SCHREIER'S TYPE FORMULAE AND TWO SCALES FOR GROWTH OF LIE ALGEBRAS AND GROUPS

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Dedicated to 70th anniversary of Vesselin Drensky

Abstract. Let G be a free group of rank n and $H \subset G$ its subgroup of finite index. Then H is also a free group and the rank m of H is determined by Schreier's formula $m-1=(n-1)\cdot|G:H|$.

Any subalgebra of a free Lie algebra is also free. But a straightforward analogue of Schreier's formula for free Lie algebras does not exist, because any subalgebra of finite codimension has an infinite number of generators.

But the appropriate Schreier's formula for free Lie algebras exists in terms of formal power series. There exists also a version in terms of exponential generating functions. This is a survey on how these formulas are applied to study 1) growth of finitely generated Lie algebras and groups and 2) the codimension growth of varieties of Lie algebras. First, these formulae allow to specify explicit formulas for generating functions of respective types for free solvable (or more generally, polynilpotent) Lie algebras. Second, these explicit formulas for generating functions are used to derive asymptotic for these two types of the growth. These results can be viewed as analogues of the Witt formula for free Lie algebras and groups. In case of Lie algebras, we obtain two scales for respective types of growth. We also shortly mention the situation on growth for other types of linear algebras.

1. Analogue of Schreier's formula for free Lie algebras

Denote the ground field by K. Let X be an at most countable set supplied with a *weight function* $\mathrm{wt}: X \to \mathbb{N}$, namely we assume that

$$X=\bigcup_{i=1}^{\infty} X_i; \quad X_i=\{x\in X \mid \mathrm{wt}\, x=i\}, \quad |X_i|<\infty,\ i\in\mathbb{N}.$$

We say that X is *finitely graded*. Assume that an algebra A is generated by X, then we naturally define the weight of a monomial $a\in A$. Let Y be a set of monomials in X, then one defines the *Hilbert-Poincaré series* of Y (with respect to X), see e.g. [4]

$$\mathcal{H}_X(Y)=\mathcal{H}_X(Y,t):=\sum_{i=1}^{\infty}|Y_i|t^i; \qquad Y_i=\{y\in Y \mid \mathrm{wt}\, y=i\},\ i\in\mathbb{N}.$$

Consider a subspace $V\subset A$, then we define $\mathcal{H}_X(V)$ using a homogeneous basis of $\mathrm{gr}\, V$; where $\mathrm{gr}\, V$ being the associated graded space.

Next, we introduce the operator $\mathcal{E}$ on power series $\phi(t)=\sum_{n=1}^{\infty} b_n t^n$, $b_n\in\{0,1,2,\dots\}$ (see [25, 27]):

$$\mathcal{E}\ :\ \phi(t)=\sum_{n=1}^{\infty} b_n t^n \ \longrightarrow\ \mathcal{E}(\phi(t)):=\sum_{n=0}^{\infty} a_n t^n=\prod_{n=1}^{\infty}\frac{1}{(1-t^n)^{b_n}}.$$

Assume that L is a Lie algebra generated by X and $U(L)$ its universal enveloping algebra. Let

$$\mathcal{H}_X(L,t)=\sum_{n=1}^{\infty} b_n t^n, \quad \mathcal{H}_X(U(L),t)=\sum_{n=0}^{\infty} a_n t^n.$$

One has a well-known formula that explains importance of the operator above $\mathcal{H}_X(U(L))=\mathcal{E}(\mathcal{H}_X(L))$ [36]. The following is the natural analogue of Schreier's formula, introduced by the author. For basic facts on free Lie (super)algebras see [1, 2].

2000 *Mathematics Subject Classification.* 16R10, 16P90, 17B01, 17B65.

Key words and phrases. Identical relations, growth, generating functions, codimension sequence, solvable Lie algebras, polynilpotent Lie algebras.

