

COMPLEXITY CLASSES OF POLISHABLE SUBGROUPS

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ABSTRACT. In this paper we further develop the theory of *canonical approximations of Polishable subgroups* of Polish groups, building on previous work of Solecki and Farah–Solecki. In particular, we obtain a characterization of such canonical approximations in terms of their *Borel complexity class*. As an application we provide a *complete list* of all the possible Borel complexity classes of Polishable subgroups of Polish groups or, equivalently, of the ranges of continuous group homomorphisms between *Polish groups*. We also provide a complete list of all the possible Borel complexity classes of the ranges of: continuous group homomorphisms between *non-Archimedean Polish groups*; continuous linear maps between separable *Fréchet spaces*; continuous linear maps between separable *Banach spaces*.

1. INTRODUCTION

The goal of this paper is to exactly pin down the possible Borel complexity classes of Polishable subgroups of Polish groups. Most of the equivalence relations studied in the context of Borel complexity theory (and mathematics in general) arise as orbit equivalence relations associated with continuous actions of Polish groups on Polish spaces. Many of these actions can be seen as the (left) *translation action* associated with a continuous group homomorphism $\varphi : H \rightarrow G$ between Polish groups. In such a case, the image $\varphi(H)$ of φ inside of G is Borel, and the *potential complexity class* (in the sense of Louveau [Lou94]) of the orbit equivalence relation associated with the translation action of H on G is essentially the same as the Borel complexity class of $\varphi(H)$ of φ inside of G ; see Theorem 3.3 for a precise statement. It is thus an interesting problem to determine what are the possible values for the Borel complexity class of such a subgroup.

The study of Polishable subgroups of Polish groups, which are precisely the ranges of continuous homomorphisms between Polish groups, has been undertaken by several authors over a number of years. The problem of determining their complexity has been considered as early as the 1970s, when Saint-Raymond proved that there exist Polishable subgroups of $\mathbb{R}^{\mathbb{N}}$ that are arbitrarily high in the Borel hierarchy [SR76]. A construction of arbitrarily complex non-Archimedean Polishable subgroups of $\mathbb{Z}_2^{\mathbb{N}}$ was presented by Hjorth, Kechris, and Louveau in [HKL98]. Hjorth constructed in [Hjo06] arbitrarily complex Polishable subgroups of any uncountable abelian Polish groups. Farah and Solecki in [FS06], building on previous work of Sain-Raymond in the context of separable Fréchet spaces [SR76], related the least multiplicative Borel class containing a given Polishable subgroup to the length of the canonical approximation of that Polishable subgroup as in [Sol99, Sol09].

In this paper, we refine the analysis from [FS06] by considering not only the multiplicative classes in the Borel hierarchy, but also the additive and difference classes. By relating the Borel complexity class of a Polishable subgroup to its canonical approximation, we completely characterize the possible Borel complexity classes of Polishable subgroups of Polish groups.

Theorem 1.1. *If H is a Polishable subgroup of a Polish group G , then the Borel complexity class of H is one of the following: $\Pi_{1+\lambda}^0$, $\Sigma_{1+\lambda+1}^0$, $D(\Pi_{1+\lambda+n+1}^0)$, $\Pi_{1+\lambda+n+2}^0$ for $\lambda < \omega_1$ either zero or a limit ordinal, and $n < \omega$. Furthermore, each of these classes is the Borel complexity class of a Polishable subgroup of $\mathbb{Z}^{\mathbb{N}}$.*

Theorem 4.1 from [HKL98, Section 5] shows that the complexity class $D(\Pi_{1+\lambda+1}^0)$ where λ is either zero or a countable limit ordinal cannot arise in the context of Theorem 1.1 if one demands H to be non-Archimedean. In this case, we have the following characterization.

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Theorem 1.2. *If H is a non-Archimedean Polishable subgroup of a Polish group G , then the Borel complexity class of H in G is one of the following: $\Pi_{1+\lambda}^0$, $\Sigma_{1+\lambda+1}^0$, $D(\Pi_{1+\lambda+n+2}^0)$, $\Pi_{1+\lambda+n+2}^0$ for $\lambda < \omega_1$ either zero or a limit ordinal, and $n < \omega$. Furthermore, each of these classes is the complexity class of a non-Archimedean Polishable subgroup of $\mathbb{Z}^{\mathbb{N}}$.*

The existence assertions in Theorem 1.1 and Theorem 1.2 are proved by providing a unified approach to the constructions in [SR76] and [HKL98, Section 5] of arbitrarily complex Polishable subgroups, together with a careful analysis of their canonical approximations in the sense of Solecki [Sol99, Sol09].

Theorem 1.1 entails in particular a negative answer to a Question 6.3(1) from [Din17]. Let X be a separable Banach space with a Schauder basis $(x_n)_{n \in \mathbb{N}}$. Then the collection $\text{coef}(X, (x_n))$ of $(\lambda_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that $\sum_{n \in \mathbb{N}} \lambda_n x_n$ converges in X is a Polishable subgroup of $\mathbb{R}^{\mathbb{N}}$. Question 6.3(1) asks whether there is an example of such a Polishable subgroup that is Δ_3^0 and not $D(\Sigma_2^0)$. By Theorem 1.1, a Δ_3^0 Polishable subgroup of a Polish group must be $D(\Sigma_2^0)$.

We also apply the techniques of this paper to provide a complete characterization of the Borel complexity classes of the ranges of continuous homomorphisms between separable Fréchet spaces and between separable Banach spaces.

Theorem 1.3. *The complexity classes in Theorem 1.1 form a complete list of all the Borel complexity classes of the ranges of continuous linear maps between separable Fréchet spaces.*

Theorem 1.4. *The following is a complete list of all the Borel complexity classes of the ranges of continuous linear maps between separable Banach spaces: Π_1^0 , $\Sigma_{1+\lambda+1}^0$, $D(\Pi_{1+\lambda+n+1}^0)$, $\Pi_{1+\lambda+n+2}^0$ for $\lambda < \omega_1$ either zero or a limit ordinal, and $n < \omega$.*

Continuous linear maps with arbitrarily complex range, with a fixed separable Banach space or separable Fréchet space as target, were constructed in [DG08, Mal08].

The rest of this paper is organized as follows. In Section 2 and Section 3 we recall some definitions and known results concerning Polishable subgroups and their Borel complexity class. In Section 4 we recall the definition of the canonical approximation of a Polishable subgroup, whose elements we call Solecki subgroups as they were originally described by Solecki in [Sol99]. In Section 5, building on the work of Farah and Solecki, we refine the analysis from [FS06] to characterize the Solecki subgroups in terms of their Borel complexity class. This is then applied in Section 6 to obtain the characterization of complexity classes of Polishable subgroups as in Theorem 1.1. Section 7 shows that the length of the canonical approximation, called the Polishable rank in [FS06], coincides with a notion of rank originally considered by Saint-Raymond in [SR76] in the context of separable Fréchet spaces. The existence assertions in Theorem 1.1 and Theorem 1.2 are proved in Section 8. Finally, Section 9 and Section 10 contain a proof of Theorem 1.3 and Theorem 1.4, respectively.

Notation. In this paper, we use \mathbb{N} to denote the set of positive integers *excluding zero*. As usual, we let ω be the first infinite ordinal, which can also be seen as the set of positive integers *including zero*.

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2. POLISHABLE SUBGROUPS

A *Polish space* is a second countable topological space whose topology is induced by a complete metric. A *Polish group* is a group in the category of Polish spaces, namely a Polish space that is endowed with a continuous group operation such that the function that maps each element to its inverse is also continuous (in fact, the latter requirement holds automatically; see the remark after [Kec95, Corollary 9.15]). A subgroup H of a Polish group G is *Polishable* if it is Borel and there exists a Polish group topology on H whose open sets are Borel in G . Notice that such a Polish topology on H , if it exists, is unique by [Kec95, Theorem 9.10]. In the following, we will regard H as a Polish group with respect to its unique Polish group topology, which is in general finer than the subspace topology induced from G . Equivalently, H is a Polishable subgroup of G if and only if there exists a Polish group \tilde{H} and a continuous group homomorphism $\varphi : \tilde{H} \rightarrow G$ with image equal to H . Noticing that one can assume without loss of generality that φ is an injection, the equivalence of the two definitions follows from [Kec95, Theorem 9.10] and the fact that if $f : X \rightarrow Y$ is an injective Borel function between standard Borel spaces, then $f(A)$ is a Borel subset of Y and $f|_A$ is a Borel isomorphism between A and $f(A)$ [Kec95, Theorem 15.1]. If G is a Polish group

and H is a Polishable subgroup of G , then G is a *Polish H -space* with respect to the left translation action of H on G [BK96, Section 2.2]. We will denote by E_H^G the corresponding orbit equivalence relation. Recall that a Polish group G is *non-Archimedean* if it admits a basis of neighborhoods of the identity consisting of open subgroups; see [Gao09, Theorem 2.4.1] for equivalent characterizations.

Lemma 2.1. *Suppose that G is a Polish group. Let $(G_n)_{n \in \omega}$ be a sequence of Polishable subgroups of G . Then $G_\omega := \bigcap_{n \in \omega} G_n$ is a Polishable subgroup of G . If G_n is non-Archimedean for every $n \in \omega$, then G_ω is non-Archimedean as well. If $A \subseteq G_\omega$ is such that A is dense in G_n for every $n \in \omega$, then A is dense in G_ω .*

Proof. We have that G_ω is the image of the Polish group

$$Z := \left\{ (x_n)_{n \in \omega} \in \prod_{n \in \omega} G_n : \forall n \in \omega, x_n = x_{n+1} \right\} \subseteq \prod_{n \in \omega} G_n$$

under the continuous injective group homomorphism $Z \rightarrow G$, $(x_n)_{n \in \omega} \mapsto x_0$. This shows that G_ω is Polishable. If G_n is non-Archimedean for every $n \in \omega$, then Z is non-Archimedean, and hence G_ω is non-Archimedean as well. By the above, the sets of the form $W \cap G_\omega$, where W is a neighborhood of the identity in G_n for some $n \in \omega$, form a basis of neighborhoods of the identity in G_ω . Thus, if A is dense in G_n for every $n \in \omega$, then A is dense in G_ω . \square

3. POTENTIAL COMPLEXITY

A *complexity class* Γ is a function $X \mapsto \Gamma(X)$ that assigns to each Polish space X a collection $\Gamma(X)$ of Borel subsets, such that if X, Y are Polish spaces and $f : X \rightarrow Y$ is a continuous function, then $f^{-1}(A) \in \Gamma(X)$ for every $A \in \Gamma(Y)$. For a complexity class Γ , we let $D(\Gamma)$ be the complexity class consisting of *differences* between sets in Γ ; see [Kec95, Section 22.E] where it is denoted by $D_2(\Gamma)$. We let $\check{\Gamma}$ be the *dual* complexity class of Γ , such that $\check{\Gamma}(X)$ comprises the *complements* of the elements of $\Gamma(X)$. We say that Γ is self-dual if $\Gamma = \check{\Gamma}$. If Γ is a complexity class that is not self-dual, then we say that Γ is the complexity class of $A \subseteq X$ if $A \in \Gamma(X)$ and $A \notin \check{\Gamma}(X)$. We will be mainly interested in the complexity classes Σ_α^0 , Π_α^0 , Δ_α^0 , and $D(\Pi_\alpha^0)$ for $\alpha \in \omega_1$; see [Kec95, Section 11.B].

If X is a standard Borel space and E is an equivalence relation on X , then E has potential complexity Γ if there exists a Polish topology τ on X that induces the Borel structure of X such that $E \in \Gamma(\tau \times \tau)$ [Lou94]. This is equivalent to the assertion that there exists a Borel equivalence relation F on a Polish space Y such that $F \in \Gamma(Y \times Y)$ and E is Borel reducible to F ; see [Gao09, Lemma 12.5.4]. The following result is essentially proved in [HKL98, Section 5].

Proposition 3.1 (Hjorth–Kechris–Louveau). *Suppose that G is a Polish group, and X is a Polish G -space. For $x \in X$, denote by $[x]$ the corresponding G -orbit. Let Γ be a complexity class, and assume that the orbit equivalence relation E_G^X is potentially Γ . Suppose that α is a countable ordinal.*

- (1) *If Γ is the class Π_α^0 for $\alpha \geq 2$, Σ_α^0 for $\alpha \geq 3$, or $D(\Pi_\alpha^0)$ for $\alpha \geq 2$, then $\{x \in X : [x] \in \Gamma\}$ is comeager in X .*
- (2) *If Γ is the class $\check{D}(\Pi_\alpha^0)$ for $\alpha \geq 3$, then $\{x \in X : [x] \text{ is either } \Pi_\alpha^0 \text{ or } \Sigma_\alpha^0\}$ is comeager in X .*

Proof. Fix a countable open basis $\{U_i : i \in \omega\}$ of G . Below we adopt the Vaught transform notation as in [Gao09, Section 3.2]. By [Kec95, Theorem 8.38], there exists a dense G_δ set $W \subseteq X$ such that $E_G^X \cap (W \times W) \in \Gamma(W \times W)$. Notice that W^* is also a dense G_δ subset of X . Fix $x \in W \cap W^*$. Thus, we have that $[x] \cap W \in \Gamma(W)$. If $\Gamma = \Sigma_\alpha^0$ for $\alpha \geq 3$, then $[x] \cap W = A \cap W$ for some $A \in \Sigma_\alpha^0(X)$. Then we have that $[x] = ([x] \cap W)^\Delta = (A \cap W)^\Delta$ is Σ_α^0 in X . If $\Gamma = \Pi_\alpha^0$ for $\alpha \geq 2$, then $[x] \cap W = B \cap W$ for some $B \in \Pi_\alpha^0(X)$. Then $[x] = (W \cap [x])^* = (B \cap W)^*$ is Π_α^0 in X . If $\Gamma = D(\Pi_\alpha^0)$ for $\alpha \geq 2$, then $W \cap [x] = A \cap B \cap W$ where $A \in \Sigma_\alpha^0(X)$ and $B \in \Pi_\alpha^0(X)$. Thus, $[x] = A^\Delta \cap (B \cap W)^* \in D(\Pi_\alpha^0)$.

If $\Gamma = \check{D}(\Pi_\alpha^0)$ for $\alpha \geq 3$, then $W \cap [x] = (A \cap W) \cup (B \cap W)$, where $A \in \Sigma_\alpha^0(X)$ and $B \in \Pi_\alpha^0(X)$. Thus, $[x] = (A \cap W)^\Delta$ or $[x] = (B \cap W)^*$. Hence, either $[x]$ is Σ_α^0 or $[x]$ is Π_α^0 . If $\Gamma = \check{D}(\Pi_2^0)$, then $A \cap W$ as above is $D(\Pi_2^0)$ in X , and hence $[x]$ is $D(\Pi_2^0)$ in X . \square

A similar proof as Proposition 3.1 gives the following.

Lemma 3.2. *Suppose that G is a Polish group, and H is a Polishable subgroup of G . Let α be a countable ordinal. If H is $\check{D}(\Pi_\alpha^0)$, then H is either Π_α^0 or Σ_α^0 .*

Proof. Adopt the notation of the Vaught transform with respect to the left translation action of H on G . We have that $H = A \cup B$ where A is Σ_α^0 and B is Π_α^0 . If $x \in H$, then we have that either $x \in A^\Delta$ or $x \in B^*$. Since A^Δ and B^* are H -invariant, we have that either $H \subseteq A^\Delta$ or $H \subseteq B^*$. Since A^Δ and B^* are contained in H , we have that either $H = A^\Delta$ or $H = B^*$. This concludes the proof. \square

Applying Proposition 3.1 to the left translation action associated with a Polishable subgroup of a Polish group, we obtain Items (1) and (3) of the following result. The proof of Item (2) is postponed to Section 6.

Theorem 3.3. *Suppose that G is a Polish group, and H is a Polishable subgroup of G . Denote by E_H^G the corresponding coset equivalence relation.*

- (1) E_H^G is potentially Π_2^0 if and only if H is closed G .
- (2) E_H^G is potentially Σ_2^0 if and only if H is $D(\Pi_2^0)$ in G .
- (3) Let Γ be one of the following complexity classes: Σ_α^0 for $\alpha \neq 2$, Π_α^0 , and $D(\Pi_\alpha^0)$. Then E_H^G is potentially Γ in G if and only if H is Γ in G .

Proof. (1): Suppose that E_H^G is potentially Π_2^0 . By [Gao09, Lemma 12.5.3] we have that E_H^G is smooth. Thus, H is closed by [Sol09, page 574].

(2): The forward implication is a particular instance of Proposition 3.1, while the converse implication follows from Lemma 6.5 in Section 6.

(3): Only the forward implication requires a proof. If $\Gamma = \Sigma_1^0$ then E_H^G has countably many classes by [Gao09, Lemma 12.5.2]. Thus, H has countable index in G , and hence it is nonmeager. Therefore, H is open by [Gao09, Theorem 2.3.2]. If Γ is Π_1^0 or Π_2^0 or $D(\Pi_1^0)$, then $H \in \Pi_1^0(G) \subseteq \Gamma(G)$ by Part (1). If Γ is Π_α^0 or Σ_α^0 for $\alpha \geq 3$, or $D(\Pi_\alpha^0)$ for $\alpha \geq 2$, the conclusion follows from Proposition 3.1. \square

We now recall some results concerning the possible complexity classes of Polishable subgroups. The following proposition is a reformulation of [FS06, Corollary 3.4].

Proposition 3.4 (Farah–Solecki). *Suppose that G is a Polish group, and H is a Polishable subgroup of G . If $\lambda < \omega_1$ is either zero or a limit ordinal, and H is $\Pi_{1+\lambda+1}^0$ in G , then H is $\Pi_{1+\lambda}^0$ in G .*

The following proposition is a consequence of [HKL98, Theorem 4.1] and Proposition 3.1.

Proposition 3.5 (Hjorth–Kechris–Louveau). *Suppose that G is a Polish group, and H is a non-Archimedean Polishable subgroup of G . Suppose that $\lambda < \omega_1$ is either zero or limit. If H is $\Sigma_{1+\lambda+2}^0$, then H is $\Sigma_{1+\lambda+1}^0$.*

4. SOLECKI SUBGROUPS

Suppose that G is a Polish group, and H is a Polishable subgroup of G . Then H admits a canonical approximation by Polishable subgroups indexed by countable ordinals. As these were originally described by Solecki in [Sol99], we call them *Solecki subgroups* of G associated with H . They have also been considered in [Sol09, FS06].

