
THE PERFECTOID COMMUTANT OF LUBIN-TATE POWER SERIES

by

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Abstract. — Let LT be a Lubin-Tate formal group attached to a finite extension of \mathbf{Q}_p . By a theorem of Lubin-Sarkis, an invertible characteristic p power series that commutes with the elements $\text{Aut}(LT)$ is itself in $\text{Aut}(LT)$. We extend this result to perfectoid power series, by lifting such a power series to characteristic zero and using the theory of locally analytic vectors in certain rings of p -adic periods. This allows us to recover the field of norms of the Lubin-Tate extension from its completed perfection.

Introduction

Let F be a finite extension of \mathbf{Q}_p , with ring of integers \mathcal{O}_F and residue field k . Let $q = \text{Card}(k)$ and let π be a uniformizer of \mathcal{O}_F . Let LT be the Lubin-Tate formal \mathcal{O}_F -module attached to π . Let $F_\infty = F(LT[\pi^\infty])$ denote the extension of F generated by the torsion points of LT , and let $\Gamma_F = \text{Gal}(F_\infty/F)$. The Lubin-Tate character χ_π gives rise to an isomorphism $\chi_\pi : \Gamma_F \rightarrow \mathcal{O}_F^\times$.

The field of norms ([Win83]) \mathbf{E}_F of the extension F_∞/F is a local field of characteristic p , endowed with an action of Γ_F , that can be explicitly described as follows. We choose a coordinate T on LT , so that for each $a \in \mathcal{O}_F$ we get a power series $[a](T) \in \mathcal{O}_F[[T]]$. We then have $\mathbf{E}_F = k((Y))$, on which Γ_F acts via the formula $\gamma(f(Y)) = f([\chi_\pi(\gamma)](Y))$. In p -adic Hodge theory, we consider the field $\tilde{\mathbf{E}}_F$, which is the Y -adic completion of the maximal purely inseparable extension $\cup_{n \geq 0} \mathbf{E}_F^{q^{-n}}$ of \mathbf{E}_F inside an algebraic closure. The action of Γ_F extends to the field $\tilde{\mathbf{E}}_F$. If $f \in \tilde{\mathbf{E}}_F$ and $\gamma \in \Gamma_F$, we still have $\gamma(f(Y)) = f([\chi_\pi(\gamma)](Y))$. The question that motivated this paper is the following.

2020 Mathematics Subject Classification. — 11S; 12J; 13J.

Key words and phrases. — Lubin-Tate group; field of norms; p -adic period; locally analytic vector; p -adic dynamical system; perfectoid field.

Question. — Can we recover \mathbf{E}_F from the data of the valued field $\tilde{\mathbf{E}}_F$ endowed with the action of Γ_F ?

If $a \in \mathcal{O}_F^\times$, then $u(Y) = [a](Y)$ is an element of \mathbf{E}_F of valuation 1 that satisfies the functional equation $u \circ [g](Y) = [g] \circ u(Y)$ for all $g \in \mathcal{O}_F^\times$. Conversely, we prove the following theorem, which answers the question, as it allows us to find a uniformizer of \mathbf{E}_F from the data of the valued field $\tilde{\mathbf{E}}_F$ endowed with the action of Γ_F .

Theorem A. — If $u \in \tilde{\mathbf{E}}_F$ is such that $\text{val}_Y(u) = 1$ and $u \circ [g] = [g] \circ u$ for all $g \in \mathcal{O}_F^\times$, then there exists $a \in \mathcal{O}_F^\times$ such that $u(Y) = [a](Y)$.

