An exponentially convergent discretization for space-time fractional parabolic equations using hp-FEM

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Dedicated to Professor Christoph Schwab on the occasion of his 60^{th} birthday.

We consider a space-time fractional parabolic problem. Combining a sinc-quadrature based method for discretizing the Riesz-Dunford integral with hp-FEM in space yields an exponentially convergent scheme for the initial boundary value problem with homogeneous right-hand side. For the inhomogeneous problem, an hp-quadrature scheme is implemented. We rigorously prove exponential convergence with focus on small times t, proving robustness with respect to startup singularities due to data incompatibilities. fractional diffusion, sinc-quadrature, Mittag-Leffler, Riesz-Dunford, hp-FEM

1 Introduction

The study of fractional partial differential equations has attracted a lot of interest in the mathematical community in recent years. Motivated by processes in physics or finance, it has become necessary to leave the realm of classical derivatives, and one encounters new difficulties, most notably one introduces non-local aspects to the problems under consideration [SZB+18].

In the context of numerical approximation, several different approaches have been proposed to handle fractional diffusion problems, although often they focused on the stationary elliptic case. For the stationary case, we mention those based on a Caffarelli-Silvestre

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extension [NOS15, BMN$^+$18, MPSV18], the integral fractional Laplacian [AB17, FMMS21] and the Riesz-Dunford (also known as Riesz-Taylor and Dunford-Taylor) functional calculus [BLP17b, BLP17a, BMS20]. See [BBN$^+$18] for a summary of discretization methods for the elliptic fractional diffusion problem. Another recent approach for discretizing both the fractional Laplace and heat equation is based on a reduced basis method [DS22, DS21]. Further methods that are based on discretizing first and computing fractional powers of the resulting stiffness matrix include [HHT08, GHK05, HLM$^+$18, HLM$^+$20]; see also the discussion in [Hof20].

For fractional ODE problems, the most common formulation is based on the Caputo derivative due to its natural behavior with respect to initial conditions. As for numerical approximation, it is common to use time-stepping methods [NOS16], but especially when already dealing with fractional operators in space, it is very natural to use a Riesz-Dunford based formulation and apply an appropriate sinc-quadrature [BLP17b]. Recently, a slight modification of the sinc-quadrature scheme based on a double exponential transformation has been proposed [Rie20]. This modified scheme can also be combined with the *hp*-FEM techniques from this article to obtain a very fast numerical scheme.

For the spatial discretization, finite element based-methods are popular. Most results focus on the *h*-version, designing schemes which provide algebraic convergence rates, under suitable compatibility conditions on the given data. Lately, *hp*-FEM based schemes have started to appear. First, only for the extended variable in a Caffarelli-Silvestre based discretization scheme [MPSV18], then for the full discretization of an elliptic fractional problem [BMN$^+$18, BMS20]. Unlike previous results, [BMN$^+$18] also removed the compatibility conditions on the initial data, relying instead on the fact that appropriately designed *hp*-FEM spaces can resolve the developing singularities at the boundary.

In the context of time-dependent problems, *hp*-FEM based approaches have been pioneered in [MR21], again focusing on an extension-based formulation for the fractional Laplacian and restricted to the classical first order derivative in time. It is proven for smooth data and geometries (but without compatibility assumption) that an *hp*-discretization can deliver exponential convergence towards the exact solution. The present work consists of transferring the methods from [MR21] to a functional calculus based discretization in the spirit of [BLP17b], generalizing the problem class to the case of (Caputo) fractional time derivatives in the process. We prove that for smooth geometries in 1D or 2D, one can design meshes and approximation spaces such that the proposed method features exponential convergence to the exact solution, even in the presence of startup singularities. Most of the paper is concerned with analyzing the convergence in the pointwise-in-time $\widetilde{H}^{\beta}(\Omega)$ Sobolev-norm, which is the natural setting for the model problem (2.1) and the considered numerical method. For small times t, these estimates suffer from a degeneracy at small times $t > 0$. In order to remedy this, we consider an appropriate space-time energy norm and prove that, given an additional abstract assumption on the initial condition, one can expect robust convergence in this weaker norm.

Compared to [BLP17a], we also improve the time dependence of the estimates for the sinc-quadrature error from $t^{-\gamma}$ to $t^{-\gamma/2}$. Similar behavior for the discretization errors was also observed in the $\gamma = 1$ case in [MR21] and is well established for discretizations of the heat-equation [Tho06].

We close with a short comment on notation. We write $a \lesssim b$ if there exists a constant $C > 0$, which is independent of the main quantities of interest (for example the number of quadrature points, the mesh size, polynomial degree employed or the time t) such that $a \leq Cb$. We write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$. The specific dependencies of the implied constants are stated in the context.

2 Model Problem and notation

We consider the numerical approximation of the following time-dependent problem. Working on a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$, we fix $\gamma, \beta \in (0, 1]$ and a final time $T > 0$. Given an initial condition $u_0 \in L^2(\Omega)$ and right-hand side $f \in L^\infty((0,T), L^2(\Omega))$ we seek $u : [0,T] \to \mathbb{R}$ satisfying

$$\partial_t^\gamma u + \mathcal{L}^\beta u = f \text{ in } \Omega \times [0,T], \qquad u|_{\partial\Omega} = 0, \qquad u(0) = u_0 \text{ in } \Omega. \tag{2.1}$$

The operator $\mathcal{L}u := -\operatorname{div}(A\nabla u) + cu$ is linear, elliptic and self adjoint. We assume that the given coefficients satisfy that $A \in L^\infty(\Omega, \mathbb{R}^{d\times d})$ is uniformly symmetric, positive definite and $c \in L^\infty(\Omega)$ satisfies $c \geq 0$. The fractional power $\mathcal{L}^\beta$ is given using the spectral representation

$$\mathcal{L}^\beta u := \sum_{j=0}^{\infty} \lambda_j^\beta (u, \varphi_j)\varphi_j, \tag{2.2}$$

where $(\lambda_j, \varphi_j)_{j=0}^\infty$ are the eigenvalues and eigenfunctions of the operator $\mathcal{L}$ with homogeneous Dirichlet boundary conditions; as is standard, the eigenfunctions are $L^2(\Omega)$-orthonormalized. The homogeneous Dirichlet boundary condition is to be understood in the sense that $u(t) \in \operatorname{dom}(\mathcal{L}^\beta)$.

For $\gamma \in (0,1)$, the fractional time derivative is taken in the sense of Caputo, i.e.,

$$\partial_t^\gamma u(t) := \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{1}{(t-r)^\gamma} \frac{\partial u(t)}{\partial r}\, dr,$$

whereas for $\gamma = 1$, $\partial_t^1 := \partial_t$ is the classical derivative.

As was shown in [BLP17b], the exact solution to (2.1) can be written in the following form using the Mittag-Leffler function $e_{\gamma,\mu}$ (see (2.6) for the precise definition):

$$u(t) = E(t)u_0 + \int_0^t W(\tau) f(t-\tau)\, d\tau := e_{\gamma,1}\big(-t^\gamma \mathcal{L}^\beta\big)u_0 + \int_0^t \tau^{\gamma-1} e_{\gamma,\gamma}\big(-\tau^\gamma \mathcal{L}^\beta\big) f(t-\tau)\, d\tau,$$

where we used the following functional calculus based on the spectral decomposition:

$$e_{\gamma,\mu}\big(-t^\gamma \mathcal{L}^\beta\big)v := \sum_{j=0}^{\infty} e_{\gamma,\mu}(-t^\gamma \lambda_j^\beta)(v, \varphi_j)_{L^2(\Omega)}\, \varphi_j \qquad \forall v \in L^2(\Omega).$$

An alternative representation, which will prove more amenable to numerical discretization is based on the Riesz-Dunford calculus:

$$e_{\gamma,\mu}\big(-t^{\gamma}\mathcal{L}^{\beta}\big)v = \frac{1}{2\pi i}\int_{\mathcal{C}} e_{\gamma,\mu}(-t^{\gamma}z^{\beta})\,\big(z-\mathcal{L}\big)^{-1}v\,dz \qquad \forall v \in L^2(\Omega), \tag{2.3}$$

where the contour $\mathcal{C}$ is taken to be parameterized by

$$z(y) := b(\cosh(y) + i\sinh(y)) \quad \text{ for } y \in \mathbb{R}. \tag{2.4}$$

The parameter $b > 0$ is chosen sufficiently small so that $z(y)$, $y \in \mathbb{R}$, is in the sector $\mathscr{S}$ given in Definition 2.1.

The natural spaces for formulating our results are given by two scales of interpolation spaces, $H^{\theta}(\Omega)$ and $\widetilde{H}^{\theta}(\Omega)$. To that end, we define for $\theta \in [0,1]$:

$$\mathbb{H}^{\theta}(\Omega) := \Big\{u \in L^2(\Omega) : \sum_{j=0}^{\infty} \lambda_j^{\theta} |(u,\varphi_j)_{L^2(\Omega)}|^2 < \infty\Big\}.$$

Another way of introducing spaces between L^2 and H_0^1 is given by the real interpolation method. That is, given two Banach spaces $X_1 \subseteq X_0$ with continuous embedding, we define for $\theta \in (0,1)$:

$$\begin{aligned}\|u\|^2_{[X_0,X_1]_{\theta,2}} &:= \int_{t=0}^{\infty} t^{-2\theta}\left(\inf_{v\in X_1} \|u-v\|_0 + t\|v\|_1\right)^2 \frac{dt}{t},\\ [X_0,X_1]_{\theta,2} &:= \Big\{u \in X_0 : \|u\|_{[X_0,X_1]_{\theta,2}} < \infty\Big\}.\end{aligned}$$

For $\theta = 0$ and $\theta = 1$ we take the convention that $[X_0,X_1]_{\theta,2} = X_{\theta}$. Using this notation, we define the fractional Sobolev spaces:

$$H^{\theta}(\Omega) := \big[L^2(\Omega), H^1(\Omega)\big]_{\theta,2}, \qquad \widetilde{H}^{\theta}(\Omega) := \big[L^2(\Omega), H_0^1(\Omega)\big]_{\theta,2}.$$

It is well known, that for $\theta \in [0,1]$ the spaces $\widetilde{H}^{\theta}(\Omega)$ and $\mathbb{H}^{\theta}(\Omega)$ coincide with equivalent norms; see [Tar07]. We will therefore use whichever definition is more convenient. Most notably we have

$$\|u\|^2_{\widetilde{H}^{\theta}(\Omega)} \sim \sum_{j=0}^{\infty} \lambda_j^{\theta} |(u,\varphi_j)_{L^2(\Omega)}|^2.$$

We will need the following multiplicative interpolation estimate (see [Tri99, Sect. 1.3.3]):

$$\|u\|_{[X_0,X_1]_{\theta,2}} \leq C_{\theta}\,\|u\|_{X_0}^{1-\theta}\,\|u\|_{X_1}^{\theta} \qquad \forall u \in X_1, \theta \in [0,1]. \tag{2.5}$$

Throughout, we take inner products to be antilinear in the second argument.