Theorem 1 ([27])**.** *Assume that L is a free Lie algebra generated by a finitely graded set X. Let H be a subalgebra and Y is a set of its free generators. Then*

$$\mathcal{H}(Y,t)-1=(\mathcal{H}(X,t)-1)\cdot\mathcal{E}(\mathcal{H}(L/H),t).$$

Let L be a Lie algebra. One defines the *lower central series* as $L^1=L$, $L^{i+1}=[L,L^i]$ for $i=1,2,\dots$. Now, L is *nilpotent* of class s iff $L^{s+1}=\{0\}$ while $L^s\neq\{0\}$. All Lie algebras nilpotent of class at most s form the variety denoted by $\mathbf{N}_s$. A Lie algebra L is called *polynilpotent* with a tuple of integers $(s_q,\dots,s_2,s_1)$ iff there exists a chain of ideals

$$\{0\}=L_{q+1}\subset L_q\subset\cdots\subset L_2\subset L_1=L,\qquad L_n/L_{n+1}\in\mathbf{N}_{s_n},\ n=1,\dots,q.$$

All polynilpotent Lie algebras with a fixed tuple form a variety denoted by $\mathbf{N}_{s_q}\dots\mathbf{N}_{s_2}\mathbf{N}_{s_1}$. In the particular case $s_q=\cdots=s_1=1$, one obtains the variety $\mathbf{A}^q$, consisting of *solvable* Lie algebras of length at most q. On the other hand, a polynilpotent Lie algebra is solvable. Moreover, the free polynilpotent Lie algebras, a tuple being fixed, provide interesting examples of solvable Lie algebras. The definitions in case of group theory are similar. Let G be a group, denote by $\{\gamma_n(G)|n=1,2,\dots\}$, terms of the *lower central series* (warning: below γ has also a different meaning!).

Suppose that L is the free Lie algebra of rank k, $L=\oplus_{n=1}^{\infty}L_n$ its natural grading, and G the free group of rank k. Then the lower central series factors $\gamma_n(G)/\gamma_{n+1}(G)$ are free abelian groups and their ranks are given by the classical Witt formula [1]:

$$\psi_k(n):=\operatorname{rank}_{\mathbb{Z}}\gamma_n(G)/\gamma_{n+1}(G)=\dim_K L_n=\frac{1}{n}\sum_{a|n}k^a\mu\left(\frac{n}{a}\right)\approx\frac{k^n}{n},\quad n\in\mathbb{N},\tag{1}$$

where $\mu(*)$ is the Möbius function. Theorem 1 allows to derive the following explicit formulas.

Theorem 2 ([27, 29])**.** *Consider the free polynilpotent Lie algebra $L=F(\mathbf{N}_{s_q}\cdots\mathbf{N}_{s_1},k)$ of finite rank $k\geq 2$, where $q\geq 1$. Set $\beta_0(z):=0$, $\alpha_0(z):=kz$, and define the following functions recursively*

$$\left.\begin{aligned}\beta_i(z)&:=\beta_{i-1}(z)+\sum_{m=1}^{s_i}\frac{1}{m}\sum_{a|m}\mu\left(\frac{m}{a}\right)\Big(\alpha_{i-1}(z^{m/a})\Big)^a,\\ \alpha_i(z)&:=1+(kz-1)\cdot\mathcal{E}(\beta_i(z)),\end{aligned}\right.\qquad 1\leq i\leq q.$$

Then $\mathcal{H}(L,z)=\beta_q(z)$.

For example, we have a particular case.

Corollary 3 ([27, 29])**.** *Let $L:=F(\mathbf{AN}_d,k)$. Then*

$$\mathcal{H}(L,z)=\psi_k(1)z+\cdots+\psi_k(d)z^d+1+\frac{kz-1}{(1-z)^{\psi_k(1)}\cdots(1-z^d)^{\psi_k(d)}}.$$

2. Scale 1 and Growth of free solvable (polynilpotent) finitely generated Lie algebras and groups

Now we describe applications of the analogue of Schreier's formula for free Lie algebras (Theorem 1 and its application Theorem 2) to specify the growth of free solvable (more generally, polynilpotent) Lie algebras and groups of finite rank.

