# PLATEAU FLOW OR THE HEAT FLOW FOR HALF-HARMONIC MAPS

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ABSTRACT. Using the interpretation of the half-Laplacian on  $S^1$  as the Dirichletto-Neumann operator for the Laplace equation on the ball B, we devise a classical approach to the heat flow for half-harmonic maps from  $S^1$  to a closed target manifold  $N \subset \mathbb{R}^n$ , recently studied by Wettstein, and for arbitrary finite-energy data we obtain a result fully analogous to the author's 1985 results for the harmonic map heat flow of surfaces and in similar generality. When N is a smoothly embedded, oriented closed curve  $\Gamma \subset \mathbb{R}^n$  the halfharmonic map heat flow may be viewed as an alternative gradient flow for a variant of the Plateau problem of disc-type minimal surfaces.

### 1. Background and results

1.1. **Half-harmonic maps and their heat flow.** Let  $N \subset \mathbb{R}^n$  be a closed submanifold, that is, compact and without boundary. The concept of a half-harmonic map  $u: S^1 \to N \subset \mathbb{R}^n$  was introduced by Da Lio-Rivière [14], who together with Martinazzi in [12], Theorem 2.9, also made the interesting observation that the harmonic extension of a half-harmonic map yields a free boundary minimal surface supported by  $N$ , a fact which also was noticed by Millot-Sire [\[29\]](#page-40-0), Remark 4.28.

In his PhD-thesis, Wettstein [\[48\]](#page-41-0),[\[49\]](#page-41-1), [\[50\]](#page-41-2), recently studied the corresponding heat flow given by the equation

<span id="page-0-2"></span>(1.1) 
$$
d\pi_N(u)(u_t + (-\Delta)^{1/2}u) = 0 \text{ on } S^1 \times [0, \infty[,
$$

where  $u_t = \partial_t u$ , and where  $\pi_N : N_\rho \to N$  is the smooth nearest neighbor projection on a  $\rho$ -neighborhood  $N_{\rho}$  of the given target manifold to N, and, with the help of a fine analysis of the fractional differential operators involved, he showed global existence for initial data of small energy.

Moser [\[32\]](#page-40-1) and Millot-Sire [\[29\]](#page-40-0) contributed important results to the study of half-harmonic maps by exploiting the fact that for any smooth  $u: S^1 \to \mathbb{R}^n$  we can represent the half-Laplacian classically in the form

<span id="page-0-1"></span>
$$
(1.2)\qquad \qquad (-\Delta)^{1/2}u = \partial_r U
$$

where  $U: B \to \mathbb{R}^n$  is the harmonic extension of u to the unit disc  $B^{-1}$  $B^{-1}$  $B^{-1}$ . Here, using the identity [\(1.2\)](#page-0-1) we are able to remove the smallness assumption in Wettstein's work and show the existence of a "global" weak solution to the heat flow  $(1.1)$ 

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<span id="page-0-0"></span><sup>&</sup>lt;sup>1</sup>The classical formula  $(1.2)$  is a special case of a much more general result, due to Caffarelli-Silvestre [\[4\]](#page-39-0), who pointed out that many nonlocal problems involving fractional powers of the Laplacian can be related to a local, possibly degenerate, elliptic equation via a suitable extension of the solution to a half-space.

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for data of arbitrarily large (but finite) energy, which is defined for all times and smooth away from finitely many "blow-up points" where energy concentrates, and whose energy is non-increasing. The solution is unique in this class in exact analogy with the classical result [\[42\]](#page-41-3) by the author on the harmonic map heat flow for maps from a closed surface to a closed target manifold  $N \subset \mathbb{R}^n$ ; see Theorem [1.2](#page-2-0) below.

In order to describe our work in more detail, let

$$
H^{1/2}(S^1;N) = \{ u \in H^{1/2}(S^1; \mathbb{R}^n); \ u(z) \in N \text{ for almost every } z \in S^1 \}.
$$

Interpreting  $S^1 = \partial B$ , where  $B = B_1(0; \mathbb{R}^2)$  and tacitly identifying a map  $u \in$  $H^{1/2}(S^1;N)$  with its harmonic extension  $U \in H^1(B;\mathbb{R}^n)$ , for a given function  $u_0 \in$  $H^{1/2}(S^1;N)$  we then seek to find a family of harmonic functions  $u(t) \in H^1(B; \mathbb{R}^n)$ with traces  $u(t) \in H^{1/2}(S^1; N)$  for  $t > 0$ , solving the equation

<span id="page-1-2"></span>(1.3) 
$$
d\pi_N(u)(u_t + \partial_r u) = u_t + d\pi_N(u)\partial_r u = 0 \text{ on } S^1 \times [0, \infty[,
$$

with initial data

(1.4) 
$$
u|_{t=0} = u_0 \in H^{1/2}(S^1;N).
$$

1.2. Energy. The half-harmonic heat flow may be regarded as the heat flow for the half-energy

<span id="page-1-3"></span>
$$
E_{1/2}(u) = \frac{1}{2} \int_{S^1} |(-\Delta)^{1/4} u|^2 d\phi
$$

of a map  $u \in H^{1/2}(S^1; N)$ . Note that the half-energy of u equals the standard Dirichlet energy

$$
E(u) = \frac{1}{2} \int_B |\nabla u|^2 dz
$$

of its harmonic extension  $u \in H^1(B; \mathbb{R}^n)$ . Indeed, integrating by parts we have

<span id="page-1-1"></span>
$$
(1.5) \qquad \int_B |\nabla u|^2 dz = \int_{S^1} u \partial_r u \, d\phi = \int_{S^1} u (-\Delta)^{1/2} u \, d\phi = \int_{S^1} |(-\Delta)^{1/4} u|^2 d\phi,
$$

where we use the Millot-Sire identity  $(1.2)$  and where the last identity easily follows from the representation of the operators  $(-\Delta)^{1/2}$  and  $(-\Delta)^{1/4}$  in Fourier space with symbols  $|\xi|$ ,  $\sqrt{|\xi|}$ , respectively, and Parceval's identity. <sup>[2](#page-1-0)</sup> Therefore, in the following for convenience we may always work with the classically defined Dirichlet energy. Moreover, we may interpret the half-harmonic heat flow as the heat flow for the Dirichlet energy in the class of harmonic functions with trace in  $H^{1/2}(S^1;N)$ ; see Section [2](#page-6-0) below for details.

1.3. Results. Identifying  $\mathbb{R}^2 \cong \mathbb{C}$ , we denote as M the 3-dimensional Möbius group of conformal transformations of the unit disc, given by

$$
M = \{ \Phi(z) = e^{i\theta} \frac{z - a}{\bar{a} + z} \in C^{\infty}(\bar{B}; \bar{B}) : |a| < 1, \ \theta \in \mathbb{R} \}.
$$

Observe that the Dirichlet energy is invariant under conformal transformations, and we have  $E(u) = E(u \circ \Phi)$  for any  $u \in H^1(B; \mathbb{R}^n)$  and any  $\Phi \in M$ .

For smooth data we then have the following result.

<span id="page-1-0"></span><sup>&</sup>lt;sup>2</sup>Conversely, via Fourier expansion we also can prove  $(1.5)$  directly. Computing the first variations of E and  $E_{1/2}$ , respectively, we then obtain [\(1.2\)](#page-0-1).

<span id="page-2-2"></span>**Theorem 1.1.** Let  $N \subset \mathbb{R}^n$  be a closed, smooth sub-manifold of  $\mathbb{R}^n$ , and suppose that the normal bundle  $T^{\perp}N$  is parallelizable. Then the following holds:

i) For any smooth  $u_0 \in H^{1/2}(S^1;N)$  there exists a time  $T_0 \leq \infty$  and a unique smooth solution  $u = u(t)$  of [\(1.3\)](#page-1-2), hence of [\(1.1\)](#page-0-2), with data [\(1.4\)](#page-1-3) for  $0 < t < T_0$ .

ii) If  $T_0 < \infty$ , we have concentration in the sense that for some  $\delta > 0$  and any  $R > 0$  there holds

<span id="page-2-1"></span>
$$
\sup_{z_0 \in B, 0 < t < T_0} \int_{B_R(z_0) \cap B} |\nabla u(t)|^2 \, dz \ge \delta,
$$

and for suitable  $t_k \uparrow T_0$  there exist finitely many points  $z_k^{(1)}$  $\mathbf{z}_k^{(1)}, \ldots, \mathbf{z}_k^{(i_0)}$  and conformal maps  $\Phi_k^{(i)} \in M$  with  $z_k^{(i)} \to z^{(i)} \in \overline{B}$  and  $\Phi_k^{(i)} \to \Phi_{\infty}^{(i)} \equiv z^{(i)}$  weakly in  $H^1(B)$  such that  $u(t_k) \circ \Phi_k^{(i)} \to \bar{u}^{(i)}$  weakly in  $H^1(B)$  as  $k \to \infty$ , where  $\bar{u}^{(i)}$  is non-constant and conformal and satisfies

(1.6) 
$$
d\pi_N(\bar{u}^{(i)})\partial_r\bar{u}^{(i)} = 0, 1 \le i \le i_0.
$$

Moreover, there exists  $\delta = \delta(N) > 0$  such that  $E(\bar{u}^{(i)}) \geq \delta$ , and  $i_0 \leq E(u_0)/\delta$ . Finally,  $u(t_k)$  smoothly converges to a limit  $u_1 \in H^{1/2}(S^1; N)$  on  $\overline{B} \setminus \{z^{(1)}, \ldots, z^{(i_0)}\}$ .

iii) If  $T_0 = \infty$ , then, as  $t \to \infty$  suitably,  $u(t)$  smoothly converges to a halfharmonic limit map away from at most finitely many concentration points where non-constant half-harmonic maps "bubble off" as in ii).

