{(N+2)(1-\beta_0)}\left(q_0^*(\alpha_0,\beta_0)-q_1\right)>0,$$

so that

$$\int_{B_R} |x|^{\frac{\alpha_0-\frac{N-2}{2}(q_1-1)}{(N+2)(1-\beta_0)}2N}dx = CR^{\frac{N-2}{2}(q_0^*(\alpha_0,\beta_0)-q_1)}.$$

On the other hand, one has $\alpha_0+N(1-\beta_0)>0$ by hypothesis. Hence as $R\to 0^+$ we have

$$\mathcal{S}_0(q_1,R)\le CR^{\frac{N-2}{2}(q_0^*(\alpha_0,\beta_0)-q_1)}+CR^{\alpha_0+N(1-\beta_0)}\le CR^{\delta},$$

where $\delta=\min\left\{\frac{N-2}{2}\left(q_0^*(\alpha_0,\beta_0)-q_1\right),\alpha_0+N(1-\beta_0)\right\}>0$.

If $\beta_0=1/2$, we have

$$\int_{B_R} K(|x|)\,|u|^{q_1}\,dx \le C\left[\left(\int_{B_R} |x|^{2\alpha_0-\frac{N-2}{2}(2q_1-1)}dx\right)^{\frac{1}{2}}+R^{\alpha_0+\frac{N}{2}}\right],$$

where

$$\int_{B_R} |x|^{2\alpha_0-\frac{N-2}{2}(2q_1-1)}dx = C\int_0^R \rho^{2\alpha_0-\frac{N-2}{2}(2q_1-1)+N-1}d\rho$$

and

$$2\alpha_0-\frac{N-2}{2}(2q_1-1)+N=2\alpha_0+\frac{3}{2}N-1-(N-2)q_1=(N-2)\left(\frac{2\alpha_0+\frac{3}{2}N-1}{N-2}-q_1\right)$$

$$= (N-2)\left(q_0^*\left(\alpha_0, \frac{1}{2}\right) - q_1\right) > 0.$$

Hence we get

$$\int_{B_R} |x|^{2\alpha_0 - \frac{N-2}{2}(2q_1-1)} dx = CR^{\frac{N-2}{2}(q_0^*(\alpha_0,1/2)-q_1)},$$

and, recalling that $\alpha_0 + \frac{N}{2} > 0$ by hypothesis, for $R \to 0^+$ we have

$$\mathcal{S}_0(q_1, R) \le CR^{\frac{N-2}{2}(q_0^*(\alpha_0,1/2)-q_1)} + CR^{\alpha_0 + \frac{N}{2}} \le CR^\delta$$

with $\delta = \min\left\{\frac{N-2}{2}\left(q_0^*\left(\alpha_0, \frac{1}{2}\right) - q_1\right), \alpha_0 + \frac{N}{2}\right\} > 0$.

If $1/2 < \beta_0 < 1$, we have

$$\int_{B_R} K(|x|)\,|u|^{q_1}\,dx \le C\left[\left(\int_{B_R} |x|^{\frac{\alpha_0 - \frac{N-2}{2}(q_1-\beta_0)}{1-\beta_0}} dx\right)^{1-\beta_0} + R^{\alpha_0 + N(1-\beta_0)}\right].$$

where

$$\int_{B_R} |x|^{\frac{\alpha_0 - \frac{N-2}{2}(q_1-\beta_0)}{1-\beta_0}} dx = C\int_0^R \rho^{\frac{\alpha_0 - \frac{N-2}{2}(q_1-\beta_0)}{1-\beta_0} + N} d\rho$$

and

$$\frac{\alpha_0 - \frac{N-2}{2}(q_1-\beta_0)}{1-\beta_0} + N = \frac{1}{2(1-\beta_0)}\left(2\alpha_0 + 2N - \beta_0(N+2) - (N-2)q_1\right)$$

$$= \frac{N-2}{2(1-\beta_0)}\left(\frac{2\alpha_0 + 2N - \beta_0(N+2)}{N-2} - q_1\right) = \frac{N-2}{2(1-\beta_0)}\left(q_0^*(\alpha_0, \beta_0) - q_1\right) > 0.$$

So

$$\left(\int_{B_R} |x|^{\frac{\alpha_0 - \frac{N-2}{2}(q_1-\beta_0)}{1-\beta_0}} dx\right)^{1-\beta_0} = CR^{\frac{N-2}{2}(q_0^*(\alpha_0,\beta_0)-q_1)}.$$

Then, as $R \to 0^+$, we obtain

$$\mathcal{S}_0(q_1, R) \le CR^{\frac{N-2}{2}(q_0^*(\alpha_0,\beta_0)-q_1)} + CR^{\alpha_0 + N(1-\beta_0)} \le CR^\delta$$

with $\delta = \min\left\{\frac{N-2}{2}\left(q_0^*(\alpha_0, \beta_0) - q_1\right), \alpha_0 + N(1-\beta_0)\right\} > 0$.

If $\beta_0 = 1$, then we have

$$\int_{B_R} K(|x|)\,|u|^{q_1}\,dx \le C\left[\left(\int_{B_R} |x|^{2\alpha_0 - (N-2)(q_1-1)} V f(u)^2 dx\right)^{\frac{1}{2}} + R^{\alpha_0}\right].$$

Notice that $\alpha_0 > -N(1-\beta_0)$ means $\alpha_0 > 0$, since $\beta_0 = 1$. Notice also that

$$2\alpha_0 - (N-2)(q_1-1) = (N-2)\left(\frac{2\alpha_0 + N - 2}{N-2} - q_1\right) = (N-2)\left(q_0^*(\alpha_0, 1) - q_1\right) > 0$$

implies

$$\left(\int_{B_R} |x|^{2\alpha_0 - (N-2)(q_1-1)} V f(u)^2 dx\right)^{\frac{1}{2}} \le R^{\alpha_0 - \frac{N-2}{2}(q_1-1)}\left(\int_{B_R} V f(u)^2 dx\right)^{\frac{1}{2}} \le CR^{\alpha_0 - \frac{N-2}{2}(q_1-1)}.$$

Hence, as $R \to 0^+$, we get

$$\mathcal{S}_0(q_1, R) \le CR^{\alpha_0 - \frac{N-2}{2}(q_1-1)} + CR^{\alpha_0} \le CR^{\alpha_0 - \frac{N-2}{2}(q_1-1)}$$

with $\alpha_0 - \frac{N-2}{2}(q_1 - 1) = \delta > 0$.

As a conclusion, in any case, we have $\mathcal{S}_0(q_1, R) \le CR^\delta$ for some $\delta = \delta(N, \alpha_0, \beta_0, q_1) > 0$ and the proof is thus complete. □

Proof of Theorem 3.3. Assume the hypotheses of the theorem and let $u \in E$ be such that $\|u\| = 1$. Let $R > R_2$. We will denote by C any positive constant which does not depend on u and R. We will separate three different cases and we will get, in each one, an inequality of the following form:

$$\int_{B_R^c} K(|x|)\, |u|^{q_2}\, dx \leq CR^{\delta}$$

with $C > 0$ and $\delta < 0$ independent from R, u. This clearly gives $\mathcal{S}_\infty(q_2, R) \leq CR^{\delta}$, and hence $\lim_{r\to+\infty} \mathcal{S}_\infty(q_2, R) = 0$. As in the proof of the previous theorem, by pointwise estimates and the fact that

$$\sup_{x\in B_R^c} \frac{K(|x|)}{|x|^{\alpha_\infty} V(|x|)^{\beta_\infty}} \leq \sup_{r>R_2} \frac{K(r)}{r^{\alpha_\infty} V(r)^{\beta_\infty}} < +\infty,$$

we can apply Lemma 5.2 with $R_0 = R_2$, $\alpha = \alpha_\infty$, $\beta = \beta_\infty$, $m = M\|u\| = M$ and $\nu = \frac{N-2}{2}$. Recall also that $||u|| = 1$ inplies $\int_{\mathbb{R}^N} V f(u)^2 dx \leq C$ with C independent from u. The computations of the present proof are essentially the same of those in the proof of Theorem 3 in [7]: the function there called q^* is the same as the function q_∞^* here. Hence, we will be a little sketchy here.