In particular, $\mathbf{E}_F = k((u))$ for any u as in theorem A. The main difficulty in the proof of theorem A is to prove that if u is as in the statement of theorem A, then there exists $n \geq 0$ such that $u \in \mathbf{E}_F^{q^{-n}}$. If $F = \mathbf{Q}_p$ and $\pi = p$, namely in the cyclotomic situation, this follows from the main result of [BR22]. However, a crucial ingredient in that paper does not generalize to $F \neq \mathbf{Q}_p$. In order to go beyond the cyclotomic case, we instead use a result of Colmez ([Col02]) to lift u to an element \hat{u} of a ring $\tilde{\mathbf{A}}_F^+$ (the Witt vectors over the ring of integers of $\tilde{\mathbf{E}}_F$, as well as a completion of $\cup_{n \geq 0} \varphi_q^{-n}(\mathcal{O}_F[[\hat{Y}]])$, where $\varphi_q(\hat{Y}) = [\pi](\hat{Y})$), that will satisfy a similar functional equation. In particular, \hat{u} is a locally analytic element of a suitable ring of p -adic periods. By previous results of the author ([Ber16]), \hat{u} belongs to $\varphi_q^{-n}(\mathcal{O}_F[[\hat{Y}]])$ for a certain n . This allows us to prove that there exists $n \geq 0$ such that $u \in \mathbf{E}_F^{q^{-n}}$. By replacing u with u^{p^k} for a well chosen k , we are led to the study of elements of $Y \cdot k[[Y]]$ under composition. We prove that u is invertible for composition, and to conclude we use a theorem of Lubin-Sarkis ([LS07]) saying that if an invertible series commutes with a nontorsion element of $\text{Aut}(\text{LT})$, then that series is itself in $\text{Aut}(\text{LT})$. We finish this paper with an explanation of why the ‘‘Tate traces’’ on $\tilde{\mathbf{E}}_F$ used in [BR22] don’t exist if $F \neq \mathbf{Q}_p$.

1. Locally analytic vectors

We use the notation that was introduced in the introduction. In order to apply lemma 9.3 of [Col02], we assume that the coordinate T on LT is chosen such that $[\pi](T)$ is a monic polynomial of degree q (for example, we could ask that $[\pi](T) = T^q + \pi T$).

Let $F_0 = \mathbf{Q}_p^{\text{unr}} \cap F$. Let $\tilde{\mathbf{E}}_F^+$ denote the ring of integers of $\tilde{\mathbf{E}}_F$ and let $\tilde{\mathbf{A}}_F^+ = \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} W(\tilde{\mathbf{E}}_F^+)$ be the \mathcal{O}_F -Witt vectors over $\tilde{\mathbf{E}}_F^+$.

Proposition 1.1. — If $u \in \tilde{\mathbf{E}}_F^+$ is such that $\gamma(u) = [\chi_\pi(\gamma)](u)$ for all $\gamma \in \Gamma_F$, then u has a lift $\hat{u} \in \tilde{\mathbf{A}}_F^+$ such that $\gamma(\hat{u}) = [\chi_\pi(\gamma)] \circ \hat{u}$ for all $\gamma \in \Gamma_F$.

Proof. — By lemma 9.3 of [Col02], there is a unique lift $\hat{u} \in \tilde{\mathbf{A}}_F^+$ of u such that $\varphi_q(\hat{u}) = [\pi](\hat{u})$ (in *ibid.*, this element is denoted by $\{u\}$). If $\gamma \in \Gamma_F$, then both $\gamma(\hat{u})$ and $[\chi_\pi(\gamma)](\hat{u})$ are lifts of u that are compatible with Frobenius as above. By unicity, they are equal. \square

Let $\log_{\text{LT}}(T)$ and $\exp_{\text{LT}}(T)$ be the logarithm and exponential series for LT. Write $\exp_{\text{LT}}(T) = \sum_{n \geq 1} e_n T^n$ and $\exp_{\text{LT}}(T)^j = \sum_{n \geq j} e_{j,n} T^n$ for $j \geq 1$.