By the Da Lio-Rivière interpretation of [\(1.6\)](#page-2-1), the "bubbles"  $\bar{u}^{(i)}$  as well as the limit  $u_{\infty}$  of the flow conformally parametrize minimal surfaces with free boundary on N, meeting N orthogonally along their free boundaries.

The hypothesis regarding the target manifold  $N$  in particular is fullfilled if  $N$ is a closed, orientable hypersurface of co-dimension 1 in  $\mathbb{R}^n$ , or if N is a smoothly embedded, closed curve  $\Gamma \subset \mathbb{R}^n$ .

It would be interesting to find examples of initial data for which the flow blows up in finite time, as in the work of Chang-Ding-Ye [\[5\]](#page-39-1) on the harmonic map heat flow.

For data in  $H^{1/2}(S^1;N)$  the following global existence result holds, which is our main result.

<span id="page-2-0"></span>**Theorem 1.2.** For  $N \subset \mathbb{R}^n$  as in Theorem [1.1](#page-2-2) the following holds: i) For any  $u_0 \in H^{1/2}(S^1;N)$  there exists a unique global weak solution of [\(1.3\)](#page-1-2) with data [\(1.4\)](#page-1-3) as in Definition [6.3,](#page-30-0) whose energy is non-increasing and which is smooth for positive time away from finitely many points in space-time where non-trivial half-harmonic maps "bubble off" in the sense of Theorem [1.1.](#page-2-2)ii).

ii) As  $t \to \infty$  suitably,  $u(t)$  smoothly converges to a half-harmonic limit map away from at most finitely many concentration points where non-constant half-harmonic maps "bubble off" as in Theorem [1.1.](#page-2-2)iii).

Note that uniqueness is only asserted within the class of partially regular weak solutions with non-increasing energy, as in the case of the harmonic map heat flow. It would be interesting to find out if the latter condition suffices, as in the work of Freire [\[18\]](#page-40-2), [\[19\]](#page-40-3), and, conversely, to explore the possibility of "backward bubbling" in [\(1.3\)](#page-1-2), as in the examples of Topping [\[45\]](#page-41-4) for the latter flow.

1.4. Key features of the proof and related flow equations. In our approach, in a similar vain as Lenzmann-Schikorra [\[27\]](#page-40-4), we uncover and exploit surprising regularity properties of the normal component  $d\pi_N^{\perp}(u)\partial_r u$  for the harmonic extension

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of u, likely related to the fractional commutator estimates for the normal projection in the work of Da Lio-Rivière  $[14]$  or the regularity estimates of Da Lio-Pigati  $[13]$ , Mazowiecka-Schikorra [\[28\]](#page-40-5), and others.

The use of the Dirichlet-to-Neumann map for the harmonic extension  $u: B \to \mathbb{R}^n$ of  $u$  instead of the half-Laplacian, and the simple identity  $(3.2)$  as well as equation [\(3.5\)](#page-7-1) allow to perform the analysis using only local, classically defined operators, avoiding fractional calculus almost entirely.

Note that equation [\(1.3\)](#page-1-2) is similar to the equation governing the (scalar) evolution problem for conformal metrics  $e^{2u}g_{\mathbb{R}^2}$  of prescribed geodesic boundary curvature and vanishing Gauss curvature on the unit disc  $B$ , studied for instance by Brendle [\[2\]](#page-39-2) or Gehrig [\[20\]](#page-40-6). In contrast to the latter flows, due to the presence of the projection operator mapping  $u_r$  to its tangent component, the flow [\(1.3\)](#page-1-2) at first sight appears to be degenerate. However, surprisingly, within our framework we are able to obtain similar smoothing properties as in the case of the harmonic map heat flow of surfaces.

A different heat flow associated with half-harmonic maps, using the half-heat operator  $(\partial_t - \Delta)^{1/2}$  instead of [\(1.1\)](#page-0-2), was suggested by Hyder et al. [\[22\]](#page-40-7), and they obtained global existence of partially regular, but possibly non-unique weak solutions for their flow, with a possibly large singular set of measure zero.

1.5. Applications to Plateau problem. In the case when  $N$  is a smoothly embedded, oriented closed curve  $\Gamma \subset \mathbb{R}^n$  the half-harmonic heat flow [\(1.3\)](#page-1-2) may furnish an alternative gradient flow for the Plateau problem of minimal surfaces of the type of the disc, which has a long and famous tradition in geometric analysis.

Having been posed by Plateau in the 1890's, Plateau's problem was finally solved independently by Douglas  $[16]$  and Radó  $[33]$  in 1930/31. In order to analyse the set of all minimal surfaces solving the Plateau problem, including saddle points of the Dirichlet integral, thereby building on Douglas' ideas, in 1939 Morse-Tompkins [\[31\]](#page-40-10) proposed a critical point theory for Plateau's problem in the sense of Morse [\[30\]](#page-40-11), attempting to characterize non-minimizing solutions as "homotopy-critical" points of Dirichlet's integral. However, in the 1980's Tromba [\[47\]](#page-41-5), [\[46\]](#page-41-6) pointed out that it was not even clear that all smooth, non-degenerate minimal surfaces would be "homotopy-critical" in the sense of Morse-Tompkins [\[31\]](#page-40-10). To overcome this problem, Tromba developed a version of degree theory that could be applied in this case and which yielded at least a proof of the "last" Morse inequality, which is an identity for the total degree.

In 1982, finally, this author [\[41\]](#page-41-7) recast the Plateau problem as a variational problem on a closed convex set and he was able to develop a version of the Palais-Smale type critical point theory for the problem within this frame-work, which allowed him to obtain all Morse inequalities in a rigorous fashion; see the monograph [\[44\]](#page-41-8) and the paper by Imbusch-Struwe [\[23\]](#page-40-12) for further details. In the papers [\[43\]](#page-41-9) by this author and [\[25\]](#page-40-13) by Jost-Struwe the approach was extended to the case of multiple boundaries and/or higher genus.

A key element of critical point theory for a variational problem is the construction of a pseudo-gradient flow for the problem at hand. In [\[41\]](#page-41-7) this was achieved in an ad-hoc way. However, starting with the work of Eells-Sampson [\[17\]](#page-40-14) on the harmonic map heat flow, it is now an established approach in geometric analysis to study the (negative)  $(L^2$ -)gradient flow related to a variational problem, similar to the standard heat equation. For Plateau's problem, such a flow was obtained

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by Chang-Liu [\[6\]](#page-39-3) within the frame-work laid out by Struwe [\[41\]](#page-41-7) in the form of a parabolic variational inequality, for which Chang-Liu obtained a solution of class  $H^2$ by means of a time-discrete minimization scheme. Rupflin [\[35\]](#page-41-10), Rupflin-Schrecker [\[36\]](#page-41-11) studied the analogous parabolic variational inequality in the case of an annulus, which again had previously been studied by this author [\[43\]](#page-41-9) by means of an ad-hoc pseudo-gradient flow.

In view of the much better regularity properties of the flow equation [\(1.3\)](#page-1-2) it would be tempting to regard this as the correct definition of the canonical gradient flow for the Plateau problem, but an important issue still needs to be addressed.

1.6. **Monotonicity.** Recall that in the classical Plateau problem  $u(t)$  is required to induce a (weakly) monotone parametrization of  $\Gamma$  for each  $t > 0$ . Even though it may seem likely that – at least for curves  $\Gamma$  on the boundary of a convex body in  $\mathbb{R}^3$ – this Plateau boundary condition will be preserved along the flow [\(1.3\)](#page-1-2) whenever it is satisfied initially, at this moment even for a strictly convex planar curve  $\Gamma \subset \mathbb{R}^2$  it is not clear whether this actually happens. However, the results that we obtain also seem to be of interest if we drop the Plateau condition. In particular, our results motivate the study of smooth minimal surfaces with continous trace covering only a part of the given boundary curve Γ; dropping the monotonicity condition also brings the parametric approach to the Plateau problem closer to the approach via geometric measure theory or level sets.

1.7. Plateau flow. It should be straightforward to extend our results to the case when the disc B is replaced by a surface  $\Sigma$  of higher genus with boundary  $\partial \Sigma \cong$  $S^1$ , if for given initial data  $u_0 \in H^{1/2}(S^1;N)$  we consider a family  $u = u(t)$  in  $H^{1/2}(S^1;N)$  solving the equation [\(1.3\)](#page-1-2), that is,

$$
u_t + d\pi_N(u)\partial_\nu u = 0
$$

instead of [\(1.1\)](#page-0-2), where for each time we harmonically extend  $u(t)$  to  $\Sigma$  and denote as  $\partial_{\nu}u$  the outward normal derivative of u along  $\partial\Sigma$ , as was proposed and analysed by Da Lio-Pigati [13] in the time-independent case. Similarly, one might study the flow [\(1.3\)](#page-1-2) on a domain  $\Sigma$  with multiple boundaries. Of course, in order for the flow to converge to a minimal surface in the case of higher genus or higher connectivity it will be necessary to couple the flow [\(1.3\)](#page-1-2) with a corresponding evolution equation for the conformal structure on  $\Sigma$ , as in the work of Rupflin-Topping [\[37\]](#page-41-12) on minimal immersions. Note that on a general domain  $\Sigma$  the flow equations [\(1.1\)](#page-0-2) and [\(1.3\)](#page-1-2) no longer agree. In order to clearly distinguish the flow equation [\(1.3\)](#page-1-2) from the equation [\(1.1\)](#page-0-2) defining the half-harmonic map heat flow, we therefore propose to say that  $(1.3)$  defines the "Plateau flow".

