ON VANDERMONDE DETERMINANTS VIA n-DETERMINANTS

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Abstract. We use earlier defined notion of n- determinant to investigate subdeterminants of an extended Vandermonde matrix. Firstly, we demonstrate our method on a number of particular cases. Then we prove that all these results may be stated in terms of Schur's polynomials. In our main result, we prove that Schur polynomials are equal to minors of a fixed matrix, which entries are formed of elementary symmetric polynomials. Such a formula is known as the second Jaccobi-Trudi identity.

1. Introduction

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of variables. We consider the following extended Vandermonde matrix:

$$V_{n,n+r} = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n & \cdots & x_1^{n+r-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^n & \cdots & x_2^{n+r-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n & \cdots & x_n^{n+r-1} \end{pmatrix}.$$

First n columns of $V_{n,n+r}$ form the standard Vandermonde matrix

$$V_n = \begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{pmatrix}.$$

We denote by $K_i, (i = 1, 2, \ldots, n+r)$ columns of $V_{n,n+r}$. By $\overline{e}_i(X), (i = 0, 1, \ldots, n)$ are denoted elementary symmetric polynomial of $x_1, x_2, \ldots, x_n$. Hence, $\overline{e}_0(X) = 1, \overline{e}_1(X) = x_1 + x_2 + \cdots + x_n, \ldots, \overline{e}_n(X) = x_1 \cdot x_2 \cdots x_n$.

We next consider the following polynomial $p_n(x) = \prod_{i=1}^n (x - x_i)$. Expanding the right-hand side we obtain

$$p_n(x) = \sum_{i=0}^{n} (-1)^{n-i} \cdot \overline{e}_{n-i}(X) \cdot x^i$$

From $p_n(x_k) = 0, (k = 1, \ldots, n)$, we obtain

$$x_k^n = \sum_{i=1}^{n} (-1)^{n-i} \overline{e}_{n-i+1}(X) \cdot x_k^{i-1}.$$

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Denoting $(-1)^{n-i}\overline{e}_{n-i+1}(X) = e_{n-i+1}(X), (i = 1, \ldots, n)$ implies

$$K_{n+1} = \sum_{i=1}^{n} e_{n-i+1}(X) \cdot K_i.$$

. In the same way, for each $p = 1, 2, \ldots, r$, we obtain

$$K_{n+p} = \sum_{i=1}^{n} e_{n-i+1}(X) \cdot K_{i+p-1}. \tag{1}$$

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Remark 1. We may consider that each column K_{n+k} is, in fact, a linear combinations of all preceding columns of $V_{n,n+r}$, by taking coefficient 0 for columns that do not appear in the equation (1).

We next consider the following matrix of order $(r+n-1) \times r$.

$$P_{n+r-1,r} = \begin{pmatrix} e_n & 0 & 0 & \cdots & 0 & 0 & 0 \\ e_{n-1} & e_n & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ e_1 & e_2 & e_3 & \cdots & 0 & 0 & 0 \\ -1 & e_1 & e_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & e_1 & e_2 \\ 0 & 0 & 0 & \cdots & 0 & -1 & e_1 \end{pmatrix}.$$

Assume that

$$\{1, 2, \ldots, n+r-1\} = \{i_1, i_2, \ldots, i_{n-1}\} \cup \{j_1, j_2, \ldots, j_r\}.$$

Clearly, the union on the right-hand side must be disjoint. We denote by

$$M = M(i_1, i_2, \ldots, i_{n-1}, n+r)$$

the sub-matrix of $V_{n,n+r}$ lying in columns $i_1, i_2, \ldots, i_{n-1}, n+r$ of $V_{n,n+r}$. We define

$$\text{sgn}M = (-1)^{\frac{n(n-1)}{2} + i_1 + \cdots + i_{n-1}}.$$

It is easy to see that the equation $\text{sgn}M = (-1)^{nr + \frac{r(r-1)}{2} + j_1 + \cdots + j_r}$ also holds.

We next denote by $Q_r = Q_r(j_1, j_2, \ldots, j_r)$ the sub-matrix of order r of P lying in rows $j_1, j_2, \ldots, j_r$ of P.

In our paper [1], the following formula is proved:

$$\frac{\begin{vmatrix} x_1^{i_1-1} & \cdots & x_1^{i_{n-1}-1} & x_1^{n+r-1} \\ x_2^{i_1-1} & \cdots & x_2^{i_{n-1}-1} & x_2^{n+r-1} \\ \vdots & \ldots & \vdots & \vdots \\ x_n^{i_1-1} & \cdots & x_n^{i_{n-1}-1} & x_n^{n+r-1} \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{vmatrix}} = \text{sgn}M \cdot \det Q_r(j_1, \ldots, j_r). \tag{2}$$

We start with examples of this result.