Lemma 1.2. — *We have $\text{val}_\pi(e_{j,n}) \geq -n/(q-1)$ for all $j, n \geq 1$.*

Proof. — Fix $\varpi \in \overline{\mathbf{Q}}_p$ such that $\text{val}_\pi(\varpi) = 1/(q-1)$ and let $K = F(\varpi)$. Recall that $\log_{\text{LT}}(T) = \lim_{n \rightarrow +\infty} [\pi^n](T)/\pi^n$. If $z \in \mathbf{C}_p$ and $\text{val}_\pi(z) \geq 1/(q-1)$, then $\text{val}_\pi([\pi](z)) \geq \text{val}_\pi(z) + 1$. This implies that $1/\varpi \cdot \log_{\text{LT}}(\varpi T) \in T + T^2 \mathcal{O}_K[[T]]$. Its composition inverse $1/\varpi \cdot \exp_{\text{LT}}(\varpi T)$ therefore also belongs to $T + T^2 \mathcal{O}_K[[T]]$. This implies the claim for $j = 1$. The claim for $j \geq 1$ follows easily. \square

We use a number of rings of p -adic periods in the Lubin-Tate setting, whose construction and properties were recalled in §3 of [Ber16]. Proposition 1.1 gives us an element $\hat{Y} \in \tilde{\mathbf{A}}_F^+$ (denoted by u in *ibid.*). Let $\tilde{\mathbf{B}}_F^+ = \tilde{\mathbf{A}}_F^+[1/\pi]$. Given an interval $I = [r; s] \subset [0; +\infty[$, a valuation $V(\cdot, I)$ on $\tilde{\mathbf{B}}_F^+[1/\hat{Y}]$ is constructed in *ibid.*, as well as various completions of that ring. We use $\tilde{\mathbf{B}}_F^I$, the completion of $\tilde{\mathbf{B}}_F^+[1/\hat{Y}]$ for $V(\cdot, I)$ and $\tilde{\mathbf{B}}_{\text{rig}, F}^{\dagger, r} = \varprojlim_{s \geq r} \tilde{\mathbf{B}}_F^{[r; s]}$. Inside $\tilde{\mathbf{B}}_{\text{rig}, F}^{\dagger, r}$, there is the ring $\mathbf{B}_{\text{rig}, F}^{\dagger, r}$ of power series $f(\hat{Y})$ with coefficients in F , where $f(T)$ converges on a certain annulus depending on r .

Lemma 1.3. — *If $s \geq 0$, then $\mathbf{B}_{\text{rig}, F}^{\dagger, s} \cap \tilde{\mathbf{A}}_F^+ = \mathbf{A}_F^+$.*

Proof. — Take $f(\hat{Y}) \in \mathbf{B}_{\text{rig}, F}^{\dagger, s}$, $t \geq s$ and let $I = [s; t]$. We have $V(f, I) \geq 0$, so that f is bounded by 1 on the corresponding annulus. This is true for all t , so that $f \in \mathbf{B}_F^{\dagger, s}$. We now have $f \in \mathbf{B}_F^{\dagger, s} \cap \tilde{\mathbf{A}}_F^+ = \mathbf{A}_F^+$. \square

Let W be a Banach space with a continuous action of Γ_F . The notion of locally analytic vector was introduced in [ST03]. Recall (see for instance §2 of [Ber16]; the definition given there is easily seen to be equivalent to the following one) that an element $w \in W$ is locally F -analytic if there exists a sequence $\{w_k\}_{k \geq 0}$ of W such that $w_k \rightarrow 0$, and an integer $n \geq 1$ such that for all $\gamma \in \Gamma_F$ such that $\chi_\pi(\gamma) = 1 + p^n c(\gamma)$ with $c(\gamma) \in \mathcal{O}_F$, we have $\gamma(w) = \sum_{k \geq 0} c(\gamma)^k w_k$. If $W = \varprojlim_i W_i$ is a Fréchet representation of Γ_F , we say that $w \in W$ is pro- F -analytic if its image in W_i is locally F -analytic for all i .