Lemma 2.3 from [Sol99, Lemma 2.3] implies that G has a smallest Π_3^0 Polishable subgroup containing H , which we denote by $s_1^H(G)$. One can explicitly describe $s_1^H(G)$ as the Π_3^0 subgroup of G defined by

$$\bigcap_V \bigcup_{z_0, z_1 \in H} z_0 \overline{V}^G \cap \overline{V}^G z_1$$

where V ranges among the open neighborhoods of the identity in H , and \overline{V}^G is the closure of V inside of G . It is proved in [Sol99, Lemma 2.3] that $s_1^H(G)$ satisfies the following properties—see also [Tsa06, Lemma 4.5] and [FS06, Section 3]:

- H is dense in $s_1^H(G)$;
- a neighborhood basis of $x \in s_1^H(G)$ consists of sets of the form $\overline{W}^G \cap s_1^H(G)$ where W is an open neighborhood of the identity in H ;
- if $A \subseteq G$ is Π_3^0 and contains H , then $A \cap s_1^H(G)$ is comeager in the Polish group topology of $s_1^H(G)$.

Lemma 4.1. *Suppose that G is a Polish group, and H is a non-Archimedean Polishable subgroup of G . Then a neighborhood basis of the identity in $s_1^H(G)$ consists of the sets of the form $\overline{W}^G \cap s_1^H(G)$ where W is an open subgroup of H . In particular, $s_1^H(G)$ is non-Archimedean.*

Proof. Since H is non-Archimedean, the first assertion follows from the remarks above. If W is an open subgroup of H , then $\overline{W}^G \cap s_1^H(G)$ is a subgroup of $s_1^H(G)$ with nonempty interior, whence it is an open subgroup by Pettis' Theorem [Pet50, Corollary 3.1]. Therefore, the second assertion follows from the first one. \square

A similar argument as in the proof of [Sol99, Lemma 2.3] gives the following.

Lemma 4.2. *Suppose that G is a Polish group, and N is a Polishable subgroup of G . Let H be a Polishable subgroup of G such that:*

- (1) $N \subseteq H$ and N is dense in the Polish group topology of H ;
- (2) for every open neighborhood V of the identity in N , $\overline{V}^G \cap H$ contains an open neighborhood of the identity in H .

If $A \subseteq G$ is Π_3^0 and contains N , then $A \cap H$ is comeager in H . In particular, $H \subseteq s_1^N(G)$. If H is furthermore Π_3^0 , then $H = s_1^N(G)$.

Proof. It suffices to consider the case when A is Σ_2^0 . In this case, there exist closed subsets L_k of G for $k \in \omega$ such that $A = \bigcup_{k \in \omega} L_k$. Suppose that $U \subseteq H$ is a nonempty open set. Since N is dense in H , $U \cap N$ is a nonempty open subset of N . By the Baire Category Theorem, there exists $k_0 \in \omega$ such that $L_{k_0} \cap U \cap N$ is not meager in N . Thus, there exist $x \in N$ and an open neighborhood V of the identity in N such that

$$Vx \subseteq L_{k_0} \cap U \cap N.$$

Since L_{k_0} is closed in G , we have that $\overline{Vx}^G \subseteq L_{k_0}$. By (2), there is an open neighborhood W of the identity in H such that $Wx \subseteq \overline{Vx}^G \subseteq L_{k_0}$. This shows that $x \in U$ is in the interior of $L_{k_0} \cap H \subseteq A \cap H$. Since this holds for every nonempty open subset of H , we have that $A \cap H$ contains a dense open subset of H , and hence it is comeager in H . This concludes the proof. \square

The sequence of *Solecki subgroups* $s_\alpha^H(G)$ for $\alpha < \omega_1$ of G associated with H is defined recursively by setting:

- $s_0^H(G) = \overline{H}^G$;
- $s_{\alpha+1}^H(G) = s_1^H(s_\alpha^H(G))$ for $\alpha < \omega_1$;
- $s_\lambda^H(G) = \bigcap_{\beta < \lambda} s_\beta^H(G)$ for a limit ordinal $\lambda < \omega_1$.

Using Lemma 2.1 at the limit stage, one can prove by induction on $\alpha < \omega_1$ that $s_\alpha^H(G)$ is a Polishable subgroup of G , and H is dense in $s_\alpha^H(G)$. Furthermore, by Lemma 4.1, if H is non-Archimedean, then $s_\alpha^H(G)$ is non-Archimedean for every $1 \leq \alpha < \omega_1$. It is proved in [Sol99, Theorem 2.1] that there exists $\alpha < \omega_1$ such that $s_\alpha^H(G) = H$. We call the least countable ordinal α such that $s_\alpha^H(G) = H$ the *Solecki rank* of H in G .

One can define the Polish groups $s_\alpha^H(G)$ solely in terms of H endowed with the subspace topology inherited from G . Indeed, $s_0^H(G)$ can be seen as the completion of H with respect to a suitable metric that induces the subspace topology inherited from G ; see [Sol99, Section 2.1]. Using Lemma 4.2 one can describe the Solecki subgroups of products, as follows.

Lemma 4.3. *Suppose that, for every $n \in \mathbb{N}$, G_n is a Polish group, and N_n is a Polishable subgroup. Define $G = \prod_{n \in \omega} G_n$ and $N = \prod_{n \in \omega} N_n$. Then we have that*

$$s_\gamma^H(G) = \prod_{n \in \omega} s_\gamma^{N_n}(G_n)$$

for every $\gamma < \omega_1$.

Proof. It suffices to consider the case when $\gamma = 1$. In this case, set

$$H_n := s_1^{N_n}(G_n)$$

for $n \in \omega$, and

$$H := \prod_{n \in \omega} H_n.$$

Then we have that H is a $\mathbf{\Pi}_3^0$ Polishable subgroup of G , $N \subseteq H$, and N is dense in H . If V is an open neighborhood of the identity in N , then there exist $n \in \omega$ and open neighborhoods V_i of the identity in N_i for $i < n$ such that V contains

$$\prod_{i < n} V_i = \{x \in N : \forall i < n, x_i \in V_i\}.$$

For $i < n$, we have that $\overline{V}_i^{G_i} \cap H_i$ contains an open neighborhood W_i of the identity in H_i . Therefore, we have that $\overline{V}^G \cap H$ contains

$$\prod_{i < n} W_i = \{x \in H : \forall i < n, x_i \in W_i\},$$

which is an open neighborhood of the identity in H . The conclusion thus follows from Lemma 4.2. \square

5. COMPLEXITY OF SOLECKI SUBGROUPS

Suppose that G is a Polish group, and H is a Polishable subgroup of G . For a complexity class Γ , we define $\Gamma(G)|_H$ to be the collection of sets of the form $A \cap H$ for $A \in \Gamma(G)$. The following results are essentially established in [FS06]. In the statements and proofs, we adopt the Vaught transform notation in reference to the action of H on G by left translation; see [Gao09, Section 3.2].

Lemma 5.1. *Suppose that G is a Polish group, H is a Polishable subgroup of G , and $\alpha, \beta < \omega_1$ are ordinals. Then*

$$\Sigma_{1+\beta}^0(s_\alpha^H(G)) \subseteq \Sigma_{1+\alpha+\beta}^0(G)|_{s_\alpha^H(G)}$$

and

$$\mathbf{\Pi}_{1+\beta}^0(s_\alpha^H(G)) \subseteq \mathbf{\Pi}_{1+\alpha+\beta}^0(G)|_{s_\alpha^H(G)}.$$

Proof. It is proved in [FS06, Theorem 3.1] by induction on α that $\Sigma_1^0(s_\alpha^H(G)) \subseteq \Sigma_{1+\alpha}^0(G)|_{s_\alpha^H(G)}$. By taking complements, we have that $\mathbf{\Pi}_1^0(s_\alpha^H(G)) \subseteq \mathbf{\Pi}_{1+\alpha}^0(G)|_{s_\alpha^H(G)}$. This is the case $\beta = 0$ of the statement above. The rest follows by induction on β . \square

Lemma 5.2. *Suppose that G is a Polish group, H is a Polishable subgroup of G , $\alpha, \beta < \omega_1$, and $U \subseteq H$ is open in H . If $A \in \Sigma_{1+\alpha+\beta}^0(G)$ and $B \in \mathbf{\Pi}_{1+\alpha+\beta}^0(G)$, then $A^{\Delta U} \cap s_\alpha^H(G) \in \Sigma_{1+\beta}^0(s_\alpha^H(G))$, and $B^{*U} \cap s_\alpha^H(G) \in \mathbf{\Pi}_{1+\beta}^0(s_\alpha^H(G))$.*

Proof. When $\beta = 0$, the assertion about A is the content of Claim 3.3 in the proof of [FS06, Theorem 3.1]. The assertion about B follows by taking complements. This concludes the proof when $\beta = 0$. The case of an arbitrary β is established by induction on β using the properties of the Vaught transform; see [Gao09, Proposition 3.2.5]. \square

Corollary 5.3. *Suppose that G is a Polish group, H is a Polishable subgroup of G , and $\alpha, \beta < \omega_1$. Let L be a Polishable subgroup of G containing H . If $L \in \Sigma_{1+\alpha+\beta}^0(G)$, then $L \cap s_\alpha^H(G) \in \Sigma_{1+\beta}^0(s_\alpha^H(G))$. If $L \in \mathbf{\Pi}_{1+\alpha+\beta}^0(G)$, then $L \cap s_\alpha^H(G) \in \mathbf{\Pi}_{1+\beta}^0(s_\alpha^H(G))$. If $L \in D(\mathbf{\Pi}_{1+\alpha+\beta}^0(G))$, then $L \cap s_\alpha^H(G) \in D(\mathbf{\Pi}_{1+\beta}^0(s_\alpha^H(G)))$.*

Proof. Observe that $L = L^* = L^\Delta$. Thus, the first two assertions follow immediately from Lemma 5.2. If $L = A \cap B$ where A is $\Sigma_{1+\beta+1}^0$ in G and B is $\mathbf{\Pi}_{1+\beta+1}^0$ in G , then we have that $L \cap s_\alpha^H(G) = A^\Delta \cap B^* \cap s_\alpha^H(G)$ where $A^\Delta \cap s_\alpha^H(G) \in \Sigma_{1+\beta}^0(s_\alpha^H(G))$ and $B \cap s_\alpha^H(G) \in \mathbf{\Pi}_{1+\beta}^0(s_\alpha^H(G))$ by Lemma 5.2. Hence, $L \cap s_\alpha^H(G) \in D(\mathbf{\Pi}_{1+\beta}^0(s_\alpha^H(G)))$. \square

Recall that, by Proposition 3.4, if α is either zero or a countable limit ordinal, and H is a $\mathbf{\Pi}_{1+\alpha+1}^0$ Polishable subgroup of a Polish group, then H is $\mathbf{\Pi}_{1+\alpha}^0$.

Theorem 5.4. *Suppose that G is a Polish group, H is a Polishable subgroup of H , and $\alpha < \omega_1$. Then $s_\alpha^H(G)$ is the smallest $\mathbf{\Pi}_{1+\alpha+1}^0$ Polishable subgroup of G containing H .*

Proof. It is established in the proof of [FS06, Theorem 3.1] that $s_\alpha(G)$ is a $\mathbf{\Pi}_{1+\alpha+1}^0$ Polishable subgroup of G . We now prove the minimality assertion by induction on α . For $\alpha = 0$ this follows from the fact that $s_0^H(G) = \overline{H}^G$. Suppose that the conclusion holds for α . We now prove that it holds for $\alpha + 1$. Let L be a $\mathbf{\Pi}_{1+\alpha+2}^0$ Polishable subgroup of G containing H . Thus, $L \cap s_\alpha^H(G)$ is a $\mathbf{\Pi}_{1+\alpha+2}^0$ Polishable subgroup of $s_\alpha^H(G)$. Then by Corollary 5.3 we have that $L \cap s_\alpha^H(G) \in \mathbf{\Pi}_3^0(s_\alpha(H))$. As $s_{\alpha+1}^H(G) = s_1^H(s_\alpha^H(G))$ is the smallest $\mathbf{\Pi}_3^0(s_\alpha^H(G))$ Polishable subgroup of $s_\alpha^H(G)$, this implies that $s_{\alpha+1}^H(G) \subseteq L \cap s_\alpha^H(G) \subseteq L$.

Suppose that α is a limit ordinal and the conclusion holds for every $\beta < \alpha$. Fix an increasing sequence (α_n) in α such that $\alpha = \sup_n \alpha_n$. Suppose that L is a $\mathbf{\Pi}_{1+\alpha}^0$ Polishable subgroup of G containing H . Since L is $\mathbf{\Pi}_{1+\alpha}^0$ in G , we can write $L = \bigcap_{n \in \omega} A_n$ where, for every $n \in \omega$, $A_n \in \mathbf{\Pi}_{1+\alpha_n}^0(G)$. Then by Lemma 5.2 we have that

$$A_n^* \cap s_{\alpha_n}^H(G) \in \mathbf{\Pi}_1^0(s_{\alpha_n}(G)).$$

Since $H \subseteq L \subseteq A_n$ we have that $H \subseteq A_n^* \cap s_{\alpha_n}^H(G)$. Since H is dense in $s_{\alpha_n}(H)$, this implies that $s_{\alpha_n}^H(G) \subseteq A_n^*$. Therefore, we have that

$$s_\alpha^H(G) = \bigcap_{n \in \omega} s_{\alpha_n}^H(G) \subseteq \bigcap_{n \in \omega} A_n^* = L^* = L.$$

This shows that $s_\alpha^H(G) \subseteq L$, concluding the proof. \square

Lemma 5.5. *Suppose that G is a Polish group, H is a Polishable subgroup of G , and $\alpha < \omega_1$. Let L be a Polishable subgroup of $s_\alpha^H(G)$.*

- (1) *If $L \in \mathbf{\Pi}_3^0(s_\alpha^H(G))$, then $L \in \mathbf{\Pi}_{1+\alpha+2}^0(G)$.*
- (2) *If $L \in D(\mathbf{\Pi}_2^0)(s_\alpha^H(G))$, then $L \in D(\mathbf{\Pi}_{1+\alpha+1}^0)(G)$.*
- (3) *If $L \in \mathbf{\Sigma}_2^0(s_\alpha^H(G))$ and α is either zero or limit, then $L \in \mathbf{\Sigma}_{1+\alpha+1}^0(G)$.*

Proof. By Lemma 5.1 we have that

$$\mathbf{\Pi}_3^0(s_\alpha^H(G)) \subseteq \mathbf{\Pi}_{1+\alpha+2}^0(G)|_{s_\alpha^H(G)}$$

and

$$D(\mathbf{\Pi}_2^0)(s_\alpha^H(G)) \subseteq D(\mathbf{\Pi}_{1+\alpha+1}^0)(G)|_{s_\alpha^H(G)}.$$

Furthermore, $s_\alpha^H(G) \in \mathbf{\Pi}_{1+\alpha+1}^0(G)$ by Theorem 5.4. Therefore, we have that

$$\mathbf{\Pi}_3^0(s_\alpha^H(G)) \subseteq \mathbf{\Pi}_{1+\alpha+2}^0(G)|_{s_\alpha^H(G)} \subseteq \mathbf{\Pi}_{1+\alpha+2}^0(G)$$

and

$$D(\mathbf{\Pi}_2^0)(s_\alpha^H(G)) \subseteq D(\mathbf{\Pi}_{1+\alpha+1}^0)(G)|_{s_\alpha^H(G)} \subseteq D(\mathbf{\Pi}_{1+\alpha+1}^0)(G).$$

This concludes the proof of (1) and (2).

When α is either zero or limit, we have by Theorem 5.4 and Proposition 3.4 that $s_\alpha^H(G) \in \mathbf{\Pi}_{1+\alpha}^0(G) \subseteq \mathbf{\Sigma}_{1+\alpha+1}^0(G)$. Therefore, in this case we have that

$$\mathbf{\Sigma}_2^0(s_\alpha^H(G)) \subseteq \mathbf{\Sigma}_{1+\alpha+1}^0(G)|_{s_\alpha^H(G)} \subseteq \mathbf{\Sigma}_{1+\alpha+1}^0(G).$$

This concludes the proof of (3). \square

Lemma 5.6. *Suppose that, for every $k \in \omega$, G_k is a Polish group and H_k is a Polishable subgroup of G_k . Define $G = \prod_{k \in \omega} G_k$ and $H = \prod_{k \in \omega} H_k$. Assume that, for every $k \in \omega$, H_k is $\mathbf{\Pi}_\alpha^0$ in G_k , and for every $\beta < \alpha$ there exist infinitely many $k \in \omega$ such that H_k is not $\mathbf{\Pi}_\beta^0$ in G_k . Then $\mathbf{\Pi}_\alpha^0$ is the complexity class of H in G .*

Proof. Write $H = \prod_{k \in \omega} H_k \subseteq \prod_{k \in \omega} G_k$. Clearly, H is $\mathbf{\Pi}_\alpha^0$. By [Kec95, Theorem 22.10], for every $k \in \omega$ and $\beta < \alpha$ such that H_k is not $\mathbf{\Pi}_\beta^0$, H_k is $\mathbf{\Sigma}_\beta^0$ -hard [Kec95, Definition 22.9]. Therefore, H is $\mathbf{\Sigma}_\alpha^0$ -hard, and hence H is not $\mathbf{\Sigma}_\alpha^0$ by [Kec95, Theorem 22.10] again. \square

Lemma 5.7. *Suppose that, for every $k \in \omega$, G_k is a Polish group, H_k is a Polishable subgroup of G_k , and $\alpha < \omega_1$. Define $G = \prod_{k \in \omega} G_k$ and $H = \prod_{k \in \omega} H_k$. If H_k has Solecki rank α in G_k for every $k \in \omega$, then $\mathbf{\Pi}_{1+\alpha+1}^0$ is the complexity class of H in G if α is a successor ordinal, and $\mathbf{\Pi}_{1+\alpha}^0$ is the complexity class of H in G if α is either zero or a limit ordinal.*

Proof. Define $\lambda = 1 + \alpha + 1$ if α is a successor ordinal, and $\lambda = 1 + \alpha$ if α is either zero or a limit ordinal. By Lemma 4.2, for every $k \in \omega$, H_k is $\mathbf{\Pi}_\lambda^0$ but not $\mathbf{\Pi}_\beta^0$ for $\beta < \lambda$ in G_k . Therefore, by Lemma 5.6, $\mathbf{\Pi}_\lambda^0$ is the complexity of H in G . \square

6. COMPLEXITY OF POLISHABLE SUBGROUPS

The goal of this section is to establish the following theorem, characterizing the possible values of the complexity class of a Polishable subgroup of a Polish group.