1.8. Outline. After a brief discussion of energy estimates in Section [2,](#page-6-0) in Section [3](#page-7-2) we present the analytic core of the argument for higher regularity in Section [4](#page-10-0) and for the blow-up analysis, later presented in Section [8.](#page-33-0) These tools are also instrumental in proving uniqueness of partially regular weak solutions in Section [7.](#page-31-0) The  $L^2$ -bounds for higher and higher derivatives which we establish in Section [4,](#page-10-0) assuming that energy does not concentrate, may be of particular interest. These bounds either concern estimates for  $\nabla \partial_{\phi}^{k} u$  on B or on  $\partial B$ , and we view the latter bounds as stronger by an order of 1/2. These bounds may be used interlaced, as we later do in Section [6,](#page-29-0) to prove uniform smooth estimates, locally in time, for smooth flows with smooth initial data converging in  $H^{1/2}(N; S^1)$ . Since the latter

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data are dense in  $H^{1/2}(N; S^1)$  we thus not only obtain existence of weak solutions for arbitrary data  $u_0 \in H^{1/2}(N; S^1)$  but also can show their smoothness for positive time and hence are able to derive Theorem [1.2](#page-2-0) from Theorem [1.1.](#page-2-2) A peculiar feature is that one set of regularity estimates can only be obtained globally, that is on all of B, whereas the other set of estimates may be localized using cut-off functions. Similar estimates for a regularized version of [\(1.3\)](#page-1-2) are employed in Section [5](#page-24-0) to prove local existence of smooth solutions of  $(1.3)$  for smooth data  $(1.4)$ . Finally, in Section [9](#page-37-0) the large-time behavior of smooth solutions to [\(1.3\)](#page-1-2) is discussed, finishing the proof of Theorem [1.1.](#page-2-2)

1.9. Notation. The letter  $C$  is used throughout to denote a generic constant, possibly depending on the "target" N and the initial energy  $E(u_0)$ .

Moreover, since  $T^{\perp}N$  by assumption is parallelizable and compact, there exists  $\rho > 0$  such that the representation

$$
T: N \times B_{\rho}(0; \mathbb{R}^m) \ni (p, y) \to p + \sum_{i=1}^{m} y^{i} \nu_i(p) \in N_{\rho}
$$

of the tubular neighborhood  $N_{\rho} = \bigcup_{p \in N} B_{\rho}(p)$  of N is a diffeomorphism, where  $\nu_1, \ldots, \nu_m$  is a suitable smooth orthonormal frame along N and where we let  $y =$  $(y^1, \ldots, y^m) \in \mathbb{R}^m$ . For  $q \in N_\rho$  then  $T^{-1}(q) = (p, h)$  with  $p = \pi_N(q)$  defines a (vector-valued) signed distance function  $h = h(q) = (h^1(q), \ldots, h^m(q))$  with  $h^i(q) =$  $\nu_i(p) \cdot (q - \pi_N(q))$  for each  $1 \leq i \leq i_0$ . Fixing a smooth function  $\eta \colon \mathbb{R} \to \mathbb{R}$  such that  $\eta(s) = s$  for  $|s| < \rho/2$ , and with  $\eta(s) = 0$  for  $|s| \geq 3\rho/4$ , we then let

$$
dist_N(q) = (dist_N^1(q), \ldots, dist_N^m(q)),
$$

with

$$
dist_N^i(q) = \eta(h^i(q)) \text{ for } q \in N_\rho, \ dist_N^i(q) = 0 \text{ else, } 1 \le i \le m.
$$

Then for any smooth  $u \in H^{1/2}(S^1; N)$  with harmonic extension  $u \in H^1(B; \mathbb{R}^n)$  we have

(1.7) 
$$
\sum_{i=1}^{m} \nu_i(u) \partial_r dist_N^i(u) = \sum_{i=1}^{m} \nu_i(u) \nu_i(u) \cdot u_r = d\pi_N^{\perp}(u) u_r \text{ on } \partial B = S^1,
$$

where for each  $p \in N$  we denote as  $d\pi_N^{\perp}(p) = 1 - d\pi_N(p)$ :  $\mathbb{R}^n \to T_p^{\perp}N$  the orthogonal projection. In the sequel, we abbreviate

$$
\sum_{i=1}^{m} \nu_i(u)\nu_i(u) \cdot u_r =: \nu(u)\nu(u) \cdot u_r = \nu(u)\partial_r dist_N(u);
$$

moreover, we extend the vector fields  $\nu_i$  to the whole ambient space by letting  $\nu_i(q) = \nabla dist_N^i(q)$  for  $q \in \mathbb{R}^n$ ,  $1 \leq i \leq m$ .

Finally, we fix a smooth cut-off function  $\varphi \in C_c^{\infty}(B)$  satisfying  $0 \leq \varphi \leq 1$  with  $\varphi \equiv 1$  on  $B_{1/2}(0)$ , and for any  $z_0 \in B$ , any  $0 < R < 1$  we scale

$$
\varphi_{z_0,R}(z)=\varphi((z-z_0)/R)\in C_c^{\infty}(B_R(z_0)).
$$

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## 2. Energy inequality and first consequences

<span id="page-6-0"></span>The half-harmonic heat flow may be regarded as the heat flow for the Dirichlet energy in the class  $H^{1/2}(S^1;N)$ . Indeed, let  $u(t)$  be a smooth solution of [\(1.3\)](#page-1-2), [\(1.4\)](#page-1-3) for  $0 < t < T_0$ . Then we have the following result.

<span id="page-6-1"></span>**Lemma 2.1.** For any  $0 < T < T_0$  there holds

$$
E(u(T)) + \int_0^T \int_{\partial B} |u_t|^2 d\phi \, dt \le E(u_0).
$$

Proof. Integrating by parts and using [\(1.3\)](#page-1-2) we compute

$$
\frac{d}{dt}E(u) = \int_B \nabla u \nabla u_t dx = \int_{\partial B} u_r \cdot u_t d\phi
$$

$$
= -\int_{\partial B} |d\pi_N(u)u_r|^2 d\phi = -\int_{\partial B} |u_t|^2 d\phi
$$

for any  $0 < t < T_0$ . The claim follows by integration.

Moreover, there holds a localized version of this energy inequality.

<span id="page-6-2"></span>**Lemma 2.2.** There exists a constant  $C > 0$  such that for any  $z_0 \in B$ , any  $0 <$  $R < 1$ , any  $\varepsilon > 0$ , and any  $0 < t_0 < t_1 \le t_0 + \varepsilon R < T_0$  there holds

$$
\int_{B} |\nabla u(t_1)|^2 \varphi_{z_0, R}^2 dz + 4 \int_{t_0}^{t_1} \int_{\partial B} |u_t|^2 \varphi_{z_0, R}^2 d\phi dt
$$
  

$$
\leq 4 \int_{B} |\nabla u(t_0)|^2 \varphi_{z_0, R}^2 dz + C \varepsilon E(u_0).
$$

*Proof.* Writing  $\varphi = \varphi_{z_0,R}$  for brevity, integrating by parts, and using Young's inequality, similar to the proof of Lemma [2.1](#page-6-1) for any  $0 < t < T_0$  we have

<span id="page-6-3"></span>
$$
\frac{d}{dt} \left( \frac{1}{2} \int_B |\nabla u|^2 \varphi^2 dz \right) = \int_{\partial B} u_t \cdot u_r \varphi^2 d\phi - \int_B u_t div(\nabla u \varphi^2) dz
$$
\n
$$
(2.1) \qquad = -\int_{\partial B} |d\pi_N(u)u_r|^2 \varphi^2 d\phi - 2 \int_B u_t \nabla u \varphi \nabla \varphi dz
$$
\n
$$
\leq -\int_{\partial B} |u_t|^2 \varphi^2 d\phi + (8\varepsilon R)^{-1} \int_B |\nabla u|^2 \varphi^2 dz + 8\varepsilon R \int_B |u_t|^2 |\nabla \varphi|^2 dz.
$$

Letting

$$
A = \sup_{t_0 < t < t_1} \left( \frac{1}{2} \int_B |\nabla u(t)|^2 \varphi^2 \, dz \right),
$$

then upon integration we find

$$
A + \int_{t_0}^{t_1} \int_{\partial B} |u_t|^2 \varphi^2 d\phi dt
$$
  
\n
$$
\leq \int_{B} |\nabla u(t_0)|^2 \varphi^2 dz + \frac{t_1 - t_0}{2\varepsilon R} A + C\varepsilon R^{-1} \int_{t_0}^{t_1} \int_{B_R(z_0) \cap B} |u_t|^2 dz dt.
$$