2. Two introducing examples

In this part, we describe two extreme case, when $Q_r(j_1,\dots,j_r)$ lies either in the first r rows of P or in the last r. We firstly take $i_1 = r+1, i_2 = r+2, \dots, i_{n-1} = r+n-1$. Then Q_r lies in the first r rows of P. Hence, it is a lower triangular matrix having all diagonal elements equal to e_n. Also, $\mathrm{sgn}M = (-1)^{(n-1)\cdot r}$, so that

Proposition 2. *The following equation holds*

$$\frac{\begin{vmatrix} x_1^r & x_1^{r+1} & \cdots & x_1^{r+n-2} & x_1^{r+n-1} \\ x_2^r & x_2^{r+1} & \cdots & x_2^{r+n-2} & x_2^{r+n-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_n^r & x_n^{r+1} & \cdots & x_n^{r+n-2} & x_n^{r+n-1} \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-2} & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-2} & x_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} & x_n^{n-1} \end{vmatrix}} = (-1)^{(n-1)\cdot r} \cdot e_n^r(X).$$

Extracting $x_1^r x_2^r \cdots x_n^r$ from nominator of the fraction, the equation becomes obvious.

Proposition 3. *Assume that $i_t = t, (t = 1,2,\dots,n-1)$. We obviously have* $\mathrm{sgn}M = 1$. *For $r > n$, we have*

$$\frac{\begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-2} & x_1^{n+r-1} \\ 1 & x_2 & \cdots & x_2^{n-2} & x_2^{n+r-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} & x_n^{n+r-1} \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-2} & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-2} & x_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} & x_n^{n-1} \end{vmatrix}} = \begin{vmatrix} e_1 & e_2 & \cdots & 0 & 0 \\ -1 & e_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & e_1 & e_2 \\ 0 & 0 & \cdots & -1 & e_1 \end{vmatrix}.$$

3. Case $r = 1$

In the case $r = 1$, the matrix P has the following form

$$P = \begin{pmatrix} e_n(X) \\ e_{n-1}(X) \\ \vdots \\ e_1(X) \\ -1 \end{pmatrix},$$

If $A_1, A_2, \dots, A_n, A_{n+1}$ are columns of $V_{n,n+1}$, then we have

$$A_{n+1} = \sum_{i=1}^{n} e_{n-i+1}(X) \cdot A_{n-i+1}.$$

We next have $\mathrm{sgn}M = \frac{n^2-n+2}{2}$. Hence,

Proposition 4. *We have*

$$\frac{\begin{vmatrix}1 & x_1 & \cdots & x_1^{j-2} & x_1^j & \cdots & x_1^n\\ 1 & x_2 & \cdots & x_1^{j-2} & x_2^j & \cdots & x_2^n\\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots\\ 1 & x_n & \cdots & x_n^{j-2} & x_n^j & \cdots & x_n^n\end{vmatrix}}{\begin{vmatrix}1 & x_1 & \cdots & x_1^{n-2} & x_1^{n-1}\\ 1 & x_2 & \cdots & x_2^{n-2} & x_2^{n-1}\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ 1 & x_n & \cdots & x_n^{n-2} & x_n^{n-1}\end{vmatrix}} = (-1)^{\frac{n^2-n+2}{2}} \cdot e_{n-j+1}(X). \tag{3}$$

4. Case $r = 2$

In this case, we have

$$V_{n,n+2} = \begin{pmatrix}1 & x_1 & x_1^2 & \cdots & x_1^n & x_1^{n+1}\\ 1 & x_2 & x_2^2 & \cdots & x_2^n & x_2^{n+1}\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^n & x_{n-1}^{n+1}\\ 1 & x_n & x_n^2 & \cdots & x_n^n & x_n^{n+1}.\end{pmatrix}$$

We denote by $\hat{x}_i$ deleted elements of $V_{n,n+2}$. If i and j are indices of deleted columns, we have

$$M = \begin{pmatrix}1 & x_1 & \cdots & \hat{x}_1^{i-1} & \cdots & \hat{x}_1^{j-1} & \cdots & x_1^{n+1}\\ 1 & x_2 & \cdots & \hat{x}_2^{i-1} & \cdots & \hat{x}_2^{j-1} & \cdots & x_2^{n+1}\\ \vdots & \vdots & \cdots & \vdots & \vdots & & & \\ 1 & x_n & \cdots & \hat{x}_n^{i-1} & \cdots & \hat{x}_n^{j-1} & \cdots & x_n^{n+1}\end{pmatrix}.$$

In this case the matrix P is of order $(n+1)\times 2$, and Q is a 2×2 sub-matrix of P. We consider the case when $Q_2 = \begin{pmatrix}e_i(X) & e_{i+1}(X)\\ e_j(X) & e_{j+1}(X)\end{pmatrix}$. Next, we have $\operatorname{sgn} M = (-1)^{nr+\frac{r(r-1)}{2}+i+j}$. We thus obtain