Assume that L is a relatively free algebra of some multihomogeneous variety of (associative, or Lie) algebras, generated by $X=\{x_1,\dots,x_k\}$. Then we have a natural grading $L=\oplus_{n=1}^{\infty}L_n$ by degree in X. One defines the *growth function* with respect to X as $\gamma_L(X,n):=\sum_{s=1}^{n}\dim_K L_s$.

Theorem 4 (Berele, [3])**.** *The growth function of a finitely generated associative PI-algebra is bounded by a polynomial function.*

For more details on proofs of this important result see [4, 16]. But the growth of finitely generated Lie PI-algebras is more complicated. In this case, the author introduced scale (2) of functions of intermediate growth and suggested that it is complete in the sense of Conjecture 1 below. Define functions

$$\left.\begin{aligned}\ln^{(0)}x&:=x, &\ln^{(s+1)}x&:=\ln(\ln^{(s)}x),\\ \exp^{(0)}x&:=x, &\exp^{(s+1)}x&:=\exp(\exp^{(s)}x),\end{aligned}\right.\qquad s=0,1,2,\dots$$

Recall standard notations $f(x)\approx g(x)$, $x\to\infty$, denotes that $\lim_{x\to\infty} f(x)/g(x)=1$; $f(x)=o(g(x))$ means that beginning with some number $f(x)=\alpha(x)g(x)$ and $\lim_{x\to\infty}\alpha(x)=0$. Also, $\zeta(x)$ is the zeta function.

Consider the **scale 1** consisting of a a series of functions $\Phi^q_\alpha(n)$, $q=1,2,3,\dots$ of a natural argument with a parameter $\alpha\in\mathbb{R}^+$:

$$\textbf{scale 1:}\qquad \begin{aligned} &\Phi^1_\alpha(n):=\alpha,\\ &\Phi^2_\alpha(n):=n^\alpha,\\ &\Phi^3_\alpha(n):=\exp(n^{\alpha/(\alpha+1)}),\\ &\Phi^q_\alpha(n):=\exp\left(\frac{n}{(\ln^{(q-3)} n)^{1/\alpha}}\right),\qquad q=4,5,\dots. \end{aligned} \tag{2}$$

Now, we specify the growth of free solvable (more generally, polynilpotent) Lie algebras of finite rank with the respect to scale (2) by giving the following asymptotic.

Theorem 5 ([25])**.** *Consider the free polynilpotent Lie algebra $L:=F(\mathbf{N}_{s_q}\cdots\mathbf{N}_{s_1},k)$ of finite rank $k\ge 2$, where $q\ge 2$, generated by $X=\{x_1,\dots,x_k\}$. Then*

$$\gamma_L(X,n)=\begin{cases} \dfrac{A+o(1)}{N!}\,n^N, & q=2,\\ \exp\Big((C{+}o(1))\,n^{N/(N+1)}\Big), & q=3,\\ \exp\left((B^{1/N}{+}o(1))\,\dfrac{n}{(\ln^{(q-3)} n)^{1/N}}\right), & q\ge 4,\end{cases}\qquad n\to\infty;$$

where the constants are:

$$N:=s_2\dim_K F(\mathbf{N}_{s_1},k),\qquad A:=\frac{1}{s_2}\left(\frac{k-1}{\prod_{q=2}^{s_1} q^{\psi_k(q)}}\right)^{s_2},$$
$$B:=s_3A\zeta(N{+}1),\qquad\qquad C:=(1+1/N)(BN)^{\frac{1}{1+N}}.$$

This result has the following application to group theory. Let G be a group. Due to Lazard, one constructs the related Lie algebra [19]:

$$L_K(G):=\mathop{\oplus}_{n=1}^{\infty}(\gamma_n(G)/\gamma_{n+1}(G))\otimes_{\mathbb{Z}}K.$$

If G is a free polynilpotent group, then $L_K(G)$ is the free polynilpotent Lie algebra of the same rank and with the same tuple, moreover, the lower central series factors are free abelian groups (A. Shmelkin [35]).