If $0 \leq \beta_\infty \leq 1/2$, we get

$$\int_{B_R^c} K(|x|)\, |u|^{q_2}\, dx \leq C \left(\int_{B_R^c} |x|^{\frac{\alpha_\infty - \frac{N-2}{2}(q_2-1)}{N+2(1-2\beta_\infty)} 2N} dx \right)^{\frac{N+2(1-2\beta_\infty)}{2N}}$$

$$= C\left(R^{\frac{2\alpha_\infty - 4\beta_\infty + 2N - (N-2)q_2}{N+2(1-2\beta_\infty)} N} \right)^{\frac{N+2(1-2\beta_\infty)}{2N}},$$

since $2\alpha_\infty - 4\beta_\infty + 2N - (N-2)q_2 = (N-2)\left(q_\infty^*(\alpha_\infty, \beta_\infty) - q_2\right) < 0$.

On the other hand, if $1/2 < \beta_\infty < 1$, then we have

$$\int_{B_R^c} K(|x|)\, |u|^{q_2}\, dx \leq C \left(\int_{B_R^c} |x|^{\frac{\alpha_\infty - \frac{N-2}{2}(q_2 - 2\beta_\infty)}{1-\beta_\infty}} dx \right)^{1-\beta_\infty} = C\left(R^{\frac{2\alpha_\infty - (N-2)(q_2-2\beta_\infty)}{2(1-\beta_\infty)}} \right)^{1-\beta_\infty},$$

since

$$\frac{2\alpha_\infty - (N-2)(q_2 - 2\beta_\infty)}{2(1-\beta_\infty)} = \frac{N-2}{2(1-\beta_\infty)} \left(q_\infty^*(\alpha_\infty, \beta_\infty) - q_2\right) < 0.$$

Finally, if $\beta_\infty = 1$, we obtain

$$\int_{B_R^c} K(|x|)\, |u|^{q_2}\, dx \leq C \left(\int_{B_R^c} |x|^{2\alpha_\infty - (N-2)(q_2-2)} V(|x|)\, f(u)^2 dx \right)^{\frac{1}{2}},$$

$$\leq C \left(R^{2\alpha_\infty - (N-2)(q_2-2)} \int_{B_R^c} V(|x|)\, f(u)^2 dx \right)^{\frac{1}{2}} \leq CR^{\frac{2\alpha_\infty - (N-2)(q_2-2)}{2}},$$

since

$$2\alpha_\infty - (N-2)(q_2 - 2) = (N-2)\left(q_\infty^*(\alpha_\infty, \beta_\infty) - q_2\right) < 0.$$

So, in any case, we get $\mathcal{S}_\infty(q_2, R) \leq CR^{\delta}$ for some $\delta = \delta(N, p, \alpha_\infty, \beta_\infty, q_2) < 0$, anf this completes the proof. □

6 Critical points in the Orlicz-Sobolev space

In this section we study the relations between the equation (2.2) and the original equation, that is

$$-\Delta w + V(|x|)\,w - w\left(\Delta w^2\right) = K(|x|)g(w) \quad \text{in } \mathbb{R}^N \tag{6.1}$$

where g satifies the assumptions stated in Section 2. For both the equations, the solutions we get must be understood in two ways, weak and classical (in $\mathbb{R}^N\backslash\{0\}$). As to (6.1) we will get weak solutions, that is, functions $w \in X$ satisfying, for all $h \in C^\infty_{c,r}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \left(1+2w^2\right)\nabla w\cdot\nabla h\,dx + \int_{\mathbb{R}^N} 2w|\nabla w|^2\,h\,dx + \int_{\mathbb{R}^N} V(|x|)\,wh\,dx = \int_{\mathbb{R}^N} K(|x|)g(w)h\,dx, \tag{6.2}$$

which is obviously a weak formulation of (6.1). We also prove that the solutions that we get are in $C^2(\mathbb{R}^N\backslash\{0\})$ and are classical solutions of (6.1) in $\mathbb{R}^N\backslash\{0\}$.

As we said in the introduction, we will obtain solutions by variational techniques, studying a functional related to the original problem by a change of variable. Let us define $I : E \to \mathbb{R}$ by setting

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(|x|)f(u)^2 dx - \int_{\mathbb{R}^N} K(|x|)G(f(u))\,dx. \tag{6.3}$$

In the following theorem, we state the main properties of I.

Theorem 6.1. *Assume $N \geq 3$ and hypothesis $(\mathbf{H})$. Assume that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $(\mathbf{g_1})$, $(\mathbf{g_2})$ and $(\mathbf{g}_{q_1,q_2})$ with q_1, q_2 satisfying $\left(\mathcal{S}'_{q_1,q_2}\right)$ (see Section 3). Then we have:*

- *I is well defined and continuous in E.*
- *I is a C^1 map on E and, for any $u \in E$, its differential $I'(u)$ is given by*

$$I'(u)h = \int_{\mathbb{R}^N} \nabla u\nabla h dx + \int_{\mathbb{R}^N} V(|x|)f(u)f'(u)h dx - \int_{\mathbb{R}^N} K(|x|)g(f(u))f'(u)h\,dx \tag{6.4}$$

for all $h \in E$.

Proof. Let us define

$$I_1(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad I_2(u) = \frac{1}{2}\int_{\mathbb{R}^N} V(|x|)f(u)^2 dx, \quad I_3(u) = \int_{\mathbb{R}^N} K(|x|)G(f(u))\,dx.$$

and study these three functionals.

As to I_1, it is a standard task to get that I_1 is C^1 in E with differential given by $I_1'(u)h = \int_{\mathbb{R}^N} \nabla u\nabla h dx$.

As to I_3, we notice that, setting $h(x,t) = K(|x|)G(f(t))$, we have $h(x,t) = \int_0^t K(|x|)g(f(s))f'(s)ds$ and

$$|K(|x|)g(f(t))f'(t)| \leq CK(|x|)\min\left\{|f(t)|^{q_1-1}, |f(t)|^{q_2-1}\right\} \leq CK(|x|)\min\left\{|t|^{q_1-1}, |t|^{q_2-1}\right\}.$$

Then we can apply the results in [8] (in particular Proposition 3.8) and the fact that $E \hookrightarrow L^{q_1}_K(\mathbb{R}^N) + L^{q_2}_K(\mathbb{R}^N)$ (see Theorem 3.1), to get that also I_3 is C^1 in E, with differential given by

$$I_3'(u)h = \int_{\mathbb{R}^N} K(|x|)g(f(u))f'(u)h\,dx.$$

As to I_2, we can repeat the arguments of proposition (2.9) of [14], which work also in our hypotheses, to get that I_2 is well defined, continuous and Gateaux differentiable, with differential I_2' given by

$$I_2'(u)h = \int_{\mathbb{R}^N} V(|x|)f(u)f'(u)h dx.$$

In order to conclude, we need to prove that the map $I_2' : E \to E'$ is continuous. Let $\{u_n\}_n$ be a sequence in E with $u_n \to u$ in E. Define

$$\alpha_n = ||I_2'(u_n) - I_2'(u)||_{E'} = \sup_{||h|| \le 1} |(I_2'(u_n) - I_2'(u))\, h| = \sup_{||h|| \le 1} \left| \int_{\mathbb{R}^N} V(|x|)\,(f(u_n)f'(u_n) - f(u)f'(u))\, h dx \right|.$$

We claim that $\alpha_n \to 0$. In proving this, we will use C to indicate different positive constants, that can change from line to line but are independent from h and n. We notice as first thing that (1) of Lemma 2.4 implies

$$\sup_{||h|| \le 1} \left\{ \int_{\mathbb{R}^N} V(|x|) f(h)^2 dx \right\} \le C.$$

Then we compute

$$\left| \int_{\mathbb{R}^N} V(|x|)\,(f(u_n)f'(u_n) - f(u)f'(u))\, h dx \right| \le \int_{B_1} V(|x|)\,|f(u_n)f'(u_n) - f(u)f'(u)|\,|h| dx$$
$$+ \int_{B_1^c} V(|x|)\,|f(u_n)f'(u_n) - f(u)f'(u)|\,|h| dx.$$