But with  $u = u(t)$  also  $u_t = u_t(t)$  is harmonic for each t. Expanding

$$
u_t(re^{i\phi}) = \sum_{k\geq 0} a_k r^k e^{ik\phi}
$$

in a Fourier series, we see that the map

$$
r \mapsto \int_{\partial B_r(0)} |u_t|^2 ds = 2\pi \sum_{k \ge 0} |a_k|^2 r^{2k+1},
$$

with ds denoting the element of length along  $\partial B_r(0)$ , is non-decreasing. Thus for any  $z_0 \in B$ , any  $0 < R < 1$ , and any  $t_0 < t < t_1$  there holds

<span id="page-7-6"></span>(2.2) 
$$
\int_{B_R(z_0)\cap B} |u_t|^2 dz \leq 2R \int_{\partial B} |u_t|^2 d\phi,
$$

<span id="page-7-2"></span>and we may use Lemma [2.1](#page-6-1) to conclude.  $\Box$ 

# <span id="page-7-3"></span>3. A regularity estimate

To illustrate the key ideas that later will allow us to prove higher regularity and analyze blow-up of solutions of [\(1.3\)](#page-1-2), we first consider smooth solutions  $u \in$  $H^{1/2}(S^1;N)$  of the equation

(3.1) 
$$
d\pi_N(u)\partial_r u + f = 0 \text{ on } \partial B = S^1,
$$

where  $f \in L^2(S^1)$ . We prove the following a-priori estimate, where we use classical estimates similar to Wettstein's [\[48\]](#page-41-0) Lemma 3.4, which in turn is a fractional version of an earlier result by Rivière [\[34\]](#page-40-15). Note that with the truncated signed distance function  $dist_N : \mathbb{R}^n \to \mathbb{R}^m$  we have the orthogonal decomposition

(3.2) 
$$
\partial_r u = d\pi_N(u)\partial_r u + d\pi_N^{\perp}(u)\partial_r u = d\pi_N(u)\partial_r u + \nu(u)\partial_r (dist_N(u))
$$

on  $\partial B = S^1$ , where we recall that we use the shorthand notation

<span id="page-7-0"></span>
$$
\nu(u)\partial_r(dist_N(u)) = \sum_{i=1}^m \nu_i(u)\partial_r(dist_N^i(u)) = \sum_{i=1}^m \nu_i(u)\nu_i(u) \cdot \partial_r u
$$

and extend  $\nu_i(p) = \nabla dist^i_N(p), p \in \mathbb{R}^n$ .

<span id="page-7-5"></span>**Proposition 3.1.** There exist constants  $C, \delta_0 = \delta_0(N) > 0$  such that for any smooth solution  $u \in H^{1/2}(S^1; N)$  of  $(3.1)$  with  $E(u) \leq \delta^2 < \delta_0^2$  there holds

(3.3) 
$$
\int_{S^1} |\partial_{\phi} u|^2 d\phi \leq C \|f\|_{L^2(S^1)}^2.
$$

*Proof.* Multiplying [\(3.2\)](#page-7-0) with  $\partial_r u$ , we find the Pythagorean identity

<span id="page-7-4"></span>
$$
(3.4) \quad |\partial_r u|^2 = |d\pi_N(u)\partial_r u|^2 + |d\pi_N^{\perp}(u)\partial_r u|^2 = |d\pi_N(u)\partial_r u|^2 + |\partial_r (dist_N(u))|^2.
$$

Note that  $dist_N(u) \in H_0^1(B)$ ; moreover, for each  $1 \leq i \leq m$  we have  $\nabla dist_N^i(u)$  =  $\nu_i(u) \cdot \nabla u$ , and there holds the equation

(3.5) 
$$
\Delta(dist_N^i(u)) = div(\nu_i(u) \cdot \nabla u) = \nabla u \cdot d\nu_i(u) \nabla u \text{ in } B.
$$

The divergence theorem now gives

<span id="page-7-1"></span>
$$
\|\partial_r (dist_N(u))\|_{L^2(S^1)}^2 = (\nabla (dist_N(u)), \nabla (dist_N(u))_r)_{L^2(B)} + (\Delta (dist_N(u)), (dist_N(u))_r)_{L^2(B)}\n\n\leq C \|\nabla u\|_{L^2(B)} \|\nabla^2 (dist_N(u))\|_{L^2(B)} \leq C\delta \|\nabla^2 (dist_N(u))\|_{L^2(B)},
$$

where the basic  $L^2$ -theory for the Laplace equation  $(3.5)$  yields the bound

 $\|\nabla^2(dist_N(u))\|_{L^2(B)} \leq C \|\Delta(dist_N^i(u))\|_{L^2(B)} \leq C \|\nabla u\|_{L^4(B)}^2.$ 

With Sobolev's embedding  $H^{1/2}(B) \hookrightarrow L^4(B)$  we then conclude

 $\|\partial_r(distr_N(u))\|_{L^2(S^1)}^2 \leq C\delta \|\nabla u\|_{H^{1/2}(B)}^2.$ 

Thus from  $(3.4)$  and  $(3.1)$  we have

<span id="page-8-0"></span>(3.6) 
$$
\|\partial_r u\|_{L^2(S^1)}^2 \le \|f\|_{L^2(S^1)}^2 + \|\partial_r (dist_N(u))\|_{L^2(S^1)}^2 \le \|f\|_{L^2(S^1)}^2 + C\delta \|\nabla u\|_{H^{1/2}(B)}^2.
$$

But Fourier expansion of the harmonic function  $u$  gives

(3.7) 
$$
\|\partial_{\phi}u\|_{L^2(S^1)}^2 = \|\partial_r u\|_{L^2(S^1)}^2 = \frac{1}{2}\|\nabla u\|_{L^2(S^1)}^2
$$

as well as the bound

<span id="page-8-1"></span>
$$
\|\nabla u\|_{H^{1/2}(B)}^2 \le C \|\nabla u\|_{L^2(S^1)}^2,
$$

and from [\(3.6\)](#page-8-0) we obtain

$$
\|\partial_r u\|_{L^2(S^1)}^2 \le \|f\|_{L^2(S^1)}^2 + C\delta \|\nabla u\|_{H^{1/2}(B)}^2 \le \|f\|_{L^2(S^1)}^2 + C\delta \|\partial_r u\|_{L^2(S^1)}^2,
$$

which for sufficiently small  $\delta > 0$  by [\(3.7\)](#page-8-1) yields the claim.

In particular, from Proposition [3.1](#page-7-5) we obtain a positive energy threshold for non-constant solutions of [\(1.6\)](#page-2-1).

<span id="page-8-5"></span>**Corollary 3.2.** Suppose  $u \in H^{1/2}(S^1; N)$  smoothly solves [\(1.6\)](#page-2-1). Then, either u is constant, or  $E(u) \geq \delta_0^2$ , with  $\delta_0 = \delta_0(N) > 0$  given by Proposition [3.1.](#page-7-5)

Combining the ideas in the proof of the previous result with ideas from the classical proof of the Courant-Lebesgue lemma in minimal surface theory, we can obtain the following local version of Proposition [3.1.](#page-7-5)

<span id="page-8-4"></span>**Proposition 3.3.** There exists a constant  $\delta > 0$  with the following property. Given any smooth solution  $u \in H^{1/2}(S^1; N)$  of  $(3.1)$  with harmonic extension  $u \in H^1(B)$ , any  $z_0 \in \partial B$ , and any  $0 < R \leq 1/2$  such that

(3.8) 
$$
\int_{B_R(z_0)\cap B} |\nabla u|^2 dz < \delta^2,
$$

with a constant  $C = C(R) > 0$  there holds

<span id="page-8-2"></span>
$$
\int_{B_{R^2}(z_0)\cap S^1} |\partial_{\phi} u|^2 d\phi \le C \|f\|_{L^2(B_R(z_0)\cap S^1)}^2 + CE(u).
$$

*Proof.* Fix any  $z_0 \in \partial B$  and  $0 \leq R \leq 1/2$  such that [\(3.8\)](#page-8-2) holds. For suitable  $\rho \in [R^2, R]$ , with s denoting arc-length along the curve  $C_\rho = \{z_0 + \rho e^{i\theta} \in B; \theta \in \mathbb{R}\}\$ with end-points  $z_j = z_0 + \rho e^{i\theta_j} = e^{i\phi_j} \in \partial B$ ,  $j = 1, 2$ , we have

$$
\rho \int_{C_{\rho}} |\nabla u|^2 ds \le 2 \inf_{R^2 < \rho' < R} \left( \rho' \int_{C_{\rho'}} |\nabla u|^2 ds \right).
$$

We can bound the latter infimum by the average over  $\rho \in [R^2, R]$  with respect to the measure with density  $1/\rho$  to obtain the bound

<span id="page-8-3"></span>(3.9) 
$$
\rho \int_{C_{\rho}} |\nabla u|^2 ds \le 2 \int_{R^2}^R \int_{C_{\rho}} |\nabla u|^2 ds d\rho \Big/ \int_{R^2}^R \frac{d\rho}{\rho} \Big/ \int_{R^2}^R \frac{d\rho}{\rho} \Big/ \int_{B} |\nabla u|^2 dz \Big/ |\log(R)| = 4E(u)/|\log(R)|.
$$