Theorem 6.1. *Suppose that G is a Polish group, and H is a Polishable subgroup of G . Let $\alpha = \lambda + n$ be the Solecki rank of H in G , where $\lambda < \omega_1$ is either zero or a limit ordinal and $n < \omega$.*

- (1) *Suppose that $n = 0$. Then $\mathbf{\Pi}_{1+\lambda}^0$ is the complexity class of H in G .*
- (2) *Suppose that $n \geq 1$. Then:*
 - (a) *if $H \in \mathbf{\Pi}_3^0(s_{\lambda+n-1}^H(G))$ and $H \notin D(\mathbf{\Pi}_2^0)(s_{\lambda+n-1}^H(G))$, then $\mathbf{\Pi}_{1+\lambda+n+1}^0$ is the complexity class of H in G ;*
 - (b) *if $n \geq 2$ and $H \in D(\mathbf{\Pi}_2^0)(s_{\lambda+n-1}^H(G))$, then $D(\mathbf{\Pi}_{1+\lambda+n}^0)$ is the complexity class of H in G ;*
 - (c) *if $n = 1$, $H \in D(\mathbf{\Pi}_2^0)(s_\lambda^H(G))$, and $H \notin \Sigma_2^0(s_\lambda^H(G))$, then $D(\mathbf{\Pi}_{1+\lambda+1}^0)$ is the complexity class of H in G ;*
 - (d) *if $n = 1$ and $H \in \Sigma_2^0(s_\lambda^H(G))$, then $\Sigma_{1+\lambda+1}^0$ is the complexity class of H in G .*

Furthermore, if H is non-Archimedean then the case (2c) is excluded.

Theorem 1.1 and Theorem 1.2 are immediate consequences of Theorem 6.1. We will obtain Theorem 6.1 as a consequence of a number of *complexity reduction* lemmas. We fix a Polish group G and a Polishable subgroup H of G . We adopt the Vaught transform notation, in reference to the left translation action of H on G .

Lemma 6.2. *If H is $\mathbf{\Delta}_3^0$, then H is $D(\mathbf{\Pi}_2^0)$.*

Proof. Since H is $\mathbf{\Pi}_3^0$ in G , we have that $H = s_1^H(G)$. Thus, H has a countable basis $\{V_n : n \in \omega\}$ of neighborhoods of the identity such that $\overline{V_n^G} \cap H = V_n$ for every $n \in \omega$. Indeed, if $\{W_n : n \in \omega\}$ is a countable basis of open neighborhood of the identity in H , then $\{\overline{W_n^G} \cap H : n \in \omega\}$ is a countable basis of neighborhoods of the identity in H . If $V_n = \overline{W_n^G} \cap H$, then we have that $W_n \subseteq V_n \subseteq \overline{W_n^G}$ and hence $\overline{V_n^G} = \overline{W_n^G}$ and $V_n = \overline{V_n^G} \cap H$.

Let also $\{U_\ell : \ell \in \omega\}$ be a countable basis for the Polish group topology of H . Since H is $\mathbf{\Sigma}_3^0$, we can write $H = \bigcup_{k \in \omega} F_k$ where F_k is $\mathbf{\Pi}_2^0$ in G . Thus, we have that $H = \bigcup_{k, \ell \in \omega} F_k^{*U_\ell}$ where, by Lemma 5.2, $F_k^{*U_\ell}$ is closed in H and $\mathbf{\Pi}_2^0$ in G . Hence, without loss of generality we can assume that F_k is closed in H for every $k \in \omega$. By the Baire Category Theorem, we can assume without loss of generality that $V_0 \subseteq F_0$. Fix a countable dense subset $\{z_m : m \in \omega\}$ of H . Since $H = s_1^H(G)$, we have that, for $x \in G$, $x \in H$ if and only if for every $k \in \omega$ there exist $m_0, m_1 \in \omega$ such that $xz_{m_0} \in \overline{V_k^G}$ and $z_{m_1}x \in \overline{V_k^G}$.

We claim that, for $x \in G$, $x \in H$ if and only (1) there exists $m \in \omega$ such that $xz_m \in \overline{V_0^G}$, and (2) for all $m \in \omega$, if $xz_m \in \overline{V_0^G}$, then $xz_m \in F_0$. This will witness that H is $D(\mathbf{\Pi}_2^0)$ in G .

Indeed, since $\{z_m : m \in \omega\}$ is dense in H , if $x \in H$, then we have that there exists $m \in \omega$ such that $xz_m \in V_0 \subseteq \overline{V_0^G}$. Furthermore, if $m \in \omega$ is such that $xz_m \in \overline{V_0^G}$, then we have $xz_m \subseteq \overline{V_0^G} \cap H = V_0 \subseteq F_0$. Conversely suppose that there exists $m_0 \in \omega$ such that $xz_{m_0} \in \overline{V_0^G}$, and for all $m \in \omega$, if $xz_m \in \overline{V_0^G}$ then $x \in F_0$. Then we have that $xz_{m_0} \in F_0 \subseteq H$ and hence $x \in H$. \square

Lemma 6.3. *If H is $\mathbf{\Pi}_4^0$ in G , then H has a countable basis of neighborhoods of the identity consisting of sets that are in $\mathbf{\Pi}_2^0(G)|_H$.*

Proof. Define $\tilde{H} = s_1^H(G)$. By Theorem 5.4 we have that $H = s_2^H(G) = s_1^H(\tilde{H})$. Thus a neighborhood basis of the identity in H consists of sets of the form $\overline{Wx}^{\tilde{H}} \cap H$ where W is an open neighborhood of the identity in H . We have that, by Lemma 5.1,

$$\overline{Wx}^{\tilde{H}} \in \mathbf{\Pi}_1^0(s_1^H(G)) \subseteq \mathbf{\Pi}_2^0(G)|_{s_1^H(G)}.$$

Therefore,

$$\overline{Wx}^{\tilde{H}} \cap H \in \mathbf{\Pi}_2^0(G)|_H.$$

This concludes the proof. \square

Lemma 6.4. *Suppose that H is $\mathbf{\Sigma}_3^0$ in G , and define $\tilde{H} = s_1^H(G)$. Then we have that:*

- (1) $H = s_1^H(\tilde{H})$;
- (2) We can write H as the union of an increasing sequence $(F_k)_{k \in \omega}$ such that F_k is $\mathbf{\Pi}_2^0$ in G and closed in \tilde{H} for every $k \in \omega$;
- (3) H has a countable family of neighborhoods of the identity consisting of sets that are in $\mathbf{\Pi}_2^0(G) \cap \mathbf{\Pi}_1^0(\tilde{H})$.

Proof. (1): This is a consequence of Theorem 5.4.

(2): We can write $H = \bigcup_{k \in \omega} F_k$ where F_k is $\mathbf{\Pi}_2^0$ in G . Fix a countable basis $\{V_n : n \in \omega\}$ for the topology of H . Let also $\{z_m : m \in \omega\}$ be a countable dense subset of H . Then we have that $H = \bigcup_{n,k \in \omega} F_k^{*V_n}$ where, by Lemma 5.2, $F_k^{*V_n}$ is closed in \tilde{H} and $\mathbf{\Pi}_2^0$ in G . Thus, without loss of generality we can assume that F_k is closed in \tilde{H} for every $k \in \omega$.

(3): Let $(F_k)_{k \in \omega}$ be as in (2). By the Baire Category Theorem, we can assume without loss of generality that there exists an open neighborhood V of the identity in H such that $V \subseteq F_0$.

Fix an open neighborhood W of the identity in H contained in V . By Lemma 6.3, there exists a neighborhood U of the identity in H that belongs to $\mathbf{\Pi}_2^0(G)|_H$ and such that $U \subseteq W$. Since $U \subseteq W \subseteq V \subseteq F_0$ and $F_0 \in \mathbf{\Pi}_2^0(G)$, we have that $U \in \mathbf{\Pi}_2^0(G)$.

Let now $U_1 \subseteq U$ be an open neighborhood of the identity in H such that $U_1 U_1 \subseteq U$ and $U_1 \in \mathbf{\Pi}_2^0(G)$. Then we have that $U_1 \subseteq U^{*U_1}$ and $U^{*U_1} \in \mathbf{\Pi}_2^0(G) \cap \mathbf{\Pi}_1^0(\tilde{H})$ by Lemma 5.2. This concludes the proof that H has a countable basis of neighborhoods of the identity consisting of sets that are in $\mathbf{\Pi}_2^0(G) \cap \mathbf{\Pi}_1^0(\tilde{H})$. \square

Lemma 6.5. *If H is Σ_3^0 in G , then the coset equivalence relation E_H^G is potentially Σ_2^0 , and H is $D(\mathbf{\Pi}_2^0)$ in G .*

Proof. By Lemma 6.4, we can fix a countable basis $\{V_m : m \in \omega\}$ of neighborhoods of the identity in H that are $\mathbf{\Pi}_2^0$ in G and closed in H . Fix also a countable dense subset $\{h_k : k \in \omega\}$ of H . Let $(U_n)_{n \in \omega}$ be an enumeration of the countable set $\{V_m h_k : m, k \in \omega\}$. We have that for every nonempty open subset W of H there exists $n \in \omega$ such that $U_n \subseteq W$.

Let $H \times G$ be the product Polish group. By [BK96, Theorem 5.1.8] applied to the continuous action $a : H \times G \curvearrowright G$ defined by $(h, g) \cdot x = hxg^{-1}$, together with [BK96, Theorem 5.1.3], there exists a Polish topology t of G such that the action $a : H \times G \curvearrowright (G, t)$ is continuous, U_n is t -closed for every $n \in \omega$, t is finer than the Polish group topology of G , and it generates the same Borel structure as the Polish group topology of G . Since the action $a : H \times G \curvearrowright (G, t)$ is continuous, we have that the left translation action $H \curvearrowright (G, t)$ and the right translation action $(G, t) \curvearrowright G$ are continuous.

Fix a metric d on G compatible with t . For a closed subset C of G and $x \in G$ we define

$$d(x, C) = \inf \{d(x, c) : c \in C\}.$$

Let $K(G, t)$ be the space of t -closed subsets of G . We regard $K(G, t)$ as endowed with the *Wijsman topology*, which is obtained by declaring a net (C_i) to converge to C if and only if, for every $x \in X$, $(d(C_i, x))$ converges to $d(C, x)$ in \mathbb{R} . This turns $K(G, t)$ into a Polish space; see [Bee91, Theorem 4.3]. The Borel σ -algebra on $K(G, t)$ is the σ -algebra generated by sets of the form

$$\{C \in K(G, t) : C \cap W \neq \emptyset\},$$

where W is some t -open subset of G [Kec95, Section 12.C]. The relation $C_0 \subseteq C_1$ for closed subsets C_0, C_1 of G is closed in $K(G, t)$, since we have that $C_0 \subseteq C_1$ if and only if $d(C_1, x) \leq d(C_0, x)$ for every $x \in G$.

Define the Borel function $G \rightarrow K(G, t)^\omega$

$$x \mapsto (U_n x)_{n \in \omega}.$$

Notice that this function is indeed Borel: if $W \subseteq G$ is t -open, then

$$\{x \in G : U_n x \cap W \neq \emptyset\} = \bigcup_{u \in U_n} u^{-1}W$$

is t -open, and hence Borel, for every $n \in \omega$.

We have that, for $x, y \in G$, $x E_H^G y$ if and only if $\exists \ell \in \omega, U_\ell x \subseteq U_0 y$. Indeed, if $x E_H^G y$ then we have that $Hx = Hy$. Thus, $U_0 y x^{-1} \subseteq H$ is closed and nonmeager in the Polish topology of H , and hence there exists $\ell \in \omega$ such that $U_\ell \subseteq U_0 y x^{-1}$. Conversely if there exists $\ell \in \omega$ such that $U_\ell x \subseteq U_0 y$ then we have that $Hx \cap Hy \neq \emptyset$ and $x E_H^G y$.

This shows that E_H^G is potentially Σ_2^0 and in particular potentially $D(\mathbf{\Pi}_2^0)$. It now follows from Proposition 3.1 that H is $D(\mathbf{\Pi}_2^0)$ in G . \square

Lemma 6.6. *If λ is a limit ordinal and H is Σ_λ^0 in G , then there exists $\mu < \lambda$ such that H is $\mathbf{\Pi}_\mu^0$ in G .*

Proof. Let α be the Solecki rank of H in G . If $\alpha < \lambda$ then H is $\mathbf{\Pi}_{1+\alpha+1}^0$ in G and hence the conclusion holds. Suppose that $\alpha \geq \lambda$. We have that $H = \bigcup_{k \in \omega} F_k$ where, for $k \in \omega$, F_k is $\Sigma_{\mu_k}^0$ in G for some $\mu_k < \lambda$. In this case, as in the proof of Lemma 6.2, by Lemma 5.2 we can assume without loss of generality that F_k is closed in H for every $k \in \omega$. By the Baire Category Theorem, without loss of generality we can assume that F_0 is nonmeager in H . Thus, we have that $H = F_0^\Delta$ is $\Sigma_{\mu_0}^0$ and in particular $\mathbf{\Pi}_{\mu_0+1}^0$ in G . \square

Lemma 6.7. *If H is $\Delta_{1+\lambda+n+1}^0$ in G for some $1 \leq n < \omega$ and $\lambda < \omega_1$ either zero or limit, then H is $D(\mathbf{\Pi}_{1+\lambda+n}^0)$ in G .*

Proof. Fix a countable open basis $\{V_n : n \in \omega\}$ for H . We have that

$$H = s_{\lambda+n}^H(G) = s_1^H(s_{\lambda+n-1}^H(G)) \in \mathbf{\Pi}_3^0(s_{\lambda+n-1}^H(G)).$$

Furthermore, we can write $H = \bigcup_{k \in \omega} F_k$ where F_k is $\mathbf{\Pi}_{1+\lambda+n}^0$ in G for every $k \in \omega$. Thus, we have that $H = \bigcup_{k, \ell \in \omega} F_k^{*V_\ell}$, where $F_k^{*V_\ell} \in \mathbf{\Pi}_2^0(s_{\lambda+n-1}^H(G))$ by Lemma 5.2. Thus, we have that $H \in \Sigma_3^0(s_{\lambda+n-1}^H(G))$. Hence, by Lemma 6.5 we have that $H \in D(\mathbf{\Pi}_2^0)(s_{\lambda+n-1}^H(G))$. Furthermore, $D(\mathbf{\Pi}_2^0)(s_{\lambda+n-1}^H(G))$ is contained in $D(\mathbf{\Pi}_{1+\lambda+n}^0)(G)$ by Lemma 5.5, concluding the proof. \square

Lemma 6.8. *If H is $\Sigma_{1+\lambda+n+1}^0$ in G for some $1 \leq n < \omega$ and $\lambda < \omega_1$ either zero or limit, then H is $D(\mathbf{\Pi}_{1+\lambda+n}^0)$ in G .*

Proof. Fix a countable open basis $\{V_n : n \in \omega\}$ for H . Let α be the Solecki rank of H in G . Since H is $\mathbf{\Pi}_{1+\lambda+n+2}^0$ we have that $\alpha \leq \lambda + n + 1$ by Theorem 5.4. If $\alpha \leq \lambda + n$ then we have that H is $\Delta_{1+\lambda+n+1}^0$ and hence H is $D(\mathbf{\Pi}_{1+\lambda+n}^0)$ by Lemma 6.7. Suppose that $\alpha = \lambda + n + 1$. Thus, we have that $H = s_1^H(s_{\lambda+n}^H(G))$. We can write $H = \bigcup_{k \in \omega} F_k$ where F_k is $\mathbf{\Pi}_{1+\lambda+n}^0$ in G for every $k \in \omega$. Thus we have that $H = \bigcup_{k, \ell \in \omega} F_k^{*V_\ell}$, where, by Lemma 5.2, $F_k^{*V_\ell}$ is $\mathbf{\Pi}_2^0$ in $s_{\lambda+n-1}^H(G)$. Thus, $H \in \Sigma_3^0(s_{\lambda+n-1}^H(G))$. By Lemma 6.5, this implies that $H \in D(\mathbf{\Pi}_2^0)(s_{\lambda+n-1}^H(G))$. By Lemma 5.5, we have that $D(\mathbf{\Pi}_2^0)(s_{\lambda+n-1}^H(G)) \subseteq D(\mathbf{\Pi}_{1+\lambda+n}^0)(G)$, concluding the proof. \square

We have now all the ingredients to present a proof of Theorem 6.1.

Proof of Theorem 6.1. (1) We have that H is closed if and only if its Solecki rank is zero. Suppose now that λ is a limit ordinal and $n = 0$. By Theorem 5.4 we have that H is $\mathbf{\Pi}_\lambda^0$ and not $\mathbf{\Pi}_\mu^0$ for $\mu < \lambda$. Thus, H is not Σ_λ^0 by Lemma 6.6.

(2a) By Lemma 5.5 we have that H is $\mathbf{\Pi}_{1+\lambda+n+1}^0$. It remains to prove that H is not $\Sigma_{1+\lambda+n+1}^0$. Suppose that H is $\Sigma_{1+\lambda+n+1}^0$. Then by Lemma 6.8 we have that H is $D(\mathbf{\Pi}_{1+\lambda+n}^0)$. Thus, by Corollary 5.3, $H \in D(\mathbf{\Pi}_2^0)(s_{\lambda+n-1}^H(G))$, contradicting the hypothesis.