Recalling Corollary 2.3, we get $|h(x)| \le C$ in B_1^c for all h with $||h|| \le 1$, and we can assume $C > 1$. Hence from (7) of Lemma 2.1 we derive

$$|h(x)| = C\,\frac{|h(x)|}{C} \le C f\left(\frac{|h(x)|}{C}\right) \le C f\left(|h(x)|\right).$$

From this, applying Hölder inequality, we get

$$\int_{B_1^c} V(|x|)\,|f(u_n)f'(u_n) - f(u)f'(u)|\,|h| dx \le C \int_{B_1^c} V(|x|)\,|f(u_n)f'(u_n) - f(u)f'(u)|\, f\left(|h(x)|\right) dx$$
$$\le C \left(\int_{B_1^c} V(|x|)\,|f(u_n)f'(u_n) - f(u)f'(u)|^2\, dx \right)^{1/2} \left(\int_{B_1^c} V(|x|) f\left(|h(x)|\right)^2 dx \right)^{1/2}$$
$$\le C \left(\int_{B_1^c} V(|x|)\,|f(u_n)f'(u_n) - f(u)f'(u)|^2\, dx \right)^{1/2}$$

As $u_n \to u$ in $D_r^{1,2}\left(\mathbb{R}^N\right)$, we can assume, up to a subsequence, that $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$. Also, from (2) of Lemma 2.4, we deduce that $V^{\frac{1}{2}} f(u_n) \to V^{\frac{1}{2}} f(u)$ in $L^2(\mathbb{R}^N)$ and hence, up to a subsequence, we can assume $V f(u_n)^2 \le k \in L^1(\mathbb{R}^N)$. Hence we have $V(|x|)\,|f(u_n)f'(u_n) - f(u)f'(u)| \to 0$ a.e. and

$$V(|x|)\,|f(u_n)f'(u_n) - f(u)f'(u)|^2 \le C V(|x|) \left[f(u_n)^2 f'(u_n)^2 + f(u)^2 f'(u)^2 \right]$$
$$\le Ck + C V f(u)^2 \in L^1(\mathbb{R}^N),$$

so, by Dominated Convergence Theorem, we have

$$\int_{B_1^c} V(|x|)\,|f(u_n)f'(u_n) - f(u)f'(u)|^2\, dx \to 0$$

which implies

$$\sup_{||h||\le 1}\int_{B_1^c} V(|x|)\,|f(u_n)f'(u_n)-f(u)f'(u)|\,|h|\,dx\to 0.$$

On the other hand, by the hypothesis on V and Lemma 2.3, we get

$$\int_{B_1} V(|x|)\,|f(u_n)f'(u_n)-f(u)f'(u)|\;|h|\,dx \le C\int_{B_1}\frac{1}{|x|^2}\,|f(u_n)f'(u_n)-f(u)f'(u)|\,\frac{1}{|x|^{\frac{N-2}{2}}}\,dx =$$

$$C\int_{B_1}|f(u_n)f'(u_n)-f(u)f'(u)|\,\frac{1}{|x|^{\frac{N}{2}+1}}\,dx.$$

As $N/2+1<N$ we have $|x|^{-N/2-1}\in L^1(B_1)$, while $|f(u_n)f'(u_n)-f(u)f'(u)|\to 0$ a.e. in $\mathbb{R}^N$ and $|f(u_n)f'(u_n)-f(u)f'(u)|\le C$ because of (9) of Lemma 2.1. Again by Dominated Convergence Theorem we get

$$\int_{B_1}|f(u_n)f'(u_n)-f(u)f'(u)|\,\frac{1}{|x|^{\frac{N}{2}+1}}dx\to 0$$

and hence

$$\sup_{||h||\le 1}\int_{B_1} V(|x|)\,|f(u_n)f'(u_n)-f(u)f'(u)|\,|h|\,dx\to 0.$$

This holds for a subsequence of any sequence $u_n\to u$, and from this it is easy to get the thesis. □

According to the above result, a critical point u of I satisfies $I'(u)h=0$, that is

$$\int_{\mathbb{R}^N}\nabla u\nabla h\,dx+\int_{\mathbb{R}^N}V(|x|)f(u)f'(u)h\,dx-\int_{\mathbb{R}^N}K(|x|)g(f(u))f'(u)h\,dx=0 \tag{6.5}$$

for all $h\in E$. This is, of course, a weak formulation of equation (2.2). We now want to show that a critical point u of I is a classical solution of equation (2.2) in $\mathbb{R}^N\backslash\{0\}$.

Theorem 6.2. *Assume the hypotheses of Theorem 6.1. Let u be a critical point of I. Then $u\in C^2(\mathbb{R}^N\backslash\{0\})$ and u is a classical solution of equation (2.2) in $\mathbb{R}^N\backslash\{0\}$.*

Proof. We deal with radial functions and for them, with a little abuse of notation, we will write $u(x)=u(|x|)=u(r)$ for $r=|x|$, so identifying u with a function defined a.e. on $\mathbb{R}_+$. Using this trick, the integral equation (6.5) becomes an integral equation in dimension 1, that is

$$\int_0^{+\infty}u'(r)h'(r)\,r^{N-1}dr+\int_0^{+\infty}V(r)f(u(r))f'(u(r))h(r)\,r^{N-1}dr \tag{6.6}$$

$$-\int_0^{+\infty}K(r)g(f(u(r)))f'(u(r))h(r)\,r^{N-1}dr=0$$

for all $h\in E$. Of course equation (6.6) can be considered as a weak formulation of the following ODE:

$$u''+\frac{N-1}{r}u'+V(r)f(u)f'(u)-K(r)g(f(u))f'(u)=0\quad\text{in}\quad\mathbb{R}_+. \tag{6.7}$$

We will now prove that u is a classical solution of (6.7). To be precise, we will prove the following claim.

> *Claim:* fix any $0<a<b<+\infty$ and let $I=(a,b)$. Then $u\in C^2(I)$ and u is a classical solution of (6.7) in I.

The proof of the claim in divided in three steps:

(i) $u \in H^1(I)$;

(ii) $u \in H^2(I)$;

(iii) u is a classical solution of (6.7) in I.

Step (i) is easily obtained with the same argument of Lemma 27 in [6]. We now prove (ii). Let us take any $\varphi \in C_c^\infty(I)$. Take $\delta > 0$ such that $a - \delta > 0$ and define $I_\delta = (a-\delta, b+\delta)$. Of course $\varphi \in C_c^\infty(I_\delta)$. Let $\psi \in C^\infty(\mathbb{R})$ such that $0 \le \psi \le 1$, $\psi(r) = 0$ if $r \le a - \delta/2$ and $\psi(r) = 1$ if $r \ge a - \delta/3$. Define also

$$v(r) := \int_a^r \frac{\varphi'(s)}{s^{N-1}}\,ds = \frac{\varphi(r)}{r^{N-1}} + (N-1)\int_a^r \frac{\varphi(s)}{s^N}\,ds.$$

Notice that $v \in C^\infty(I_\delta)$ and $v(r) = 0$ if $r \in (a-\delta, a)$. Let $\varepsilon > 0$ be such that $\operatorname{supp}\varphi \subset (a+\varepsilon, b-\varepsilon)$. Then for $r \in (b-\varepsilon, b+\delta)$ one has

$$v(r) = v(b) = (N-1)\int_a^b \frac{\varphi(s)}{s^N}\,ds =: \overline{v}.$$

Define now $w(r) = v(r) - \overline{v}\psi(r)$. Then $w \in C^\infty(I_\delta)$ and $\operatorname{supp} w \subseteq [a - \frac{\delta}{2}, b-\varepsilon]$, whence $w \in C_c^\infty(I_\delta)$. Hence by the equation (6.6) we easily get that

$$\int_{I_\delta} u'(r)w'(r)r^{N-1}dr = \int_{I_\delta} \eta(r)w(r)dr,$$

where $\eta \in L^\infty(I_\delta)$. As $w'(r) = \varphi'(r)r^{1-N} - \overline{v}\psi'(r)$, from the above equation we deduce

$$\int_I u'(r)\varphi'(r)dr = \int_{I_\delta} \eta(r)w(r)dr + \overline{v}\int_{I_\delta} u'(r)\psi'(r)\,r^{N-1}dr = \int_{I_\delta} \eta(r)\frac{\varphi(r)}{r^{N-1}}dr$$
$$+(N-1)\int_{I_\delta} \eta(r)\left(\int_a^r \frac{\varphi(s)}{s^N}\,ds\right)dr + \overline{v}\int_{I_\delta} u'(r)\psi'(r)\,r^{N-1}dr.$$