Let  $\Phi_0: B \to B$  be the conformal map fixing the circular arc  $C_\rho$  and mapping the point  $z_0$  to the point  $-z_0$ , obtained as composition  $\Phi_0 = \pi_0^{-1} \circ \Psi_0 \circ \pi_0$  of stereographic projection  $\pi_0: B \to \mathbb{R}^2_+$  from the point  $-z_0$  and reflection  $\Psi_0: \mathbb{R}^2_+ \to$  $\mathbb{R}^2_+$  of the upper half-plane  $\mathbb{R}^2_+$  in the half-circle  $\pi_0(C_\rho)$ . Replacing u by the map  $u \circ \Phi_0$  in  $B \setminus B_\rho(z_0)$  we obtain a piecewise smooth map  $v_1 : B \to \mathbb{R}^n$  which is harmonic on  $B \setminus C_\rho$  and continuous on all of B. Let  $v_0 \in H^1(B)$  be harmonic with  $w := v_1 - v_0 \in H_0^1(B)$ . Note that by the variational characterization of harmonic functions and conformal invariance of the Dirichlet integral we have

(3.10) 
$$
E(v_0) \le E(v_1) \le \int_{B_R(z_0) \cap B} |\nabla u|^2 dz \le \delta^2.
$$

Moreover, for any smooth  $\varphi \in H_0^1(B)$  by  $(3.9)$  we can estimate

<span id="page-9-0"></span>
$$
\begin{aligned} \left| \int_{B} \nabla w \nabla \varphi dz \right| &= \left| \int_{B} \nabla v_{1} \nabla \varphi dz \right| = \left| \int_{C_{\rho}} [\partial_{\nu} v_{1}] \varphi ds \right| \\ &\leq \left( \int_{C_{\rho}} |\nabla u|^{2} ds \right)^{1/2} \left( \int_{C_{\rho}} |\varphi|^{2} ds \right)^{1/2} \leq C(R) E(u)^{1/2} \|\varphi\|_{H^{1/2}(B)}, \end{aligned}
$$

where  $[\partial_\nu v_1]$  denotes the difference of the outer and inner normal derivatives of  $v_1$ along  $C_\rho$ . Thus we have  $\Delta w \in H^{-1/2}(B)$ , and the basic  $L^2$ -theory for the Laplace equation gives  $w \in H^{3/2} \cap H_0^1(B)$  with

<span id="page-9-1"></span>
$$
||w||_{H^{3/2}(B)} \le \sup_{\varphi \in H_0^1(B), ||\varphi||_{H^{1/2}(B)} \le 1} \left( \int_B \nabla w \nabla \varphi dz \right) \le C(R) E(u)^{1/2}
$$

and then also

$$
(3.11) \t\t\t ||\partial_r w||_{L^2(S^1)}^2 \le C||w||_{H^{3/2}(B)}^2 \le C(R)E(u).
$$

In view of [\(3.10\)](#page-9-0), for sufficiently small  $\delta > 0$  from Proposition [3.1](#page-7-5) we obtain the estimate

$$
(3.12) \t\t\t ||\partial_{\phi}v_0||_{L^2(S^1)}^2 \leq C||d\pi_N(v_0)\partial_r v_0||_{L^2(S^1)}^2.
$$

Observe that since  $v_0 = v_1$  on  $\partial B = S^1$  and since we also have  $v_1 = u$  on  $B \cap B_\rho(z_0)$ ,  $v_1 = u \circ \Phi_0$  on  $B \setminus B_\rho(z_0)$ , respectively, we can bound

<span id="page-9-2"></span>
$$
||d\pi_N(v_0)\partial_r v_0||^2_{L^2(S^1)} = ||d\pi_N(v_1)\partial_r v_0||^2_{L^2(S^1)}
$$
  
\n
$$
\leq 2||d\pi_N(v_1)\partial_r v_1||^2_{L^2(S^1)} + 2||\partial_r w||^2_{L^2(S^1)}
$$

and

$$
||d\pi_N(v_1)\partial_r v_1||_{L^2(S^1)}^2 \leq C(R)||d\pi_N(u)\partial_r u||_{L^2(S^1 \cap B_\rho(z_0))}^2.
$$

Thus from [\(3.11\)](#page-9-1) we obtain

$$
||d\pi_N(v_0)\partial_r v_0||^2_{L^2(S^1)} \leq C(R)||d\pi_N(u)\partial_r u||^2_{L^2(S^1 \cap B_\rho(z_0)} + C||\partial_r w||^2_{L^2(S^1)}\leq C(R)||f||^2_{L^2(S^1 \cap B_\rho(z_0))} + C(R)E(u),
$$

and from [\(3.12\)](#page-9-2) there results the bound

$$
\|\partial_{\phi}u\|_{L^{2}(S^{1}\cap B_{\rho}(z_{0}))}^{2}=\|\partial_{\phi}v_{0}\|_{L^{2}(S^{1}\cap B_{\rho}(z_{0}))}^{2}\leq \|\partial_{\phi}v_{0}\|_{L^{2}(S^{1})}^{2}\leq C\|d\pi_{N}(v_{0})\partial_{r}v_{0}\|_{L^{2}(S^{1})}^{2}\leq C(R)\|f\|_{L^{2}(S^{1}\cap B_{R}(z_{0}))}^{2}+C(R)E(u),
$$

as claimed.  $\Box$ 

The local estimate Proposition [3.3](#page-8-4) also implies the following global bound.

$$
\Box
$$

**Proposition 3.4.** There exists a constant  $\delta > 0$  with the following property. Given any smooth solution  $u \in H^{1/2}(S^1; N)$  of  $(3.1)$ , any  $0 < R \leq 1/2$  with

(3.13) 
$$
\sup_{z_0 \in B} \int_{B_R(z_0) \cap B} |\nabla u|^2 dz < \delta^2,
$$

there holds

<span id="page-10-4"></span>
$$
\int_{S^1} |\partial_{\phi} u|^2 d\phi \le C(R) \|f\|_{L^2(S^1)}^2 + C(R)E(u).
$$

*Proof.* Covering ∂B with balls  $B_{R^2}(z_i)$ ,  $1 \le i \le i_0$ , from Proposition [3.3](#page-8-4) we obtain the claim. the claim.  $\Box$ 

<span id="page-10-0"></span>*Remark* 3.5. The proofs of the above propositions only require  $u \in H^1(S^1; N)$  with harmonic extension  $u \in H^{3/2}(B)$ .

### 4. Higher regularity

Again let  $u(t)$  be a smooth solution of the half-harmonic heat flow [\(1.3\)](#page-1-2) for  $0 < t < T_0$  with smooth initial data [\(1.4\)](#page-1-3). We show that as long as the flow does not concentrate energy in the sense of Theorem [1.1.](#page-2-2)ii) the solution remains smooth and can be a-priori bounded in any  $H^k$ -norm in terms of the data.

4.1.  $H^2$ -bound. In a first step we show an  $L^2$ -bound in space-time for the second derivatives of our solution to the flow [\(1.3\)](#page-1-2). Recall that by harmonicity, writing  $u = u(t)$ ,  $\partial_{\phi} u = u_{\phi}$ , and so on, for any  $0 < t < T_0$  we have [\(3.7\)](#page-8-1), that is,

<span id="page-10-3"></span>
$$
\int_{\partial B} |u_{\phi}|^2 d\phi = \int_{\partial B} |u_r|^2 d\phi,
$$

as Fourier expansion shows, with similar identities for partial derivatives of  $u$  of higher order. Indeed, writing

(4.1) 
$$
\Delta u = \frac{1}{r} (r u_r)_r + \frac{1}{r^2} u_{\phi\phi}
$$

we see that also  $\partial_{\phi}^{j}u$  and then also  $\nabla^{k-j}\partial_{\phi}^{j}u$  is harmonic for any  $j \leq k$  in  $\mathbb{N}_{0}$ , where  $\nabla u = (u_x, u_y)$  in Euclidean coordinates  $z = x + iy$ . Thus by induction we obtain

<span id="page-10-1"></span>(4.2) 
$$
\int_{\partial B} |\nabla^k u|^2 d\phi = 2 \int_{\partial B} |\nabla^{k-1} u_{\phi}|^2 d\phi = \dots = 2^k \int_{\partial B} |\partial_{\phi}^k u|^2 d\phi
$$

for any  $k \in \mathbb{N}$ . Similarly, for any  $1/4 < r < 1$  with uniform constants  $C > 0$  we have

$$
\int_{\partial B_r(0)} |\nabla^k u|^2 dz \le C \int_{\partial B_r(0)} |\nabla^{k-1} u_\phi|^2 dz \le \dots \le C \int_{\partial B_r(0)} |\partial_\phi^k u|^2 dz.
$$

Integrating, and using the mean value property of harmonic functions together with [\(4.2\)](#page-10-1) to bound

<span id="page-10-2"></span>
$$
\sup_{B_{1/4}(0)} |\nabla^k u|^2 \le C \int_{B \setminus B_{1/4}(0)} |\nabla^k u|^2 dz \le C \int_B |\nabla \partial_{\phi}^{k-1} u|^2 dz,
$$

in particular, for any  $k \in \mathbb{N}$  we have the bound

(4.3) 
$$
\int_{B} |\nabla^{k} u|^{2} dz \leq C \int_{B} |\nabla \partial_{\phi}^{k-1} u|^{2} dz
$$

with an absolute constant  $C > 0$ .

The following lemma is strongly reminiscent of analogous results for the harmonic map heat flow in two space dimensions.