(2b) By Lemma 5.5 we have that H is $D(\mathbf{\Pi}_{1+\lambda+n}^0)$. It remains to prove that H is not $\check{D}(\mathbf{\Pi}_{1+\lambda+n}^0)$. Suppose by contradiction that H is $\check{D}(\mathbf{\Pi}_{1+\lambda+n}^0)$. Then by Lemma 3.2 we have that H is either $\mathbf{\Pi}_{1+\lambda+n}^0$ or $\Sigma_{1+\lambda+n}^0$. If H is $\mathbf{\Pi}_{1+\lambda+n}^0$ then by Theorem 5.4 and Proposition 3.4 we have that $\lambda + n - 1$ is the Solecki rank of H in G , contradicting the hypothesis. If H is $\Sigma_{1+\lambda+n}^0$, then $H \in \Sigma_2^0(s_{\lambda+n-1}^H(G))$ by Corollary 5.3, contradicting the hypothesis.

(2c) By Lemma 5.5 we have that H is $D(\mathbf{\Pi}_{1+\lambda+n}^0)$. The same proof as (2b) shows that H is not $\check{D}(\mathbf{\Pi}_{1+\lambda+n}^0)$.

(2d) By Lemma 5.5 we have that H is $\Sigma_{1+\lambda+n}^0$. The same proof as (2b) shows that H is not $\mathbf{\Pi}_{1+\lambda+n}^0$.

When H is non-Archimedean, the case (2c) is excluded by Proposition 3.5. \square

7. THE SAINT-RAYMOND RANK

Saint-Raymond introduced in [SR76, Definition 18] a notion of rank (therein called *degree*) for Fréchetable subspaces of Fréchet spaces; see Section 9. We consider in this section the natural generalization of such a notion to Polishable subgroups of Polish groups. Recall that for a complexity class Γ , and a Polishable subgroup H of a Polish group G , we define $\Gamma(G)|_H$ to be the collection of sets of the form $A \cap H$ for $A \in \Gamma(G)$.

Definition 7.1. Suppose that G is a Polish group, and $H \subseteq G$ is a Polishable subgroup. The *Saint-Raymond rank* of H is the least countable ordinal α such that every open subset in the Polish group topology of H belongs to $\Sigma_{1+\alpha}^0(G)|_H$.

Suppose that X, Y are Polish spaces, and α is a countable ordinal. As in [SR76, page 216], one can define $\mathcal{B}_\alpha(X, Y)$ to be the set of Borel functions that have class α as in [Kur66, Section 31], namely are $\Sigma_{1+\alpha}^0$ -measurable; see [Kec95, Definition 24.2]. By definition, the Saint-Raymond rank of H is the least countable ordinal α such that the identity function of H belongs to $\mathcal{B}_\alpha(X, Y)$, where X is equal to H endowed with the subspace topology inherited from G , and Y is equal to H endowed with its Polish group topology. Adapting an argument of Tsankov from [Tsa06], we now show that the Saint-Raymond rank and the Solecki rank of a Polishable subgroup of G coincide.

Theorem 7.2. *Suppose that G is a Polish group, and $H \subseteq G$ is a Polishable subgroup. Then the Saint-Raymond rank is equal to the Solecki rank.*

Proof. By Lemma 5.1 we have that the Saint-Raymond rank is less than or equal to the Solecki rank. We prove the converse inequality as in the proof of [Tsa06, Proposition 4.6]. If H has Saint-Raymond rank α , then every open set in H belongs to $\Sigma_{1+\alpha}^0(G)|_H$. Suppose that U is an open neighborhood of the identity in H , and let V be an open neighborhood of the identity in H such that $V^{-1}V \subseteq U$. Then there exists $A \in \Sigma_{1+\alpha}^0(G)$ such that $A \cap H = V$. Thus, $1 \in A^{\Delta V}$, where $A^{\Delta V} \cap s_\alpha^H(G)$ is open in $s_\alpha^H(G)$ by Lemma 5.2. Furthermore, we have that $A^{\Delta V} \cap H \subseteq V^{-1}V \subseteq U$. This shows that U contains a neighborhood of the identity with respect to the topology on H induced by $s_\alpha^H(G)$. This shows that the Polish topology on H is the subspace topology induced by $s_\alpha^H(G)$, whence H is closed in $s_\alpha^H(G)$. As H is dense in $s_\alpha^H(G)$, we have that $H = s_\alpha^H(G)$. \square

8. POLISHABLE SUBGROUPS IN EACH COMPLEXITY CLASS

The goal of this section is to prove the following theorem. Recall that a Polish group is *CLI* if it admits a compatible complete left-invariant metric or, equivalently, its left uniformity is complete [Mal11].

Theorem 8.1. *Let Γ be one of the possible complexity classes of Polishable subgroups from Theorem 1.1. Suppose that G is a nontrivial CLI Polish group. Then there exists a CLI Polishable subgroup of $G^{\mathbb{N}}$ whose complexity class is Γ .*

Remark 8.2. After replacing G with $G^{\mathbb{N}}$, we can assume that G is not discrete. We will assume that G is not discrete in the rest of this section.

Recall that a pseudo-length function on a group H is a function $L : H \rightarrow [0, +\infty)$ such that, for $h, h' \in H$:

- $L(1_H) = 0$;
- $L(h^{-1}) = L(h)$;
- $L(hh') \leq L(h) + L(h')$.

A length function is a pseudo-length function L such that $L(h) = 0 \Rightarrow h = 1_H$ for $h \in H$. A (pseudo-)length function L gives rise to a left-invariant (pseudo-)metric d defined by setting $d(h, h') = L(h^{-1}h')$, and every left-invariant metric arises in this fashion.

Suppose that G is a CLI Polish group, and let L_G be a length function on G that induces the Polish topology on G . We define the length functions L_1 and L_∞ on $G^{\mathbb{N}}$, with corresponding left-invariant metrics d_1 and d_∞ , by setting

$$L_1((g_n)_{n \in \mathbb{N}}) := \sum_{n \in \mathbb{N}} L_G(g_n)$$

and

$$L_\infty((g_n)_{n \in \mathbb{N}}) := \sup_{n \in \mathbb{N}} L_G(g_n).$$

for a sequence $(g_n)_{n \in \mathbb{N}} \in G^{\mathbb{N}}$. We say that $(g_n)_{n \in \mathbb{N}}$ is L_G -summable if $L_1((g_n)_{n \in \mathbb{N}}) < \infty$, and has bounded (left) L_G -variation if $\sum_{k \in \mathbb{N}} L_G(g_{n+1}^{-1}g_n) < \infty$. We let $\ell_1(G, L_G) \subseteq G^{\mathbb{N}}$ to be the CLI Polishable subgroup of L_G -summable sequences, $\text{bv}_0(G, L_G) \subseteq G^{\mathbb{N}}$ to be the CLI Polishable subgroup of vanishing sequences of bounded L_G -variation, and $\text{c}(G) \subseteq G^{\mathbb{N}}$ be the CLI Polishable subgroup of convergent sequences.

Fix, for each limit ordinal $\lambda < \omega_1$, an increasing cofinal sequence $(\lambda_i)_{i \in \mathbb{N}}$ in λ . If $\gamma = \delta + 1$ is a successor ordinal, define $\gamma_i = \delta$ for every $i \in \mathbb{N}$. Define by recursion on $\alpha < \omega_1$, $I_0^\alpha = \{(\emptyset, \emptyset)\}$ where \emptyset is the empty tuple., and I_0^α to be the set of tuples $(n_0, \dots, n_d; \beta_0, \dots, \beta_d)$ for $d \in \omega$, $n_0, \dots, n_d \in \mathbb{N}$, $0 = \beta_0 < \dots < \beta_d = \alpha_{n_d}$, and $(n_0, \dots, n_{d-1}; \beta_0, \dots, \beta_{d-1}) \in I_0^{\alpha_{n_d}}$.

Similarly for a fixed $\gamma < \omega_1$ we define I_γ^α by recursion on $\alpha \geq \gamma$, by setting $I_\gamma^\gamma = \{(\emptyset, \emptyset)\}$, and I_γ^α to be the set of tuples $(n_0, \dots, n_d; \beta_0, \dots, \beta_d)$ for $d \in \omega$, $n_0, \dots, n_d \in \mathbb{N}$, $\gamma = \beta_0 < \dots < \beta_d = \alpha_{n_d}$, and $(n_0, \dots, n_{d-1}; \beta_0, \dots, \beta_{d-1}) \in I_\gamma^{\alpha_{n_d}}$. Notice that, by definition, if $(n_0, \dots, n_d; \beta_0, \dots, \beta_d) \in I_\gamma^\alpha$ for some $\gamma \geq 1$, then $(m, n_0, \dots, n_d; \gamma_m, \beta_0, \dots, \beta_d) \in I_{\gamma_m}^\alpha$ for every $m \in \mathbb{N}$.

Thus, for example we have that, for $1 \leq k < \omega$, $I_\gamma^{\gamma+k}$ is the set of tuples

$$(n_0, \dots, n_{k-1}; \gamma, \gamma + 1, \dots, \gamma + k - 1)$$

for $n_0, \dots, n_{k-1} \in \mathbb{N}$, and $I_\gamma^{\gamma+\omega}$ is the set of tuples

$$(n_0, \dots, n_{(\gamma+\omega)_d}; \gamma, \gamma + 1, \dots, (\gamma + \omega)_d)$$

for $d \in \omega$ such that $(\gamma + \omega)_d \geq \gamma$, and $n_0, \dots, n_{(\gamma+\omega)_d} \in \mathbb{N}$.

We also define $I_\alpha^\alpha = \{(\emptyset, \emptyset)\}$. We denote by I^α the union of I_γ^α for $\gamma \leq \alpha$. If $\gamma \leq \alpha$, we denote by $I_{\leq \gamma}^\alpha$ the union of I_δ^α for $\delta \leq \gamma$, and by $I_{< \gamma}^\alpha$ to be the union of I_δ^α for $\delta < \gamma$. For $(n; \beta)$ and $(m; \tau)$ in I^α we define $(n; \beta) \leq (m; \tau)$ if and only if there exist $\gamma_0 \leq \gamma_1 \leq \alpha$ such that $(n; \beta) \in I_{\gamma_0}^\alpha$, $(m; \tau) \in I_{\gamma_1}^\alpha$, m is a tail of n , and τ is a tail of β , i.e. we have that, for some $\ell \leq d < \omega$, $(n; \beta) = (n_0, \dots, n_d; \beta_0, \dots, \beta_d)$, $(m; \tau) = (m_0, \dots, m_\ell; \tau_0, \dots, \tau_\ell)$, and for $0 \leq i \leq \ell$, $m_i = n_{i+d-\ell}$ and $\tau_i = \beta_{i+d-\ell}$. We regard I^α as an ordered set with respect to this order relation. Observe that $I_{\leq \gamma}^\alpha$ and $I_{< \gamma}^\alpha$ are downward-closed. For a subset F of I^α , we denote by F_\downarrow its downward closure. Notice also that if $F \subseteq I^\alpha$ is finite, and $(n; \beta) \in I_\gamma^\alpha$ for some $\gamma \geq 1$ is such that $(k, n; \gamma_k, \beta) \in F_\downarrow$ for infinitely many $k \in \mathbb{N}$, then $(n; \beta) \in F_\downarrow$.

Fix a countable ordinal α . We define, by recursion on $\gamma < \alpha$, a decreasing sequence $(P_\gamma)_{\gamma < \alpha}$ of CLI Polishable subgroups of $G^{I_0^\alpha}$. Furthermore, for $x \in P_\gamma$, we define the values $x(n; \beta) \in G$ for $(n; \beta) \in I_\gamma^\alpha$. If $\gamma \geq 1$ and $(n; \beta) \in I_\gamma^\alpha$, then we let $\mathbf{x}(n; \beta)$ be the convergent sequence $(x(k, n; \gamma_k, \beta))_{k \in \omega}$ in G with limit $x(n; \beta)$. If $(n; \beta) \in I_0^\alpha$, then we let $\mathbf{x}(n; \beta)$ be the sequence constantly equal to $x(n; \beta)$.

Define P_0 to be $G^{I_0^\alpha}$. This is a CLI Polish group with topology induced by the pseudo-length functions

$$L_0^{(n; \beta)}(x) = L_G(x(n; \beta))$$

for $(n; \beta) \in I_0^\alpha$. Suppose that $1 \leq \gamma \leq \alpha$, and that P_δ has been defined for every $\delta < \gamma$. Define $P_{< \gamma} = \bigcap_{\delta < \gamma} P_\delta$, and $P_\gamma \subseteq P_{< \gamma}$ to contain those $x \in P_{< \gamma}$ such that, for every $(n; \beta) \in I_\gamma^\alpha$, the sequence $\mathbf{x}(n; \beta) := (x(i, n; \gamma_i, \beta))_{i \in \mathbb{N}}$ is convergent. For $x \in P_\gamma$ and $(n; \beta) \in I_\gamma^\alpha$, we define $x(n; \beta)$ to be the limit of $\mathbf{x}(n; \beta)$. Then we have that the Polish topology on P_γ is induced by the restriction to P_γ of the continuous pseudo-length functions on P_δ for $\delta < \gamma$, together with the pseudo-length functions $L_\gamma^{(n; \beta)}(x) = L_\infty(\mathbf{x}(n; \beta))$ for $(n; \beta) \in I_\gamma^\alpha$. This concludes the recursive definition of the CLI Polishable subgroups P_γ of $G^{I_0^\alpha}$ for $\gamma \leq \alpha$. Notice that in particular P_α contains the elements $x \in P_{< \alpha}$ such that the sequence $\mathbf{x}(\alpha) := (x(n, \alpha_n))_{n \in \mathbb{N}}$ belongs to $\mathfrak{c}(G)$. We also define S_α and D_α to be the subgroups of P_α containing the elements $x \in P_{< \alpha}$ such that the sequence $\mathbf{x}(\alpha)$ belongs to $\ell_1(G, L_G)$ and $\text{bv}_0(G, L_G)$, respectively. Theorem 8.1 will be a consequence of the following.

Theorem 8.3. *Fix $\alpha = 1 + \lambda + n < \omega_1$, where λ is a limit ordinal or zero and $n < \omega$:*

- (1) *if $n = 0$ and λ is limit, then $P_{< \lambda}$ has Solecki rank λ in $G^{I_0^\alpha}$, and complexity class $\mathbf{\Pi}_\lambda^0$;*
- (2) *if $n = 0$, then $S_{1+\lambda}$, $D_{1+\lambda}$, and $P_{1+\lambda}$ have Solecki rank $\lambda + 1$ in $G^{I_0^\alpha}$, and complexity class $\mathbf{\Sigma}_{1+\lambda+1}^0$, $D(\mathbf{\Pi}_{1+\lambda+1}^0)$, and $\mathbf{\Pi}_{1+\lambda+1}^0$ respectively;*
- (3) *if $n \geq 1$, then $S_{1+\lambda+n}$, $D_{1+\lambda+n}$, and $P_{1+\lambda+n}$ have Solecki rank $\lambda + n + 1$ in $G^{I_0^\alpha}$, and complexity class $D(\mathbf{\Pi}_{1+\lambda+n+1}^0)$, $D(\mathbf{\Pi}_{1+\lambda+n+1}^0)$, and $\mathbf{\Pi}_{1+\lambda+n+1}^0$ respectively.*

We will obtain Theorem 8.3 as a consequence of a number of lemmas.

Lemma 8.4. *Suppose that $\gamma < \alpha$, F is a finite subset of $I_{\leq \gamma}^\alpha$, and $x \in P_\gamma$. Define $y \in G^{I_0^\alpha}$ by setting, for $(n; \beta) \in I_0^\alpha$,*

$$y(n; \beta) := \begin{cases} x(n; \beta) & \text{if } (n; \beta) \in F_\downarrow; \\ 1_G & \text{otherwise.} \end{cases} \quad (1)$$

Then we have that $y \in S_\alpha$, and (1) holds for every $(n; \beta) \in I^\alpha$.

Proof. We prove by induction on $\sigma < \alpha$ that $y \in P_\sigma$, and that (1) holds for every $(n; \beta) \in I_\sigma^\alpha$. For $\sigma = 0$, this holds by definition. Suppose that the conclusion holds for every $\delta < \sigma$. Fix $(n; \beta) \in I_\sigma^\alpha$. If $(n; \beta) \in F_\downarrow$ then necessarily $\sigma \leq \gamma$, and for every $k \in \mathbb{N}$, $(k, n; \sigma_k, \beta) \in I_{\sigma_k}^\alpha \cap F_\downarrow$ and hence by the inductive hypothesis, we have that $y(k, n; \sigma_k, \beta) = x(k, n; \sigma_k, \beta)$. Since $x \in P_\gamma \subseteq P_{\sigma_k}$, we have that the sequence $\mathbf{y}(n; \beta) = \mathbf{x}(n; \beta)$ converges to $x(n; \beta)$. Thus, $y(n; \beta) = x(n; \beta)$. If $(n; \beta) \notin F_\downarrow$ then we have that there exists k_0 such that, for all $k \geq k_0$, $(k, n; \sigma_k, \beta) \notin F_\downarrow$. Therefore, we have that the sequence $\mathbf{y}(n; \beta)$ is eventually equal to 1_G , and thus $y(n; \beta) = 1_G$. This shows that $y \in P_\sigma$. This concludes the proof by induction.