2.1 The Mittag-Leffler function

Since it plays a major role in the numerical method, we briefly introduce the Mittag-Leffler function and summarize its most important properties. See for example [KST06, Sect. 1.8] for a more detailed treatment. Given parameters $\gamma > 0$, $\mu \in \mathbb{R}$, the Mittag-Leffler function is analytic on $\mathbb{C}$ and given by the power series

$$e_{\gamma,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\gamma + \mu)}. \tag{2.6}$$

For $0 < \gamma < 1, \mu \in \mathbb{R}$ and $\frac{\gamma\pi}{2} < \zeta < \gamma\pi$, there exists a constant C only depending on γ, μ, ζ such that

$$|e_{\gamma,\mu}(z)| \leq \frac{C}{1 + |z|} \qquad \text{for } \zeta \leq |\operatorname{Arg} z| \leq \pi. \tag{2.7}$$

For $\gamma = \mu = 1$, the Mittag-Leffler function $e_{1,1}$ is the usual exponential function. Estimate (2.7) also holds in this case for $\pi/2 < \zeta < \pi$.

The derivative of the Mittag-Leffler function can be expressed as:

$$\frac{d}{dt} e_{\gamma,1}(-t^{\gamma}\lambda^{\beta}) = -\lambda^{\beta}\, t^{\gamma-1} e_{\gamma,\gamma}(-t^{\gamma}\lambda^{\beta}). \tag{2.8}$$

2.2 Assumptions on the discretization in space

When considering the Riesz-Dunford representation of u, the contour lies in the set of values for which $\mathcal{L} - z$ is invertible. Therefore we consider the set of complex numbers up to a cone which contains an interval $[a, \infty) \subseteq \mathbb{R}$ that in turn contains the eigenvalues of $\mathcal{L}$.

Definition 2.1. *Let C_P denote the Poincaré constant of Ω, i.e., the smallest constant such that $\|u\|_{L^2(\Omega)} \leq C_P \|\nabla u\|_{L^2(\Omega)}$ for all $u \in H_0^1(\Omega)$. With $\lambda_{\min}(A)$ the smallest eigenvalue of A and fixed $0 < \varepsilon_0 < z_0 \leq \min\left(\frac{\lambda_{\min}(A)}{2C_P}, 1\right)^2$, we define*

$$\mathscr{S} := \mathbb{C} \setminus \left[\left\{z_0 + z : |\operatorname{Arg}(z)| \leq \frac{\pi}{8}, \operatorname{Re}(z) \geq 0\right\} \cup B_{\varepsilon_0}(0)\right].$$

Remark 2.2. *The set $\mathscr{S}$ is chosen in such a way that it contains the contour $\mathcal{C}$ used in the functional calculus, given by $z(y) := b\big(\cosh(y) + i\sinh(y)\big)$. See Figure 2.1.* ∎

An important role will be played by the resolvent operator $R(z) := (z - \mathcal{L})^{-1}$ and its discrete counterpart, where we replace it with a Galerkin solver $R_h(z)$. Let $\mathbb{V}_h \subseteq H_0^1(\Omega)$ be a closed subspace. Recalling that $\mathcal{L}u = -\operatorname{div}(A\nabla u) + cu$, we define $R_h(z)f := u_h$ as the solution $u_h \in \mathbb{V}_h$ satisfying

$$\big((z - c)u_h, v_h\big)_{L^2(\Omega)} - \big(A\nabla u_h, \nabla v_h\big)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)} \qquad \forall v_h \in \mathbb{V}_h. \tag{2.9}$$

For z in the set $\mathscr{S}$ the following stability estimate holds for the resolvent operator:

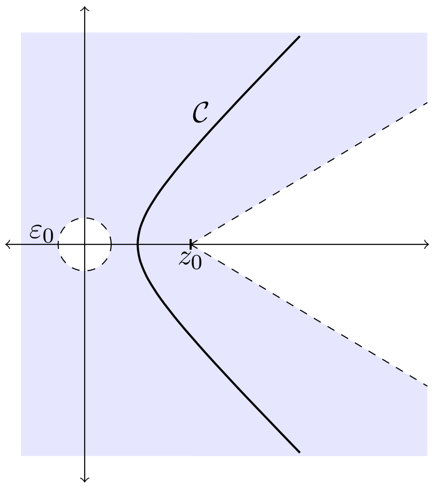


Figure 2.1: Geometric configuration of Definition 2.1

Proposition 2.3. *Let $z \in \mathscr{S}$ and $u_0 \in L^2(\Omega)$. Then, for $\alpha \in [0,1]$:*

$$\|R(z)u_0\|_{\widetilde{H}^\alpha(\Omega)} \lesssim |z|^{-1+\frac{\alpha}{2}} \|u_0\|_{L^2(\Omega)} \quad \text{and} \quad \|R_h(z)u_{h,0}\|_{\widetilde{H}^\alpha(\Omega)} \lesssim |z|^{-1+\frac{\alpha}{2}} \|u_{h,0}\|_{L^2(\Omega)}. \tag{2.10}$$

Proof. This is basically [MR21, Lem. A.2] with the substitution $\zeta = -z + c$. We include the proof for completeness.

Writing $u := R_h(z)u_0$, and testing in (2.9) with $v_h := \overline{\beta}u$ for $\beta \in \mathbb{C}$ with $|\beta| = 1$ to be chosen later, we can show

$$\begin{aligned}\operatorname{Re}(\beta)\big(\|A^{1/2}\nabla u\|^2_{L^2(\Omega)} + \|c^{1/2}u\|^2_{L^2(\Omega)}\big) - \operatorname{Re}(\beta z)\, \|u\|^2_{L^2(\Omega)} \\ \lesssim \big|(u_0, u)_{L^2(\Omega)}\big| \leq \big(\,|z|^{-1/2}\, \|u_0\|_{L^2(\Omega)}\big)\big(\,|z|^{1/2}\, \|u\|_{L^2(\Omega)}\big).\end{aligned} \tag{2.11}$$

For small $|z| < 2z_0$, we can use $\beta = 1$ and bound using the Poincaré estimate:

$$|z|\, \|u\|^2_{L^2(\Omega)} \leq 2z_0 C_P^2 \|\nabla u\|^2_{L^2(\Omega)} < \frac{1}{2}\|A\nabla u\|^2_{L^2(\Omega)},$$

to easily get the a priori estimate for $\alpha \in \{0,1\}$.

For larger $|z| > 2z_0$, we may assume that $|\operatorname{Arg}(z)| \geq \delta > 0$ for some $\delta > 0$. (Graphically speaking, we exclude a slightly thinner cone starting at 0 instead of the cone starting at z_0.) It is then sufficient, if we can pick β such that $\operatorname{Re}(\beta) > 0$ and $\operatorname{Arg}(-z\beta) \in (-\pi/2, \pi/2)$. Just as in [MR21, Lem. A.2] one can check that if $\operatorname{Im}(z) \leq 0$, $\beta := e^{-i\frac{\pi-\delta}{2}}$ satisfies the necessary conditions that $\operatorname{Re}(\beta) > 0$ and $-\operatorname{Re}(\beta z) > 0$. If $\operatorname{Im}(z) \geq 0$, then $\beta := e^{i\frac{\pi-\delta}{2}}$ does. This shows as before $\|R_h(z)u_0\|_{H^1(\Omega)} \lesssim |z|^{-1/2}\|u_0\|_{L^2(\Omega)}$ and also $\|R_h(z)u_0\|_{L^2(\Omega)} \lesssim |z|^{-1}\|u_0\|_{L^2(\Omega)}$.

By interpolation, we deduce that for $\alpha \in [0,1]$:

$$\|R_h(z)u_0\|_{\widetilde{H}^\alpha(\Omega)} \lesssim |z|^{-1+\alpha/2} \|u_0\|_{L^2(\Omega)} . \tag{2.12}$$

The discrete estimate follows verbatim. □

Definition 2.4. *A function $f : [0,T] \to L^\infty(\Omega)$ is said to be* uniformly analytic *if:*

(i) For all $t \in [0,T]$, $f(t)$ is analytic in a fixed neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$;

(ii) there exist constants $C_f, \gamma_f > 0$, the analyticity constants of f, *such that for all $t \in [0,T]$ and $p \in \mathbb{N}_0$,*

$$\|\nabla^p f(t)\|_{L^\infty(\widetilde{\Omega})} \le C_f \gamma^p p! ;$$

(iii) there exists an open set $\mathcal{O} \subseteq \mathbb{C}$, containing a positive sector $\{z \in \mathbb{C} : z \neq 0, |\mathrm{Arg}\, z| < \delta\}$ for some constant $\delta > 0$, and there exists a constant $C_f > 0$ such that, for each $v \in L^2(\Omega)$, the function $t \mapsto (f(t), v)_{L^2(\Omega)}$ has an analytic extension to $\mathcal{O}$ and

$$\sup_{s \in \mathcal{O}} \frac{|(f(s), v)_{L^2(\Omega)}|}{\|v\|_{L^2(\Omega)}} \le C_f < \infty.$$

The main assumption on the discretization space $\mathbb{V}_h$ is taken such that it approximates the solutions to singularly perturbed problems exponentially well. The same assumption has already been used in [MR21].

Assumption 2.5. *A function space $\mathbb{V}_h$ is said to resolve down to the scale $\varepsilon > 0$ if for all $z \in \mathscr{S}$ with $|z|^{-1/2} \ge \varepsilon$ and for all $f \in L^2(\Omega)$ that are analytic on a fixed neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$, the solutions to the elliptic problem*

$$-z^{-1}\mathcal{L}u + u = f$$

can be approximated exponentially well from it. That is, there exist constants $C(f)$, ω and $\mu > 0$ such that

$$\inf_{v_h \in \mathbb{V}_h} \left[|z|^{-1} \|\nabla u - \nabla v\|^2_{L^2(\Omega)} + \|u - v\|^2_{L^2(\Omega)} \right] \lesssim C(f) e^{-\omega \mathcal{N}_\Omega^\mu},$$

where $\mathcal{N}_\Omega := \dim(\mathbb{V}_h)$. The constant $C(f)$ may depend only on $\widetilde{\Omega}$, the analyticity constants of f, on A, c, Ω, z_0, and ε_0, while the constants ω and μ may depend only on A, c, $\widetilde{\Omega}$, Ω, z_0 and ε_0 Most notably the constants are independent of z, ε, and $\mathcal{N}_\Omega$.