<span id="page-11-1"></span>**Lemma 4.1.** With a constant  $C > 0$  depending only on N there holds

$$
\frac{d}{dt}\left(\int_{\partial B}|u_{\phi}|^{2}d\phi\right)+\int_{B}|\nabla u_{\phi}|^{2}dz\leq C\int_{B}|\nabla u|^{2}|u_{\phi}|^{2}dz.
$$

*Proof.* Writing  $d\pi_N(u) = 1 - d\pi_N^{\perp}(u)$  with

$$
d\pi_N^{\perp}(u)X = \nu(u)\nu(u) \cdot X = \sum_{i=1}^m \nu_i(u)\nu_i(u) \cdot X
$$

for any  $X \in \mathbb{R}^n$ , we compute

$$
\frac{1}{2}\frac{d}{dt}\left(\int_{\partial B}|u_{\phi}|^{2}d\phi\right) = \int_{\partial B}u_{\phi}\cdot u_{\phi,t}d\phi = -\int_{\partial B}u_{\phi\phi}\cdot u_{t}d\phi
$$
\n
$$
= \int_{\partial B}u_{\phi\phi}\cdot d\pi_{N}(u)u_{r}d\phi = -\int_{\partial B}\left(u_{\phi}\cdot u_{r\phi} - u_{\phi}\cdot\partial_{\phi}(\nu(u)\nu(u)\cdot u_{r})\right)d\phi
$$
\n
$$
= -\frac{1}{2}\int_{\partial B}\partial_{r}(|u_{\phi}|^{2})d\phi - \int_{\partial B}u_{\phi}\cdot d\nu(u)u_{\phi}\nu(u)\cdot u_{r}d\phi,
$$

where we use orthogonality  $u_{\phi} \cdot \nu_i(u) = 0$  on  $\partial B$ ,  $1 \leq i \leq m$ , in the last step. But  $u_{\phi}$  is harmonic. So with  $\Delta |u_{\phi}|^2 = 2|\nabla u_{\phi}|^2$ , from Gauss' theorem we obtain

$$
\frac{1}{2} \int_{\partial B} \partial_r (|u_\phi|^2) d\phi = \int_B |\nabla u_\phi|^2 dz.
$$

On the other hand, by Young's inequality we can estimate

$$
\int_{\partial B} u_r \cdot \nu(u) u_{\phi} \cdot d\nu(u) u_{\phi} d\phi = \int_B \nabla u \cdot \nabla (\nu(u) u_{\phi} \cdot d\nu(u) u_{\phi}) dz
$$
  
\n
$$
\leq C \int_B |\nabla u_{\phi}| |\nabla u| |u_{\phi}| dz + C \int_B |\nabla u|^2 |u_{\phi}|^2 dz
$$
  
\n
$$
\leq \frac{1}{2} \int_B |\nabla u_{\phi}|^2 dz + C \int_B |\nabla u|^2 |u_{\phi}|^2 dz,
$$

and our claim follows.  $\hfill \square$ 

Combining the previous result with a quantitative bound for the concentration of energy, we obtain a space-time bound for the second derivatives of  $u$ . Note that since u is smooth by assumption, for any  $\delta > 0$ , any  $T < T_0$  there exists a number  $R = R(T, u) > 0$  such that

<span id="page-11-0"></span>(4.4) 
$$
\sup_{z_0 \in B, 0 < t < T} \int_{B_R(z_0) \cap B} |\nabla u(t)|^2 \, dz < \delta.
$$

<span id="page-11-2"></span>**Proposition 4.2.** There exist constants  $\delta = \delta(N) > 0$  and  $C > 0$  such that for any  $T < T_0$  with  $R > 0$  as in [\(4.4\)](#page-11-0) there holds

<span id="page-11-3"></span>(4.5) 
$$
\sup_{0
$$
\leq C \int_{\partial B} |u_{0,\phi}|^2 d\phi + CTR^{-2}E(u_0).
$$
$$

*Proof.* For given  $T < T_0$  and  $\delta > 0$  to be determined we fix  $R > 0$  such that [\(4.4\)](#page-11-0) holds. Let  $B_{R/2}(z_i)$ ,  $1 \leq i \leq i_0$ , be a cover of B such that any point  $z_0 \in B$  belongs to at most L of the balls  $B_R(z_i)$ , where  $L \in \mathbb{N}$  is independent of  $R > 0$ . We then split

$$
\int_{B} |\nabla u|^2 |u_{\phi}|^2 dz \leq \sum_{i=1}^{i_0} \int_{B_{R/2}(z_i)} |\nabla u|^4 dz \leq \sum_{i=1}^{i_0} \int_{B} |\nabla (u \varphi_{z_i, R})|^4 dz.
$$

Using the multiplicative inequality  $(10.2)$  in the Appendix for each i we can bound

$$
\int_B |\nabla(u\varphi_{z_i,R})|^4 dz \leq C\delta \int_{B_R(z_i)} \left( |\nabla^2 u|^2 + R^{-2} |\nabla u|^2 \right) dz.
$$

Summing over  $1 \leq i \leq i_0$ , we thus obtain the bound

$$
\int_B |\nabla u|^2 |u_\phi|^2 dz \leq CL\delta \int_B |\nabla^2 u|^2 dz + CL\delta R^{-2} E(u)
$$
  

$$
\leq CL\delta \int_B |\nabla u_\phi|^2 dz + CL\delta R^{-2} E(u_0),
$$

and for sufficiently small  $\delta > 0$  from Lemma [4.1](#page-11-1) we obtain the claim.

With the help of Proposition [4.2](#page-11-2) we can now bound u in  $H^2(B)$  also uniformly in time. For this, we first note the following estimate, which also will be useful later for bounding higher order derivatives.

<span id="page-12-0"></span>**Lemma 4.3.** For any  $k \in \mathbb{N}$ , with a constant  $C > 0$  depending only on k and N, for the solution  $u = u(t)$  to [\(1.3\)](#page-1-2), [\(1.4\)](#page-1-3) for any  $0 < t < T_0$  there holds

$$
\frac{d}{dt} \left( \|\nabla \partial_{\phi}^{k} u\|_{L^{2}(B)}^{2} \right) + \|\partial_{\phi}^{k} u_{r}\|_{L^{2}(S^{1})}^{2} \n\leq C \sum_{1 \leq j_{i} \leq k+1, \ \Sigma_{i} j_{i} \leq k+2} \|\nabla \partial_{\phi}^{k} u\|_{L^{2}(B)} \|\Pi_{i} \nabla^{j_{i}} u\|_{L^{2}(B)}.
$$

*Proof.* For any  $k \in \mathbb{N}$  we use harmonicity of  $\partial_{\phi}^{2k} u$  to compute

$$
\frac{1}{2} \frac{d}{dt} \left( \|\nabla \partial_{\phi}^{k} u\|_{L^{2}(B)}^{2} \right) = (-1)^{k} \int_{B} \nabla \partial_{\phi}^{2k} u \nabla u_{t} dx
$$
\n
$$
(4.6) \qquad = (-1)^{k} (\partial_{\phi}^{2k} u_{r}, u_{t})_{L^{2}(S^{1})} = (-1)^{k+1} (\partial_{\phi}^{2k} u_{r}, d\pi_{N}(u) u_{r})_{L^{2}(S^{1})}
$$
\n
$$
= -(\partial_{\phi}^{k} u_{r}, \partial_{\phi}^{k} u_{r})_{L^{2}(S^{1})} + (\partial_{\phi}^{k} u_{r}, \partial_{\phi}^{k} (\nu(u) \nu(u) \cdot u_{r}))_{L^{2}(S^{1})}
$$
\n
$$
= -\|\partial_{\phi}^{k} u_{r}\|_{L^{2}(S^{1})}^{2} + I,
$$

where we split  $I = \sum_{j=0}^{k} \binom{k}{j} I_j$  with

$$
I_j = (\partial_{\phi}^k u_r, \partial_{\phi}^j (\nu(u) \nu(u)) \partial_{\phi}^{k-j} u_r)_{L^2(S^1)}
$$
  
= 
$$
(\nabla \partial_{\phi}^k u, \nabla (\partial_{\phi}^j (\nu(u) \nu(u)) \cdot \partial_{\phi}^{k-j} u_r))_{L^2(B)}.
$$

Hence for any  $1 \leq j \leq k$  we can bound

$$
\begin{split} |I_j| &\leq C\sum_{0\leq i\leq j}\|\nabla\partial_{\phi}^ku\|_{L^2(B)}\|\nabla\partial_{\phi}^{j-i}\nu(u)\partial_{\phi}^i\nu(u)\partial_{\phi}^{k-j}u_r\|_{L^2(B)}\\ &+C\sum_{0\leq i\leq j}\|\nabla\partial_{\phi}^ku\|_{L^2(B)}\|\partial_{\phi}^{j-i}\nu(u)\partial_{\phi}^i\nu(u)\nabla\partial_{\phi}^{k-j}u_r\|_{L^2(B)}\\ &\leq C\sum_{1\leq j_i\leq k+1,\,\Sigma_i j_i=k+2}\|\nabla\partial_{\phi}^ku\|_{L^2(B)}\|\Pi_i\nabla^{j_i}u\|_{L^2(B)}, \end{split}
$$

as claimed. It remains to bound the term  $I_0 = ||\partial_{\phi}^k u_r \cdot \nu(u)||_{L^2(S^1)}^2$ . With the signed distance function we can express

$$
\nu(u) \cdot u_{\phi r} = (\nu(u) \cdot u_r)_{\phi} - u_r \cdot d\nu(u)u_{\phi} = (dist_N(u))_{\phi r} - u_r \cdot d\nu(u)u_{\phi},
$$

so that

$$
I_0 = \|\partial_{\phi}^k u_r \cdot \nu(u)\|_{L^2(S^1)}^2 = (\partial_{\phi}^k u_r \cdot \nu(u), \partial_{\phi}^k (dist_N(u))_r)_{L^2(S^1)} + II
$$
  