Corollary 6 ([25, 29])**.** *Let $G=G(\mathbf{N}_{s_q}\dots\mathbf{N}_{s_1},k)$, $q\ge 2$, be the free polynilpotent group of rank k. Consider ranks of the lower central series factors, namely, denote $b_n:=\operatorname{rank}_{\mathbb{Z}}\gamma_n(G)/\gamma_{n+1}(G)$, $n\ge 1$. Then we get the asymptotic:*

$$b_n=\begin{cases} \dfrac{A+o(1)}{(N-1)!}\,n^{N-1}, & q=2,\\ \exp\Big((C{+}o(1))\,n^{N/(N+1)}\Big), & q=3,\\ \exp\left((B^{1/N}{+}o(1))\,\dfrac{n}{(\ln^{(q-3)} n)^{1/N}}\right), & q\ge 4,\end{cases}\qquad n\to\infty;$$

where N,A,B,C are the same as in Theorem 5.

Observe that just by setting $s_q=\cdots=s_1=1$, Theorem 5 and its Corollary 6 are turned into results on the free solvable Lie algebra and group of rank k and length q. A similar observation is valid concerning the results (e.g. Theorem 20) on the codimension growth below.

M.I.Kargapolov raised problem 2.18 in [17] to specify the lower central series ranks for free polynilpotent finitely generated groups. Exact recursive formulae were given by Egorychev [6]. We suggest another answer to this problem by specifying the asymptotic behaviour of that ranks. We consider to view the asymptotic of Theorem 5 and its Corollary 6 as an analogue of the Witt formula (1), now for the free solvable (more generally, polynilpotent) Lie algebras and groups.

Since the growth of finitely generated solvable Lie algebras is intermediate [20], the respective generating function converges in the unit circle. It is important to study an asymptotic of the generating function when we approach the unit circumference from inside. Since the coefficients of the series are real nonnegative numbers, it is sufficient to study this behaviour while $z = t \to 1-0$. The crucial step to prove Theorem 5 is the following asymptotic of the generating function $\mathcal{H}_X(L,t)$.

Theorem 7 ([25])**.** *Let $L := F(\mathbf{N}_{s_q}\cdots\mathbf{N}_{s_1},k)$ be the free polynilpotent Lie algebra of finite rank $k\ge 2$, where $q\ge 2$, generated by $X=\{x_1,\dots,x_k\}$. Then*

$$\lim_{t\to 1-0}(1-t)^N\mathcal{H}_X(L,t)=A, \qquad q=2,$$
$$\lim_{t\to 1-0}(1-t)^N\ln^{(q-2)}(\mathcal{H}_X(L,t))=s_3\,\zeta(N+1)\,A, \qquad q\ge 3;$$

where N, A are the same as in Theorem 5.

Now, we describe an important idea to prove Theorem 7. The explicit formula of the generating function (Theorem 2) shows that $\mathcal{H}_X(L,t)$ is "roughly speaking" a $(q-1)$-iteration of $\mathcal{E}$ applied to $\beta_1(z)$. We can easily describe the first application:

$$\beta_1(z)=\psi_k(1)z+\psi_k(2)z^2+\dots+\psi_k(s_1)z^{s_1};$$
$$\mathcal{E}(\beta_1(z))=\frac{1}{(1-z)^{\psi_k(1)}\cdots(1-z^{s_1})^{\psi_k(s_1)}}\approx\frac{\mu}{(1-t)^M},\quad \text{as}\quad t\to 1{-}0;$$
$$\text{where}\quad M=\psi_k(1)+\dots+\psi_k(s_1)=\dim_K F(\mathbf{N}_{s_1},k),\qquad \mu=\prod_{q=2}^{s_1}q^{-\psi_k(q)}.$$

Next, we use the fact that $\mathcal{E}$ is "approximately" the exponent, thus $\mathcal{H}_X(L,t)$ behaves like

$$\exp^{(q-2)}\left(\frac{\bar\mu+o(1)}{(1-t)^N}\right),\qquad t\to 1{-}0\,.$$

Another idea in the proof of Theorem 5 is to specify a connection between the growth of a function analytic in the unit circle with asymptotic of its coefficients [25].

A similar version of Schreier's formula for free Lie *super*algebras was established as well [27, 29]. Also, the asymptotic of Theorem 5 was extended to the case of free solvable (polynilpotent) Lie *super*algebras of finite rank [13].