We state and prove all our results under the general assumption that $\mathbb{V}_h$ resolves specific scales. In Section 6, we will give one construction of such a space using *hp* finite elements; see also [BMN+18] and [BMS20] for similar constructions, focused on real valued parameters z.

3 The pure initial value problem

We first focus on discretizing the homogeneous problem, i.e., we only consider the case $f = 0$. In this case, the exact solution can be written as

$$u(t) := \frac{1}{2\pi i} \int_{\mathcal{C}} e_{\gamma,1}(-t^\gamma z^\beta) \left(z - \mathcal{L}\right)^{-1} u_0 \, dz. \tag{3.1}$$

We discretize this function in two steps. First, we replace the contour integral with a sinc-quadrature rule, analogous to what was done in [BLP17b]. For a fixed number of quadrature points $\mathcal{N}_{\mathrm{q}} \in \mathbb{N}$ and grid size $k > 0$, we write, setting $y_n := nk$:

$$u^q(t) := \frac{k}{2\pi i} \sum_{n=-\mathcal{N}_{\mathrm{q}}}^{\mathcal{N}_{\mathrm{q}}} e_{\gamma,1}\big(-t^\gamma z(y_n)^\beta\big)\, z'(y_n)\big(z(y_n) - \mathcal{L}\big)^{-1} u_0. \tag{3.2}$$

As a second step, we replace the resolvent operator $R(z) = (z - \mathcal{L})^{-1}$ by the discrete counterpart $R_h(z)$ as defined in (2.9). We obtain the fully discrete approximation

$$u^{q,h}(t) := \frac{k}{2\pi i} \sum_{n=-\mathcal{N}_{\mathrm{q}}}^{\mathcal{N}_{\mathrm{q}}} e_{\gamma,1}\big(-t^\gamma z(y_n)^\beta\big)\, z'(y_n) R_h\big(z(y_n)\big) u_0. \tag{3.3}$$

Lemma 3.1. *Assume that u_0 is analytic on a neighborhood of the closure of Ω. Let $\mathcal{N}_q \in \mathbb{N}$, $k > 0$ be given. Let $\varepsilon_0 < b < z_0$, with ε_0 and z_0 as defined in Definition 2.1 and b as in* (2.4)*. Then, if $\mathbb{V}_h$ resolves the scales down to $\varepsilon = b^{-1/2} e^{-\frac{\mathcal{N}_q k}{2}}$, the error due to the spatial discretization can be bounded by*

$$\left\| u^q(t) - u^{q,h}(t) \right\|_{\widetilde{H}^\beta(\Omega)} \leq C t^{-\gamma/2} \mathcal{N}_q k \, e^{-\omega \mathcal{N}_\Omega^\mu}. \tag{3.4}$$

The constants C, ω, μ depend on the constants from Assumption 2.5 and on the initial condition u_0.

Proof. We note the following, easy to verify estimates, with the generic constants only depending on b (as used in the definition of the contour $\mathcal{C}$):

$$|z(y_j)| \leq b e^{|j|k} \quad \text{and} \quad |z(y_j)| \sim \left|z'(y_j)\right| \sim e^{|j|k}. \tag{3.5}$$

For fixed $z \in \mathscr{S}$ with $|z|^{1/2} \leq \sqrt{b}\, e^{\mathcal{N}_{\mathrm{q}} k/2}$, Assumption 2.5 gives using (2.5):

$$\|R(z)u_0 - R_h(z)u_0\|_{\widetilde{H}^\beta(\Omega)} \lesssim |z|^{\beta/2-1} e^{-\omega \mathcal{N}_\Omega^\mu}.$$

Thus, using (2.7), as long as $\mathbb{V}_h$ resolves all the scales $|z(y_j)|^{-1/2}$, the error can be estimated by

$$\begin{aligned}
\left\| u^q(t) - u^{q,h}(t) \right\|_{\widetilde{H}^\beta(\Omega)} &\lesssim k \sum_{n=-\mathcal{N}_{\mathrm{q}}}^{\mathcal{N}_{\mathrm{q}}} \left|e_{\gamma,1}(-t^\gamma z(y_n)^\beta)\right| \left|z'(y_n)\right| |z(y_n)|^{-1+\beta/2} e^{-\omega \mathcal{N}_\Omega^\mu} \\
&\lesssim k \sum_{n=-\mathcal{N}_{\mathrm{q}}}^{\mathcal{N}_{\mathrm{q}}} \frac{1}{1 + t^\gamma |z(y_n)|^\beta} |z(y_n)|^{\beta/2} e^{-\omega \mathcal{N}_\Omega^\mu}.
\end{aligned}$$

The following simple calculation then concludes the proof:

$$\begin{aligned}\frac{1}{1+t^{\gamma}\left|z(y_n)\right|^{\beta}}\left|z(y_n)\right|^{\beta/2} &= \left(\frac{1}{1+t^{\gamma}\left|z(y_n)\right|^{\beta}}\right)^{1/2}\left(\frac{1}{1+t^{\gamma}\left|z(y_n)\right|^{\beta}}\left|z(y_n)\right|^{\beta}\right)^{1/2} \\ &\lesssim \left(\frac{1}{\left|z(y_j)\right|^{-\beta}+t^{\gamma}}\right)^{1/2} \lesssim t^{-\gamma/2}.\end{aligned} \tag{3.6}$$

□

Next, we consider the discretization error due to replacing the contour integral by the sinc-quadrature formula. This can be done along the same lines as in [BLP17b]. We take their definition for the class of functions which can be approximated well by the sinc quadrature.

Definition 3.2. *Given $H > 0$, we define $S(B_H)$ as the set of functions f, defined on $\mathbb{R}$, satisfying the following assumptions:*

(i) f can be extended to an analytic function on the infinite strip

$$B_H := \{z \in \mathbb{C} : |\mathrm{Im}(z)| < H\} \tag{3.7}$$

that is also continuous on $\overline{B_H}$.

(ii) There exists a constant $C > 0$ independent of $y \in \mathbb{R}$ such that

$$\int_{-H}^{H} |f(y + iw)|\, dw \leq C \qquad \forall y \in \mathbb{R}. \tag{3.8}$$

(iii) We have

$$N(B_H) := \int_{-\infty}^{\infty} |f(y + iH)| + |f(y - iH)|\, dy < \infty. \tag{3.9}$$

The following error estimate is proved in [LB92], and is also the basis for the convergence result [BLP17b, Lem. 4.2].

Proposition 3.3 ([LB92, Thm. 2.20])**.** *If $f \in S(B_H)$ and $k > 0$, then*

$$\left|\int_{-\infty}^{\infty} f(x)\, dx - k \sum_{n=-\infty}^{\infty} f(k\,n)\right| = \frac{N(B_H)}{2\sinh(\pi d/k)} e^{-\pi d/k}. \tag{3.10}$$

The behavior of the exact solution, most notably the blowup of the energy norm for small times is determined by the regularity and boundary conditions of the initial condition u_0, formalized using the spaces $\widetilde{H}^{\theta}(\Omega)$. Even if the boundary conditions are not met, u_0 is slightly better than just $L^2(\Omega)$. This is the content of the following proposition.

Proposition 3.4. *For $\theta \in [0, 1/2)$, and $u_0 \in H^\theta(\Omega)$, we can bound*

$$\|u_0\|_{\widetilde{H}^\theta(\Omega)} \leq C \|u_0\|_{H^\theta(\Omega)} .$$

Proof. For $\theta < 1/2$, the two families of spaces H^θ and $\widetilde{H}^\theta$ coincide with equivalent norms, see [Tri06, Sect. 1.11.6] or [McL00, Thms. 3.33, B.9, 3.40]). □

Lemma 3.5. *For $\lambda \geq \lambda_1 > 0$, we define the function*

$$g_\lambda(y,t) := \frac{1}{2\pi i} e_{\gamma,1}(-t^\gamma z^\beta) z'(y)(z(y) - \lambda)^{-1}. \tag{3.11}$$

Then, for $H \in (0, \pi/4)$ and $\varepsilon \in (0, \beta/2)$, we have the estimate

$$|g_\lambda(y,t)| \lesssim t^{-\gamma/2} \lambda^{-\beta/2+\varepsilon} e^{-\varepsilon \operatorname{Re}(y)} \qquad \forall y \in B_H, \tag{3.12}$$

with implied constant depending on λ_1, β , γ, and H.

Additionally, it holds that $g_\lambda \in S(B_H)$ with

$$N(B_H) \leq C(\beta, H, b, \varepsilon, \gamma) t^{-\gamma/2} \lambda^{-\beta/2+\varepsilon}.$$

Proof. We begin by noting the following estimates for $y \in B_H$, with implied constants depending on λ_1, b, and H:

$$|z(y)| + |z'(y)| \lesssim e^{|\operatorname{Re}(y)|}, \tag{3.13a}$$

$$\operatorname{Re}(z(y)^\beta) \gtrsim e^{\beta|\operatorname{Re}(y)|}, \tag{3.13b}$$

$$|z'(y)(z(y) - \lambda)^{-1}| \lesssim 1. \tag{3.13c}$$

The first estimate follows from the definition, the others can be found in [BLP17b, Lem. B.1] and were proven in the course of the proof [BLP17a, Thm.4.1].