By the above, we have that $y \in P_{<\alpha}$. For $k \in \mathbb{N}$ such that $\alpha_k > \gamma$ we have that $y(k, \alpha_k) = 1_G$ and hence $y \in S_\alpha$. \square

Lemma 8.5. *For every $\gamma < \alpha$, S_α is dense in P_γ .*

Proof. Suppose that $x \in P_\gamma$, and let V be a neighborhood of x in P_γ . Then we have that there exist $\varepsilon > 0$ and a finite subset F of $I_{\leq \gamma}^\alpha$ such that

$$\bigcap_{(n; \beta) \in F} \{z \in P_\gamma : d_\infty(\mathbf{x}(n; \beta), \mathbf{z}(n; \beta)) < \varepsilon\}$$

is contained in V . Define $z \in G^{I_\sigma^\alpha}$ by setting, for $(n; \beta) \in I_0^\alpha$,

$$z(n; \beta) = \begin{cases} x(n; \beta) & \text{if } (n; \beta) \in F_\downarrow \\ 1_G & \text{otherwise.} \end{cases}$$

Then by Lemma 8.4, we have that $z \in S_\alpha$ and, for $(n; \beta) \in I^\alpha$ we have that

$$z(n; \beta) = \begin{cases} x(n; \beta) & \text{if } (n; \beta) \in F_\downarrow; \\ 1_G & \text{otherwise.} \end{cases}$$

In particular, we have that $z \in V$. \square

Lemma 8.6. *For every $\gamma < \alpha$, for every open neighborhood V of the identity in S_α , $\overline{V}^{P_{<\gamma}} \cap P_\gamma$ contains an open neighborhood of the identity in P_γ .*

Proof. Let V be a neighborhood of the identity in S_α . Fix a finite subset F of I^α and $\varepsilon > 0$ such that

$$\{w \in S_\alpha : L_1(\mathbf{w}(\alpha)) < \varepsilon\} \cap \bigcap_{(n; \beta) \in F} \{w \in S_\alpha : L_\infty(\mathbf{w}(n; \beta)) < \varepsilon\}$$

is contained in V . Define

$$N = \max_{\substack{\gamma < \delta \leq \alpha \\ F \cap I_\delta^\alpha \neq \emptyset}} \max \{n \in \mathbb{N} : \delta_n \leq \gamma\}$$

Define also the finite subset

$$B = (F \cap I_{\leq \gamma}^\alpha) \cup \{(k, n; \delta_k, \beta) : \gamma < \delta \leq \alpha, k \leq N, (n; \beta) \in F \cap I_\delta^\alpha\}$$

of $I_{\leq \gamma}^\alpha$. Consider the open neighborhood W of the identity in P_γ defined by

$$W = \left\{ x \in P_\gamma : \sum_{n \leq N} L_G(x(n; \alpha_n)) < \varepsilon \right\} \cap \bigcap_{(n; \beta) \in B} \{x \in P_\gamma : L_\infty(\mathbf{x}(n; \beta)) < \varepsilon\}.$$

We claim that $W \subseteq \overline{V}^{P_{<\gamma}} \cap P_\gamma$. Suppose that $x \in W$. Let U be an open neighborhood of x in $P_{<\gamma}$. Then there exist a finite subset A of $I_{<\gamma}^\alpha$ containing $B \cap I_{<\gamma}^\alpha$ and $\varepsilon_1 > 0$ such that

$$\bigcap_{(n; \beta) \in A} \{z \in P_{<\gamma} : d_\infty(\mathbf{x}(n; \beta), \mathbf{z}(n; \beta)) < \varepsilon_1\}$$

is contained in U . We need to prove that $U \cap V \neq \emptyset$.

We define $z \in G^{I_0^\alpha}$ by setting, for $(n; \beta) \in I_0^\alpha$,

$$z(n; \beta) := \begin{cases} x(n; \beta) & \text{if } (n; \beta) \in A_\downarrow; \\ 1_G & \text{otherwise.} \end{cases}$$

Then by Lemma 8.4, we have that $z \in S_\alpha$ and, for $(n; \beta) \in I^\alpha$, we have that

$$z(n; \beta) = \begin{cases} x(n; \beta) & \text{if } (n; \beta) \in A_\downarrow; \\ 1_G & \text{otherwise.} \end{cases}$$

In particular, we have that $z \in U$. We need to show that $z \in V$, i.e., that $L_1(z(\alpha)) < \varepsilon$ and if $(n, \beta) \in F$, then $L_\infty(z(n; \beta)) < \varepsilon$. We have that

$$L_1(z(\alpha))_1 = \sum_{n \in \mathbb{N}} L_G(z(n; \alpha_n)) \leq \sum_{n \leq N} L_G(x(n; \alpha_n)) < \varepsilon.$$

If $(n, \beta) \in F \cap I_{<\gamma}^\alpha$, then $z(n; \beta) = x(n; \beta)$. As $(n; \beta) \in B$ and $x \in W$, this implies that $L_\infty(z(n; \beta)) = L_\infty(x(n; \beta)) < \varepsilon$. If $(n, \beta) \in F \cap I_\gamma^\alpha$, then we have that

$$\begin{aligned} L_\infty(z(n; \beta)) &= \sup_{k \in \mathbb{N}} L_G(z(k, n; \gamma_k, \beta)) \\ &\leq \sup_{k \in \mathbb{N}} L_G(x(k, n; \gamma_k, \beta)) = L_\infty(x(n; \beta)) < \varepsilon \end{aligned}$$

since $(n; \beta) \in B$ and $x \in W$. If $(n; \beta) \in F \cap I_\delta^\alpha$ for some $\delta > \gamma$, then

$$\begin{aligned} L_\infty(z(n; \beta)) &= \sup_{k \in \mathbb{N}} L_G(z(k, n; \delta_k, \beta)) \\ &\leq \max_{k \leq N} L_G(x(k, n; \delta_k, \beta)) \leq \max_{k \leq N} L_\infty(x(k, n; \delta_k, \beta)) < \varepsilon \end{aligned}$$

since $(k, n; \delta_k, \beta) \in B$ for $k \leq N$, and $x \in W$. This shows that $z \in V$, concluding the proof. \square

Proposition 8.7. *For $\gamma < \alpha$ we have that*

$$s_\gamma^{S_\alpha}(G^{I_0^\alpha}) = s_\gamma^{D_\alpha}(G^{I_0^\alpha}) = s_\gamma^{P_\alpha}(G^{I_0^\alpha}) = s_\gamma^{P_{<\alpha}}(G^{I_0^\alpha}) = P_{<(1+\gamma)}$$

Proof. Since $S_\alpha \subseteq D_\alpha \subseteq P_\alpha \subseteq P_{<(1+\gamma)}$, it suffices to prove that $s_\gamma^{S_\alpha}(G^{I_0^\alpha}) = P_{<(1+\gamma)}$. We do this by induction on $\gamma < \alpha$. For $\gamma = 0$, we have that S_α is dense in $G^{I_0^\alpha} = P_0$ by Lemma 8.5, and hence $s_0^{S_\alpha}(G^{I_0^\alpha}) = P_0$. Suppose that the conclusion holds for every $\delta < \gamma$. If γ is limit, then we have that

$$s_\gamma^{S_\alpha}(G^{I_0^\alpha}) = \bigcap_{\delta < \gamma} s_{\delta+1}^{S_\alpha}(G^{I_0^\alpha}) = \bigcap_{\delta < \gamma} P_{1+\delta} = P_{<\gamma} = P_{<(1+\gamma)}.$$

Suppose that $\gamma = \delta + 1$ is a successor ordinal. Then we have that, by the inductive hypothesis

$$s_\gamma^{S_\alpha}(G^{I_0^\alpha}) = s_{\delta+1}^{S_\alpha}(G^{I_0^\alpha}) = s_1^{S_\alpha}(s_\delta^{S_\alpha}(G^{I_0^\alpha})) = s_1^{S_\alpha}(P_{<(1+\delta)}).$$

Thus, it remains to prove that

$$s_1^{S_\alpha}(P_{<(1+\delta)}) = P_{1+\delta}.$$

Notice that $P_{1+\delta}$ is a $\mathbf{\Pi}_3^0$ subspace of $P_{<(1+\delta)}$. Thus, the conclusion follows from Lemma 4.2, in view of Lemma 8.5 and Lemma 8.6. \square

Lemma 8.8. *For every $\gamma \leq \alpha$, there exists a continuous group homomorphism $\Phi : G^{I_{<\gamma}^\alpha} \rightarrow P_{<\gamma}$ such that $\Phi(z)(k, n; \gamma_k, \beta) = z(k, n; \gamma_k, \beta)$ for every $z \in G^{I_{<\gamma}^\alpha}$, $(n; \beta) \in I_\gamma^\alpha$, and $k \in \mathbb{N}$.*

Proof. For $z \in G^{I_{<\gamma}^\alpha}$, define $\Phi(z) := x \in G^{I_0^\alpha}$ by setting, for $(m; \tau) \in I_0^\alpha$,

$$x(m; \tau) := \begin{cases} z(k, n; \gamma_k, \beta) & \text{if } (m, \tau) \leq (k, n; \gamma_k, \beta) \text{ for some } k \in \mathbb{N} \text{ and } (n; \beta) \in I_\gamma^\alpha; \\ 1_G & \text{otherwise.} \end{cases}$$

It is clear that $\Phi : G^{I_{<\gamma}^\alpha} \rightarrow G^{I_0^\alpha}$ is a continuous group homomorphism. One can prove by induction on $\delta < \gamma$ that $x \in P_\delta$, and for $(m, \tau) \in I_\delta^\alpha$,

$$x(m; \tau) = \begin{cases} z(k, n; \gamma_k, \beta) & \text{if } (m, \tau) \leq (k, n; \gamma_k, \beta) \text{ for some } k \in \mathbb{N} \text{ and } (n; \beta) \in I_\gamma^\alpha; \\ 1_G & \text{otherwise.} \end{cases}$$

This concludes the proof. \square

Lemma 8.9. *We have that:*

- (1) Σ_2^0 is the complexity class of $\ell_1(G, L_G)$ in $G^{\mathbb{N}}$;
- (2) $D(\Pi_2^0)$ is the complexity class of $\text{bv}_0(G, L_G)$ in $G^{\mathbb{N}}$;
- (3) Π_3^0 is the complexity class of $c(G)$ in $G^{\mathbb{N}}$.

Proof. (1): It is clear that $\ell_1(G, L_G)$ is Σ_2^0 in $G^{\mathbb{N}}$. Since $\ell_1(G, L_G)$ is a dense, proper subgroup of $G^{\mathbb{N}}$, it is not closed.

(2): We have that $(g_n)_{n \in \mathbb{N}} \in \text{bv}_0(G, L_G)$ if and only if

$$\sum_{n \in \mathbb{N}} L_G(g_{n+1}^{-1}g_n) < +\infty$$

and for all $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ there exists $n \geq n_0$ such that $L_G(g_n) < \varepsilon$. This shows that $\text{bv}_0(G, L_G)$ is $D(\Pi_2^0)$ in $G^{\mathbb{N}}$. It remains to prove that it is not Σ_2^0 . Suppose by contradiction that

$$\text{bv}_0(G, L_G) = \bigcup_{k \in \mathbb{N}} F_k$$

where $F_k \subseteq G^{\mathbb{N}}$ is closed for every $k \in \mathbb{N}$. By the Baire Category Theorem, we can assume without loss of generality that F_0 contains a neighborhood of the identity. Thus, there exists $\varepsilon > 0$ such that

$$\left\{ (g_n)_{n \in \mathbb{N}} \in \text{bv}_0(G, L_G) : \sum_{n \in \mathbb{N}} L_G(g_{n+1}^{-1}g_n) < \varepsilon \text{ and } \sup_{n \in \mathbb{N}} L_G(g_n) < \varepsilon \right\} \subseteq F_0.$$

Since we are assuming that G is not discrete—see Remark 8.2—there exists $g \in G$ such that $0 < L_G(g) < \varepsilon$. Define for $N \in \mathbb{N}$, $x^{(N)} \in \text{bv}_0(G, L_G)$ by setting

$$x_k^{(N)} = \begin{cases} g & \text{if } k \leq N; \\ 1_G & \text{otherwise.} \end{cases}$$

Then we have that $x^{(N)} \in F_0$ for every $N \in \mathbb{N}$. The sequence $(x^{(N)})_{N \in \mathbb{N}}$ converges in $G^{\mathbb{N}}$ to the element $x \in G^{\mathbb{N}}$ that is the sequence constantly equal to g . Since F_0 is closed in $G^{\mathbb{N}}$, we have that $x \in \text{bv}_0(G, L_G)$, which is a contradiction to the fact that $L_G(g) > 0$.

(3): By definition, we have that $c(G)$ is Π_3^0 in $G^{\mathbb{N}}$. By Theorem 3.3, it suffices to prove that $c(G)$ is not potentially Σ_2^0 . Let E_0 be the relation of tail equivalence in $2^{\mathbb{N}}$, and let $E_0^{\mathbb{N}}$ be the corresponding product equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}} = 2^{\mathbb{N} \times \mathbb{N}}$. Then we have that Π_3^0 is the potential complexity class of $E_0^{\mathbb{N}}$, for example by Lemma 5.7 and Theorem 3.3.

Thus, it suffices to define a Borel function $2^{\mathbb{N} \times \mathbb{N}} \rightarrow G^{\mathbb{N}}$ that is a Borel reduction from $E_0^{\mathbb{N}}$ to the coset relation of $c(G)$ inside $G^{\mathbb{N}}$. We argue as in [Gao09, Lemma 8.5.3]. Fix a bijection $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \rightarrow \omega$ such that, if $n \leq n'$ and $m \leq m'$, then $\langle n, m \rangle \leq \langle n', m' \rangle$. Fix also a sequence $(g_n)_{n \in \mathbb{N}}$ in G such that $0 < L_G(g_n) < 2^{-(n+1)}$ for every $n \in \mathbb{N}$. Define $\Xi : 2^{\mathbb{N} \times \mathbb{N}} \rightarrow G^{\omega}$, $\varphi \mapsto a$ by setting

$$a_{\langle n, m \rangle} = \begin{cases} g_n & \varphi(n, m) = 1; \\ 1_G & \varphi(n, m) = 0. \end{cases}$$

Fix $\varphi, \psi \in 2^{\mathbb{N} \times \mathbb{N}}$. Define $\Xi(\varphi) = a$ and $\Xi(\psi) = b$.

Suppose that $\varphi E_0^{\mathbb{N}} \psi$. Thus we have that for every $n \in \mathbb{N}$ there exists $M_n \in \mathbb{N}$ such that $\varphi(n, m) = \psi(n, m)$ for $m \geq M_n$. Fix $\varepsilon > 0$ and fix $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon$. Define then

$$M = \max \{M_n : n < N\}.$$

We claim that for $k \geq \langle N, M \rangle$ we have that $L_G(a_k^{-1}b_k) < \varepsilon$. Indeed, suppose that $k \geq \langle N, M \rangle$. Then $k = \langle n, m \rangle$ for some $n, m \in \mathbb{N}$. If $n \geq N$ then we have that

$$\begin{aligned} L_G(a_k^{-1}b_k) &\leq L_G(a_k) + L_G(b_k) \\ &\leq 2L_G(g_n) < 2^{-n} \leq 2^{-N} < \varepsilon. \end{aligned}$$

If $n < N$ then we must have that $m \geq M \geq M_n$, and hence $\varphi(n, m) = \psi(n, m)$ and $a_k = b_k$. This shows that $a^{-1}b \in c(G)$.

Conversely, suppose that $a^{-1}b \in c(G)$. Fix $n_0 \in \mathbb{N}$. Then we have that there exists $k_0 \in \mathbb{N}$ such that for $k \geq k_0$, $L_G(a_k^{-1}b_k) < L_G(g_{n_0})$. Thus, for $k \geq k_0$, if $k = \langle n_0, m \rangle$ for some $m \in \mathbb{N}$, we must have $a_k = b_k$ and $\varphi(n_0, m) = \psi(n_0, m)$. Thus, if $m_0 \in \mathbb{N}$ is such that $\langle n_0, m_0 \rangle \geq k_0$, we must have that $\varphi(n_0, m) = \psi(n_0, m)$ for all $m \geq m_0$. As this holds for every $n_0 \in \mathbb{N}$, $\varphi E_0^{\mathbb{N}} \psi$, concluding the proof. \square

Corollary 8.10. *For every $\gamma < \alpha$, P_γ is a proper subgroup of $P_{<\gamma}$. The complexity class of $S_\alpha, D_\alpha, P_\alpha$, respectively, inside $P_{<\alpha}$ is $\Sigma_2^0, D(\Pi_2^0)$, and Π_3^0 , respectively.*

Proof. Fix $\gamma < \alpha$ and $(n; \beta) \in I_\gamma^\alpha$. By Lemma 8.8 there exists $x \in P_{<\gamma}$ such that $\mathbf{x}(n; \beta)$ is not convergent. Such an x does not belong to P_γ , thus showing that P_γ is a proper subgroup of $P_{<\gamma}$.

We now prove the assertion about P_α , as the other assertions are proved in a similar fashion. Recall that $I_\alpha^\alpha = \{(\emptyset; \emptyset)\}$. Define

$$H = \left\{ x \in G^{I_\alpha^\alpha} : (x(k; \alpha_k))_{k \in \mathbb{N}} \in c(G) \right\}.$$

By Lemma 8.9 we have that Π_3^0 is the complexity class of H in $G^{I_\alpha^\alpha}$ and of $c(G)$ in $G^{\mathbb{N}}$.

By Lemma 8.8 there exists a continuous group homomorphism $\Phi : G^{I_\alpha^\alpha} \rightarrow P_{<\alpha}$ such that $\Phi(x)(k; \alpha_k) = x(k; \alpha_k)$ for every $k \in \mathbb{N}$, and hence $\Phi^{-1}(P_\alpha) = H$. Similarly, the function $\Psi : P_{<\alpha} \rightarrow G^{\mathbb{N}}$, $x \mapsto \mathbf{x}(\alpha)$ is a continuous group homomorphism such that $\Psi^{-1}(c(G)) = P_\alpha$. Thus, Π_3^0 is the complexity class of P_α in $P_{<\alpha}$. \square

We are now in position to present the proof of Theorem 8.3.

Proof of Theorem 8.3. By the first assertion in Corollary 8.10 and Proposition 8.7, we have that $S_\alpha, D_\alpha, P_\alpha$ have Solecki rank $\alpha + 1$ in P_0 , and $P_{<\alpha}$ has Solecki rank α in P_0 if α is limit. The conclusion now follows by applying Theorem 6.1 and the second assertion in Corollary 8.10. \square

Recall that a (pseudo-)ultralength function on a group H is a (pseudo-)length function L such that $L(hh') \leq \max\{L(h), L(h')\}$ for $h, h' \in H$. A Polish group G is non-Archimedean if and only if it admits a compatible ultralength function [Gao09, Theorem 2.4.1]. In a similar fashion as above, one can prove the following statement; see also [HKL98, Section 5].