Below we see that the scale (4) for the superexponential codimension growth for varieties of Lie algebras is rather complete (Theorem 12). So, we conjecture that the scale (2) for the intermediate growth of finitely generated Lie PI-algebras is complete as well.

Conjecture 1 ([22])**.** *Let L be a finitely generated Lie PI-algebra. Then there exist numbers q, N_0 such that*

$$\gamma_L(n)\le\exp\left(\frac{n}{\ln^{(q)}n}\right),\quad n\ge N_0.$$

This bound was confirmed for almost solvable (more generally, almost polynilpotent) Lie algebras [14, 15].

3. Exponential analogue of Schreier's formula for free Lie algebras

Let us consider *complexity functions*, refereed to also as *exponential generating functions* in combinatorics. Assume that we are given a set A of monomials in $X=\{x_i\,|\,i\in\mathbb{N}\}$. Consider a set of distinct elements $\widetilde X=\{x_{i_1},\dots,x_{i_n}\}\subset X$, denote by $P_n(A,\widetilde X)$ the set of all multilinear elements of degree n on $\widetilde X$ belonging to A. Suppose that the number of these elements $c_n(A,\widetilde X)$ does not depend on the choice of $\widetilde X$, but depends only on n. In this case, we denote $c_n(A):=c_n(A,\widetilde X)$ and say that A is X-*uniform* and define the *complexity function with respect to X*:

$$\mathcal{C}_X(A,z):=\sum_{n=1}^{\infty}\frac{c_n(A)}{n!}z^n,\quad z\in\mathbb{C}. \tag{3}$$

(the sum is taken from $n=0$, $c_0=1$ for associative algebras and groupoids with unity). Remark that A need not consist of multilinear elements. We also omit the variable z and (or) the set X and write $\mathcal{C}_X(A,z)=\mathcal{C}(A)$. These definitions are naturally extended to algebras, subspaces, and their dimensions with respect to their generating sets. We illustrate this notion by examples. Let X be a countable set and $A = A(X)$, $L = L(X)$ the free associative and Lie algebras, respectively. Then

$$\mathcal{C}_X(A,z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z},$$

$$\mathcal{C}_X(L,z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = -\ln(1-z),$$

$$\mathcal{C}_X(\mathbf{N}_s,z) = \sum_{n=1}^{s} \frac{z^n}{n}.$$

The author established an exponential analogue of Schreier's formula for free Lie algebras as follows.

Theorem 8 ([26])**.** *Assume that L is the free Lie algebra generated by a countable generating set X. Assume that H is an X-uniform subalgebra. Then H has an X-uniform set of free generators Y and*

$$\mathcal{C}_X(Y,z) - 1 = (z-1)\cdot \exp(\mathcal{C}_X(L/H,z)).$$

4. Scale 2 for the codimension growth of Lie PI-algebras

For the theory of varieties of associative and Lie algebras see [1, 4, 10]. Let $\mathbf{V}$ be a variety of Lie algebras, and $F(\mathbf{V},X)$ its free algebra generated by $X = \{x_i | i \in \mathbb{N}\}$. Let $P_n(\mathbf{V}) \subset F(\mathbf{V},X)$ be the subspace of all multilinear elements in $\{x_1,\dots,x_n\}$ and consider the *codimension growth sequence* $c_n(\mathbf{V}) = c_n(F(\mathbf{V},X),X) := \dim_K P_n(\mathbf{V})$, $n = 1,2,\dots$.

In case of associative algebras the fundamental fact is as follows:

Theorem 9 (Regev, [34]; Latyshev [18])**.** *Let an associative algebra A satisfies a nontrivial identical relation of degree d. Then $c_n(A) \le C^n$, $n \ge 1$; where $C := (d-1)^2$.*

Another crucial fact on the codimension growth of associative algebras in characteristic zero is that the *exponent*, defined as $\operatorname{Exp}(A) := \lim\limits_{n\to\infty} \sqrt[n]{c_n(A)}$ always exists and is integral [8].