We assert the following estimate for $\varepsilon \in [0, \beta/2]$:

$$\lambda^\beta \left| z'(y) \left(z(y) - \lambda\right)^{-1} \right| \lesssim \lambda^{2\varepsilon} e^{(\beta - 2\varepsilon)|\operatorname{Re}(y)|}.$$

To see this, we compute using estimates (3.13a) and (3.13c):

$$\left| \lambda\, z'(y) \left(z(y) - \lambda\right)^{-1} \right| = \left| z'(y) \left(-1 + \frac{z(y)}{z(y) - \lambda} \right) \right| \lesssim |z'(y)| + |z(y)| \lesssim e^{|\operatorname{Re}(y)|}.$$

Interpolation with (3.13c) then gives the general estimate:

$$\begin{aligned} \lambda^\beta \left| z'(y) \left(z(y) - \lambda\right)^{-1} \right| &= \lambda^{2\varepsilon} \left(\left| z'(y) \left(z(y) - \lambda\right)^{-1} \right| \right)^{1-\beta+2\varepsilon} \left(\lambda \left| z'(y) \left(z(y) - \lambda\right)^{-1} \right| \right)^{\beta - 2\varepsilon} \\ &\lesssim \lambda^{2\varepsilon} e^{(\beta - 2\varepsilon)|\operatorname{Re}(y)|}. \end{aligned}$$

To show (3.12), we use (2.7) to get

$$\begin{aligned}
\lambda^{\beta/2}\left|g_\lambda(y,t)\right| &\lesssim \Big(\left|g_\lambda(y,t)\right|\Big)^{1/2}\Big(\lambda^{\beta}\left|g_\lambda(y,t)\right|\Big)^{1/2} \\
&\lesssim \Big(\frac{1}{1+t^\gamma e^{\beta|\mathrm{Re}(y)|}}\Big)^{1/2}\Big(\frac{1}{1+t^\gamma e^{\beta|\mathrm{Re}(y)|}}\lambda^{2\varepsilon}e^{(\beta-2\varepsilon)|\mathrm{Re}(y)|}\Big)^{1/2} \\
&\lesssim t^{-\gamma/2}\lambda^{\varepsilon}e^{-\varepsilon|\mathrm{Re}(y)|}.
\end{aligned}$$

From estimate (3.12), we easily deduce parts (ii) and (iii) of Definition 3.2. Part (i) was already shown in [BLP17b]. ☐

Lemma 3.6. *For $\mathcal{N}_q \in \mathbb{N}$ and $k \sim \mathcal{N}_q^{-1/2}$, the following estimate holds for $0 \le \varepsilon < \min\left(\frac{\beta}{2}, \frac{1}{4}\right)$:*

$$\|u(t)-u^q(t)\|_{\widetilde{H}^\beta(\Omega)} \lesssim t^{-\gamma/2}e^{-\omega\mathcal{N}_q^{1/2}}\,\|u_0\|_{H^{2\varepsilon}(\Omega)}$$

for some constant $\omega > 0$, depending on ε, γ, β.

Proof. Using the error function

$$\mathcal{E}(\lambda,t) := \int_{-\infty}^{\infty} g_\lambda(y,t)\,dy - k\sum_{n=-\infty}^{\infty} g_\lambda(n\,k,t)$$

and the spectral decomposition $u_0 = \sum_{j=0}^{\infty} u_{0,j}\varphi_j$, we can write the quadrature error as

$$u(t)-u^q(t) = \sum_{j=0}^{\infty}\Big(\mathcal{E}(\lambda_j,t) + k\sum_{|n|\ge\mathcal{N}_{\mathrm{q}}+1} g_{\lambda_j}(n\,k,t)\Big)\,u_{0,j}\,\varphi_j.$$

For the error in the $\widetilde{H}^\beta(\Omega)$-norm, this means:

$$\|u(t)-u^q(t)\|^2_{\widetilde{H}^\beta(\Omega)} \lesssim \sum_{j=0}^{\infty}\lambda_j^\beta\left|\mathcal{E}(\lambda_j,t)\right|^2\left|u_{0,j}\right|^2 + \Big(k\sum_{|n|\ge\mathcal{N}_{\mathrm{q}}+1}\lambda_j^{\beta/2}\left|g_{\lambda_j}(n\,k,t)\right|\Big)^2\left|u_{0,j}\right|^2.$$

The terms can be estimated by Proposition 3.3 and Lemma 3.5.

$$\begin{aligned}
\|u(t)-u^q(t)\|^2_{\widetilde{H}^\beta(\Omega)} &\lesssim t^{-\gamma}e^{-4\pi d/k}\,\|u_0\|^2_{\widetilde{H}^{2\varepsilon}(\Omega)} + t^{-\gamma}\,\|u_0\|^2_{\widetilde{H}^{2\varepsilon}(\Omega)}\Big(k\sum_{|n|\ge\mathcal{N}_{\mathrm{q}}+1} e^{-\varepsilon nk}\Big)^2 \\
&\lesssim t^{-\gamma}\,\|u_0\|^2_{\widetilde{H}^{2\varepsilon}(\Omega)}\Big(e^{-4\pi d/k} + e^{-2\varepsilon\mathcal{N}_{\mathrm{q}}k}\Big).
\end{aligned}$$

Picking $k \sim \mathcal{N}_{\mathrm{q}}^{-1/2}$ and using Proposition 3.4 gives the stated result. ☐

4 The inhomogeneous problem

In this section, we focus on the inhomogeneous problem with homogeneous initial condition, i.e., (2.1) with $u_0 = 0$ and general f. The representation formula in this case reads

$$u_i(t) = \int_0^t \tau^{\gamma-1} e_{\gamma,\gamma}(-\tau^\gamma \mathcal{L}^\beta) f(t-\tau)\, d\tau, \tag{4.1}$$

or, using the Riesz-Dunford calculus,

$$u_i(t) = \frac{1}{2\pi i} \int_0^t \tau^{\gamma-1} \left(\int_{\mathcal{C}} e_{\gamma,\gamma}(-\tau^\gamma z^\beta) R(z) f(t-\tau)\, dz \right) d\tau. \tag{4.2}$$

Our approximation scheme will be based on a sinc-type quadrature for the contour integral and an *hp*-type quadrature for the convolution in time.

4.1 *hp*-quadrature

In this section, we briefly summarize the theory and notation for using *hp*-methods to approximate integrals. Given an interval $I = (a, b)$ and function $g : I \to \mathbb{C}$, we are interested in approximating

$$\int_I g(\tau)\, d\tau,$$

where g may have an algebraic singularity at the left endpoint $\tau = a$.

For a given degree $p \in \mathbb{N}_0$, we denote the Gauss quadrature points and weights on $(-1, 1)$ by $(x_j^p, w_j^p) \in (-1, 1) \times \mathbb{R}_+$. See [DR84, Sect. 2.7] for details. For $I = (-1, 1)$, the Gauss-quadrature approximation is then given by

$$Q_I^p g := \sum_{j=0}^{p} w_j^p g(x_j^p).$$

For general $I = (a, b)$, the approximation is obtained by an affine change of variables.

In order to get a method that adequately handles a singularity at the left endpoint, we consider a mesh on the interval $(0, 1)$ with a mesh grading factor $\sigma \in (0, 1)$ and parameter $L \in \mathbb{N}$, $L \leq p$ given by

$$K_0 := (0, \sigma^L),\ K_1 := (\sigma^L, \sigma^{L-1}),\ \ldots,\ K_L := (\sigma, 1).$$

On each one of these elements, we apply a Gauss quadrature, reducing the order as we approach the singularity, i.e.

$$\int_I g(\tau)\, d\tau \approx Q_I^{hp} g := \sum_{\ell=0}^{L} Q_{K_\ell}^{p-L+\ell} g.$$

For general intervals (a, b) we again apply an affine change of variables.

The main result on such composite Gauss-quadrature is the following proposition, versions of which are well-known, see for example [Sch94, Sch92, CvPS11].

Proposition 4.1. *Fix $T > 0$. Assume that $g : I := (0,T) \to \mathbb{C}$ can be extended holomorphically to a function $g : \mathcal{O} \to \mathbb{C}$ such that*

(i) $\mathcal{O}$ contains a positive sector $\{z \in \mathbb{C} : z \neq 0, |\operatorname{Arg} z| < \delta\}$ for some constant $\delta > 0$;

(ii) there exist constants $\alpha \in [0,1)$ and $C_g > 0$ such that

$$|g(z)| \leq C_g \, |z|^{-\alpha} \qquad \forall z \in \mathcal{O}. \tag{4.3}$$

Fix $\sigma > 0$ and consider the composite Gauss quadrature rule with L layers of refinement with degree $p = L$. Then there exist positive constants C_q, ω, and C_{stab} such that

$$\left| \int_0^T g(\tau)\, d\tau - Q_I^{hp} g \right| \leq C_q T^{1-\alpha} e^{-\omega p} \qquad \textit{and} \qquad \left| Q_I^{hp} g \right| \leq C_{stab} \, T^{1-\alpha}. \tag{4.4}$$

Proof. To prove the estimate (4.4), we assume $T = 1$. The more general result follows by an affine transformation. We distinguish two cases. On the element K_0, we use the bound on g and the fact that K_0 is small to get

$$\left| \int_0^{\sigma^L} g(\tau)\, d\tau - Q^0_{K_0} g \right| \lesssim C_g \int_0^{\sigma^L} \tau^{-\alpha}\, d\tau + \sigma^L |g(\sigma^L/2)| \lesssim C_g \sigma^{(1-\alpha)L}.$$

For any other element $K_{L-\ell} = (\sigma^{\ell+1}, \sigma^\ell)$, we map to the reference interval $(-1,1)$:

$$\left| \int_{\sigma^{\ell+1}}^{\sigma^\ell} g(\tau)\, d\tau - Q^{p-\ell}_{K_\ell} g \right| \sim \sigma^\ell \left| \int_{-1}^{1} \widetilde{g}(\widetilde{\tau})\, d\widetilde{\tau} - Q^{p-\ell}_{(-1,1)} \widetilde{g} \right|, \tag{4.5}$$

where $\widetilde{g} := g\big(\frac{\sigma^\ell}{2}(1-\sigma)\tau + (1+\sigma)\big)$ is the pull-back of g to $(-1,1)$. By geometric considerations, $\widetilde{g}$ is analytic on an ellipse $\mathcal{E}_\rho$ with semiaxis sum $\rho > 1$, where ρ depends on σ and the opening angle of the sector. We apply standard results on Gauss-quadrature (see [DR84, Eqn. (4.6.1.11)]) to estimate:

$$\Big| \int_{-1}^{1} \widetilde{g}(\widetilde{\tau})\, d\widetilde{\tau} - Q^{p-\ell}_{(-1,1)} \widetilde{g} \Big| \lesssim \rho^{-2(p-\ell)} \sup_{z \in \mathcal{E}_\rho} |\widetilde{g}(z)| \lesssim C_g \rho^{-2(p-\ell)} \sigma^{-\alpha\ell},$$

where we used Assumption (4.3) to bound $\widetilde{g}$. Going back to (4.5) we conclude

$$\Big| \int_{\sigma^{\ell+1}}^{\sigma^\ell} g(\tau)\, d\tau - Q^{hp} g \Big| \lesssim C_g \max(\sigma^{1-\alpha}, \rho^{-2})^p.$$

For general intervals $(0,T)$ we note that C_g behaves like $C_g \sim T^{-\alpha}$ under affine transformations.