Theorem 8.11. *Let Γ be one of the possible complexity classes of non-Archimedean Polishable subgroups from Theorem 1.2. Suppose that G is a countable discrete group. Then there exists a non-Archimedean CLI Polishable subgroup of $G^{\mathbb{N}}$ whose complexity class is Γ .*

Define $H := G^{\mathbb{N}}$. This is a non-Archimedean CLI group. The topology on H is induced by the ultralength function

$$L_H((g_n)_{n \in \mathbb{N}}) = \exp(-\min\{n \in \mathbb{N} : g_n \neq 1_G\}).$$

Notice that the subgroup $c(H)$ of $H^{\mathbb{N}}$ convergent sequences is a non-Archimedean CLI Polishable subgroup of $H^{\mathbb{N}}$ of complexity class Π_3^0 with topology induced by the ultralength function

$$L_\infty((h_n)_{n \in \mathbb{N}}) = \max\{L_H(h_n) : n \in \mathbb{N}\}.$$

The subgroup $\sigma(H)$ of $H^{\mathbb{N}}$ consisting of sequences $(h_n)_{n \in \mathbb{N}}$ such that the sequence $(h_n(0))_{n \in \mathbb{N}}$ in G is eventually equal to 1_G is a non-Archimedean CLI Polishable subgroup of $H^{\mathbb{N}}$ of complexity class Σ_2^0 .

Fix $\alpha < \omega_1$. We define by recursion on $\gamma \leq \alpha$ a decreasing sequence $(F_\gamma)_{\gamma < \alpha}$ of non-Archimedean Polishable subgroups of $H^{I_\alpha^\alpha}$. We also recursively define, for $x \in F_\gamma$ and $(n; \beta) \in I_0^\beta$, the values $x(n; \beta) \in H$. We set $F_0 = H^{I_0^\alpha}$. If F_δ has been defined for every $\delta < \gamma$, define $F_{<\gamma} = \bigcap_{\delta < \gamma} F_\delta$, F_γ to contain those $x \in F_{<\gamma}$ such that, for every $(n; \beta) \in I_\gamma^\alpha$, the sequence $\mathbf{x}(n; \beta) := (x(k, n; \gamma_k, \beta))_{k \in \omega}$ is convergent in H . For $x \in F_\gamma$ and $(n; \beta) \in I_\gamma^\alpha$, we define $x(n; \beta)$ to be the limit of $\mathbf{x}(n; \beta)$. Then we have that the non-Archimedean Polish group topology on F_γ is induced by the restriction of the continuous pseudo-ultralength functions on F_δ for $\delta < \gamma$ together with the pseudo-ultralength function

$$L_\gamma^{(n; \beta)}(x) = L_\infty(\mathbf{x}(n; \beta))$$

for $(n; \beta) \in I_\gamma^\alpha$. This concludes the recursive definition of the non-Archimedean Polishable subgroups F_γ of $H^{I_\alpha^\alpha}$ for $\gamma \leq \alpha$. Notice that in particular F_α contains the elements $x \in F_{<\alpha}$ such that $\mathbf{x}(\alpha) := (x(n, \alpha_n))_{n \in \mathbb{N}}$ belongs to $c(H)$. Define Z_α to contain those elements $x \in F_{<\alpha}$ such that $\mathbf{x}(\alpha)$ belongs to $\sigma(H)$. The same argument as above, gives the following.

Theorem 8.12. *Adopt the notations above. Suppose that $\alpha = 1 + \lambda + n$ where $\lambda < \omega_1$ is either limit or zero and $n < \omega$.*

- (1) *If $n = 0$ and λ is limit, then $F_{<\lambda}$ has Solecki rank λ and complexity class Π_λ^0 in $H^{I_\lambda^\alpha}$.*
- (2) *If $n = 0$, then $Z_{1+\lambda}$ and $F_{1+\lambda}$ have Solecki rank $\lambda + 1$ in $H^{I_\alpha^0}$, and complexity class $\Sigma_{1+\lambda+1}^0$ and $\Pi_{1+\lambda+2}^0$, respectively;*
- (3) *if $n \geq 1$, then $Z_{1+\lambda}$ and $F_{1+\lambda}$ have Solecki rank $\lambda + n + 1$ in $H^{I_\alpha^0}$, and complexity class $D(\Pi_{1+\lambda+n+1}^0)$ and $\Pi_{1+\lambda+n+2}^0$, respectively.*

9. FRÉCHETABLE SUBSPACES

In this and the following section, we assume all the vector spaces to be over the reals. Similar considerations apply to complex vector spaces. Recall that a *Fréchet space* is a locally convex topological vector space whose topology is given by a complete, translation-invariant metric. Thus, the additive group of a separable Fréchet space is a Polish group. In analogy with the notion of Polishable subgroup of a Polish group, we consider the notion of Fréchetable subspace of a separable Fréchet space.

Definition 9.1. Suppose that X is a separable Fréchet space, and Y is a subspace of X . Then we say that Y is *Fréchetable* if it is Borel, and there exists a separable Fréchet space topology on Y whose open sets are Borel in X .

This notion was considered by Saint-Raymond in [SR76]: a subspace Y of X is Fréchetable if and only if *it has a separable model* according to [SR76, Definition 1]. Notice that a Fréchetable subspace of X is, in particular, a Polishable subgroup of the additive group of X . Thus, if it exists, the separable Fréchet space topology on Y as in Definition 9.1, is unique; see also [Osb14, Corollary 4.38]. A subspace Y of a separable Fréchet space X is Fréchetable if and only if there exists a separable Fréchet space Z and a continuous linear map $\varphi : Z \rightarrow X$ with image equal to Y [SR76, Proposition 4]. If Y is a Fréchetable subspace of X , then the separable Fréchet space topology on Y is the *finest* locally convex topological vector space topology on Y that makes all the Borel linear functionals on Y continuous [SR76, Theoreme 9]. Furthermore, we have a subspace Y of X is Fréchetable if and only if it is a Polishable subgroup of the additive group of X , and the Polish topology on Y has a basis of neighborhoods of zero consisting of convex, balanced sets; see [Osb14, Proposition 3.33 and Corollary 3.36]

Lemma 9.2. *Suppose that X is a separable Fréchet space, and Y a Fréchetable subspace of X . The first Solecki subgroup $s_1^Y(X)$ of X relative to Y , where X and Y are regarded as additive groups, is a Fréchetable subspace of X .*

Proof. By definition, we have that, for $x \in X$, $x \in Y$ if and only if for every open neighborhood V of zero in Y there exists $z \in Y$ such that $x + z \in \overline{V}^G$. If $x \in Y$, $\lambda \in \mathbb{R}$ is nonzero, and V is an open neighborhood of zero in Y , then there exists $z \in Y$ such that $x + z \in \overline{\lambda^{-1}V}^G$, whence $\lambda x + \lambda z \in \overline{V}^G$. This shows that $\lambda x \in s_1^Y(X)$, whence $s_1^Y(X)$ is a subspace of X .

We now show that $s_1(Y)$ is Fréchetable. Since Y is a separable Fréchet space, by the remarks above it has a basis $(V_n)_{n \in \omega}$ of neighborhoods of zero consisting of convex, balanced sets. Thus, $(\overline{V}_n^G \cap s_1^Y(X))_{n \in \omega}$ is a basis of neighborhoods of zero in $s_1(Y)$ consisting of convex, balanced sets. Thus, $s_1^Y(X)$ is a Fréchetable subspace of X by the remarks above again. \square

As an immediate consequence of Lemma 9.2 and Theorem 5.4 by induction on $\alpha < \omega_1$ we have the following.

Theorem 9.3. *Suppose that X is a separable Fréchet space, Y is a Fréchetable subspace of X , and $\alpha < \omega_1$. Then the α -th Solecki subgroup $s_\alpha^Y(X)$ of X relative to Y , where X and Y are regarded as additive groups, is the smallest $\Pi_{1+\alpha+1}^0$ Fréchetable subspace of X containing Y .*

A similar proof as Theorem 8.1 gives the following.

Theorem 9.4. *Let Γ be one of the possible complexity classes of Polishable subgroups from Theorem 1.1. Suppose that X is a nontrivial separable Fréchet space. Then there exists a Fréchetable subspace of $X^{\mathbb{N}}$ whose complexity class is Γ .*

10. BANACHABLE SUBSPACES

Let V be a separable Fréchet space. A subspace $X \subseteq V$ is *Banachable* if it is the image of a continuous linear map $T : Z \rightarrow V$ for some separable Banach space Z . Equivalently, X is a Borel subspace of V that is also a separable Banach space such that the inclusion map $X \rightarrow V$ is continuous. We have that the Solecki subgroups whose index is a successor associated with a Banachable subspace of a separable Fréchet space are also Banachable.

Proposition 10.1. *Suppose that V is a separable Fréchet space, $X \subseteq V$ is Banachable. Then $s_{\alpha+1}^X(V) \subseteq V$ is Banachable for every $\alpha < \omega_1$.*

Proof. It suffices to consider the case $\alpha = 0$. Suppose that $\|\cdot\|_X$ is a compatible norm on X and B is the corresponding unit ball. Define $C := \overline{B}^V \cap s_1^X(V)$. As $(2^{-n}B)_{n \in \omega}$ is a basis of neighborhoods of the identity in X , $(2^{-n}C)_{n \in \omega}$ is a basis of neighborhoods of the identity in $s_1^X(V)$. Thus $s_1^X(V)$ is a normed space, and hence Banach space (being complete). \square

In this section we will prove using the methods from Section 8 and Theorem 8.3 the following characterization of the possible complexity class of Banachable subspaces.

Theorem 10.2. *The following is a complete list of all the possible complexity classes of Banachable subspaces of separable Fréchet spaces: Π_1^0 , $\Pi_{1+\lambda+n+1}^0$, $D(\Pi_{1+\lambda+n}^0)$, and $\Sigma_{1+\lambda+1}^0$ for $\lambda < \omega_1$ either zero or limit and $1 \leq n < \omega$. Furthermore, for every complexity class Γ in this list and nontrivial separable Banach space Z , there exists a Banachable subspace of $c_0(\mathbb{N}, Z)$ that has complexity class Γ .*

We begin with showing that a Banachable subspace of a separable Fréchet space cannot have complexity class Π_λ^0 for some countable limit ordinal λ .

Proposition 10.3. *Suppose that V is a separable Fréchet space, and $X \subseteq V$ is a Fréchetable subspace. Suppose that $s_\alpha^X(V)$ is Banachable for some limit ordinal α . Then X has Solecki rank less than α .*

Proof. Without loss of generality, we can assume that $X = s_\alpha^X(V)$. Since X is Banachable, there exists $B \subseteq X$ such that $(2^{-n}B)_{n \in \omega}$ forms a basis of neighborhoods of zero in X . Since $X = \bigcap_{\beta < \alpha} s_\beta^X(V)$, there exists $\beta < \alpha$ and a neighborhood C of 0 in $s_\beta^X(V)$ such that $B = C \cap s_\alpha^X(V)$. Thus, X is endowed with the subspace topology inherited from $s_\beta^X(V)$. Whence, X is closed in $s_\beta^X(V)$. Since X is also dense in $s_\beta^X(V)$, we have that $X = s_\beta^X(V)$. Hence, X has Solecki rank at most β . \square

Corollary 10.4. *Suppose that V is a separable Fréchet space, and $X \subseteq V$ is a Banachable subspace. If X is Π_λ^0 for some limit ordinal $\lambda < \omega_1$, then X is Π_β^0 for some $\beta < \lambda$.*

Proof. By Theorem 6.1 we have that $X = s_\lambda^X(V)$ is Banachable. Thus, by Proposition 10.3 we have that X has Solecki rank β for some $\beta < \lambda$, and hence X is $\Pi_{1+\beta+1}^0$ by Theorem 6.1 again. \square

In order to conclude the proof of Theorem 10.2 it remains to prove that all the complexity classes from the statement of Theorem 10.2 can arise. Fix a countable ordinal α . We adopt the notation from Section 8. We regard I_γ^α as a set *fibred* over α , with respect to the map $I^\alpha \rightarrow \alpha$, $(n; \beta) \mapsto \gamma$ such that $(n; \beta) \in I_\gamma^\alpha$. For $\gamma < \alpha$ we define

$$J_\gamma^\alpha = \{(k, \sigma) \in \mathbb{N} \times (\alpha + 1) : \sigma_k < \gamma < \sigma \leq \alpha\}$$

We also regard J_γ^α as a set fibred over α with respect to the function $J_\gamma^\alpha \rightarrow \alpha$, $(k, \sigma) \mapsto \sigma$. We then define the *fibred product*

$$J_\gamma^\alpha * I^\alpha = \{((k, \sigma), (n; \beta)) : (k, \sigma) \in J_\gamma^\alpha, (n; \beta) \in I_\sigma^\alpha\}.$$

Notice that the projection map $J_\gamma^\alpha * I^\alpha \rightarrow I^\alpha$ is finite-to-one. Indeed, suppose that $((k, \sigma), (n; \beta)) \in J_\gamma^\alpha * I^\alpha$. Then we have that $\gamma < \sigma$, and hence $\{k \in \mathbb{N} : \sigma_k < \gamma\}$ is finite.

Fix a nontrivial separable Banach space Z . We denote the norm of $z \in Z$ by $|z|$. We consider the Banach spaces

$$\ell_1(Z) = \left\{ (x_n) \in Z^\mathbb{N} : \sum_{n \in \mathbb{N}} |x_n| < +\infty \right\}$$

and

$$\text{bv}_0(Z) = \left\{ (x_n) \in Z^{\mathbb{N}} : \sum_{n \in \mathbb{N}} |z_n - z_{n+1}| < +\infty \text{ and } (z_n)_{n \in \mathbb{N}} \text{ is vanishing} \right\}.$$

Define $X_0 = c_0(I_0^\alpha, Z)$. We now define by recursion on $\gamma \leq \alpha$, Banachable subspaces X_γ and Fréchetable subspaces $X_{<\gamma}$ of X_0 such that $X_\gamma \subseteq X_{<\gamma} \subseteq X_\delta$ for $\delta < \gamma \leq \alpha$. Furthermore, for $x \in X_\gamma$, we define the values $x(n; \beta) \in Z$ for $(n; \beta) \in I_\gamma^\alpha$, such that the linear functional $x \mapsto x(n; \beta)$ on X_γ is continuous. If $\gamma \geq 1$ and $(n; \beta) \in I_\gamma^\alpha$, then we let $\mathbf{x}(n; \beta)$ be the convergent sequence $(kx(k; n; \gamma_k, \beta))_{k \in \omega}$ with limit $x(n; \beta)$. If $(n; \beta) \in I_0^\alpha$, then we let $\mathbf{x}(n; \beta)$ be the sequence constantly equal to $x(n; \beta)$. Suppose that $1 \leq \gamma \leq \alpha$, and that X_δ has been defined for $\delta < \gamma$, in such a way that X_δ is a separable Banach space with norm $\|\cdot\|_{X_\delta}$.

Define $X_{<\gamma}$ to be the intersection of X_δ for $\delta < \gamma$. Consider the continuous linear map

$$T_\gamma^0 : X_{<\gamma} \rightarrow (Z^{\mathbb{N}})^{I_{\leq \gamma}^\alpha}$$

defined by

$$T_\gamma^0(x) = (\mathbf{x}(n; \beta))_{(n; \beta) \in I_{\leq \gamma}^\alpha}.$$

Consider also the continuous linear map

$$T_\gamma^1 : X_{<\gamma} \rightarrow Z^{J_\gamma^\alpha * I^\alpha}$$

defined by

$$T_\gamma^1(x) = (kx(k; n; \sigma_k, \beta))_{((k, \sigma), (n; \beta)) \in J_\gamma^\alpha * I^\alpha}.$$

Define $X_\gamma \subseteq X_{<\gamma}$ to be the intersection of the preimage of

$$c_0(I_{\leq \gamma}^\alpha, c(\mathbb{N}, Z)) \subseteq (Z^{\mathbb{N}})^{I_\gamma^\alpha}$$

under T_γ^0 and the preimage of

$$c_0(J_\gamma^\alpha * I^\alpha, Z) \subseteq Z^{J_\gamma^\alpha * I^\alpha}$$

under T_γ^1 . It follows from Lemma 10.6 below that X_γ is a separable Banach space with respect to the norm

$$\|x\|_{X_\gamma} = \max \left\{ \|T_\gamma^0(x)\|_{c_0(I_{\leq \gamma}^\alpha, c(\mathbb{N}, Z))}, \|T_\gamma^1(x)\|_{c_0(J_\gamma^\alpha * I^\alpha, Z)} \right\}$$

for $x \in X_\gamma$. Observe that in particular

$$X_\alpha = \{x \in X_{<\alpha} : \mathbf{x}(\alpha) \in c(\mathbb{N}, Z)\}$$

and

$$\|x\|_{X_\alpha} = \max \left\{ \sup_{\gamma < \alpha} \|x\|_{X_\gamma}, \|\mathbf{x}(\alpha)\|_{c(\mathbb{N}, Z)} \right\}$$

for $x \in X_\alpha$, where $\mathbf{x}(\alpha) := (kx(k; \alpha_k))_{k \in \mathbb{N}}$. Define also $S_\alpha \subseteq D_\alpha \subseteq X_\alpha$ by setting

$$S_\alpha = \left\{ x \in X_{<\alpha} : \sup_{\gamma < \alpha} \|x\|_{X_\gamma} < +\infty \text{ and } \mathbf{x}(\alpha) \in \ell_1(Z) \right\}$$

and

$$D_\alpha = \left\{ x \in X_{<\alpha} : \sup_{\gamma < \alpha} \|x\|_{X_\gamma} < +\infty \text{ and } \mathbf{x}(\alpha) \in \text{bv}_0(Z) \right\}$$

where

$$\mathbf{x}(\alpha) = (x(k; \alpha_k))_{k \in \mathbb{N}}.$$

Then we have that S_α is a separable Banach space with respect to the norm

$$\|x\|_{S_\alpha} = \max \left\{ \|x\|_{X_\alpha}, \|\mathbf{x}(\alpha)\|_{\ell_1(Z)} \right\}$$

and D_α is a separable Banach space with respect to the norm

$$\|x\|_{D_\alpha} = \max \left\{ \|x\|_{X_\alpha}, \|\mathbf{x}(\alpha)\|_{\text{bv}_0(Z)} \right\}.$$

The existence statement in Theorem 10.2 will be a consequence of the following result.