Proposition 1.4. — *If $r \geq 0$ and $x \in \tilde{\mathbf{A}}_F^+$ is such that $\text{val}_Y(\bar{x}) > 0$ and $\gamma(x) = [\chi_\pi(\gamma)](x)$ for all $\gamma \in \Gamma_F$, then x is a pro- F -analytic element of $\tilde{\mathbf{B}}_{\text{rig}, F}^{\dagger, r}$.*

Proof. — We prove that for all $s \geq r$, x is a locally F -analytic vector of $\tilde{\mathbf{B}}_F^{[r;s]}$. The proposition then follows, since $\tilde{\mathbf{B}}_{\text{rig},F}^{\dagger,r} = \varprojlim_{s \geq r} \tilde{\mathbf{B}}_F^{[r;s]}$ as Fréchet spaces.

Let $S(X, Y) = \sum_{i,j} s_{i,j} X^i Y^j \in \mathcal{O}_F[[X, Y]]$ be the power series that gives the addition in LT. We have $\log_{\text{LT}}(x) \in \tilde{\mathbf{B}}_F^{[r;s]}$. Take $n \geq 1$ such that $V(p^{n-1} \log_{\text{LT}}(x), [r; s]) > 0$. We have $[a](T) = \exp_{\text{LT}}(a \log_{\text{LT}}(T))$, so that $[1 + p^n c](T) = S(T, \exp_{\text{LT}}(p^n c \log_{\text{LT}}(T)))$. If $\chi_\pi(\gamma) = 1 + p^n c(\gamma)$, then

$$\begin{aligned} \gamma(x) &= \sum_{k \geq 0} c(\gamma)^k \sum_{j \leq k} p^{nk} e_{j,k} \log_{\text{LT}}(x)^k \sum_{i \geq 0} s_{i,j} x^i \\ &= \sum_{k \geq 0} c(\gamma)^k \sum_{j \leq k} p^k e_{j,k} \cdot (p^{n-1} \log_{\text{LT}}(x))^k \cdot \sum_{i \geq 0} s_{i,j} x^i. \end{aligned}$$

We have $p^k e_{j,k} \in \mathcal{O}_F$ by lemma 1.2, $V(p^{n-1} \log_{\text{LT}}(x), [r; s]) > 0$ by hypothesis, $s_{i,j} \in \mathcal{O}_F$ and $V(x, [r; s]) > 0$. This implies the claim. \square

Proposition 1.5. — *If $r > 0$ and $x \in \tilde{\mathbf{A}}_F^+$ is a pro- F -analytic element of $\tilde{\mathbf{B}}_{\text{rig},F}^{\dagger,r}$, then there exists $n \geq 0$ such that $x \in \varphi_q^{-n}(\mathbf{A}_F^+)$.*

Proof. — By item (3) of theorem 4.4 of [Ber16] (applied with $K = F$), there exists $n \geq 0$ and $s > 0$ such that $x \in \varphi_q^{-n}(\mathbf{B}_{\text{rig},F}^{\dagger,s})$. The proposition now follows from lemma 1.3 applied to $\varphi_q^n(x)$. \square

2. Composition of power series

Recall that a power series $f(Y) \in k[[Y]]$ is separable if $f'(Y) \neq 0$. If $f(Y) \in Y \cdot k[[Y]]$, we say that f is invertible if $f'(0) \in k^\times$, which is equivalent to f being invertible for composition (denoted by \circ). We say that $w(Y) \in Y \cdot k[[Y]]$ is nontorsion if $w^{\circ n}(Y) \neq Y$ for all $n \geq 1$. If $w(Y) = \sum_{i \geq 0} w_i Y^i \in k[[Y]]$ and $m \in \mathbf{Z}$, let $w^{(m)}(Y) = \sum_{i \geq 0} w_i^{p^m} Y^i$. Note that $(w \circ v)^{(m)} = w^{(m)} \circ v^{(m)}$.