We now show the stability estimate. On each subinterval K_ℓ, for $\ell > 0$ we have

$$\sigma t < T\sigma^{L-\ell+1} \leq t \leq T\sigma^{L-\ell}.$$

Since all the weights of Gauss-methods are positive and the quadrature integrates constants exactly, we can calculate

$$\left|Q_I^{hp} g\right| \lesssim Q^{hp}(\tau^{-\alpha}) \lesssim \Big(\frac{T\sigma^L}{2}\Big)^{-\alpha} + \sum_{\ell=0}^{L-1} Q^{p-\ell}(T\sigma^{\ell+1})^{-\alpha} = \Big(\frac{T\sigma^L}{2}\Big)^{-\alpha} + \sum_{\ell=0}^{L-1} \int_{T\sigma^{\ell+1}}^{T\sigma^{\ell}} \big(T\sigma^{\ell+1}\big)^{-\alpha}\,d\tau$$
$$\lesssim \Big(\frac{T\sigma^L}{2}\Big)^{-\alpha} + \sum_{\ell=0}^{L-1} \int_{T\sigma^{\ell+1}}^{T\sigma^{\ell}} (\sigma\tau)^{-\alpha}\,d\tau \lesssim \big(\sigma^{-\alpha L} + \sigma^{-\alpha}\big)T^{1-\alpha}. \qquad \square$$

4.2 Analysis of the inhomogeneous problem

We are now in the position to define and analyze the discretization scheme for the inhomogeneous problem. The discretization is done on multiple levels. We define the operator-valued functions

$$W(\tau) := \tau^{\gamma-1} e_{\gamma,\gamma}(-\tau^\gamma \gamma \mathcal{L}^\beta) = \tau^{\gamma-1}\frac{1}{2\pi i}\int_{\mathcal{C}} e_{\gamma,\gamma}(-\tau^\gamma z^\beta)\, R(z)\,dz,$$
$$W^q(\tau) := \tau^{\gamma-1}\frac{k}{2\pi i}\sum_{n=-\mathcal{N}_{\mathrm{q}}}^{\mathcal{N}_{\mathrm{q}}} e_{\gamma,\gamma}(-\tau^\gamma z^\beta(n\,k)) z'(n\,k) R(z(n\,k)),$$
$$W^{q,h}(\tau) := \tau^{\gamma-1}\frac{k}{2\pi i}\sum_{n=-\mathcal{N}_{\mathrm{q}}}^{\mathcal{N}_{\mathrm{q}}} e_{\gamma,\gamma}(-\tau^\gamma z^\beta(n\,k)) z'(n\,k) R_h(z(n\,k)).$$

We get the first layer of approximation by replacing the convolution in time with an hp-quadrature:

$$u_i^q(t) := Q_{(0,t)}^{hp}\left[W(\cdot)f(t-\cdot)\right].$$

Instead of relying on the exact evaluation of $\mathcal{L}^\beta$ we then switch to the sinc-quadrature approximation of the Riesz-Dunford integral by defining

$$u_i^{q,q}(t) := Q_{(0,t)}^{hp}\left[W^q(\cdot)f(t-\cdot)\right].$$

Finally, we do the discretization in space to obtain the fully discrete function

$$u_i^{q,q,h}(t) := Q_{(0,t)}^{hp}\left[W^{q,h}(\cdot)f(t-\cdot)\right].$$

We now step by step estimate the discretization errors, starting with the one in space.

Lemma 4.2. *Assume that $f:(0,T)\to L^2(\Omega)$ is uniformly analytic (cf. Definition 2.4), and $\mathbb{V}_h$ resolves the scales down to $\varepsilon = b^{-1/2}e^{-\mathcal{N}_q k/2}$. Then*

$$\left\|u_i^{q,q}(t) - u_i^{q,q,h}(t)\right\|_{\widetilde{H}^\beta(\Omega)} \lesssim t^{\gamma/2} k\,\mathcal{N}_q\, e^{-\omega \mathcal{N}_\Omega^\mu}.$$

Proof. The proof is very similar to Lemma 3.1, we only have to account for the extra quadrature step. Since the proof only relied on the analyticity of the right-hand side f and the estimate (2.7), we can analogously estimate for all $\tau \in (0,t)$

$$\left\| W^q(\tau) f(t-\tau) - W^{q,h}(\tau) f(t-\tau) \right\|_{\widetilde{H}^\beta(\Omega)} \lesssim \tau^{\gamma/2-1} \mathcal{N}_{\mathrm{q}} k e^{-\omega \mathcal{N}_\Omega^\mu}.$$

Using the stability estimate for *hp*-quadrature in (4.4), the stated result follows. ☐

Next, we consider the error due to the sinc-quadrature:

Lemma 4.3. *For* $0 < \varepsilon < \min\left(\frac{\beta}{2}, \frac{1}{4}\right)$, $\mathcal{N}_q \in \mathbb{N}$ *and* $k \sim \mathcal{N}_q^{-1/2}$ *we can estimate*

$$\|u_i^q(t) - u_i^{q,q}(t)\|_{\widetilde{H}^\beta(\Omega)} \lesssim t^{\gamma/2} e^{-\omega \mathcal{N}_q^{1/2}} \sup_{\tau \in (0,t)} \|f(\tau)\|_{H^{2\varepsilon}(\Omega)}. \tag{4.6}$$

Proof. We again focus on pointwise estimates of $W - W^q$, this time proceeding analogously to Lemma 3.6. We get for $\tau \in (0,t)$:

$$\|W(\tau) f(t-\tau) - W^q f(t-\tau)\|_{\widetilde{H}^\beta(\Omega)} \lesssim \tau^{\gamma/2-1} e^{-\omega \mathcal{N}_{\mathrm{q}}^{1/2}} \|f(t-\tau)\|_{H^{2\varepsilon}(\Omega)}.$$

Using the stability of the *hp*-quadrature then gives (4.6). ☐

The final step is analyzing the error due to *hp*-quadrature.

Lemma 4.4. *Assume that* $f : [0,T] \to L^2(\Omega)$ *is uniformly analytic. Assume we are using an hp-quadrature with* $\mathcal{N}_{hp}$ *refinement layers and maximum order* $p = \mathcal{N}_{hp}$*. Then we can estimate*

$$\|u_i(t) - u_i^q(t)\|_{\widetilde{H}^\beta(\Omega)} \lesssim t^{\gamma/2} e^{-\omega \mathcal{N}_{hp}}.$$

The constant depends on f *and the mesh grading factor* σ *used for the hp-quadrature.*

Proof. Using the notation $W_\lambda(\tau) := \tau^{\gamma-1} e_{\gamma,\gamma}(-\tau^\gamma \lambda^\beta)$, we write via the spectral decomposition

$$u_i(t) = \sum_{j=0}^{\infty} \int_0^t W_{\lambda_j}(\tau) \left(f(t-\tau), \varphi_j\right)_{L^2(\Omega)} d\tau\, \varphi_j,$$

$$u_i^q(t) = \sum_{j=0}^{\infty} Q_{(0,t)}^{hp}\Big(W_{\lambda_j}(\cdot) \left(f(t-\cdot), \varphi_j\right)_{L^2(\Omega)}\Big) \varphi_j.$$

By Definition 2.4 (iii), the integrands are analytic on a set $\mathcal{O}$ containing a positive sector. Using Proposition 4.1, it is sufficient to show for all $\tau \in \mathcal{O}$ and $\lambda > 0$ that $\left|\lambda^{\beta/2} W_\lambda(\tau)\right| \leq |\tau|^{\gamma/2-1}$ with a constant independent of λ. We note that $-\tau^\gamma \lambda^\beta$ is always in the sector required for (2.7). This can be seen since $-\tau^\gamma$ is always in the left-half plane with argument between $-\gamma\pi/4$ and $\gamma\pi/4$. Multiplying with $\lambda^\beta > 0$ does not change the argument of the complex number.

Therefore, we can estimate using (2.7):

$$\left|\lambda^{\beta/2} W_\lambda(\tau)\right| \lesssim |\tau|^{\gamma-1} \frac{\lambda^{\beta/2}}{1 + |\tau|^\gamma \lambda^\beta} \lesssim |\tau|^{\gamma-1} \left(\frac{1}{1+|\tau|^\gamma \lambda^\beta}\right)^{1/2} \left(\frac{\lambda^\beta}{1+|\tau|^\gamma \lambda^\beta}\right)^{1/2} \lesssim |\tau|^{\gamma/2-1}.$$

Applying Proposition 4.1 then concludes the proof. ☐

5 The general result

Combining the approximations for the inhomogeneous and homogeneous scheme, we define the fully discrete approximation as

$$u^{\mathrm{fd}}(t) := u^{q,h}(t) + u_i^{q,q,h}(t).$$

The corresponding error analysis is then a simple combination of the individual error estimates. Since we will use it later on, we also include a stability result with respect to the initial data.

Theorem 5.1. *Assume that u_0 is analytic on a neighborhood of $\overline{\Omega}$ and assume that f is uniformly analytic (cf. Definition 2.4). Use $\mathcal{N}_q \in \mathbb{N}$ quadrature points for the sinc-quadrature with $k := \kappa \mathcal{N}_q^{-1/2}$, $\mathcal{N}_{hp} \sim \sqrt{\mathcal{N}_q}$ layers for the hp-quadrature. Assume that $\mathbb{V}_h$ resolves the scales down to $b^{-1/2}\, e^{-(\kappa/2)\sqrt{\mathcal{N}_q}}$. Then, the following estimate holds:*

$$\left\| u(t) - u^{\mathrm{fd}}(t) \right\|_{\widetilde{H}^\beta(\Omega)} \lesssim \left(t^{-\gamma/2} + t^{\gamma/2}\right) \left(e^{-\omega\sqrt{\mathcal{N}_q}} + \sqrt{\mathcal{N}_q}\, e^{-\omega \mathcal{N}_\Omega^\mu}\right).$$

In addition, the following stability estimate holds for any $\varepsilon \in (0, 1/2)$:

$$\|u^{\mathrm{fd}}(t)\|_{\widetilde{H}^\beta(\Omega)} \lesssim C(\varepsilon)\, \min\left(\sqrt{\mathcal{N}_q}, t^{-\varepsilon\gamma}\right) \Big(t^{-\gamma/2}\, \|u_0\|_{L^2(\Omega)} + t^{\gamma/2} \max_{0\leq\tau\leq t} \|f(\tau)\|_{L^2(\Omega)}\Big). \tag{5.1}$$

Proof. For the convergence result, we just collect the different convergence results of Sections 3 and 4.