Now we start discussing the codimension growth for Lie algebras. The integrality of the exponent of the codimension growth for finite dimensional Lie algebras over a field of characteristic zero was proved by Zaitsev [38]. In general, the exponents for Lie algebras are not always integral [39, 9]. Moreover, the codimension growth in case of Lie algebras is more versatile. Unlike the associative case, the codimension growth of a rather small variety $\mathbf{AN}_2$ is overexponential (Volichenko [37]). On the other hand, the following upper bound was found.

Theorem 10 (Grishkov [12])**.** *Let L be a Lie algebra satisfying a nontrivial identity. Then for any $r > 1$ there exists N_0 such that*

$$c_n(L) \le \frac{n!}{r^n}, \quad n \ge N_0.$$

Razmyslov introduced the *complexity functions* (3) and reformulated the upper bound as follows.

Theorem 11 (Razmyslov [33])**.** *Let $\mathbf{V}$ be a nontrivial variety of Lie algebras. Then the complexity function $\mathcal{C}(\mathbf{V},z)$ is an entire function of complex variable.*

The author established a better and optimal general bound for the series that allowed to prove an upper bound for the codimension growth sequence as well. The estimate of the first item was recently obtained in [31].

Theorem 12 ([21, 23, 24, 31])**.** *Let L be a Lie algebra satisfying a nontrivial identity of degree $m \ge 4$. Then*

(1) *The following coefficientwise bound for the series holds:*

$$\mathcal{C}(L,z) \prec z\underbrace{\exp(z\exp(\dots(z\exp(z\exp}_{m-2 \text{ times } \exp}(z)))\dots)).$$

(2) *there exists an infinitesimal such that*

$$c_n(L) \le \frac{n!}{(\ln^{(m-3)} n)^n}(1+o(1))^n, \quad n \to \infty.$$

Thus, we have a vast area of overexponential growths for Lie algebras, lying between the exponent and the factorial functions. To describe such a growth we introduce the **scale 2** consisting of a series of functions $\Psi^q_\alpha(n)$, $q=2,3,\dots$, with a real parameter α [21]:

$$\textbf{scale 2:}\qquad \Psi^q_\alpha(n) := \begin{cases} \big(n!\big)^{\frac{\alpha-1}{\alpha}}, & \alpha>1, \quad q=2;\\ \dfrac{n!}{(\ln^{(q-2)} n)^{n/\alpha}}, & \alpha>0, \quad q=3,4\dots \end{cases} \tag{4}$$

The upper bounds of Theorem 12 are "adequate" and the scale (4) for the codimension growth of Lie PI-algebras is complete, because the free solvable Lie algebras do have such an asymptotic behaviour, see Theorem 20 below. Actually we obtain a more fine scale formed by a series of functions with two real parameters α,β:

$$\textbf{scale 2}':\qquad \Psi^q_{\alpha,\beta}(n) := \begin{cases} \big(n!\big)^{\frac{\alpha-1}{\alpha}}\beta^{n/\alpha}, & \alpha\ge 1,\ \beta>0; \quad q=2;\\ \dfrac{n!\cdot(\beta/\alpha)^{n/\alpha}}{(\ln^{(q-2)} n)^{n/\alpha}}, & \alpha>0,\ \beta>0; \quad q=3,4\dots \end{cases} \tag{5}$$

Observe that in terms of scale (5) the exponential growth is a subcase of level $q=2$ when $\alpha=1$.

4.1. **The codimension growth for another classes of** ***linear algebras*****.** The codimension growth of arbitrary linear algebras can be weird [7]. The varieties of absolutely free (commutative, or anticommutative) algebras have Schreier's type formulae in terms of generating function of both types, the regular generating functions and exponential generating functions (i.e. complexity functions), i.e. we have natural analogues of both, Theorem 1 and Theorem 8), see [30]. We describe both, the generating functions and the growth functions, for two kinds of growth, for different versions of nilpotency and solvability for three types of linear algebras above [30]. But we do not get an analogue of Theorem 13 and these results do not lead us to something like scale 1 and scale 2 for the respective types of growth [30].