Theorem 10.5. *Fix $\alpha = 1 + \lambda + n < \omega_1$, where λ is a limit ordinal or zero and $n < \omega$:*

(1) *if $n = 0$ and λ is limit, then $X_{<\lambda}$ has Solecki rank λ in X_0 , and complexity class $\mathbf{\Pi}_\lambda^0$;*

- (2) if $n = 0$, then $S_{1+\lambda}$, $D_{1+\lambda}$, and $X_{1+\lambda}$ have Solecki rank $\lambda + 1$ in X_0 , and complexity class $\Sigma_{1+\lambda+1}^0$, $D(\Pi_{1+\lambda+1}^0)$, and $\Pi_{1+\lambda+1}^0$ respectively;
- (3) if $n \geq 1$, then $S_{1+\lambda+n}$, $D_{1+\lambda+n}$, and $X_{1+\lambda+n}$ have Solecki rank $\lambda + n + 1$ in X_0 , and complexity class $D(\Pi_{1+\lambda+n+1}^0)$, $D(\Pi_{1+\lambda+n+1}^0)$, and $\Pi_{1+\lambda+n+1}^0$ respectively.

The rest of this section contains the proof of Theorem 10.5.

Lemma 10.6. *We have that*

$$\|x\|_{X_\delta} \leq \|x\|_{X_\gamma}$$

for $\delta < \gamma \leq \alpha$ and $x \in X_\gamma$.

Proof. It suffices to prove that

$$\|T_\delta^1(x)\|_{c_0(J_\delta^\alpha * I^\alpha, Z)} \leq \|x\|_{X_\gamma}.$$

Suppose that $((k, \sigma), (n; \beta)) \in J_\delta^\alpha * I^\alpha$. Then we have that $\sigma_k < \delta < \sigma$ and $(n; \beta) \in I_\sigma^\alpha$. Suppose initially that $\sigma \leq \gamma$. Then we have that $(n; \beta) \in I_{\leq \gamma}^\alpha$ and hence

$$|kx(k, n; \sigma_k, \beta)| \leq \|\mathbf{x}(n; \beta)\|_\infty \leq \|x\|_{X_\gamma}.$$

Suppose now that $\gamma < \sigma$. Then we have that $((k, \sigma), (n; \beta)) \in J_\gamma^\alpha * I^\alpha$ and hence

$$|kx(k, n; \sigma_k, \beta)| \leq \|T_\gamma^1(x)\|_{c_0(J_\gamma^\alpha * I^\alpha, \mathbb{R})} \leq \|x\|_{X_\gamma}.$$

This concludes the proof. \square

Lemma 10.7. *Fix $\gamma < \alpha$ and $x \in X_\gamma$. Let $F \subseteq I_{\leq \gamma}^\alpha$ be a finite set. Define $z \in Z^{I_0^\alpha}$ by setting, for $(n; \beta) \in I_0^\alpha$,*

$$z(n; \beta) = \begin{cases} x(n; \beta) & \text{if } (n; \beta) \in F_\downarrow; \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Then we have that $z \in S_\alpha$ and (2) holds for every $(n; \beta) \in I^\alpha$.

Proof. We prove by induction on $\sigma \leq \alpha$ that $z \in X_\sigma$ and (2) holds for every $(n; \beta) \in I_\sigma^\alpha$. Define

$$\tilde{F} = \{(m; \gamma) \in I^\alpha : \exists (n; \beta) \in F, (n; \beta) \leq (m; \gamma)\}.$$

Case $\sigma = 0$: We have that (2) holds for $(n; \beta) \in I_0^\alpha$ by definition of z . As $x \in X_\gamma$, for every $\varepsilon > 0$ there exists a finite subset E of $I_{\leq \gamma}^\alpha$ such that $\|\mathbf{x}(n; \beta)\|_\infty < \varepsilon$ for $(n; \beta) \in I_{\leq \gamma}^\alpha \setminus E$. Thus, if $(n; \beta) \in I_0^\alpha \setminus E$, then

$$\|z(n; \beta)\|_\infty = |z(n; \beta)| \leq |x(n; \beta)| < \varepsilon.$$

Case $1 \leq \sigma \leq \gamma$: Fix $(n; \beta) \in I_\sigma^\alpha$. If $(n; \beta) \in F_\downarrow$ then $(k, n; \sigma_k, \beta) \in F_\downarrow$ for every $k \in \mathbb{N}$. Thus, by the inductive hypothesis,

$$kz(k, n; \sigma_k, \beta) = kx(k, n; \sigma_k, \beta)$$

for every $k \in \mathbb{N}$, and hence

$$\mathbf{z}(n; \beta) = \mathbf{x}(n; \beta).$$

Since by assumption $\mathbf{x}(n; \beta)$ is a convergent sequence with limit $x(n; \beta)$, we have that $\mathbf{z}(n; \beta)$ is a convergent sequence with limit $z(n; \beta) = x(n; \beta)$. If $(n; \beta) \notin F_\downarrow$ then there exists $N \in \mathbb{N}$ such that for $k > N$, $(k, n; \sigma_k, \beta) \notin F_\downarrow$. By the inductive hypothesis, we have that $kz(k, n; \sigma_k, \beta) = 0$ for $k > N$. Thus, $\mathbf{z}(n; \beta)$ is a sequence eventually zero with limit $z(n; \beta) = 0$.

Fix $\varepsilon > 0$. Since $x \in X_\gamma$, there exist a finite subset $E \subseteq I_{\leq \gamma}^\alpha$ such that $\|\mathbf{x}(n; \beta)\|_\infty < \varepsilon$ for $(n; \beta) \in I_{\leq \gamma}^\alpha \setminus E$, and a finite subset $E' \subseteq J_\gamma^\alpha * I^\alpha$ such that $|kx(k, n; \tau_k, \beta)| < \varepsilon$ for $((k, \tau), (n; \beta)) \in (J_\gamma^\alpha * I^\alpha) \setminus E'$. Define

$$E'' = E' \cup \{((k, \tau), (n; \beta)) \in (J_\sigma^\alpha * I^\alpha) : (n; \beta) \in E\}$$

If $(n; \beta) \in I_{\leq \sigma}^\alpha \setminus E$ then we have that

$$\|z(n; \beta)\|_\infty \leq \|\mathbf{x}(n; \beta)\|_\infty \leq \varepsilon.$$

If $((k, \tau), (n; \beta)) \in (J_\sigma^\alpha * I^\alpha) \setminus E''$ then we have that $\tau_k < \sigma < \tau$ and $(n; \beta) \in I_\tau^\alpha$. If $\tau \leq \gamma$ then we have that $(n; \beta) \in I_{\leq \gamma}^\alpha \setminus E$ and hence

$$|kz(k, n; \tau_k, \beta)| \leq \|\mathbf{x}(n; \beta)\|_\infty \leq \varepsilon.$$

If $\gamma < \tau$ then $((k, \tau), (n; \beta)) \in (J_\gamma^\alpha * I) \setminus E'$ and hence

$$|kz(k, n; \tau_k, \beta)| \leq |kx(k, n; \tau_k, \beta)| \leq \varepsilon.$$

Case $\sigma > \gamma$: Fix $(n; \beta) \in I_\sigma^\alpha$. Then by the inductive assumption we have that $z(k, n; \sigma_k, \beta) = 0$ for $k > N$. Thus, the sequence $z(n; \beta)$ is eventually zero, and $z(n; \beta) = 0$.

Fix $\varepsilon > 0$. Since $x \in X_\gamma$, there exist a finite set $E \subseteq I_{\leq \gamma}^\alpha$ such that

$$\|\mathbf{x}(n; \beta)\|_\infty < \varepsilon$$

for $(n; \beta) \in I_{\leq \gamma}^\alpha \setminus E$, and a finite set $E' \subseteq (J_\gamma^\alpha * I^\alpha)$ such that

$$|kx(k, n; \tau_k, \beta)| < \varepsilon$$

for $((k, \tau), (n; \beta)) \in (J_\gamma^\alpha * I^\alpha) \setminus E'$. Define

$$\begin{aligned} \tilde{E} &= E \cup \tilde{F} \\ \tilde{E}' &= \tilde{F} \cup E' \cup \left\{ ((k, \tau), (n; \beta)) \in J_\sigma^\alpha * I^\alpha : (n; \beta) \in \tilde{E} \right\}. \end{aligned}$$

Fix $(n; \beta) \in I_{\leq \sigma}^\alpha \setminus \tilde{E}$. Fix $\delta \leq \sigma$ such that $(n; \beta) \in I_\delta^\alpha$. If $\delta \leq \gamma$, then we have that $(m; \beta) \in I_{\leq \gamma}^\alpha \setminus E$ and hence

$$\|z(n; \beta)\|_\infty \leq \|\mathbf{x}(n; \beta)\|_\infty \leq \varepsilon.$$

Suppose that $\delta > \gamma$, and fix $k \in \mathbb{N}$. If $z(k, n; \delta_k, \beta) \neq 0$ then we have that $(k, n; \delta_k, \beta) \in F$, whence $(n; \beta) \in \tilde{F} \subseteq \tilde{E}$, contradicting the hypothesis. Thus, $z(n; \beta)$ is the sequence constantly equal to zero. Fix $((k, \tau), (n; \beta)) \in (J_\sigma^\alpha * I^\alpha) \setminus \tilde{E}'$. Thus, $\tau_k < \sigma < \tau$ and $(n; \beta) \in I_\tau^\alpha$. If $\tau_k < \gamma$, then $((k, \tau), (n; \beta)) \in (J_\gamma^\alpha * I^\alpha) \setminus E'$ and hence

$$|kz(k, n; \tau_k, \beta)| \leq |kx(k, n; \tau_k, \beta)| \leq \varepsilon.$$

Suppose that $\tau_k = \gamma$. If $z(k, n; \tau_k, \beta)$ is nonzero, then $(k, n; \tau_k, \beta) \in F$ and hence $(n; \beta) \in \tilde{F} \subseteq \tilde{E}$, contradicting the assumption that $(k, n; \tau_k, \beta) \notin \tilde{E}'$. If $\tau_k > \gamma$ then we have that $(k, n; \tau_k, \beta) \notin F_\downarrow$ and hence $z(k, n; \tau_k, \beta) = 0$. This concludes the inductive proof.

Finally, to see that $z \in S_\alpha$ observe that if $N = \max\{k \in \mathbb{N} : \alpha_k \leq \gamma\}$ then we have that

$$\sum_{k \in \mathbb{N}} |z(k; \alpha_k)| \leq \sum_{k \leq N} |x(k; \alpha_k)| < +\infty.$$

□

The next lemma is similar to the previous one, with the difference that the finite set F is supposed to be a subset of $I_{< \gamma}^\alpha$ instead of $I_{\leq \gamma}^\alpha$.

Lemma 10.8. Fix $\gamma < \alpha$ and $x \in X_\gamma$. Let $F \subseteq I_{< \gamma}^\alpha$ be a finite set. Define $z \in Z^{I_0^\alpha}$ by setting, for $(n; \beta) \in I_0^\alpha$,

$$z(n; \beta) = \begin{cases} x(n; \beta) & \text{if } (n; \beta) \in F_\downarrow; \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Then we have that $z \in S_\alpha$, $\|z\|_{X_\alpha} \leq \|x\|_{X_\gamma}$, and furthermore (3) holds for every $(n; \beta) \in I^\alpha$. Furthermore

$$\|z\|_{S_\alpha} \leq \max \left\{ \sum_{k \leq N} \|x(k; \alpha_k)\|, \|x\|_{X_\gamma} \right\}$$

where $N = \max\{k \in \mathbb{N} : \alpha_k < \gamma\}$.

Proof. It follows from Lemma 10.7 that $z \in S_\alpha$ and (3) holds for every $(n; \beta) \in I_\sigma^\alpha$. We now prove by induction on $\sigma \leq \alpha$ that $\|z\|_{X_\sigma} \leq \|x\|_{X_\gamma}$. Suppose that the conclusion holds for every $\delta < \sigma$.

Case $\sigma = 0$: If $(n; \beta) \in I_0^\alpha$, then we have that $|z(n; \beta)| \leq |x(n; \beta)|$. This shows that $\|z\|_{X_0} \leq \|x\|_{X_0} \leq \|x\|_{X_\gamma}$.

Case $1 \leq \sigma \leq \gamma$: For $(n; \beta) \in I_{\leq \sigma}^\alpha$, we have

$$\|z(n; \beta)\|_\infty \leq \|\mathbf{x}(n; \beta)\|_\infty \leq \|x\|_{X_\gamma}.$$

Fix $((k, \tau), (n; \beta)) \in J_\sigma^\alpha * I^\alpha$. Thus, we have that $\tau_k < \sigma < \tau$ and $(n; \beta) \in I_\tau^\alpha$. If $\tau \leq \gamma$, then $(n; \beta) \in I_{\leq \gamma}^\alpha$ and hence

$$|kz(k, n; \tau_k, \beta)| \leq |kx(k, n; \tau_k, \beta)| \leq \|\mathbf{x}(n; \beta)\|_\infty \leq \|x\|_{X_\gamma}.$$

If $\gamma < \tau$, then $((k, \tau), (n; \beta)) \in J_\gamma^\alpha * I^\alpha$ and hence

$$|kz(k, n; \tau_k, \beta)| \leq |kx(k, n; \tau_k, \beta)| \leq \|x\|_{X_\gamma}.$$

Case $\sigma > \gamma$. Suppose that $(n; \beta) \in I_\delta^\alpha$ for some $\delta \leq \sigma$. If $\delta \leq \gamma$, then

$$\|z(n; \beta)\|_\infty \leq \|\mathbf{x}(n; \beta)\|_\infty \leq \|x\|_{X_\gamma}.$$

If $\gamma < \delta$ then for every $k \in \mathbb{N}$ we have that either $\delta_k < \gamma$, in which case $((k, \delta), (n; \beta)) \in J_\gamma^\alpha * I^\alpha$ and hence

$$|kz(k, n; \delta_k, \beta)| \leq |kx(k, n; \delta_k, \beta)| \leq \|x\|_{X_\gamma}$$

or $\gamma \leq \delta_k$, in which case $(k, n; \delta_k, \beta) \notin F_\downarrow$ and

$$z(k, n; \delta_k, \beta) = 0.$$

Thus

$$\|z(n; \beta)\|_\infty \leq \|x\|_{X_\gamma}.$$

Suppose now that $((k, \tau), (n, \beta)) \in J_\sigma^\alpha * I^\alpha$. Thus $\tau_k < \sigma < \tau$. If $\tau_k < \gamma$ then we have that $((k, \tau), (n, \beta)) \in J_\gamma^\alpha * I^\alpha$ and hence

$$|kz(k, n; \tau_k, \beta)| \leq |kx(k, n; \tau_k, \beta)| \leq \|x\|_{X_\gamma}$$

If $\gamma \leq \tau_k$ then $(k, n; \tau_k, \beta) \notin F_\downarrow$ and hence

$$z(k, n; \tau_k, \beta) = 0.$$

This concludes the inductive proof that $\|z\|_{X_\sigma} \leq \|x\|_{X_\gamma}$ for every $\sigma \leq \alpha$.

Finally, we have that

$$\sum_{k \in \mathbb{N}} |z(k; \alpha_k)| \leq \sum_{k \leq N} |x(k; \alpha_k)|.$$

This shows that $z \in S_\alpha$ and

$$\|z\|_{S_\alpha} = \max \left\{ \|z\|_{X_\alpha}, \sum_{k \in \mathbb{N}} |z(k; \alpha_k)| \right\} \leq \max \left\{ \|x\|_{X_\alpha}, \sum_{k \leq N} |x(k; \alpha_k)| \right\}.$$

This concludes the proof. \square

Lemma 10.9. *For every $\gamma < \alpha$, S_α is dense in X_γ .*

Proof. Suppose that $x \in X_\gamma$, and $\varepsilon > 0$. We need to prove that there exists $z \in S_\alpha$ such that $\|x - z\|_{X_\gamma} \leq \varepsilon$. Since $x \in X_\gamma$, there exists a finite subset $E \subseteq I_{\leq \gamma}^\alpha$ such that $\|\mathbf{x}(n; \beta)\|_\infty < \varepsilon$ for $(n; \beta) \in I_{\leq \gamma}^\alpha \setminus E$, and a finite subset $E' \subseteq J_\gamma^\alpha * I^\alpha$ such that $|kx(k, n; \tau_k, \beta)| < \varepsilon$ for $((k, \tau), (n; \beta)) \in (J_\gamma^\alpha * I^\alpha) \setminus E'$.

Suppose that $z \in X_\alpha$ is obtained from $x \in X_\gamma$ and

$$F = (E \cap I_{\leq \gamma}^\alpha) \cup \{(k, n; \tau_k, \beta) : ((k, \tau), (n; \beta)) \in E'\}$$

as in Lemma 10.7.

Fix $(n; \beta) \in I_{\leq \gamma}^\alpha$. If $(n; \beta) \in F_\downarrow$ then $z(n; \beta) = \mathbf{x}(n; \beta)$, while if $(n; \beta) \notin F_\downarrow$, then

$$\|z(n; \beta) - \mathbf{x}(n; \beta)\|_\infty \leq \|\mathbf{x}(n; \beta)\|_\infty \leq \varepsilon.$$

Consider $((k, \tau), (n; \beta)) \in J_\gamma^\alpha * I^\alpha$. Thus $\tau_k < \gamma < \tau$ and $(n; \beta) \in I_\tau^\alpha$. If $(k, n; \tau_k, \beta) \in F$ then $kz(k, n; \tau_k, \beta) = kx(k, n; \tau_k, \beta)$. If $(k, n; \tau_k, \beta) \notin F$, then $(k, n; \tau_k, \beta) \in (J_\gamma^\alpha * I^\alpha) \setminus E'$ and hence

$$|kz(k, n; \tau_k, \beta) - kx(k, n; \tau_k, \beta)| \leq |kx(k, n; \tau_k, \beta)| \leq \varepsilon.$$

This concludes the proof that $\|z - x\|_{X_\gamma} \leq \varepsilon$. \square

Lemma 10.10. *Fix $\gamma < \alpha$. If V is a neighborhood of zero in S_α , then $\overline{V}^{X < \gamma} \cap X_\gamma$ contains an open neighborhood of zero in X_γ .*

Proof. Define

$$N = \max \{k \in \mathbb{N} : \alpha_k \leq \gamma\}.$$

Suppose that V is a neighborhood of zero in X_α . Without loss of generality, we can assume that

$$V = \left\{ z \in S_\alpha : \|z\|_{X_\alpha} \leq \varepsilon, \sum_{k \in \mathbb{N}} |z(k; \alpha_k)| \leq \varepsilon \right\}.$$

We claim that $\overline{V}^{X_{<\gamma}} \cap X_\gamma$ contains

$$W := \left\{ x \in X_\gamma : \|x\|_{X_\gamma} \leq \varepsilon, \sum_{k \leq N} |x(k; \alpha_k)| \leq \varepsilon \right\}.$$

Indeed, suppose that $x \in W$. Let U be an open neighborhood of x in $X_{<\gamma}$. Without loss of generality, we can assume that

$$U = \{z \in X_{<\gamma} : \|x - z\|_{X_\delta} \leq \varepsilon_1\}$$

for some $\delta < \gamma$ and $\varepsilon_1 > 0$. We need to prove that $U \cap V \neq \emptyset$.