It is easy to see that the following estimates hold, with a constant $C > 0$ depending on u but not on φ:

$$\left|\int_{I_\delta} \eta(r)\frac{\varphi(r)}{r^{N-1}}dr\right| \le C||\varphi||_{L^2(I)}, \quad \left|(N-1)\int_{I_\delta} \eta(r)\left(\int_a^r \frac{\varphi(s)}{s^N}\,ds\right)dr\right| \le C||\varphi||_{L^2(I)},$$
$$\left|\overline{v}\int_{I_\delta} u'(r)\psi'(r)dr\right| \le C||\varphi||_{L^2(I)}.$$

The last inequality derives from the definition of $\overline{v}$ and the fact that $\left|\int_{I_\delta} u'(r)\psi'(r)dr\right| \le C||u||$, where $||u||$ is the norm of u in E. Then we get

$$\left|\int_I u'(r)\varphi'(r)dr\right| \le C||\varphi||_{L^2(I)}.$$

As this holds for every $\varphi \in C_c^\infty(I)$, standard Sobolev space theory gives $u' \in H^1(I)$ and hence $u \in H^2(I)$. As to (iii), once we have $u \in H^2(I)$, it is a standard task to get that $u \in C^2(I)$ and that the equation (6.7) is satisfied in the classical sense in I. This concludes the proof of the claim.

Now it is easy to get the thesis of the theorem. The claim holds for every $I = (a,b)$ with $0 < a < b < +\infty$, hence we have that $u \in C^2(\mathbb{R}_+)$ and that it satisfies equation (6.7) in $\mathbb{R}_+$. Coming back to dimension N, it is then obvious that $u \in C^2(\mathbb{R}^N \setminus \{0\})$ and

$$-\Delta u + V(|x|)f(u)f'(u) = K(|x|)g(f(u))f'(u) \quad \text{in } \mathbb{R}^N\setminus\{0\}. \tag{6.8}$$

□

Now we show that $w = f(u)$ is a classical solution of equation (6.1) in $\mathbb{R}^N \backslash \{0\}$.

Theorem 6.3. *Assume the hypotheses of Theorem 6.1. Let $u \in E$ be a critical point of I and set $w = f(u)$. Then $w \in C^2\left(\mathbb{R}^N \backslash \{0\}\right)$ and it is a classical solution of equation (6.1) in $\mathbb{R}^N \backslash \{0\}$.*

Proof. From Theorem 6.2 and the fact that $f \in C^\infty$, it is obvious that $w \in C^2\left(\mathbb{R}^N \backslash \{0\}\right)$. Direct computations then show that

$$\Delta w + w\Delta\left(w^2\right) = \frac{1}{f'(u)}\Delta u.$$

It is then easy to get the result by substituting in equation (6.7). □

To complete our analysis, we now prove that if u is a critical point of I and $w = f(u)$, then w also satisfies (6.2), that is, w is a weak solution of (6.1).

Theorem 6.4. *Assume the hypotheses of Theorem 6.1. Let $u \in E$ be a critical point of I and set $w = f(u)$. Then $w = f(u) \in X$ and for all $h \in C^\infty_{c,r}(\mathbb{R}^N)$ one has*

$$\int_{\mathbb{R}^N} \left(1 + 2w^2\right) \nabla w \cdot \nabla h \, dx + \int_{\mathbb{R}^N} 2w|\nabla w|^2 \, h \, dx + \int_{\mathbb{R}^N} V\left(|x|\right) wh \, dx = \int_{\mathbb{R}^N} K(|x|)g(w)h \, dx \quad (6.9)$$

Proof. It is obvious by our definitions that $\int_{\mathbb{R}^N} V\left(|x|\right) \, w^2 \, dx < +\infty$. Moreover, we have $\nabla w = f'(u)\nabla u$ and thus $\int_{\mathbb{R}^N} |\nabla w|^2 dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx < +\infty$. This gives $w \in X$. To prove (6.9), we start by noticing that easy computations give

$$\left(f^{-1}\right)'(t) = \sqrt{1 + 2t^2}, \qquad \left(f^{-1}\right)''(t) = \frac{2t}{\sqrt{1 + 2t^2}}.$$

Hence, as $u = f^{-1}(w)$, we derive $\nabla u = \left(f^{-1}\right)'(w)\, \nabla w = \sqrt{1 + 2w^2}\, \nabla w$. Let us now fix $h \in C^\infty_{c,r}(\mathbb{R}^N)$ and define $\varphi = \left(f^{-1}\right)'(w)\, h = \sqrt{1 + 2w^2}\, h$. We want to prove that $\varphi \in E$. Of course φ is radial, so what we actually need to prove are the following statements:

i) $\int_{\mathbb{R}^N} V\left(|x|\right) \, f\left(\varphi\right)^2 dx < +\infty$;

ii) $\int_{\mathbb{R}^N} |\nabla \varphi|^2 \, dx < +\infty$.

In order to prove $i)$, we use the properties of f and f^2 (see lemma 2.1). In particular, from (11) it is easy to obtain that for all $C > 1$ there is a constant $k = k(C) > 0$ such that $f(Ct)^2 \leq kf(t)^2$ for al $t > 0$. Recalling that $h \in C^\infty_{c,r}(\mathbb{R}^N)$ we can assume $|h(x)| \leq C$ and $\operatorname{supp} h \subseteq B_R$. Hence we can compute

$$\int_{\mathbb{R}^N} V\left(|x|\right) \, f\left(\varphi\right)^2 dx = \int_{B_R} V\left(|x|\right) \, f\left(|\varphi|\right)^2 dx = \int_{B_R} V\left(|x|\right) \, f\left(\sqrt{1 + 2w^2}\, |h|\right)^2 dx$$

$$\leq \int_{B_R} V\left(|x|\right) \, f\left(\sqrt{1 + 2w^2}\, C\right)^2 dx \leq k(C) \int_{B_R} V\left(|x|\right) \, f\left(\sqrt{1 + 2w^2}\right)^2 dx$$

$$= k(C) \int_{B_R \cap \{|w| \geq 1\}} V\left(|x|\right) \, f\left(\sqrt{1 + 2w^2}\right)^2 dx + k(C) \int_{B_R \cap \{|w| < 1\}} V\left(|x|\right) \, f\left(\sqrt{1 + 2w^2}\right)^2 dx.$$

On the one hand, we easily get

$$\int_{B_R \cap \{|w| < 1\}} V\left(|x|\right) \, f\left(\sqrt{1 + 2w^2}\right)^2 dx \leq \int_{B_R \cap \{|w| < 1\}} V\left(|x|\right) \, f\left(\sqrt{3}\right)^2 dx$$

$$\leq f\left(\sqrt{3}\right)^2 \int_{B_R} V\left(|x|\right)\, dx < +\infty.$$

On the other hand, when $|w| \geq 1$ one has $\sqrt{1+2w^2} \leq 2|w|$ and hence

$$\int_{B_R \cap \{|w|\geq 1\}} V\left(|x|\right)\, f\left(\sqrt{1+2w^2}\right)^2 dx \leq \int_{B_R \cap \{|w|\geq 1\}} V\left(|x|\right)\, f\left(2|w|\right)^2 dx$$

$$\leq k \int_{B_R \cap \{|w|\geq 1\}} V\left(|x|\right)\, f\left(|w|\right)^2 dx \leq k \int_{B_R} V\left(|x|\right)\, |w|^2\, dx < +\infty$$

because $w \in X$. So $i)$ is proved.