To prove the stability result, we start with only the homogeneous contribution. By an analogous computation to (3.6) we get from the definition of $u^{q,h}$ and the estimate on the resolvent (2.10) for $\varepsilon \in (0, 1/2)$:

$$\begin{aligned} \left\| u^{q,h}(t) \right\|_{\widetilde{H}^\beta(\Omega)} &\lesssim k \sum_{n=-\mathcal{N}_\mathrm{q}}^{\mathcal{N}_\mathrm{q}} t^{-\gamma(1/2+\varepsilon)}\, |z(y_n)|^{-\beta(1/2+\varepsilon)} \left|z'(y_n)\right| \|R_h(y_n) u_0\|_{\widetilde{H}^\beta(\Omega)} \\ &\lesssim t^{-\gamma(1/2+\varepsilon)} \|u_0\|_{L^2(\Omega)}. \end{aligned}$$

From which the stated estimate follows readily via (3.5). If $\varepsilon = 0$, the same computation can be made, but we end up with an additional factor $\mathcal{N}_\mathrm{q} k$ compensating for the summation. The inhomogeneous contribution follows along the same lines, but using the stability of the hp-Quadrature (4.4). ◻

5.1 Space-time and time robust estimates

Up to now, we have looked at the error in the $\widetilde{H}^\theta$-norm pointwise in time. While such an approach is natural, the resulting estimates deteriorate for small times t close to zero like $\mathcal{O}(t^{-\gamma/2})$. If the initial condition does not satisfy the boundary conditions, we cannot hope to derive t-robust estimates in the energy norm $\widetilde{H}^{\beta}$. In this section, we derive a different estimate which does not suffer from this deterioration.

For the following results to hold, we have to make an additional assumption on $\mathbb{V}_h$, a version of which is also already present in [MR21]:

Assumption 5.2. *There exists a fixed neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$ and constants θ, ω, $\mu > 0$ such that for all u_0 that are analytic on $\widetilde{\Omega}$, there exists a function $u_{h,0} \in \mathbb{V}_h$ and constants C_{stab}, $C_{approx} > 0$ (depending on u_0) such that*

$$\|u_{h,0}\|_{\mathbb{V}_h^\theta} \leq C_{stab} \qquad \text{and} \qquad \|u_0 - u_{h,0}\|_{L^2(\Omega)} \leq C_{approx} e^{-\omega \mathcal{N}_\Omega^\mu},$$

where $\mathcal{N}_\Omega := \dim\big(\mathbb{V}_h\big)$ and $\mathbb{V}_h^\theta := \big[\big(\mathbb{V}_h, L^2(\Omega)\big), \big(\mathbb{V}_h, H^1(\Omega)\big)\big]_{\theta,2}$.

We start by refining the estimates on the resolvent operator from Proposition (2.3), establishing that if we insert more regularity than L^2 in the argument, we get improved damping properties.

Lemma 5.3. *Let $z \in \mathscr{S}$. Assume $u_0 \in \widetilde{H}^\theta(\Omega)$ and $u_{h,0} \in \mathbb{V}_h^\theta$ for some $\theta \in [0,1]$. Then the following estimates hold for $\alpha \in [\theta, 1]$:*

$$\|R(z)u_0\|_{\widetilde{H}^\alpha(\Omega)} \lesssim |z|^{-1+\frac{\alpha-\theta}{2}} \|u_0\|_{\widetilde{H}^\theta(\Omega)} \quad \text{and} \quad \|R_h(z)u_{h,0}\|_{\widetilde{H}^\alpha(\Omega)} \lesssim |z|^{-1+\frac{\alpha-\theta}{2}} \|u_{h,0}\|_{\mathbb{V}_h^\theta}. \tag{5.2}$$

The implied constant depends on δ, A, c, and the constants in the definition of $\mathscr{S}$.

Proof. We only consider the discrete case. The continuous one follows analogously, replacing $\mathbb{V}_h$ with $H^1_0(\Omega)$. We consider the function $w := [R_h(z) - z^{-1}]u_{h,0}$. By elementary computations we get from (2.9) that w solves for all $v \in \mathbb{V}_h$:

$$\big((z-c)w, v_h\big)_{L^2(\Omega)} - \big(A\nabla w, \nabla v_h\big)_{L^2(\Omega)} = z^{-1}\big(\,(c\,u_{h,0}, v_h)_{L^2(\Omega)} + (A\nabla u_{h,0}, \nabla v_h)_{L^2(\Omega)}\,\big).$$

Thus, w solves the same variational problem as $R_h(z)u_{h,0}$ with modified right-hand side. If $u_{h,0} \in \mathbb{V}_h$, then $w \in \mathbb{V}_h$ is a valid test function, and we get fixing $\beta \in \mathbb{C}$ as in Proposition 2.3 that

$$\begin{aligned}\operatorname{Re}(\beta)\|A^{1/2}\nabla w\|^2_{L^2(\Omega)} &- \operatorname{Re}(\beta z)\|w\|^2_{L^2(\Omega)} \\ &\lesssim |z|^{-1}\big(\|u_{h,0}\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} + \|\nabla u_{h,0}\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}\big).\end{aligned}$$

From this and the already established bounds on $R_h(z)u_{h,0}$ from (2.10), we readily get the estimates

$$\|w\|_{H^1(\Omega)} \lesssim |z|^{-1} \|u_{h,0}\|_{H^1(\Omega)} \quad \text{and} \quad \|w\|_{L^2(\Omega)} \lesssim |z|^{-1} \|u_{h,0}\|_{L^2(\Omega)}.$$

By interpolation, this gives

$$\|w\|_{H^\alpha(\Omega)} \lesssim |z|^{-1} \|u_{h,0}\|_{\mathbb{V}_h^\alpha}.$$

From the definition of w and (2.10), we get the estimates:

$$\|R_h(z)u_{h,0}\|_{\widetilde{H}^\alpha(\Omega)} \lesssim |z|^{-1} \|u_{h,0}\|_{\mathbb{V}_h^\alpha}, \quad \text{and} \quad \|R_h(z)u_{h,0}\|_{\widetilde{H}^\alpha(\Omega)} \lesssim |z|^{-1+\alpha/2} \|u_{h,0}\|_{L^2(\Omega)}.$$

By the reiteration theorem [Tar07, Thm. 26.3], we can write $\mathbb{V}_h^\theta = [L^2(\Omega), \mathbb{V}_h^\alpha]_{\frac{\theta}{\alpha}}$. Thus, we further interpolate the previous estimates to get for $\theta \in [0, \alpha]$:

$$\|R_h(z)u_{h,0}\|_{\widetilde{H}^\alpha(\Omega)} \lesssim |z|^{-\frac{\theta}{\alpha}+(-1+\frac{\alpha}{2})(1-\frac{\theta}{\alpha})} \|u_{h,0}\|_{\mathbb{V}_h^\theta}, = |z|^{-1+\frac{\alpha}{2}-\frac{\theta}{2}} \|u_{h,0}\|_{\mathbb{V}_h^\theta}. \qquad \square$$

We use this estimate to prove a time-robust estimate. We weaken the statement from an estimate which is pointwise in time to a space-time Sobolev norm. Such norms were also considered in [MR21].

Theorem 5.4. *Let the Assumptions of Theorem 5.1 hold. In addition, let Assumption 5.2 hold, and use $u_{h,0}$ for the initial condition of the discrete method.*

Use $\mathcal{N}_q \in \mathbb{N}$ quadrature points for the sinc-quadrature with $k := \kappa \mathcal{N}_q^{-1/2}$, $\mathcal{N}_{hp} \sim \sqrt{\mathcal{N}_q}$ layers for the hp-quadrature. Assume that $\mathbb{V}_h$ resolves the scales down to $b^{-1/2}\, e^{-(\kappa/2)\sqrt{\mathcal{N}_q}}$. Then, the following estimate holds:

$$\int_0^T t^{\gamma-1} \big\|u(t) - u^{\mathrm{fd}}(t)\big\|^2_{\widetilde{H}^\beta(\Omega)}\, dt \lesssim C(T)\big(e^{-\omega\sqrt{\mathcal{N}_q}} + \sqrt{\mathcal{N}_q}\, e^{-\omega \mathcal{N}_\Omega^\mu}\big).$$

The constants depends on the end time T, the domain Ω, the data u_0 and f, the constants from Assumption 2.5 and 5.2, and on the details of the discretization, i.e., mesh grading, the factor κ, and the ratio $\mathcal{N}_{hp}/\sqrt{\mathcal{N}_q}$, but is independent of the accuracy parameters $\mathcal{N}_q$, k, $\mathcal{N}_{hp}$ or $\mathcal{N}_\Omega$.

Proof. We only consider the homogeneous part, the other one is even simpler as the singular behavior for small times is not present. In this setting we have $u^{\mathrm{fd}} = u^{q,h}$. Let $t_0 > 0$ to be fixed later. In (5.1) we have seen that u^{fd} depends continuously on the initial condition like $t^{-\gamma/2}\sqrt{\mathcal{N}_{\mathrm{q}}}\, \|u_{h,0}\|_{L^2(\Omega)}$. Replacing the discrete initial condition with u_0 allows us to apply Theorem 5.1 and we get:

$$\int_{t_0}^T t^{\gamma-1} \big\|u(t) - u^{\mathrm{fd}}(t)\big\|^2_{\widetilde{H}^\beta(\Omega)}\, dt \lesssim \log(t_0)\big(e^{-\omega\sqrt{\mathcal{N}_{\mathrm{q}}}} + \sqrt{\mathcal{N}_{\mathrm{q}}}\, \|u_0 - u_{h,0}\|^2_{L^2(\Omega)} + \sqrt{\mathcal{N}_{\mathrm{q}}}\, e^{-\omega \mathcal{N}_\Omega^\mu}\big).$$

For $t < t_0$, it is easy to derive the stability estimates

$$\big\|u(t)\big\|_{\widetilde{H}^\beta(\Omega)} \lesssim t^{-\gamma/2+\varepsilon} \|u_0\|_{H^{2\varepsilon}(\Omega)} \qquad \text{and} \qquad \big\|u^{q,h}(t)\big\|_{\widetilde{H}^\beta(\Omega)} \lesssim t^{-\gamma/2+\varepsilon} \|u_{0,h}\|_{\mathbb{V}_h^{2\varepsilon}(\Omega)}$$

from their representation formulas and the stability estimates on the resolvents in Lemma 5.3.

Thus, the stated theorem follows if we pick $t_0 \sim e^{-(\omega/\varepsilon)\min(\sqrt{\mathcal{N}_{\mathrm{q}}}, \mathcal{N}_\Omega^\mu)}$ and absorb all polynomial terms into the exponential. $\square$

Remark 5.5. *In the case $\gamma = 1$, the requirement that $u_{h,0}$ needs to be used for the discrete initial condition can be dropped. This is because the space-time norm depends continuously on the L^2-norm of the initial condition.* ■

6 hp-FEM

In this section, we provide a construction for $\mathbb{V}_h$ that satisfies Assumption 2.5. It is based on an hp-finite element method on a suitably refined grid towards the boundary $\partial\Omega$. The construction is the same as in [MR21] and has similarly also already appeared in [BMN+18] in the context of stationary elliptic problems. In [BMS20], the construction of [BMN+18] is generalized to polygonal domains.