Since $x \in X_\gamma$, there exists a finite subset E of X_γ such that $\|\mathbf{x}(n; \beta)\|_\infty \leq \varepsilon_1$ for $(n; \beta) \in I_{\leq \gamma}^\alpha \setminus E$, and a finite subset E' of $J_\gamma^\alpha * I^\alpha$ such that $|kx(k, n; \tau_k, \beta)| \leq \varepsilon_1$ for $((k, \tau), (n; \beta)) \in (J_\gamma^\alpha * I^\alpha) \setminus E'$. Define

$$E'' = E' \cup \{((k, \tau), (n; \beta)) \in J_\gamma^\alpha * I^\alpha : (n; \beta) \in E\}$$

Let $z \in X_\alpha$ be obtained from x and

$$F := (E \cap I_{\leq \delta}^\alpha) \cup \{(k, n; \tau_k, \beta) : ((k, \tau), (n; \beta)) \in E''\} \cup \{(k; \alpha_k) : k \leq N\}$$

as in Lemma 10.8. Then we have that $z \in S_\alpha$,

$$\|z\|_{X_\alpha} \leq \|x\|_{X_\gamma} \leq \varepsilon,$$

and

$$\sum_{k \leq N} |z(k; \alpha_k)| \leq \sum_{k \leq N} |x(k; \alpha_k)| \leq \varepsilon$$

and hence $z \in V$. It remains to prove that $\|z - x\|_{X_\delta} \leq \varepsilon_1$. For $(n; \beta) \in I_{\leq \delta}^\alpha$, if $(n; \beta) \in F$ then

$$\mathbf{z}(n; \beta) = \mathbf{x}(n; \beta)$$

while if $(n; \beta) \notin F$, then $(n; \beta) \in I_{\leq \gamma}^\alpha \setminus E$ and we have that

$$\|\mathbf{z}(n; \beta) - \mathbf{x}(n; \beta)\|_\infty \leq \|\mathbf{x}(n; \beta)\|_\infty \leq \varepsilon_1$$

by the choice of E .

For $((k, \tau), (n; \beta)) \in J_\delta^\alpha * I^\alpha$, we have that $\tau_k < \delta < \tau$ and $(n; \beta) \in I_\tau^\alpha$. Suppose that $\gamma < \tau$, in which case we have that $((k, \tau), (n; \beta)) \in J_\gamma^\alpha * I^\alpha$. If $((k, \tau), (n; \beta)) \in F$, then

$$kz(k, n; \tau_k, \beta) = kx(k, n; \tau_k, \beta);$$

if $((k, \tau), (n; \beta)) \notin F$ then $((k, \tau), (n; \beta)) \in (J_\gamma^\alpha * I^\alpha) \setminus E'$ and hence

$$|kz(k, n; \tau_k, \beta) - kx(k, n; \tau_k, \beta)| = |kx(k, n; \tau_k, \beta)| \leq \varepsilon_1.$$

Suppose now that $\tau \leq \gamma$, in which case $\tau_k < \delta < \tau \leq \gamma$. If $((k, \tau), (n; \beta)) \in F$, then

$$kz(k, n; \tau_k, \beta) = kx(k, n; \tau_k, \beta);$$

while if $((k, \tau), (n; \beta)) \notin F$, then we have that $(n; \beta) \in I_{\leq \gamma}^\alpha \setminus E$ and hence

$$|kz(k, n; \tau_k, \beta) - kx(k, n; \tau_k, \beta)| \leq |kx(k, n; \tau_k, \beta)| \leq \|\mathbf{x}(n; \beta)\|_\infty \leq \varepsilon_1.$$

This concludes the proof that $\|z - x\|_{X_\delta} \leq \varepsilon_1$. \square

Using Lemma 10.10 and Lemma 10.9, one can prove Proposition 10.11, similarly as Proposition 8.7 is proved from Lemma 8.6 and Lemma 8.5

Proposition 10.11. *For $\gamma < \alpha$ we have that*

$$s_\gamma^{S_\alpha}(X_0) = s_\gamma^{D_\alpha}(X_0) = s_\gamma^{X_\alpha}(X_0) = s_\gamma^{X_{<\alpha}}(X_0) = X_{<(1+\gamma)}$$

Recall that, for $(n; \beta)$ and $(m; \tau)$ in I^α , we define $(n; \beta) \leq (m; \tau)$ if and only if there exist $\gamma_0 \leq \gamma_1 \leq \alpha$ such that $(n; \beta) \in I_{\gamma_0}^\alpha$, $(m; \tau) \in I_{\gamma_1}^\alpha$, m is a tail of n , and τ is a tail of β , i.e. we have that, for some $\ell \leq d < \omega$, $(n; \beta) = (n_0, \dots, n_d; \beta_0, \dots, \beta_d)$, and $(m; \tau) = (n_{d-\ell}, \dots, n_d; \beta_{d-\ell}, \dots, \beta_d)$. In this case, we set

$$\pi_{(m; \tau)}^{(n; \beta)} := \frac{1}{n_0 \cdots n_{d-\ell-1}}.$$

Lemma 10.12. *Fix $\gamma \leq \alpha$ and $(m; \tau) \in I_\gamma^\alpha$. There exists a continuous group homomorphism $\Phi : c_0(\mathbb{N}, Z) \rightarrow X_{<\gamma}$ such that $\Phi(t)(k, m; \gamma_k, \tau) = t_k$ for every $t \in c_0(\mathbb{N}, Z)$ and $k \in \mathbb{N}$.*

Proof. For $\varepsilon > 0$, let $K_\varepsilon \in \mathbb{N}$ be such that, for $k > K_\varepsilon$ one has that $|t_k| \leq \varepsilon$. For $t \in (Z^\mathbb{N})^{I_\gamma^\alpha}$, define $\Phi(t) := x \in Z^{I_\alpha^\alpha}$ by setting, for $(n; \beta) \in I_0^\alpha$,

$$x(n; \beta) := \begin{cases} \pi_{(n; \beta)}^{(k, m; \gamma_k, \tau)} t_k & \text{if } (n; \beta) \leq (k, m; \gamma_k, \tau) \text{ for some } k \in \mathbb{N}; \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

It is clear that $\Phi : Z^\mathbb{N} \rightarrow Z^{I_\alpha^\alpha}$ is a continuous group homomorphism. We now prove by induction on $\sigma < \gamma$ that $x \in X_\sigma$, and that (4) holds for every $(n; \beta) \in I^\alpha$. Suppose that the conclusion holds for all $\delta < \sigma$.

Case $\sigma = 0$: We need to prove that $x \in c_0(I_0^\alpha, Z)$. Fix $\varepsilon > 0$. Consider

$$F = \{(n; \beta) \in I_0^\alpha : (n; \beta) \leq (k, m; \gamma_k, \tau) \text{ for some } k \leq K_\varepsilon\}$$

Then $F \subseteq I_0^\alpha$ is finite and for $(n; \beta) \in I_0^\alpha \setminus F$ one has that either

$$x(n; \beta) = 0$$

or $(n; \beta) \leq (k, m; \gamma_k, \tau)$ for some $k > K_\varepsilon$, in which case

$$|x(n; \beta)| = \left| \pi_{(n; \beta)}^{(k, m; \gamma_k, \tau)} t_k \right| \leq |t_k| \leq \varepsilon$$

Case $1 \leq \sigma < \gamma$: Fix $(n; \beta) \in I_\delta^\alpha$ for some $\delta \leq \sigma$. If $(n; \beta) \leq (k, m; \gamma_k, \tau)$ for some $k \in \mathbb{N}$, then we have that for every $\ell \in \mathbb{N}$, $(\ell, n; \delta_\ell, \beta) \leq (k, m; \gamma_k, \tau)$. Thus,

$$x(\ell, n; \delta_\ell, \beta) = \pi_{(\ell, n; \delta_\ell, \beta)}^{(k, m; \gamma_k, \tau)} t_k$$

and

$$\ell x(\ell, n; \delta_\ell, \beta) = \pi_{(n; \beta)}^{(k, m; \gamma_k, \tau)} t_k$$

Thus, the sequence $x(n; \beta)$ is constantly equal to $\pi_{(n; \beta)}^{(k, m; \gamma_k, \tau)} t_k$. This shows that

$$x(n; \beta) = \pi_{(n; \beta)}^{(k, m; \gamma_k, \tau)} t_k.$$

Suppose that there does not exist $k \in \mathbb{N}$ such that $(n; \beta) \leq (k, m; \gamma_k, \tau)$. Fix $\ell \in \mathbb{N}$. If $(\ell, n; \delta_\ell, \beta) \leq (k, m; \gamma_k, \tau)$ for some $k \in \mathbb{N}$, then we have that (k, m) is a tail of (ℓ, n) and (γ_k, τ) is a tail of (δ_ℓ, β) . If the length of (k, m) is strictly less than the length of (ℓ, n) , then m is a tail of n and τ is a tail of β , and hence $(n; \beta) \leq (k, m; \gamma_k, \tau)$, contradicting the assumption. Therefore, we have that $(\ell, n; \delta_\ell, \beta) = (k, m; \gamma_k, \tau)$. In particular, we have that $(n; \beta) = (m; \tau) \in I_\gamma^\alpha$ contradicting the assumption that $(n; \beta) \in I_\delta^\alpha$ and $\delta \leq \sigma < \gamma$. Thus, the sequence $z(n; \beta)$ is constantly zero, and hence $z(n; \beta) = 0$.

We now prove that $x \in X_\sigma$. Fix $\varepsilon > 0$. Define

$$N = \max\{K_\varepsilon, \max\{k \in \mathbb{N} : \gamma_k \leq \sigma\}\}.$$

Consider

$$E = \{(n; \beta) \in I_{\leq \sigma}^\alpha : (n; \beta) \leq (k, m; \gamma_k, \tau) \text{ for some } k \leq N\}.$$

If $(n; \beta) \in I_{\leq \sigma}^\alpha \setminus E$ and $x(n; \beta) \neq 0$ then, by the argument above, $(n; \beta) \leq (k, m; \gamma_k, \tau)$ for some $k > N \geq K_\varepsilon$ and hence

$$|x(n; \beta)| \leq |t_k| \leq \varepsilon.$$

Consider the finite set

$$E' = \{((\ell, \rho); (n; \beta)) \in J_\sigma^\alpha * I^\alpha : (\ell, n; \rho_\ell, \beta) \in E\}$$

If $(\ell, \rho; (n; \beta)) \in (J_\sigma^\alpha * I^\alpha) \setminus E'$ and $x(\ell, n; \rho_\ell, \beta) \neq 0$, then $\rho_\ell < \sigma < \rho$ and

$$(\ell, n; \rho_\ell, \beta) \leq (k, m; \gamma_k, \tau)$$

for some $k \in \mathbb{N}$. Since $(\ell, \rho; (n; \beta)) \notin E'$, we have that $k > N$ and hence $\gamma_k > \sigma > \rho_\ell$ and

$$\pi_{(\ell, n; \rho_\ell, \beta)}^{(k, m; \gamma_k, \tau)} \leq \frac{1}{\ell}.$$

Thus,

$$|x(\ell, n; \rho_\ell, \beta)| \leq \left| \ell \pi_{(\ell, n; \rho_\ell, \beta)}^{(k, m; \gamma_k, \tau)} t_k \right| \leq |t_k| \leq \varepsilon$$

since $k > N \geq N_\varepsilon$. This concludes the proof. \square

Lemma 10.13. *Consider the continuous function $c_0(\mathbb{N}, Z) \rightarrow Z^\mathbb{N}$, $(x_n)_{n \in \mathbb{N}} \mapsto (nx_n)_{n \in \mathbb{N}}$. We have that:*

- Σ_2^0 is the complexity class of $\tau^{-1}(\ell_1(Z))$ in $c_0(\mathbb{N}, Z)$;
- $D(\Pi_2^0)$ is the complexity class of $\tau^{-1}(\text{bv}_0(Z))$ in $c_0(\mathbb{N}, Z)$;
- Π_3^0 is the complexity class of $\tau^{-1}(c(\mathbb{N}, Z))$ and of $\tau^{-1}(c(\mathbb{N}, Z))$ in $c_0(\mathbb{N}, Z)$.

Proof. (1) Since $\ell_1(Z)$ is Σ_2^0 in $Z^\mathbb{N}$, we have that $\tau^{-1}(\ell_1)$ is a Σ_2^0 Polishable subgroup of $c_0(\mathbb{N}, Z)$ that is not closed. Thus, Σ_2^0 is the complexity class of $\tau^{-1}(\ell_1(Z))$.

(2) Since $\text{bv}_0(Z)$ is $D(\Pi_2^0)$ in $Z^\mathbb{N}$, and $\tau^{-1}(\text{bv}_0(Z))$ is a Polishable subgroup of $c_0(\mathbb{N}, Z)$, by Theorem 3.3 it suffices to prove that $\tau^{-1}(\text{bv}_0(Z))$ is not Σ_2^0 in $c_0(\mathbb{N}, Z)$. Suppose by contradiction that $\tau^{-1}(\text{bv}_0(Z)) = \bigcup_{k \in \omega} F_k$ where $F_k \subseteq c_0(\mathbb{N}, Z)$ is closed. Observe that a compatible norm on $\text{bv}_0(Z)$ is given by

$$\|x\|_{\text{bv}_0(Z)} = \sum_{n \in \mathbb{N}} |x_{n+1} - x_n| + \sup_{n \in \mathbb{N}} |x_n|.$$

By the Baire category theorem without loss of generality we can assume that

$$\{\mathbf{a} \in \tau^{-1}(\text{bv}_0(Z)) : \|\tau((a_n))\|_{\text{bv}_0} \leq 2\} \subseteq F_0.$$

Define then for $N \in \mathbb{N}$, $\mathbf{a}^{(N)} \in \tau^{-1}(\text{bv}_0(Z))$ by setting

$$a_n^{(N)} = \begin{cases} \frac{1}{n} & \text{if } n \leq N; \\ 0 & \text{otherwise.} \end{cases}$$

Then we have that $\|\tau(\mathbf{a}^{(N)})\|_{\text{bv}_0(Z)} \leq 2$ and $\mathbf{a}^{(N)} \in F_0$ for every $N \in \mathbb{N}$. Furthermore, the sequence $(\mathbf{a}^{(N)})_{N \in \mathbb{N}}$ converges in $c_0(\mathbb{N}, Z)$ to the sequence \mathbf{a} defined by $a_n = \frac{1}{n}$ for every $n \in \mathbb{N}$. Since F_0 is closed in $c_0(\mathbb{N}, Z)$, we must have that $\mathbf{a} \in F_0 \subseteq \tau^{-1}(\text{bv}_0(Z))$. However, $\tau(\mathbf{a})$ is not vanishing, and so $\tau(\mathbf{a}) \notin \text{bv}_0(Z)$.

(3) Since $c(\mathbb{N}, Z)$ is Π_3^0 in $Z^\mathbb{N}$, and $\tau^{-1}(c(\mathbb{N}, Z))$ is a Polishable subgroup of c_0 , by Theorem 3.3 it suffices to prove that $\tau^{-1}(c(\mathbb{N}, Z))$ is not potentially Σ_2^0 . Let E_0 be the relation of tail equivalence in $2^\mathbb{N}$, and let $E_0^\mathbb{N}$ be the corresponding product equivalence relation on $(2^\mathbb{N})^\mathbb{N} = 2^{\mathbb{N} \times \mathbb{N}}$. Then we have that Π_3^0 is the potential complexity class of $E_0^\mathbb{N}$, for example by Lemma 5.7 and Theorem 3.3.

Thus, it suffices to define a Borel function $2^{\mathbb{N} \times \mathbb{N}} \rightarrow c_0(\mathbb{N}, Z)$ that is a Borel reduction from $E_0^\mathbb{N}$ to the coset relation of $\tau^{-1}(c(\mathbb{N}, Z))$ inside $c_0(\mathbb{N}, Z)$. Fix a bijection $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that, if $n \leq n'$ and $m \leq m'$, then $\langle n, m \rangle \leq \langle n', m' \rangle$. Define $2^{\mathbb{N} \times \mathbb{N}} \rightarrow Z^\mathbb{N}$, $x \mapsto a$ by setting $a_{\langle n, m \rangle} = \frac{1}{\langle n, m \rangle} 2^{-n} x_{n, m}$. Then the argument in [Gao09, Lemma 8.5.3] shows that $x E_0^\mathbb{N} x'$ if and only if $\tau(\mathbf{a}) - \tau(\mathbf{a}') = \tau(\mathbf{a} - \mathbf{a}') \in c(\mathbb{N}, Z)$, if and only if $\mathbf{a} - \mathbf{a}' \in \tau^{-1}(c(\mathbb{N}, Z))$.

The same argument shows that Π_3^0 is the complexity class of $\tau^{-1}(c_0(\mathbb{N}, Z))$ in $c_0(\mathbb{N}, Z)$. \square

The same proof as Corollary 8.10, where Lemma 10.13 replaces Lemma 8.9, gives the proof of Corollary 10.14 below.

Corollary 10.14. *For every $\gamma < \alpha$, X_γ is a proper subspace of $X_{< \gamma}$. The complexity class inside $X_{< \alpha}$ of S_α , D_α , X_α , respectively, is Σ_2^0 , $D(\Pi_2^0)$, and Π_3^0 , respectively.*

Finally, Theorem 10.5 is proved using Corollary 10.14 and Proposition 10.11 similarly as Theorem 8.3 is proved using Corollary 8.10 and Proposition 8.7.

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