As to $ii)$, we compute

$$\nabla\varphi = \sqrt{1+2w^2}\, \nabla h + 2\, h\, \frac{w}{\sqrt{1+2w^2}}\, \nabla w$$

and we easily get

$$\left| 2\, h\, \frac{w}{\sqrt{1+2w^2}}\, \nabla w \right| \leq C|\nabla w| \in L^2\left(\mathbb{R}^N\right).$$

On the other hand, as $w \in D_r^{1,2}\left(\mathbb{R}^N\right)$, we have $w \in L^2_{loc}\left(\mathbb{R}^N\right)$ and hence

$$\int_{\mathbb{R}^N} \left| \sqrt{1+2w^2}\, \nabla h \right|^2\, dx \leq C \int_{B_R} \left(1+2w^2\right)\, dx < +\infty.$$

So also $ii)$ is proved.

We now conclude the proof of the lemma. As $\varphi \in E$ and $I'(u) = 0$, exploiting the computations above we get

$$0 = I'(u)\varphi = \int_{\mathbb{R}^N} \sqrt{1+2w^2}\, \nabla w \cdot \sqrt{1+2w^2}\, \nabla h\, dx + \int_{\mathbb{R}^N} \sqrt{1+2w^2}\, \nabla w \cdot \frac{2hw}{\sqrt{1+2w^2}}\, \nabla w\, dx$$

$$+ \int_{\mathbb{R}^N} V f(u) f'(u) \left(f^{-1}\right)'(w) h\, dx - \int_{\mathbb{R}^N} K g(f(u)) f'(u) \left(f^{-1}\right)'(w) h\, dx$$

$$= \int_{\mathbb{R}^N} \left(1+2w^2\right)\, \nabla w \cdot \nabla h\, dx + \int_{\mathbb{R}^N} 2hw\, |\nabla w|^2\, dx + \int_{\mathbb{R}^N} V\, w\, h\, dx - \int_{\mathbb{R}^N} K\, g(w)\, h\, dx.$$

Therefore (6.9) is satisfied and the theorem is proved. □

7 Existence of solutions

This section is devoted to our main existence result, which is the following.

Theorem 7.1. *Assume $N \geq 3$, $(\mathbf{H})$ and that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $(\mathbf{g_1})$, $(\mathbf{g_2})$, $(\mathbf{g}_{q_1,q_2})$. Assume the hypotheses of Theorems 3.2 and 3.3 with q_1, q_2 satisfying respectively (3.4) and (3.6). Then the functional $I : E \to \mathbb{R}$ has a nonnegative critical point $u \neq 0$.*

Remark 7.2. In Theorem 7.1, as we look for non negative solutions, we can assume $g(t) = 0$ for all $t \leq 0$. Indeed, if we have a nonlinearity g satisfying the hypotheses, we can replace g with $\chi_{\mathbb{R}_+}(t)\, g(t)$ ($\chi_{\mathbb{R}_+}$ is the characteristic function of $\mathbb{R}_+$), and the new nonlinearity still satisfies the hypotheses.

Remark 7.3. Thanks to Theorems 3.2 and 3.3, the hypotheses of Theorem 7.1 imply that E is compactly embedded into $L^{q_1}_K(\mathbb{R}^N) + L^{q_2}_K(\mathbb{R}^N)$. This is one of the main devices to get our existence result.

Remark 7.4. As concerns examples of nonlinearities satisfying the hypotheses of Theorem 7.1, the simplest $g \in C(\mathbb{R};\mathbb{R})$ such that $(\mathbf{g}_{q_1,q_2})$ holds is

$$g(t) = \min\left\{ |t|^{q_1-2}\, t, |t|^{q_2-2}\, t \right\},$$

which also ensures $(\mathbf{g_1})$ if $q_1, q_2 > 4$ (with $\theta = \min\left\{\frac{q_1}{2}, \frac{q_2}{2}\right\}$). Another model example is

$$g(t) = \frac{|t|^{q_2-2}\, t}{1+|t|^{q_2-q_1}} \quad \text{with } 1 < q_1 \leq q_2,$$

which ensures $(\mathbf{g_1})$ if $q_1 > 4$ (with $\theta = \frac{q_1}{2}$). Note that, in both these cases, also $(\mathbf{g_2})$ holds true. Moreover, both of these functions g become $g(t) = |t|^{q-2}\, t$ if $q_1 = q_2 = q$.

We will get Theorem 7.1 by applying a version of the well-known Mountain-Pass Lemma (see chapter 2 in [2]). Let us first recall the so-called Palais-Smale condition.

Definition 7.5 (Palais-Smale condition)**.** Let Y be a Banach space and $\Phi : Y \to \mathbb{R}$ a C^1 functional. We say that Φ satisfies the Palais-Smale condition if for any sequence $\{x_n\}_n$ sucht that $\Phi(x_n)$ is bounded in $\mathbb{R}$ and $\Phi'(x_n) \to 0$ in Y', there is a subsequence $\{x_{n_k}\}_k$ converging in Y.

Theorem 7.6 (Mountain Pass Lemma)**.** *Let Y be a Banach space and $\Phi : Y \to \mathbb{R}$ a C^1 functional with $\Phi(0) = 0$. Assume that Φ satisfies the Palais-Smale condition and that there are a subset $S \subseteq Y$ and a real number $\alpha > 0$ such that:*

(1) $Y \backslash S$ is not arcwise connected;

(2) $\Phi(x) \geq \alpha$ for all $x \in S$;

(3) there exists $y \in Y \backslash (C_0 \cup S)$ such that $\Phi(y) < 0$, where C_0 is the connected component of $Y \backslash S$ such that $0 \in C_0$.

Then Φ has a critical point $u \in Y$ such that $\Phi(u) \geq \alpha$.

To prove Theorem 7.1 we will prove that the functional $I : E \to \mathbb{R}$ satisfies the hypotheses of the Mountain Pass Lemma. It is obvious that $I(0) = 0$. The other hypotheses of the Mountain Pass Lemma are proved in the following lemmas. More precisely, assumptions *(1)* and *(2)* are proved in Lemma 7.7, while assumption *(3)* is proved in Lemma 7.8. In Lemmas 7.9 and 7.10 we show that I satisfies the Palais-Smale condition.

Recall the three functionals I_1, I_2, I_3 introduced in the proof of Lemma 6.1 and define, for $u \in E$,

$$J(u) = I_1(u) + I_2(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^N} V(|x|) f(u)^2\, dx.$$

Then, for any $\rho > 0$, define

$$S_\rho = \{u \in E \,|\, J(u) = \rho\}.$$

Lemma 7.7. *Assume the hypotheses of Theorem 7.1. Then there is $\rho^* > 0$ such that for all $\rho \in (0, \rho^*)$ the set $E \backslash S_\rho$ is not arcwise connected and there exists $\alpha = \alpha(\rho) > 0$ such that $I(u) \geq \alpha$ for all $u \in S_\rho$.*

Proof. Fix any $v \in E\backslash\{0\}$ and set $\rho_1 = J(v) > 0$. Then for all $\rho \in (0, \rho_1)$ the set $E\backslash S_\rho$ is not arcwise connected, because J is a continuous functional on E and any continuous path joining 0 and v must intersect S_ρ. To get $S = S_\rho$ and α as in Mountain Pass Lemma, we recall first that, by Corollary 2.5, one has $X \hookrightarrow E$ and therefore there exists $C > 0$ such that $||u|| \leq C||u||_X$ for all $u \in X$. Also, we know that if $u \in E$ then $f(u) \in X$, so that, for all $u \in E$, we have

$$||f(u)|| \leq C||f(u)||_X.$$

In the hypotheses of Theorem 7.1, we can choose $0 < R_1 < R_2$ such that $\mathcal{S}_0(q_1, R_1) < +\infty$ and $\mathcal{S}_\infty(q_2, R_2) < +\infty$. Hence

$$\left|\int_{\mathbb{R}^N} K(|x|)G(f(u))dx\right| \leq M\int_{\mathbb{R}^N} K(|x|)\min\{|f(u)|^{q_1}, |f(u)|^{q_2}\}\,dx$$
$$\leq M\int_{B_{R_1}} K(|x|)|f(u)|^{q_1}\,dx + M\int_{B_{R_2}\backslash B_{R_1}} K(|x|)|f(u)|^{q_1}\,dx + M\int_{B_{R_2}^c} K(|x|)|f(u)|^{q_2}\,dx$$
$$\leq M\mathcal{S}_0(q_1, R_1)||f(u)||^{q_1} + MC_{R_1,R_2}||f(u)||^{q_1} + M\mathcal{S}_\infty(q_2, R_2)||f(u)||^{q_2}.$$