For this, we need to make the following simplifying assumption throughout this section:

Assumption 6.1. *Let $\Omega \subset \mathbb{R}^d$ for $d = 1, 2$ have analytic boundary. Assume that A and c are analytic on a neighborhood $\widetilde{\Omega} \supset \overline{\Omega}$.*

Just as for the hp-quadrature in Section 4.1, the construction for $\mathbb{V}_h$ is based on a geometric grid that is refined towards the lower-dimensional manifolds where singularities are expected. In 1d, this means a geometric grid towards the end points, for 2d we make the following definitions, following [MS98], see also [BMN+18] and [Mel02, Def. 2.4.4].

We first introduce the (shape regular) reference mesh, used to resolve the geometry of Ω.

Definition 6.2 (reference mesh)**.** *Let $\widehat{S} := (0,1)^2$ be the reference square, and $\mathcal{T}_\Omega := \{K_i\}_{i=0}^{|\mathcal{T}_\Omega|}$ a mesh of curved quadrilaterals with bijective element maps $F_K : \overline{\widehat{S}} \to \overline{K}$ satisfying*

(M1) The elements K_i partition Ω, i.e., $\bigcup_{K_i \in \mathcal{T}} \overline{K_i} = \overline{\Omega}$;

(M2) for $i \neq j$, $\overline{K}_i \cap \overline{K}_j$ is either empty, a vertex or an entire edge;

(M3) the element maps $F_K : \widehat{S} \to K$ are analytic diffeomorphisms;

(M4) the common edge of two neighboring elements K_i, K_j has the same parametrization from both sides, i.e., if γ_{ij} is the common edge with endpoints P_1, P_2, then for $P \in \gamma_{i,j}$ we have

$$\operatorname{dist}(F_{K_i}^{-1} P, F_{K_i}^{-1} P_\ell)) = \operatorname{dist}(F_{K_j}^{-1} P, F_{K_j}^{-1} P_\ell) \qquad \textit{for } \ell = 1, 2.$$

In order to be able to resolve boundary layers on small scales, we now refine the reference mesh geometrically towards the boundary. This is captured in the next definition.

Definition 6.3 (anisotropic geometric mesh)**.** *Let $\mathcal{T}_\Omega$ be a reference mesh, and assume that K_i, $i = 0, \dots, n < |\mathcal{T}_\Omega|$ are the elements at the boundary. Also assume that the left edge $e := \{0\} \times (0,1)$ is mapped to $\partial\Omega$, i.e., $F_{K_i}(e) \subseteq \partial\Omega$ and $F_{K_i}(\partial S \setminus \overline{e}) \cap \partial\Omega = \emptyset$ for $i = 0, \dots, n$. The remaining elements are taken to satisfy $\overline{K_i} \cap \partial\Omega = \emptyset$, $i = n+1, \dots, |\mathcal{T}_\Omega|$.*

For $L \in \mathbb{N}$ and a mesh grading factor $\sigma \in (0,1)$, we subdivide the reference square

$$\widehat{S}^0 := (0, \sigma^L) \times (0,1), \qquad \widehat{S}^\ell := (\sigma^\ell, \sigma^{\ell-1}) \times (0,1), \ \ell = 1, \dots . L.$$

The anisotropic geometric mesh $\mathcal{T}_\Omega^L$ *is then given by the push-forwards of the refinements in the boundary region, plus the unrefined interior elements:*

$$\mathcal{T}_\Omega^L := \Big\{ F_{K_i}(\widehat{S}^\ell),\ \ell = 0, \dots, L,\ \ i = 0, \dots, n \Big\} \cup \bigcup_{i=n+1}^{|\mathcal{T}_\Omega|} \{K_i\}.$$

Definition 6.4. *In one dimension, for $\Omega = (-1,1)$, the reference mesh is given by the single element $\mathcal{T}_\Omega := \{(-1,1)\}$ and the anisotropic geometric mesh is given by the nodes*

$$x_0 := -1,\ x_i := -1 + \sigma^{L-i+1},\ i = 1, \dots, L,$$
$$x_i := 1 - \sigma^{i-L},\ i = L+1, \dots, 2L, \qquad x_{2L+1} := 1.$$

For general $\Omega = (a,b)$ it is given by an affine transformation of the mesh on $(-1,1)$.

Using these meshes, we can now give an exemplary construction for $\mathbb{V}_h$, which satisfies Assumptions 2.5 and 5.2.

Definition 6.5 ($\mathbb{V}_h$ via *hp*-FEM)**.** *Let $\mathcal{T}_\Omega^L$ be an anisotropic geometric mesh refined towards $\partial\Omega$ and fix $p \in \mathbb{N}$. We write $Q^p := \operatorname{span}_{0 \le i_1, \dots, i_d \le p} \big\{ x_1^{i_1} \, \dots \, x_d^{i_d} \big\}$ for the space of tensor product polynomials and set*

$$\mathbb{V}_h := S_0^{p,1}(\mathcal{T}_\Omega^L) := \Big\{ u \in H_0^1(\Omega) :\ u \circ F_K \in Q^p \quad \forall K \in \mathcal{T}_\Omega^L \Big\}. \tag{6.1}$$

In [MR21], it was shown that such spaces are able to reliably resolve small scales. We collect the result in the following proposition.

Proposition 6.6 ([MR21, Thm.3.33])**.** *Let $\mathcal{T}_\Omega^L$ be an anisotropic mesh on Ω that is geometrically refined towards $\partial\Omega$ with grading factor $\sigma \in (0,1)$ and L layers. Use polynomial degree $p \sim L$.*

Then $\mathbb{V}_h$ defined in (6.1) resolves the scales down to σ^L, i.e., there exist constants $C, \omega > 0$, such that for $z \in \mathscr{S}$ with $|z|^{-1/2} > \sigma^L$ and every f that is analytic on a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$, the solution u_z to $(\mathcal{L} - z)u = zf$ can be approximated by $v_h \in \mathbb{V}_h$ satisfying

$$|z|^{-1} \left\| \nabla u - \nabla v_h \right\|_{L^2(\Omega)}^2 + \left\| u - v_h \right\|_{L^2(\Omega)}^2 \le C e^{-\omega' p} \le C' e^{-\omega \mathcal{N}_\Omega^{\frac{1}{d+1}}}.$$

The constant ω depends only on σ, Ω and $\widetilde{\Omega}$. The constants C, C' also depend on the constants of analyticity of f.

If we are interested in the estimate from Section 5.1, we also need that the *hp*-FEM space can approximate the initial condition in a way that is stable in the discrete interpolation norm $\mathbb{V}_h^\theta$. Again following what was done in [MR21], we split the construction into two steps, first computing the H^1-best approximation in a space without boundary conditions (this can be done on the reference mesh) and then performing a cutoff procedure to correct the boundary conditions on the anisotropic geometric mesh. We first recall the properties of the cutoff operator.

Proposition 6.7 ([MR21, Lem. 3.35]). *Let $\mathcal{T}_\Omega^L$ denote an anisotropic geometric mesh with reference mesh $\mathcal{T}_\Omega$. Given $\ell \in \mathbb{N}_0$, $\ell \leq L$, there exists a linear operator $\mathscr{C}_\ell$: $\mathcal{S}^{p,1}(\mathcal{T}_\Omega) \to \mathcal{S}_0^{p,1}(\mathcal{T}_\Omega^L)$ such that for $\theta \in [0, 1/2)$*

$$\|\mathscr{C}_\ell v\|_{H^1(\Omega)} \lesssim \sigma^{-\ell/2} \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)} \quad \text{and} \quad \|v - \mathscr{C}_\ell v\|_{L^2(\Omega)} \lesssim \sigma^{\theta\ell} \|v\|_{H^\theta(\Omega)}. \tag{6.2}$$

Lemma 6.8. *Let u_0 be analytic in a neighborhood $\widetilde{\Omega} \supset \overline{\Omega}$, and let $0 \leq \theta < 1/2$. Then there exists a function $u_{h,0} \in \mathbb{V}_h$:*

$$\|u_{h,0}\|_{\mathbb{V}_h^\theta} \lesssim \|u_0\|_{H^1(\Omega)} \qquad \text{and} \qquad \|u_{h,0} - u_0\|_{L^2(\Omega)} \lesssim e^{-\omega' p}. \tag{6.3}$$

In other words, if the number of refinement layers $L \sim p$, then $\mathbb{V}_h$ satisfies Assumption 5.2 with $\mu := 1/(d+1)$.

Proof. A similar result has appeared in [MR21, Lem. 3.36], although in a more complicated setting. Let $\Pi_{H^1} : H^1(\Omega) \to \mathcal{S}^{p,1}(\mathcal{T}_\Omega)$ denote the best approximation operator in the $H^1(\Omega)$-norm (note that we do not enforce boundary conditions). Set $\widehat{u}_0 := \Pi_{H^1} u_0$. We define $u_{h,0} := \mathscr{C}_L \widehat{u}_0 \in \mathbb{V}_h$ and the piecewise constant function $v(t) \in \mathbb{V}_h$ as

$$v(t) := \begin{cases} \mathscr{C}_L \widehat{u}_0 & t \in (0, \sigma^L), \\ \mathscr{C}_\ell \widehat{u}_0 & t \in (\sigma^\ell, \sigma^{\ell-1}),\ \ell = 1, \dots, L, \\ 0 & t > 1. \end{cases}$$

For $\varepsilon > 0$ sufficiently small, we then estimate using (6.2):

$$\begin{aligned}
\|u_{h,0}\|_{\mathbb{V}_h^\theta}^2 &\leq \int_0^1 t^{-2\theta} \big(\|\mathscr{C}_L \widehat{u}_0 - v(t)\|_{L^2(\Omega)}^2 + t^2 \|v(t)\|_{H^1(\Omega)}^2 \big) \frac{dt}{t} + \frac{1}{2\theta} \|\mathscr{C}_L \widehat{u}_0\|_{L^2(\Omega)}^2 \\
&\overset{(6.2)}{\lesssim} \Big(1 + \sum_{\ell=0}^{L-1} \sigma^{(1-\varepsilon)\ell} \int_{\sigma^{\ell+1}}^{\sigma^\ell} t^{-2\theta-1} dt\Big) \|\widehat{u}_0\|_{H^{1/2-\frac{\varepsilon}{2}}(\Omega)}^2 + \Big(\sum_{\ell=0}^{L-1} \sigma^{-\ell} \int_{\sigma^{\ell+1}}^{\sigma^\ell} t^{-2\theta+1} dt\Big) \|\widehat{u}_0\|_{H^1(\Omega)}^2 \\
&\lesssim \Big(1 + \sum_{\ell=0}^{L-1} \sigma^{\ell - 2\theta\ell - \varepsilon\ell}\Big) \|\widehat{u}_0\|_{H^{1/2}(\Omega)}^2 + \Big(\sum_{\ell=0}^{L-1} \sigma^{-\ell + 2\ell - 2\theta\ell}\Big) \|\widehat{u}_0\|_{H^1(\Omega)}^2 \lesssim \|\widehat{u}_0\|_{H^1(\Omega)}^2,
\end{aligned}$$

where in the last step we used the fact that $2\theta < 1$ and a geometric series.