Proposition 2.1. — *Let $w(Y) \in Y + Y^2 \cdot k[[Y]]$ be an invertible nontorsion series, and let $f(Y) \in Y \cdot k[[Y]]$ be a separable power series. If $w^{(m)} \circ f = f \circ w$, then f is invertible.*

Proof. — This is a slight generalization of lemma 6.2 of [Lub94]. Write

$$\begin{aligned} f(Y) &= f_n Y^n + \mathcal{O}(Y^{n+1}) \\ f'(Y) &= g_k Y^k + \mathcal{O}(Y^{k+1}) \\ w(Y) &= Y + w_r Y^r + \mathcal{O}(Y^{r+1}), \end{aligned}$$

with $f_n, g_k, w_r \neq 0$. Since w is nontorsion, we can replace w by $w^{\circ p^\ell}$ for $\ell \gg 0$ and assume that $r \geq k + 1$. We have

$$\begin{aligned} w^{(m)} \circ f &= f(Y) + w_r^{(m)} f(Y)^r + \mathcal{O}(Y^{n(r+1)}) \\ &= f(Y) + w_r^{(m)} f_n^r Y^{nr} + \mathcal{O}(Y^{nr+1}). \end{aligned}$$

If $k = 0$, then $n = 1$ and we are done, so assume that $k \geq 1$. We have

$$\begin{aligned} f \circ w &= f(Y + w_r Y^r + \mathcal{O}(Y^{r+1})) \\ &= f(Y) + w_r Y^r f'(Y) + \mathcal{O}(Y^{2r}) \\ &= f(Y) + w_r g_k Y^{r+k} + \mathcal{O}(Y^{r+k+1}). \end{aligned}$$

This implies that $nr = r + k$, hence $(n - 1)r = k$, which is impossible if $r > k$ unless $n = 1$. Hence $n = 1$ and f is invertible. \square

We now prove theorem A. Take $u \in \tilde{\mathbf{E}}_F$ such that $\text{val}_Y(u) = 1$ and $u \circ [g] = [g] \circ u$ for all $g \in \mathcal{O}_F^\times$. By proposition 1.1, u has a lift $\hat{u} \in \tilde{\mathbf{A}}_F^+$ such that $\gamma(\hat{u}) = [\chi_\pi(\gamma)] \circ \hat{u}$ for all $\gamma \in \Gamma_F$. By proposition 1.4, \hat{u} is a pro- F -analytic element of $\tilde{\mathbf{B}}_{\text{rig}, F}^{\dagger, r}$. By proposition 1.5, there exists $n \geq 0$ such that $\hat{u} \in \varphi_q^{-n}(\mathbf{A}_F^+)$. This implies that $u \in \varphi_q^{-n}(\mathbf{E}_F^+)$. Hence there is an $m \in \mathbf{Z}$ such that $f(Y) = u(Y)^{p^m}$ belongs to $Y \cdot k[[Y]]$ and is separable. Note that $\text{val}_Y(f) = p^m$. Take $g \in 1 + \pi \mathcal{O}_F$ such that g is nontorsion, and let $w(Y) = [g](Y)$ so that $u \circ w = w \circ u$. We have $f \circ w = w^{(m)} \circ f$. By proposition 2.1, f is invertible. This implies that $\text{val}_Y(f) = 1$, so that $m = 0$ and u itself is invertible. Since $u \circ [g] = [g] \circ u$ for all $g \in \mathcal{O}_F^\times$, theorem 6 of [LS07] implies that $u \in \text{Aut}(\text{LT})$. Hence there exists $a \in \mathcal{O}_F^\times$ such that $u(Y) = [a](Y)$.