These inequalities derive from the hypotheses on g, the definitions of $\mathcal{S}_0$ and $\mathcal{S}_\infty$, and Lemmas 4.1 and 2.7. So we get

$$\left|\int_{\mathbb{R}^N} K(|x|)G(f(u))dx\right| \leq C_1||f(u)||^{q_1} + C_2||f(u)||^{q_2} \leq C_3||f(u)||_X^{q_1} + C_4||f(u)||_X^{q_2}.$$

Now we have

$$||f(u)||_X^2 = \int_{\mathbb{R}^N} |\nabla f(u)|^2 dx + \int_{\mathbb{R}^N} V(|x|)f(u)^2 dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} V(|x|)f(u)^2 dx = 2J(u).$$

and therefore, for $u \in S_\rho$, we get

$$\left|\int_{\mathbb{R}^N} K(|x|)G(f(u))dx\right| \leq C_5\rho^{q_1/2} + C_6\rho^{q_2/2}.$$

Hence, for $u \in S_\rho$ we conclude that

$$I(u) = J(u) - \int_{\mathbb{R}^N} K(|x|)G(f(u))dx \geq \rho - C_5\rho^{q_1/2} - C_6\rho^{q_2/2}.$$

As $\frac{q_1}{2}, \frac{q_2}{2} > 2$, it is obvious that for $\rho > 0$ small enough we have $\alpha = \alpha(\rho) = \rho - C_5\rho^{q_1/2} - C_6\rho^{q_2/2} > 0$, and this concludes the proof of the lemma. □

Lemma 7.8. *Take $\rho > 0$ as in Lemma 7.7. Then there exists $v \in E$ such that $J(v) > \rho$ and $I(v) < 0$.*

Proof. From assumption $(\mathbf{g_1})$ and $(\mathbf{g_2})$ we infer that $G(t) \geq 0$ for all t and, for every $t_+ > t_0$ and all $t > t_+$,

$$G\left(t\right) \geq \frac{G\left(t_+\right)}{t_+^{2\theta}} t^{2\theta} > 0. \tag{7.1}$$

Clearly it is not restrictive to assume $t_0 \geq 1$. Now we fix $t_1 > 1$ such that $f(t_1) > t_0$ and then we pick a non negative function $u_0 \in C_{c,r}^\infty(\mathbb{R}^N)$ such that the set $\{x \in \mathbb{R}^N : u_0\left(x\right) \geq t_1\}$ has positive Lebesgue measure. Hence for every $\lambda > 1$, using (7.1) with $t_+ = f(t_1)$, we get

$$\int_{\mathbb{R}^N} K(|x|)G\left(f\left(\lambda u_0\right)\right)dx \geq \int_{\{\lambda u_0 \geq t_1\}} K(|x|)G\left(f\left(\lambda u_0\right)\right)dx \geq \frac{G(f(t_1))}{f(t_1)^{2\theta}} \int_{\{\lambda u_0 \geq t_1\}} K(|x|)\left(f\left(\lambda u_0\right)\right)^{2\theta} dx$$

$$\geq C_1\lambda^\theta \int_{\{u_0\geq t_1\}} K(|x|)\, u_0^\theta dx = C_2\lambda^\theta,$$

where $C_2 = C_1 \int_{\{u_0\geq t_1\}} K(|x|)\, u_0^\theta dx > 0$. On the other hand

$$\int_{\mathbb{R}^N} V(|x|) f(\lambda u_0)^2 dx \leq \lambda^2 \int_{\mathbb{R}^N} V(|x|)\, u_0^2 dx,$$

so that

$$I(\lambda u_0) = \frac{1}{2}\int_{\mathbb{R}^N} |\lambda \nabla u_0|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^N} V(|x|) f\left(\lambda u_0\right)^2 dx - \int_{\mathbb{R}^N} K(|x|) G\left(f\left(\lambda u_0\right)\right) dx \leq C_3\lambda^2 - C_2\lambda^\theta.$$

As $\theta > 2$, we deduce that $I(\lambda u_0) \to -\infty$ when $\lambda \to +\infty$. As it is obvious that $J(\lambda u_0) \to +\infty$ when $\lambda \to +\infty$, the proof is concluded by choosing $v = \lambda u_0$ for λ large enough. □

Lemma 7.9. *Under the assumptions of Theorem 7.1, let $\{u_n\}_n \subseteq E$ be a Palais-Smale sequence for I, that is, a sequence such that $\{I(u_n)\}_n$ is bounded and $I'(u_n) \to 0$ in E'. Then $\{u_n\}_n$ in bounded in E.*

Proof. We start with the following computation:

$$I(u_n) - \frac{1}{\theta} I'(u_n)u_n = \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(|x|) f(u_n)^2 dx - \int_{\mathbb{R}^N} K(|x|) G(f(u_n)) dx$$

$$-\frac{1}{\theta}\int_{\mathbb{R}^N} V(|x|) f(u_n) f'(u_n) u_n dx + \frac{1}{\theta}\int_{\mathbb{R}^N} K(|x|) g(f(u_n)) f'(u_n) u_n dx.$$

Since *(5)* of Lemma 2.1 implies $f(t)^2 - f(t)f'(t)t \geq 0$ for all t, we have

$$\int_{\mathbb{R}^N} V(|x|) \left(f(u_n)^2 - f(u_n)f'(u_n)u_n\right) dx \geq 0$$

and this implies

$$\frac{1}{2}\int_{\mathbb{R}^N} V(|x|) f(u_n)^2 dx - \frac{1}{\theta}\int_{\mathbb{R}^N} V(|x|) f(u_n) f'(u_n) u_n dx = \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} V(|x|) f(u_n)^2 dx$$

$$+\frac{1}{\theta}\int_{\mathbb{R}^N} V(|x|) \left(f(u_n)^2 - f(u_n)f'(u_n)u_n\right) dx \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} V(|x|) f(u_n)^2 dx.$$

On the other hand, using the hypotheses on g and *(5)* of Lemma 2.1 again, we have

$$\frac{1}{\theta}\int_{\mathbb{R}^N} K(|x|) g(f(u_n)) f'(u_n) u_n dx - \int_{\mathbb{R}^N} K(|x|) G(f(u_n)) dx$$

$$\geq\ \frac{1}{\theta}\int_{\mathbb{R}^N} K(|x|) g(f(u_n)) f'(u_n) u_n dx - \frac{1}{2\theta}\int_{\mathbb{R}^N} K(|x|) g(f(u_n)) f(u_n) dx$$

$$\geq \frac{1}{2\theta}\int_{\mathbb{R}^N} K(|x|) g(f(u_n)) f(u_n) dx - \frac{1}{2\theta}\int_{\mathbb{R}^N} K(|x|) g(f(u_n)) f(u_n) dx = 0.$$

Therefore we get

$$I(u_n) - \frac{1}{\theta} I'(u_n)u_n \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} V(|x|) f(u_n^2) dx\right).$$

By definition, we have $\int_{\mathbb{R}^N} |\nabla u_n|^2 dx = ||u_n||_{1,2}^2$ and $\int_{\mathbb{R}^N} V(|x|) f(u_n)^2 dx + 1 \geq ||u_n||_o$. Hence, if $||u_n||_{1,2} \leq 1$ we get

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} V(|x|) f(u_n)^2 dx \geq ||u_n||_o - 1 \geq ||u_n||_o + ||u_n||_{1,2} - 2 = ||u_n|| - 2.$$

On the other hand, if $||u_n||_{1,2} > 1$ then $||u_n||_{1,2}^2 > ||u_n||_{1,2}$ and hence

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} V(|x|) f(u_n)^2 dx \geq ||u_n||_{1,2} + ||u_n||_o - 1 \geq ||u_n|| - 2.$$

So in any case we conclude

$$I(u_n) - \frac{1}{\theta} I'(u_n) u_n \geq \left(\frac{1}{2} - \frac{1}{\theta}\right) ||u_n|| - 2\left(\frac{1}{2} - \frac{1}{\theta}\right).$$