It was recently shown that the same scale (4) stratifies the ordinary codimension growth of *Poisson PI-algebras*, but here it is essential to assume that a Poisson algebra satisfies a nontrivial Lie identical relation [31]. If a Poisson algebra is satisfying so called mixed identities only, then the ordinary codimension growth has a factorial behaviour [32, 31].

The codimension growth for *Jordan PI-algebras* can be overexponential [5, 11]. We conjecture that something like scale 2 (see (4)) should appear in case of Jordan PI-algebras as well.

5. Explicit formulae for complexity functions and asymptotic for Lie PI-algebras

Let $\mathbf{M},\mathbf{V}$ be varieties of Lie algebras. Their *product* $\mathbf{M}\cdot\mathbf{V}$ is the class of all Lie algebras L such that there exists an ideal $H\subset L$ satisfying $H\in\mathbf{M}$ and $L/H\in\mathbf{V}$, see [1]. Using the exponential Schreier's formula (Theorem 8) the following explicit formula was proved.

Theorem 13 ([26])**.** *Let* $\mathbf{M}\cdot\mathbf{V}$ *be the product of varieties of Lie algebras, where* $\mathbf{M}$ *is multihomogeneous. Then*

$$\mathcal{C}(\mathbf{M}\cdot\mathbf{V},z)=\mathcal{C}(\mathbf{V},z)+\mathcal{C}(\mathbf{M},1+(z-1)\exp(\mathcal{C}(\mathbf{V},z))).$$

Roughly speaking, the formula says that $\mathcal{C}(\mathbf{M}\cdot\mathbf{V},z)$ is "almost" a composition of three functions $\mathcal{C}(\mathbf{M})\circ\exp\circ\,\mathcal{C}(\mathbf{V})$. The variety $\mathbf{V}=\mathbf{N}_{s_q}\cdots\mathbf{N}_{s_1}$ can be viewed as the product $\mathbf{V}=\mathbf{N}_{s_q}\cdots\mathbf{N}_{s_2}\cdot\mathbf{N}_{s_1}$. As application, the following explicit formula was derived.

Theorem 14 ([26, 29])**.** *Consider the variety of polynilpotent Lie algebras* $\mathbf{V}:=\mathbf{N}_{s_q}\cdots\mathbf{N}_{s_1}$, $q\ge 1$. *Define functions*

$$\beta_1(z):=\sum_{m=1}^{s_1}\frac{z^m}{m},$$
$$\beta_i(z):=\beta_{i-1}(z)+\sum_{m=1}^{s_i}\frac{(1+(z-1)\exp(\beta_{i-1}(z)))^m}{m},\qquad 2\le i\le q.$$

Then $\mathcal{C}(\mathbf{V},z)=\beta_q(z)$.

Consider particular cases.

Corollary 15. *Fix $d \in \mathbb{N}$. Then*

$$\mathcal{C}(\mathbf{AN}_d, z) = z + \frac{z^2}{2} + \cdots + \frac{z^d}{d} + 1 + (z-1)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^d}{d}\right).$$

Corollary 16 ([23])**.** *Fix $q \in \mathbb{N}$. Consider the variety of solvable Lie algebras of length q, denoted as $\mathbf{A}^q$. Set $\beta_1(z) = z$, and $\beta_{i+1}(z) = \beta_i(z) + 1 + (z-1)\exp(\beta_i(z))$ for $i = 1, 2, \ldots, q-1$. Then*

$$\mathcal{C}(\mathbf{A}^q, z) = \beta_q(z).$$

Corollary 17 ([23])**.** *Fix $c \in \mathbb{N}$. Then*

$$\mathcal{C}(\mathbf{N}_c\mathbf{A}, z) = z + \sum_{m=1}^{c} \frac{1}{m}\Big(1 + (z-1)\exp(z)\Big)^m.$$

Complexity functions are useful for computation of the codimension growth. For example, the last result yields an asymptotic.

Corollary 18 ([23])**.** *$c_n(\mathbf{N}_c\mathbf{A}) \approx c^{n-c-1} n^c$, as $n \to \infty$.*

Let $f(z)$ be an entire function of complex variable, denote $\mathrm{M}_f(r) := \max_{|z|=r} |f(z)|$. Observe that in case of complexity functions, we have $\mathrm{M}_f(r) = f(r), r \in \mathbb{R}^+$, since all coefficients are nonnegative.