Proposition 5. *The following formula holds*

$$\frac{\begin{vmatrix}1 & \cdots & \hat{x}_1^{i-1} & \cdots & \hat{x}_1^{j-1} & \cdots & x_1^{n+1}\\ 1 & \cdots & \hat{x}_2^{i-1} & \cdots & \hat{x}_2^{j-1} & \cdots & x_2^{n+1}\\ \vdots & \cdots & \vdots & \vdots & & & \\ 1 & \cdots & \hat{x}_n^{i-1} & \cdots & \hat{x}_n^{j-1} & \cdots & x_n^{n+1}\end{vmatrix}}{\begin{vmatrix}1 & x_1 & \cdots & x_1^{n-2} & x_1^{n-1}\\ 1 & x_2 & \cdots & x_2^{n-2} & x_2^{n-1}\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ 1 & x_n & \cdots & x_n^{n-2} & x_n^{n-1}\end{vmatrix}} = (-1)^{nr+\frac{r(r-1)}{2}+i+j} \cdot \begin{vmatrix}e_i(X) & e_j(X)\\ e_{i+1}(X) & e_{j+1}(X)\end{vmatrix}.$$

5. Case $i_1 = 1, i_2 = 2, \ldots, i_{n-1} = n-1$

In this case, we have

$$M = \begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-2} & x_1^{n+r} \\ 1 & x_2 & \cdots & x_2^{n-2} & x_2^{n+r} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{r+n-2} & x_n^{n+r} \end{pmatrix},$$

and $\mathrm{sgn} M = 1$. Next, we have

$$Q_r = \begin{pmatrix} e_1(X) & e_2(X) & \cdots & e_n(X) & 0 & \cdots & 0 & 0 & 0 \\ -1 & \overline{e}_1(X) & \cdots & e_{n-1}(X) & e_n(X) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & -1 & e_1(X) & e_2(X) \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & \overline{e}_1(X) \end{pmatrix}.$$

We see that Q_r is a Hessenberg matrix, and also a Toeplitz matrix. It is easy to see that $\det Q_r$ satisfies the following recurrence:

$$\det Q_r = e_1(X) \cdot Q_{r-1} + e_2(X) \cdot \det Q_{r-2}(X) + \cdots + e_n(X) \cdot \det Q_{r-n}, (r > n).$$

Being obviously $\mathrm{sgn} M = 1$, we obtain

Proposition 6. *The following formula holds*

$$\frac{\begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-2} & x_1^{n+r} \\ 1 & x_2 & \cdots & x_2^{n-2} & x_2^{n+r} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{r+n-2} & x_n^{n+r} \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-2} & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-2} & x_2^{n-1} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} & x_n^{n-1} \end{vmatrix}} = \det Q_r.$$

6. A relation with Schur polynomials

In this section, we relate obtained results with the Schur polynomials. We transform M in the following way: Firstly, we interchange rows $(1, n), (2, n-1), \ldots$. For this we need $\lfloor \frac{n}{2} \rfloor$ transposition. Hence

$$N = \begin{pmatrix} x_1^{n+r-1} & x_2^{n+r-1} & \cdots & x_n^{n+r-1} \\ x_1^{i_{n-1}-1} & x_2^{i_{n-1}-1} & \cdots & x_n^{i_{n-1}-1} \\ \vdots & \vdots & \ldots & \vdots \\ x_1^{i_1-1} & x_2^{i_1-1} & \cdots & x_n^{i_1-1} \end{pmatrix},$$

and $\det N = (-1)^{\lfloor \frac{n}{2} \rfloor} \cdot \det M$. By denoting $\lambda_1 = r, \lambda_2 = i_{n-1} - n + 1, \ldots, \lambda_n = i_1 - 1$, we obtain

$$N = \begin{pmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \cdots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \cdots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ldots & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \cdots & x_n^{\lambda_n} \end{pmatrix},$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 1$. Hence, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a partition. We see that M is uniquely determined by this partition

We thus connect M with the Schur polynomial $s_\lambda(x_1, x_2, \ldots, x_n)$. In this way all proved results becomes identities for Schur polynomials. Consider the following partition

$$\{j_1, j_2, \ldots, j_r\} = \{1, 2, \ldots, n+r\} \setminus \{\lambda_n, \lambda_{n-1}+1, \ldots, \lambda_1+n-1\}.$$

We see that the set $\{j_1, j_2, \ldots, j_r\}$ is uniquely determined by λ and r. Finally, We denote $nr + j_1 + \cdots + j_r + \frac{r(r-1)}{2} + \lfloor \frac{n}{2} \rfloor = \nu(\lambda, r)$.

Hence, the following formula is true:

$$s_\lambda(x_1, \ldots, x_n) = (-1)^{\nu(\lambda,r)} \cdot \det Q(\lambda, r).$$

The expression on the left-hand side of this equation is the Schur polynomial $s_\lambda(x_1, x_2, \ldots, x_n)$.

Remark 7. Hence, each Schur polyinomial is, up to the sign, a sub-determinant of order r of P.

Remark 8. We obtained formula in which the Schur polynomials are obtained in terms of elementary symmetric polynomials. Such a formula is known as the second Jacobi-Trudi identity.

References

[1] M, Janjic, Determinants and Recurrence Sequences, J. Integer Sequences, 15 (2012), Article 12.3.5.

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