As $\{u_n\}_n$ is a Palais-Smale sequence, we can assume $I(u_n) \leq C$ and we can fix $\delta > 0$ such that $\delta < \theta\left(\frac{1}{2} - \frac{1}{\theta}\right)$ and $|I'(u_n)u_n| \leq \delta ||u_n||$ for large n's. Hence we get

$$C + \frac{\delta}{\theta} ||u_n|| \geq I(u_n) - \frac{1}{\theta} I'(u_n) u_n \geq \left(\frac{1}{2} - \frac{1}{\theta}\right) ||u_n|| - 2\left(\frac{1}{2} - \frac{1}{\theta}\right),$$

that is,

$$C + 2\left(\frac{1}{2} - \frac{1}{\theta}\right) \geq \left(\frac{1}{2} - \frac{1}{\theta} - \frac{\delta}{\theta}\right) ||u_n||.$$

As $\frac{1}{2} - \frac{1}{\theta} - \frac{\delta}{\theta} > 0$, this implies that $\{||u_n||\}_n$ in bounded. □

Lemma 7.10. *Under the assumptions of Theorem 7.1, the functional $I : E \to \mathbb{R}$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\}_n$ be a sequence in E such that $\{I(u_n)\}_n$ is bounded and $I'(u_n) \to 0$ in E'. By Lemma 7.9, $\{u_n\}_n$ is bounded in E and therefore there exists a subsequence, that we still call $\{u_n\}_n$, such that $u_n \rightharpoonup u$ in $D_r^{1,2}(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ for a.e. x. Recall that we have introduced the three functionals I_1, I_2, I_3 (see Theorem 6.1) and we have defined $J = I_1 + I_2$, so that $I = J - I_3$. We know that I_3 is od class C^1 on $L_K^{q_1} + L_K^{q_2}$. By compactness of the embedding of E into $L_K^{q_1} + L_K^{q_2}$, up to a subsequence we have that $u_n \to u$ in $L_K^{q_1} + L_K^{q_2}$, whence $I_3'(u_n) \to I_3'(u)$ in the dual space of $L_K^{q_1} + L_K^{q_2}$ and $I_3'(u_n)(u - u_n) \to 0$ in $\mathbb{R}$. We notice now that, as f^2 is a convex function, J is a convex functional on E, so that

$$J(u) - J(u_n) \geq J'(u_n)(u - u_n) = I'(u_n)(u - u_n) + I_3'(u_n)(u - u_n).$$

As $I'(u_n) \to 0$ in E' by hypothesis and $\{u - u_n\}$ is bounded in E, we have $I'(u_n)(u - u_n) \to 0$ and thus

$$J(u) \geq J(u_n) + o(1).$$

Taking the liminf, this gives

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} V(|x|) f(u)^2 dx \geq \liminf_n \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} V(|x|) f(u_n)^2 dx\right) \tag{7.2}$$

$$\geq \liminf_n \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \liminf_n \int_{\mathbb{R}^N} V(|x|) f(u_n)^2 dx.$$

By semicontinuity of the norm, we have

$$\liminf_n \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \geq \int_{\mathbb{R}^N} |\nabla u|^2 dx,$$

so that (7.2) gives

$$\int_{\mathbb{R}^N} V(|x|) f(u)^2 dx \geq \liminf_n \int_{\mathbb{R}^N} V(|x|) f(u_n)^2 dx.$$

As Fatou's Lemma obviously implies $\int_{\mathbb{R}^N} V(|x|)f(u)^2dx \leq \liminf_n \int_{\mathbb{R}^N} V(|x|)f(u_n)^2dx$, we deduce

$$\int_{\mathbb{R}^N} V(|x|)f(u)^2dx = \liminf_n \int_{\mathbb{R}^N} V(|x|)f(u_n)^2dx. \tag{7.3}$$

So, passing to a subsequence that we still label $\{u_n\}_n$, we can assume

$$\int_{\mathbb{R}^N} V(|x|)f(u)dx = \lim_n \int_{\mathbb{R}^N} V(|x|)f(u_n)^2dx. \tag{7.4}$$

Then by *(3)* of Lemma 2.4 we get $||u - u_n||_o \to 0$. Now, repeating the previous argument for this subsequence, we get again (7.2), which now gives

$$\int_{\mathbb{R}^N} |\nabla u|^2dx \geq \liminf_n \int_{\mathbb{R}^N} |\nabla u_n|^2dx$$

and hence

$$\int_{\mathbb{R}^N} |\nabla u|^2dx = \liminf_n \int_{\mathbb{R}^N} |\nabla u_n|^2dx.$$

Up to a subsequence again, we can assume

$$\int_{\mathbb{R}^N} |\nabla u|^2dx = \lim_n \int_{\mathbb{R}^N} |\nabla u_n|^2dx.$$

Since $u_n \rightharpoonup u$ in $D_r^{1,2}\left(\mathbb{R}^N\right)$, we obtain that $u_n \to u$ in $D_r^{1,2}\left(\mathbb{R}^N\right)$, i.e., $||u - u_n||_{1,2} \to 0$. Hence $||u - u_n|| = ||u - u_n||_o + ||u - u_n||_{1,2} \to 0$. □

Proof of Theorem 7.1. Taking ρ and v as in Lemmas 7.7 and 7.8, we have $0 = J(0) < \rho < J(v)$, so that v and 0 are in two distinct connected components of $E \backslash S_\rho$. Hence, the previous lemmas show that all the hypoteses of Mountain Pass Lemma 7.6 are satisfied, and thus we get a critical point u of I, with $u \neq 0$. Let u^- be the negative part of u. It is easy to see that $u^- \in E$, so that $I'(u)u^- = 0$. The additional assumption $g(t) = 0$ for $t < 0$ implies

$$0 = I'(u)\, u^- = -\int_{\mathbb{R}^N} |\nabla u^-|^2dx + \int_{\mathbb{R}^N} V(|x|)f(u)f'(u)u^-dx, \tag{7.5}$$

where, by the properties of V and f, we have

$$\int_{\mathbb{R}^N} V(|x|)f(u)f'(u)u^-dx = \int_{\mathbb{R}^N} V(|x|)f(-u^-)f'(u)u^-dx = -\int_{\mathbb{R}^N} V(|x|)f(u^-)f'(u)u^-dx \leq 0.$$

Hence (7.5) implies $\int_{\mathbb{R}^N} |\nabla u^-|^2dx = 0$. One concludes that $u^- = 0$, because $u^- \in D_r^{1,2}\left(\mathbb{R}^N\right)$, and therefore u is nonnegative. □

8 Examples

In this section we give some examples of application of our results, obtaining some existence results which are not included, as far as we know, in the previous literature. More precisely, we will make a comparison between our results and those of [31], which inspired the present study. In that paper the authors prove some existence results for equation (1.1), assuming that g grows like a power and that V, K are controlled by suitable powers of $|x|$. Here we show some situations where the results of [31] do not apply, while ours give existence of solutions.

In all the examples, we will consider the model nonlinearity $g(t) = \min\{t^{q_1-1}, t^{q_2-1}\}$ for simplicity, and we will let $4 < q_1 \leq q_2$. As the throughout the paper, we will also assume $N \geq 3$ and hypothesis $(\mathbf{H})$.

Example 8.1. Assume that there exist $c_1, c_2, c_3, c_4 > 0$ such that

$$c_1\, r^{2N} \leq K(r) \leq c_2\, r^{2N} \quad \text{as } r \to 0^+, \qquad c_3 r^{3N} \leq K(r) \leq\, c_4 r^{3N} \quad \text{as } r \to +\infty.$$

Computing the coefficients b, b_0 in [31], one gets $b_0 = 2N$ and $b \geq 3N$, so the results in [31] cannot be applied, because they need $b_0 \geq b$. If we let $\beta_0 = \beta_\infty = 0$, $\alpha_0 = 2N$ and $\alpha_\infty = 3N$ in Theorems 3.2 and 3.3, we get

$$q_0^*(\alpha_0, 0) = 2\,\frac{\alpha_0 + N}{N-2} = \frac{6N}{N-2} > 6, \quad q_\infty^*(\alpha_\infty, 0) = 2\,\frac{\alpha_\infty + N}{N-2} = \frac{8N}{N-2}.$$

Hence we can apply our existence results to nonlinearities $g(t) = \min\{t^{q_1-1}, t^{q_2-1}\}$ with

$$4 < q_1 < \frac{6N}{N-2} < \frac{8N}{N-2} < q_2.$$

Notice that we are not assuming that V has a power-like behavior at zero or at infinity (in this regard we just need hypothesis ($\mathbf{H}$)).