By Theorem (14), $\mathcal{C}(\mathbf{N}_{s_q} \cdots \mathbf{N}_{s_1}, z)$ is "almost" $q-1$ iterations of $\exp(*)$ applied to $\beta_1(z) = \sum_{m=1}^{s_1} z^m/m$, thus one has something like $\exp^{(q-1)}(z^{s_1}/s_1)$. More, precisely, we derive the following asymptotic.

Theorem 19 ([23])**.** *Consider the variety of polynilpotent Lie algebras $\mathbf{V} := \mathbf{N}_{s_q} \cdots \mathbf{N}_{s_1}$, $q \geq 2$, and its complexity function $f(z) := \mathcal{C}(\mathbf{V}, z)$. Then*

$$\lim_{r \to \infty} \frac{\ln^{(q-1)} \mathrm{M}_f(r)}{r^{s_1}} = \frac{s_2}{s_1}.$$

Next, we establish a relation between the growth of fast growing entire functions and an asymptotic of their coefficients. This fact helped us to match the upper bounds in the next result. But a connection between the lower bounds require more direct estimates.

Theorem 20 ([23])**.** *Consider the variety of polynilpotent Lie algebras $\mathbf{V} := \mathbf{N}_{s_q} \cdots \mathbf{N}_{s_1}$, $q \geq 2$. Then there exists an infinitesimal such that*

$$c_n(\mathbf{V}) = \begin{cases} (n!)^{\frac{s_1-1}{s_1}} \big(s_2 + o(1)\big)^{n/s_1}, & q = 2; \\ \dfrac{n!}{(\ln^{(q-2)} n)^{n/s_1}} \Big(\dfrac{s_2 + o(1)}{s_1}\Big)^{n/s_1}, & q = 3, 4, \ldots; \end{cases} \qquad n \to \infty.$$

In the following two cases we obtain somewhat more precise asymptotic, but they look rather complicated.

Theorem 21 ([26])**.** *Fix $d \in \mathbb{N}$. Then*

$$c_n(\mathbf{AN}_d) \approx \mu\,(n!)^{1-1/d} \exp\left(\sum_{k=1}^{d-1} \lambda_k n^{1-k/d}\right) n^{\frac{3-d}{2d}}, \quad n \to \infty, \quad \textit{where}$$

$$\lambda_k := \frac{\left(\frac{k}{d}+1\right)\cdots\left(\frac{k}{d}+k-1\right)}{k!(d-k)}, \qquad k = 1, \ldots, d-1;$$

$$\mu := \exp\left(-\frac{1}{d}\sum_{k=2}^{d} \frac{1}{k}\right)(2\pi)^{\frac{1-d}{2d}} d^{-1/2}.$$

It is well known that $c_n(\mathbf{A}^2) = n - 1 \approx n$, this coincides with our asymptotic. The particular cases are.

$$\begin{aligned} c_n(\mathbf{AN}_2) &\approx \sqrt{n!}\,\frac{\exp(\sqrt{n})\,\sqrt[4]{n}}{\sqrt[4]{8\pi e}}, \\ c_n(\mathbf{AN}_3) &\approx (n!)^{2/3}\,\frac{\exp\left(\frac{1}{2}n^{2/3} + \frac{5}{6}n^{1/3}\right)}{\sqrt{3}\sqrt[3]{2\pi}e^{5/18}}, \end{aligned} \qquad n \to \infty.$$

Theorem 22 ([26])**.** *Consider the variety* $\mathbf{A}^3$ *of solvable Lie algebras of length 3 and its codimension growth sequence* $c_n := c_n(\mathbf{A}^3)$*. Then*

$$c_n = \frac{n!}{(\ln n)^n} \exp\left(\frac{n}{\ln n}\left(2\ln\ln n + 1 + \frac{2(\ln\ln n)^2 - 2\ln\ln n - 1}{\ln n} + o\left(\frac{1}{\ln n}\right)\right)\right), \quad n \to \infty.$$

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