=  $(\nabla \partial_{\phi}^k u, \nabla (\nu(u)\partial_{\phi}^k (dist_N(u))_r))_{L^2(B)} + II,$ 

where all terms in II can be dealt with as in the case  $1 \leq j \leq k$ . Finally, we have

$$
\begin{split} & \left( \nabla \partial_{\phi}^{k} u, \nabla \big( \nu(u) \partial_{\phi}^{k} (dist_{N}(u))_{r} \big) \right)_{L^{2}(B)} \\ &\leq \|\nabla \partial_{\phi}^{k} u\|_{L^{2}(B)} \big( \|\nabla^{2} \partial_{\phi}^{k} (dist_{N}(u))\|_{L^{2}(B)} + \|\nabla \nu(u) \partial_{\phi}^{k} (dist_{N}(u))_{r} \|_{L^{2}(B)} \big). \end{split}
$$

But by the chain rule we can bound

$$
\|\nabla \nu(u)\partial_{\phi}^k(dist_N(u))_r\|_{L^2(B)}) \leq C\|\nabla u\nabla^{k+1}(dist_N(u))\|_{L^2(B)})
$$
  
\n
$$
\leq C\sum_{1\leq j_i\leq k+1,\,\Sigma_i j_i=k+2}\|\Pi_i\nabla^{j_i}u\|_{L^2(B)}.
$$

Moreover, by [\(3.5\)](#page-7-1) and elliptic regularity theory, there holds

$$
\begin{aligned} \|\nabla^{k+2}(dist_N(u))\|_{L^2(B)}^2 &\leq C \|\Delta dist_N(u))\|_{H^k(B)}^2 \leq C \|\nabla u \cdot d\nu_i(u)\nabla u\|_{H^k((B)}^2\\ &\leq C \sum_{1 \leq j_i \leq k+1, \ \sum_i j_i \leq k+2} \|\Pi_i \nabla^{j_i} u\|_{L^2(B)}, \end{aligned}
$$

which gives the claim.  $\Box$ 

For  $k = 1$ , from Proposition [4.2](#page-11-2) we now easily derive a uniform  $L^2$ -bound for the second derivatives of the flow.

<span id="page-13-0"></span>**Proposition 4.4.** For any smooth  $u_0 \in H^{1/2}(S^1; N)$  and any  $T < T_0$  with  $R > 0$ as in Proposition [4.2](#page-11-2) with a constant  $C_1 = C_1(T, R, u_0) > 0$  depending on the right hand side of [\(4.5\)](#page-11-3) there holds

$$
\sup_{0
$$

*Proof.* For  $k = 1$  by Lemma [4.3](#page-12-0) we need to bound the term

$$
J = \sum_{1 \le j_i \le 2, \, \Sigma_i j_i \le 3} \|\Pi_i \nabla^{j_i} u\|_{L^2(B)} \le C \|\nabla^2 u\| \nabla u\| + |\nabla u|^3\|_{L^2(B)} + J_1,
$$

where  $J_1$  contains all terms of lower order. By the maximum principle and Sobolev's embedding  $H^1(\partial B) \hookrightarrow L^{\infty}(\partial B)$  we can estimate

$$
\|\nabla u\|_{L^{\infty}(B)}^2 \le \|\nabla u\|_{L^{\infty}(\partial B)}^2 \le C\|\nabla u\|_{H^1(\partial B)}^2 \le C\|u_{\phi r}\|_{L^2(\partial B)}^2 + C_1,
$$

where we have also used [\(3.7\)](#page-8-1) and Proposition [4.2.](#page-11-2) Also bounding

$$
\|\nabla u\|_{L^6(B)}^3 \le \|\nabla u\|_{L^4(B)}^2 \|\nabla u\|_{L^\infty(B)}\n\n\le C(\|\nabla^2 u\|_{L^2(B)} \|\nabla u\|_{L^2(B)} + E(u)) \|\nabla u\|_{L^\infty(B)}
$$

via [\(10.2\)](#page-39-4), and again using [\(3.7\)](#page-8-1) (and with similar, but simpler bounds for  $J_1$ ), we arrive at the estimate

$$
J \leq C \|\nabla^2 u\|\nabla u\| + |\nabla u|^3\|_{L^2(B)} + C_1
$$
  
\n
$$
\leq C \|\nabla^2 u\|_{L^2(B)} + E(u)\|\nabla u\|_{L^\infty(B)} + C_1
$$
  
\n
$$
\leq C \left(1 + \|\nabla u_\phi\|_{L^2(B)} + E(u_0)\right) \left(\|u_{\phi r}\|_{L^2(\partial B)} + C_1\right).
$$

With Lemma [4.3](#page-12-0) and Young's inequality we then have

<span id="page-14-0"></span>
$$
\frac{d}{dt} \left( 1 + \|\nabla u_{\phi}\|_{L^{2}(B)}^{2} \right) + \|u_{\phi r}\|_{L^{2}(S^{1})}^{2}
$$
\n
$$
\leq C \|\nabla u_{\phi}\|_{L^{2}(B)} \left( \|\nabla u_{\phi}\|_{L^{2}(B)} + E(u_{0}) \right) \left( \|u_{\phi r}\|_{L^{2}(\partial B)} + C_{1} \right)
$$
\n
$$
\leq \frac{1}{2} \|u_{\phi r}\|_{L^{2}(\partial B)}^{2} + C \left( 1 + \|\nabla u_{\phi}\|_{L^{2}(B)}^{2} \right) \left( \|\nabla u_{\phi}\|_{L^{2}(B)}^{2} + C_{1} \right).
$$

Absorbing the first term on the right on the left hand side of this inequality and dividing by  $1 + \|\nabla u_{\phi}\|_{L^2(B)}^2$  we obtain

$$
\frac{d}{dt} \big( \log \big( 1 + ||\nabla u_{\phi}||_{L^2(B)}^2 \big) \big) \leq C ||\nabla u_{\phi}||_{L^2(B)}^2 + C_1,
$$

and from Proposition [4.2](#page-11-2) we obtain the bound

$$
\sup_{0
$$

The claim then follows from  $(4.7)$ .

4.2. 
$$
H^3
$$
-**bounds.** The derivation of a-priori  $L^2$ -bounds for third derivatives of the solution  $u$  to the flow (1.3), (1.4) requires special care, which is why we highlight this case.

<span id="page-14-1"></span>**Proposition 4.5.** For any smooth  $u_0 \in H^{1/2}(S^1; N)$  and any  $T < T_0$  there holds

$$
\sup_{0
$$

where we denote as  $C_2 = C_2(T, R, u_0) > 0$  a constant bounded by the terms on the right hand side in the statements of Propositions [4.2](#page-11-2) and [4.4.](#page-13-0)

*Proof.* For  $k = 2$  by Lemma [4.3](#page-12-0) we need to bound the term

$$
J = \sum_{1 \leq j_i \leq 3, \ \Sigma_i j_i = 4} \|\Pi_i \nabla^{j_i} u\|_{L^2(B)}
$$
  
 
$$
\leq C \||\nabla u|^4 + |\nabla u|^2 |\nabla^2 u| + |\nabla^2 u|^2 + |\nabla u| |\nabla^3 u| \|_{L^2(B)}
$$

and corresponding terms involving at most 3 derivatives in total, which we will omit.

In dealing with the first term, by the multiplicative inequality [\(10.2\)](#page-39-4) and Sobolev's embedding  $H^2(B) \hookrightarrow L^{\infty}(B)$  we can estimate

$$
\|\nabla u\|_{L^8(B)}^4 \le \|\nabla u\|_{L^4(B)}^2 \|\nabla u\|_{L^\infty(B)}^2 \le C \|\nabla u\|_{H^1(B)} \|\nabla u\|_{L^2(B)} \|\nabla u\|_{L^\infty(B)}^2
$$
  
\n
$$
\le C (\|\nabla^2 u\|_{L^2(B)}^2 + E(u)) \|\nabla u\|_{L^\infty(B)}^2 \le C_2 \|\nabla u\|_{L^\infty(B)}^2
$$
  
\n
$$
\le C_2 (\|\nabla^3 u\|_{L^2(B)} + \|\nabla u\|_{L^2(B)}) \|\nabla u\|_{L^\infty(B)}
$$

with a constant  $C_2 = C_2(T, R, u_0) > 0$  as in the statement of the proposition. Similarly there holds

$$
\begin{aligned} \|\nabla^2 u\|_{L^4(B)}^2 &\leq C \|\nabla^2 u\|_{H^1(B)} \|\nabla^2 u\|_{L^2(B)} \\ &\leq \|\nabla^3 u\|_{L^2(B)} \|\nabla^2 u\|_{L^2(B)} + \|\nabla^2 u\|_{L^2(B)}^2 \leq C_2 (1 + \|\nabla^3 u\|_{L^2(B)}). \end{aligned}
$$