3. Tate traces in the Lubin-Tate setting

If $F = \mathbf{Q}_p$ and $\pi = p$ (namely in the cyclotomic situation) the fact that, in the proof of theorem A, there exists $n \geq 0$ such that $u \in \varphi_q^{-n}(\mathbf{E}_F^+)$ follows from the main result of [BR22]. We now explain why the methods of ibid don't extend to the Lubin-Tate case. More precisely, we prove that there is no Γ_F -equivariant k -linear projector $\tilde{\mathbf{E}}_F \rightarrow \mathbf{E}_F$ if $F \neq \mathbf{Q}_p$. Choose a coordinate T on LT such that $\log_{\text{LT}}(T) = \sum_{n \geq 0} T^{q^n} / \pi^n$, so that $\log'_{\text{LT}}(T) \equiv 1 \pmod{\pi}$. Let $\partial = 1 / \log'_{\text{LT}}(T) \cdot d/dT$ be the invariant derivative on LT.

Lemma 3.1. — *We have $d\gamma(Y)/dY \equiv \chi_\pi(\gamma)$ in \mathbf{E}_F for all $\gamma \in \Gamma_F$.*

Proof. — Since $\log'_{\text{LT}} \equiv 1 \pmod{\pi}$, we have $\partial = d/dY$ in \mathbf{E}_F . Applying $\partial \circ \gamma = \chi_\pi(\gamma) \gamma \circ \partial$ to Y , we get the claim. \square

Lemma 3.2. — *If $\gamma \in \Gamma_F$ is nontorsion, then $\mathbf{E}_F^{\gamma=1} = k$.*

Proposition 3.3. — *If $F \neq \mathbf{Q}_p$, there is no Γ_F -equivariant map $R : \mathbf{E}_F \rightarrow \mathbf{E}_F$ such that $R(\varphi_q(f)) = f$ for all $f \in \mathbf{E}_F$.*

Proof. — Suppose that such a map exists, and take $\gamma \in \Gamma_F$ nontorsion and such that $\chi_\pi(\gamma) \equiv 1 \pmod{\pi}$. We first show that if $f \in \mathbf{E}_F$ is such that $(1 - \gamma)f \in \varphi_q(\mathbf{E}_F)$, then $f \in \varphi_q(\mathbf{E}_F)$. Write $f = f_0 + \varphi_q(R(f))$ where $f_0 = f - \varphi_q(R(f))$, so that $R(f_0) = 0$ and $(1 - \gamma)f_0 = \varphi_q(g) \in \varphi_q(\mathbf{E}_F)$. Applying R , we get $0 = (1 - \gamma)R(f_0) = g$. Hence $g = 0$ so that $(1 - \gamma)f_0 = 0$. Since $\mathbf{E}_F^{\gamma=1} = k$ by lemma 3.2, this implies $f_0 \in k$, so that $f \in \varphi_q(\mathbf{E}_F)$.

However, lemma 3.1 and the fact that $\chi_\pi(\gamma) \equiv 1 \pmod{\pi}$ imply that $\gamma(Y) = Y + f_\gamma(Y^p)$ for some $f_\gamma \in \mathbf{E}_F$, so that $\gamma(Y^{q/p}) = Y^{q/p} + \varphi_q(g_\gamma)$. Hence $(1 - \gamma)(Y^{q/p}) \in \varphi_q(\mathbf{E}_F)$ even though $Y^{q/p}$ does not belong to $\varphi_q(\mathbf{E}_F)$. Therefore, no such map R can exist. \square

Corollary 3.4. — *If $F \neq \mathbf{Q}_p$, there is no Γ_F -equivariant k -linear projector $\varphi_q^{-1}(\mathbf{E}_F) \rightarrow \mathbf{E}_F$. A fortiori, there is no Γ_F -equivariant k -linear projector $\tilde{\mathbf{E}}_F \rightarrow \mathbf{E}_F$.*

Proof. — Given such a projector T , we could define R as in prop 3.3 by $R = T \circ \varphi_q^{-1}$. \square

Acknowledgements. I thank Juan Esteban Rodríguez Camargo for asking me the question that motivated both this paper and [BR22].

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February 7, 2022

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