The statement then follows from the fact that the space $\mathcal{S}^{p,1}(\mathcal{T}_\Omega)$ can approximate analytic functions exponentially fast (see for example [Mel02, Prop. 3.2.21]), and the stability of the H^1-best approximation operator Π_{H^1}. □

Remark 6.9. *The use of the H^1-norm on the right hand side of (6.3) is mostly for convenience. Using more involved results about interpolation spaces of polynomials, we expect that this requirement can be lowered to the $H^{\beta+\varepsilon}(\Omega)$-norm.* ■

Collecting the previous results, we arrive at the pointwise error estimate:

Corollary 6.10. *Assume that u_0 is analytic on a neighborhood of $\overline{\Omega}$ and assume that f is uniformly analytic.*

Use $\mathcal{N}_q \in \mathbb{N}$ quadrature points for the sinc-quadrature with $k \sim 1/\sqrt{\mathcal{N}_q}$, $\mathcal{N}_{hp} \sim \sqrt{\mathcal{N}_q}$ layers and $p = \mathcal{N}_{hp}$ for the hp-quadrature. Let $\mathcal{T}_\Omega^L$ be an anisotropic geometric grid with grading factor $\sigma \in (0,1)$ such that $L \gtrsim \sqrt{\mathcal{N}_q}$ and use the polynomial degree $p \sim L$ for $\mathbb{V}_h$.

Then, the following estimate holds for a constant $\omega > 0$:

$$\left\| u(t) - u^{\mathrm{fd}}(t) \right\|_{\widetilde{H}^\beta(\Omega)} \lesssim \left(t^{-\gamma/2} + t^{\gamma/2}\right)\left(e^{-\omega\sqrt{\mathcal{N}_q}} + e^{-\omega \mathcal{N}_\Omega^{\frac{1}{d+1}}}\right),$$

Proof. We apply Theorem 5.1. By Proposition 6.6, the space $\mathbb{V}_h$ satisfies the necessary Assumption 2.5. The factor $\sqrt{\mathcal{N}_\mathrm{q}}$ from Theorem 5.1 is absorbed into the exponential for convenience. □

The estimate for the space-time energy norm takes the following form:

Corollary 6.11. *Fix $T > 0$, assume that u_0 is analytic on a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$ and assume that f is uniformly analytic. Use $\mathcal{N}_q \in \mathbb{N}$ quadrature points for the sinc-quadrature with $k \sim 1/\sqrt{\mathcal{N}_q}$, $\mathcal{N}_{hp} \sim \sqrt{\mathcal{N}_q}$ layers and degree $p \sim \mathcal{N}_{hp}$ for the hp-quadrature. Let $\mathcal{T}_\Omega^L$ be an anisotropic geometric grid with grading factor $\sigma \in (0,1)$ such that $L \gtrsim \frac{\sqrt{\mathcal{N}_q}}{|\ln(\sigma)|}$, and use the polynomial degree $p \sim L$ for $\mathbb{V}_h$.*

Let $u_{h,0}$ be given by $u_{h,0} := \mathcal{C}_L \Pi_{H^1} u_0$ where $\mathcal{C}_L$ is the cutoff operator from Proposition 6.7 and Π_{H^1} is the H^1-orthogonal projection onto the space $\mathcal{S}^{p,1}(\mathcal{T}_\Omega)$.

Then, the following estimate holds for a constant $\omega > 0$:

$$\int_0^T t^{\gamma-1} \left\| u(t) - u^{\mathrm{fd}}(t) \right\|^2_{\widetilde{H}^\beta(\Omega)} \, dt \lesssim C(T)\left(e^{-\omega\sqrt{\mathcal{N}_q}} + e^{-\omega \mathcal{N}_\Omega^{\frac{1}{d+1}}}\right).$$

The constants depends on the end time T, the domain Ω, $\widetilde{\Omega}$, $\mathcal{O}$, the data u_0 and f, the constants from Assumption 2.5 and 5.2 and on the details of the discretization like mesh grading or the ratio $k/\sqrt{\mathcal{N}_q}$, but is independent of the accuracy parameters $\mathcal{N}_q$, k, $\mathcal{N}_{hp}$ or $\mathcal{N}_\Omega$.

Proof. Follows from 5.4. The necessary assumptions on $\mathbb{V}_h$ and $u_{h,0}$ are satisfied by Proposition 6.6 and Lemma 6.8. □

7 Numerical Results

In order to confirm our theoretical findings, we implemented the method using the NGSolve software package [Sch14, Sch22] for the finite element discretization in Ω. The geometric meshes in 2d were generated using the *hp*-refinement feature of Netgen, the integrated mesh generator of NGSolve. A sample mesh can be seen in Figure 7.1. We note that these meshes include geometric refinements towards each of the corners of the domain.

Figure 7.1: Example of a geometric mesh with 5 refinement layers.

Remark 7.1. *When implementing the presented method in practice, the dominant cost is to numerically solve the singularly perturbed problems* $(z_j - \mathcal{L})^{-1}$. *Observing that these problems only depend on the quadrature point, and therefore appear several times throughout the algorithm, most notably inside the hp-quadrature for the time convolution, it is therefore beneficial to reorder the operations in order to minimize the number of systems that need to be solved. This leads to a method which only requires* $\mathcal{N}_q$ *linear system solves, making it very efficient.* ∎

Example 7.2. *As a first example, we consider the unit square* $\Omega = (0,1)^2$ *and fix the end time* $t = 1$. *We set the parameters* $\beta := 0.75$ *and* $\gamma := 0.6$. *In order to validate our implementation, we choose the initial condition and right-hand side in a way that gives rise to a known exact solution. Namely, similarly to [BLP17b], we set* $u(t,x,y) := e_{\gamma,1}(-t^{\gamma}(8\pi^2)^{\beta})\sin(2\pi x)\sin(2\pi y) + t^3\sin(\pi x)\sin(\pi y)$. *This means the data are given by*

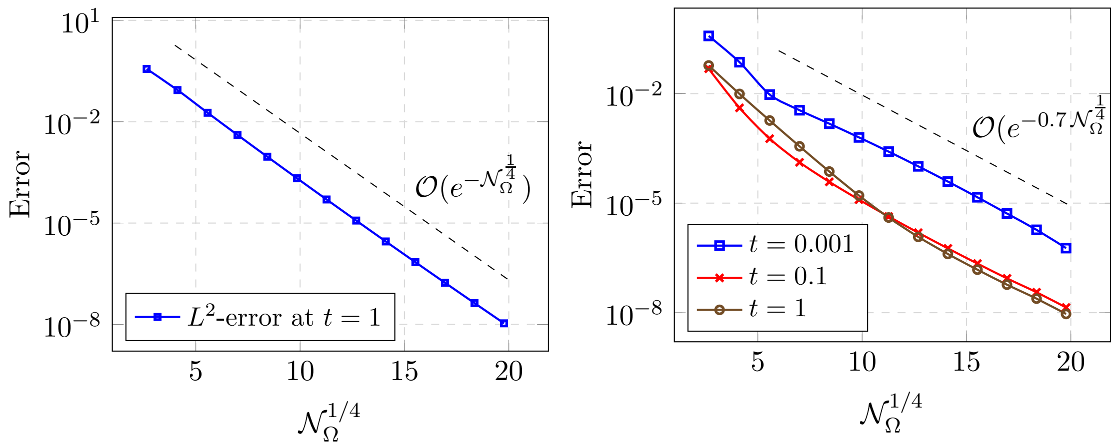
$$u_0(x,y) := \sin(2\pi x)\sin(2\pi y), \quad \text{and} \quad f(t,x,y) := \left(\frac{\Gamma(4)}{\Gamma(4-\gamma)}t^{3-\gamma} + t^3(2\pi^2)^{\beta}\right)\sin(\pi x)\sin(\pi y).$$

We look at the convergence of the $L^2(\Omega)$*-error as we increase the polynomial degree* p *used for our finite element discretization, the number of geometric refinements in the underlying grid as well as our quadrature parameters* $\mathcal{N}_q := 6\,p^2$, $k := \pi\sqrt{1/(5\beta\mathcal{N}_q)} \sim 1/p$ *and* $\mathcal{N}_{hp} := p$. *The grading factor* $\sigma := 0.125$ *is used for both the geometric mesh on* Ω *and the hp-quadrature.*

Since we are working in 2d *and we are dealing with a geometry with corners that need to be resolved, the number of degrees of freedom scale like* $\mathcal{N}_\Omega \sim p^4$. *We plot the error compared to* $\mathcal{N}_\Omega^{1/4}$. *As expected, we observe exponential convergence in this very simple case of a known smooth exact solution; see Figure 7.2a.*

Remark 7.3. *The choice of* k *is motivated by [BLP17a, Rem. 4.1], where the optimal choice is given by* $k = \sqrt{\frac{\pi H}{\beta N}}$, *where* $H < \pi/4$ *is determined by the region of analyticity in Lemma 3.5. In our case, we used* $H := \frac{\pi}{5}$. ∎

Example 7.4. *We continue with the unit square as the computational domain. But now, we take an initial condition and right-hand side that violates the boundary condition. This will lead to the formation of singularities. Namely, we fix* $u_0 :\equiv 1$ *and* $f(x,y,t) := \sin(t)$, *and choose* $\gamma = \sqrt{2}/2$ *and* $\beta := \sqrt{3}/3$. *We plot the convergence of the* $L^2(\Omega)$*-error as we increase the polynomial degree* p *for different end times* t. *Again, we keep the other discretization parameters proportional, this time using* $\mathcal{N}_q := 6\,p^2$, $k := \pi\sqrt{1/(5\beta\mathcal{N}_q)} \sim 1/p$, $\mathcal{N}_{hp} := p$, *and again setting* $\sigma := 0.125$ *for all geometric grids. Since the exact*



(a) Compatible data; Example 7.2

(b) Incompatible data; Example 7.4

Figure 7.2: Convergence at $t = 1$ in the $L^2(\Omega)$-norm for compatible and incompatible data

solution is not known, we used the approximation on the finest grid as our reference solution.

As expected, we again observe exponential convergence with respect to $\mathcal{N}_\Omega^{1/3} \sim p$*, confirming that our numerical method can resolve all the appearing singularities.*

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