Example 8.2. Let $N = 3$ and assume that there exist $c_1, c_2, c_3, c_4 > 0$ such that

$$\frac{c_1}{r^{1/2}} \leq K(r) \leq \frac{c_2}{r^{1/2}} \quad \text{as } r \to 0^+, \qquad \frac{c_3}{r^{1/3}} \leq K(r) \leq \frac{c_4}{r^{1/3}} \quad \text{as } r \to +\infty.$$

In this case the coefficients b, b_0 in [31] are $b_0 = -\frac{1}{2}$ and $b \geq -\frac{1}{3}$, so again $b > b_0$ and the results of [31] cannot be applied. If we let $\beta_0 = \beta_\infty = 0$, $\alpha_0 = -\frac{1}{2}$ and $\alpha_\infty = -\frac{1}{3}$ in Theorems 3.2 and 3.3, we get

$$q_0^*(\alpha_0, 0) = 2\,\frac{\alpha_0 + 3}{3-2} = -1 + 6 = 5, \quad q_\infty^*(\alpha_\infty, 0) = 2\,\frac{\alpha_\infty + 3}{3-2} = \frac{16}{3}.$$

Hence we can apply our existence results with

$$4 < q_1 < 5 < \frac{16}{3} < q_2.$$

As for the previous example, the only assumption we need on the asymptotic behavior of V is ($\mathbf{H}$).

Example 8.3. Assume that there exist $c, \delta > 0$ such that $V(r) \leq c\, e^{-\delta r}$ as $r \to +\infty$. The results of [31] cannot be applied because they require $\liminf_{r\to\infty} \frac{V(r)}{r^a} > 0$ for some $a \in \mathbb{R}$. Instead, we can give several existence results with different hypotheses on K. For example, assume $K(r) = r^N$. Then we take $\beta_0 = \beta_\infty = 0$ and $\alpha_0 = \alpha_\infty = N$ in Theorems 3.2 and 3.3, and we get

$$q_0^*(\alpha_0, 0) = q_\infty^*(\alpha_\infty, 0) = \frac{4N}{N-2} > 4.$$

Hence we get an existence result by choosing

$$4 < q_1 < \frac{4N}{N-2} < q_2.$$

Assume now $K(r) = \min\left\{r^N, r^{2N}\right\}$ and choose $\beta_0 = \beta_\infty = 0$, $\alpha_0 = 2N$ and $\alpha_\infty = N$ in Theorems 3.2 and 3.3. We get

$$q_0^*(\alpha_0, 0) = \frac{6N}{N-2}, \quad q_\infty^*(\alpha_\infty, 0) = \frac{4N}{N-2} > 4$$

where $q_0^*(\alpha_0, 0) > q_\infty^*(\alpha_\infty, 0)$, so that we can choose $q_1 = q_2 = q$ and get existence of solutions for power nonlinearities $g(t) = \min\{t^{q_1-1}, t^{q_2-1}\} = t^{q-1}$ with

$$4 < \frac{4N}{N-2} < q < \frac{6N}{N-2}.$$

Example 8.4. Assume that there exist $c, \delta > 0$ such that $K(r) \geq c\, e^{\delta r}$ as $r \to +\infty$. The results in [31] cannot be applied because they require $\limsup_{r\to\infty} \frac{K(r)}{r^b} < +\infty$ for some $b \in R$. To give an explicit example, assume $K(r) = r^N e^r$ and $V(r) = e^{2r}$. Then we take $\beta_0 = 0$, $\beta_\infty = 1/2$ and $\alpha_0 = \alpha_\infty = N$ in Theorems 3.2 and 3.3, and we get

$$q_0^*(\alpha_0, 0) = \frac{4N}{N-2} > q_\infty^*(\alpha_\infty, 1/2) = 2\,\frac{2N-1}{N-2} > 4.$$

So we can choose $q_1 = q_2 = q$ and this gives existence results for$g(t) = t^{q-1}$ with

$$4 < 2\,\frac{2N-1}{N-2} < q < \frac{4N}{N-2}.$$

Example 8.5. Assume that $K(r) = o(r^N)$ for all N, as $r \to 0^+$. For example, $K(r) = ce^{-\delta/r}$ for r near zero, with $c, \delta > 0$. As before, the results of [31] cannot be applied because they need a power-like behavior of K near zero. Assume also that, for $r \to +\infty$, it holds $K(r) = r^\alpha V(r)$ for some $\alpha \in \mathbb{R}$. Notice that this does not require any specific asymptotic behavior at ∞ for V and K, and the only hypothesis on the behavior of V at 0 is, again, $(\mathbf{H})$. Fix $\alpha_\infty = \alpha$ and $\beta_\infty = 1$ in Theorem 3.2. Hence

$$q_\infty^*(\alpha_\infty, 1) = 2\,\frac{\alpha + N - 2}{N-2} = \frac{2\alpha}{N-2} + 2.$$

and we can choose $q_2 > \max\left\{4, \frac{2\alpha}{N-2} + 2\right\}$. Once we have fixed such a q_2, we let $\beta_0 = 0$ and α_0 such that $2\,\frac{\alpha_0+N}{N-2} > q_2$. This means $q_0^*(\alpha_0, 0) > q_2$ in Theorem 3.3, so that we can take $q = q_1 = q_2$ and get an existence result with $g(t) = t^{q-1}$ for any $q > \max\left\{4, \frac{2\alpha}{N-2} + 2\right\}$. As another example of the same kind, assume $K(r) = e^{-1/r}$ and $V(r) = 1/r^2$ for all $r > 0$. It is easy to see that the best choice in Theorem 3.3 is $\alpha_\infty = \beta_\infty = 0$, which gives

$$q_\infty^*(\alpha_\infty, \beta_\infty) = \frac{2N}{N-2}.$$

As before, for any fixed $q_2 > \max\left\{4, \frac{2N}{N-2}\right\}$ we can let $\beta_0 = 0$ and α_0 large enough in such a way that $q_0^*(\alpha_0, 0) > q_2$, so that we can take $q_1 = q_2 = q$. Hence we get a solution for $g(t) = t^{q-1}$ with any $q > \max\left\{4, \frac{2N}{N-2}\right\}$. Notice that this means $q > 6$ for $N = 3$ and $q > 4$ for $N \geq 4$.

Example 8.6. Let $V(r) = 1/r^2$ for all $r > 0$, and assume $K(r) = cr^N$ for r near zero ($c > 0$) and $K(r) \leq C$ for $r \to +\infty$. For example $K(r) = \min\left\{r^N, 1\right\}$. Hence the coefficients a_0, b_0 of [31] are given by $b_0 = N$ and $a_0 = -2$, and the results of [31] cannot be applied because they need $a_0 \geq b_0$. We fix $\alpha_0 = N$ and $\beta_0 = 0 = \alpha_\infty = \beta_\infty$ in Theorems 3.2 and 3.3, so that

$$q_0^*(\alpha_0, \beta_0) = q_0^*(N, 0) = \frac{4N}{N-2} > 4, \quad q_\infty^*(\alpha_\infty, \beta_\infty) = q_\infty^*(0, 0) = \frac{2N}{N-2}.$$

Hence we can take $q = q_1 = q_2$ and get existence results for power nonlinearities $g(t) = t^{q-1}$ with $q \in \left(4, \frac{4N}{N-2}\right)$ if $N \geq 4$, and $q \in (6, 12)$ if $N = 3$.

Remark 8.7. As a final remark, we observe that most of existence results we can formulate for explicit potentials concern potentials K's decaying fast enough as $r \to 0$. This is the major limitation of our work. Nevertheless we believe that it might be overcome by a careful analysis of our estimates, and we hope to do this in a future paper.

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