Hence we can also bound

$$
\|\nabla u|^2 |\nabla^2 u|\|_{L^2(B)} \leq \|\nabla u\|_{L^8(B)}^4 + \|\nabla^2 u\|_{L^4(B)}^2
$$
  

$$
\leq C_2 (1 + \|\nabla^3 u\|_{L^2(B)})(1 + \|\nabla u\|_{L^\infty(B)}).
$$

Finally, we estimate

$$
\||\nabla u||\nabla^3 u|\|_{L^2(B)} \le \|\nabla^3 u\|_{L^2(B)} \|\nabla u\|_{L^\infty(B)}
$$

to obtain

$$
J \leq C_2(1 + \|\nabla^3 u\|_{L^2(B)})(1 + \|\nabla u\|_{L^\infty(B)}).
$$

But with the inequality

$$
||f||_{L^{\infty}(B)} \leq C||f||_{H^{1}(B)}(1 + \log^{1/2}(1 + ||f||_{H^{2}(B)}/||f||_{H^{1}(B)})
$$

for  $f \in H^2(B)$  due to Brezis-Gallouet [\[1\]](#page-39-5) (see also Brezis-Wainger [\[3\]](#page-39-6) for a more general version) we have

$$
\|\nabla u\|_{L^{\infty}(B)}^2 \le C \|\nabla u\|_{H^1(B)}^2 \left(1 + \log(1 + \|\nabla u\|_{H^2(B)} / \|\nabla u\|_{H^1(B)})\right)
$$
  

$$
\le C_2 (1 + \log(1 + \|\nabla^3 u\|_{L^2(B)})),
$$

and Lemma [4.3](#page-12-0) yields the differential inequality

$$
\frac{d}{dt} \left( \|\nabla \partial_{\phi}^2 u\|_{L^2(B)}^2 \right) + \|u_{\phi\phi r}\|_{L^2(\partial B)}^2
$$
\n
$$
\leq C_2 \|\nabla \partial_{\phi}^2 u\|_{L^2(B)} (1 + \|\nabla^3 u\|_{L^2(B)}) \left(1 + \log(1 + \|\nabla^3 u\|_{L^2(B)})\right)
$$

Simplifying, and recalling that  $\|\nabla^3 u\|_{L^2(B)}^2 \leq C \|\nabla \partial^2_{\phi} u\|_{L^2(B)}^2$  by [\(4.3\)](#page-10-2), we then find

$$
\frac{d}{dt} \left( 1 + \|\nabla \partial_{\phi}^2 u\|_{L^2(B)} \right) \n\leq C_2 (1 + \|\nabla \partial_{\phi}^2 u\|_{L^2(B)}) \left( 1 + \log(1 + \|\nabla \partial_{\phi}^2 u\|_{L^2(B)}) \right);
$$

that is, we have

$$
\frac{d}{dt}\big(1+\log(1+\|\nabla\partial^2_{\phi}u\|_{L^2(B)})\big)\leq C_2\big(1+\log(1+\|\nabla\partial^2_{\phi}u\|_{L^2(B)})\big).
$$

Arguing as in the proof of Proposition [4.4](#page-13-0) we then obtain the claim.  $\Box$ 

.

$$
16\quad
$$

4.3.  $H^m$ -bounds,  $m \geq 4$ . In view of Proposition [4.5](#page-14-1) we can now use induction to prove the following result.

<span id="page-16-1"></span>**Proposition 4.6.** For any  $k \geq 3$ , any smooth  $u_0 \in H^{1/2}(S^1; N)$ , and any  $T < T_0$ there holds

$$
\sup_{0
$$

where we denote as  $C_k = C_k(T, R, u_0) > 0$  a constant bounded by the terms on the right hand side in the statement of the proposition for  $k - 1$ .

*Proof.* By Proposition [4.5](#page-14-1) the claimed result holds true for  $k = 2$ . Suppose the claim holds true for some  $k_0 \geq 2$  and let  $k = k_0 + 1$ . Note that by Sobolev's embedding  $H^2(B) \hookrightarrow W^{1,4} \cap C^0(\overline{B})$  and [\(4.3\)](#page-10-2) for  $0 \leq t < T$  we then have the uniform bounds

<span id="page-16-0"></span>
$$
\|\nabla^{k_0+1}u\|_{L^2(B)}^2 + \|\nabla^{k_0}u\|_{L^4(B)}^2 + \sum_{1 \le j \le k_0-1} \|\nabla^j u\|_{L^\infty(B)}^2
$$
  

$$
\le C_{k_0} \|\nabla^{k_0+1}u_0\|_{L^2(B)}^2 + C_{k_0} \le C_k < \infty
$$

with a constant of the type  $C_k$ , as defined above.

By Lemma [4.3](#page-12-0) again we only need to bound the term

$$
J = \sum_{1 \le j_i \le k+1, \, \Sigma_i j_i \le k+2} \|\Pi_i \nabla^{j_i} u\|_{L^2(B)}.
$$

Clearly we have

$$
J \leq \|\nabla^{k+1}u\|_{L^{2}(B)}\|\nabla u\|_{L^{\infty}(B)} + \|\nabla^{k}u\|_{L^{2}(B)}\|\nabla u\|_{L^{\infty}(B)}^{2} + \|\nabla^{k}u\nabla^{2}u\|_{L^{2}(B)} + \|\nabla^{k-1}u\nabla^{3}u\|_{L^{2}(B)} + \|\nabla^{k-1}u\nabla^{2}u\|_{L^{2}(B)}\|\nabla u\|_{L^{\infty}(B)} + C_{k}
$$
  
\n
$$
\leq C_{k}\|\nabla^{k+1}u\|_{L^{2}(B)} + \|\nabla^{k}u\nabla^{2}u\|_{L^{2}(B)} + \|\nabla^{k-1}u\nabla^{3}u\|_{L^{2}(B)} + C_{k}.
$$

We now distinguish the following cases: If  $k - 1 = k_0 \geq 3$  by [\(4.8\)](#page-16-0) we can bound  $\|\nabla^k u \nabla^2 u\|_{L^2(B)} \le \|\nabla^k u\|_{L^2(B)} \|\nabla^2 u\|_{L^{\infty}(B)} \le C_{k_0} \|\nabla^{k_0+1} u\|_{L^2(B)}^2 + C_{k_0} \le C_k$ 

as well as

$$
\|\nabla^{k-1}u\nabla^3u\|_{L^2(B)} \le \|\nabla^{k-1}u\|_{L^4(B)}\|\nabla^3u\|_{L^4(B)} \le C_{k_0}\|\nabla^{k_0}u\|_{L^4(B)}^2 + C_{k_0} \le C_k
$$

to obtain the estimate

$$
J \leq C_k \|\nabla^{k+1} u\|_{L^2(B)} + C_k.
$$

If, on the other hand,  $k_0 = k - 1 = 2$ , by our induction hypothesis [\(4.8\)](#page-16-0) we have

$$
\begin{aligned} \|\nabla^{k-1} u \nabla^3 u\|_{L^2(B)} &= \|\nabla^2 u \nabla^k u\|_{L^2(B)}^2 \le \|\nabla^k u\|_{L^4(B)} \|\nabla^2 u\|_{L^4(B)} \\ &\le C_k \|\nabla^k u\|_{H^1(B)} \le C_k \|\nabla^{k+1} u\|_{L^2(B)} + C_k, \end{aligned}
$$

and we find

$$
J \le C_k ||\nabla^{k+1} u||_{L^2(B)} + C_k
$$

as before.

In any case, inequality [\(4.3\)](#page-10-2) and Lemma [4.3](#page-12-0) now may be invoked to obtain

$$
\frac{d}{dt} \left( \|\nabla \partial_{\phi}^k u\|_{L^2(B)}^2 \right) \leq C_k \|\nabla \partial_{\phi}^k u\|_{L^2(B)}^2 + C_k,
$$

and our claim follows.  $\Box$ 

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4.4. Local  $H^2$ -bounds. The bounds established so far all require the initial data to be sufficiently smooth for the estimate at hand and do not yet allow to show smoothing of the flow. For the latter purpose we next prove a second set of "intermediate" estimates that in combination with the first set of estimates later will allow boot-strapping. Moreover, in contrast to the estimates established so far, the following estimates may be localized. This will be important for showing regularity of the flow at blow-up times away from concentration points of the energy on  $\partial B$ .

For the localized estimates, fix a point  $z_0 \in \partial B$  and some radius  $0 < R_0 < 1/4$ and for  $k \in \mathbb{N}$  set  $R_k = 2^{-k}R_0$ ,  $\varphi_k = \varphi_{z_0, R_k}$ . Set  $\varphi_k = 1$  for each  $k \in \mathbb{N}$  for the analogous global bounds.

We first establish the following localized version of Lemma [4.1.](#page-11-1)

<span id="page-17-0"></span>**Lemma 4.7.** With a constant  $C > 0$  depending only on N there holds

$$
\frac{d}{dt}\left(\int_{\partial B} |u_{\phi}|^2 \varphi_1^2 d\phi\right) + \int_B |\nabla u_{\phi}|^2 \varphi_1^2 dz \le C \int_B |\nabla u|^2 |u_{\phi}|^2 \varphi_1^2 dz + CR_0^{-2} E(u_0).
$$

Proof. Similar to the proof of Lemma [4.1,](#page-11-1) we compute

$$
\frac{1}{2} \frac{d}{dt} \left( \int_{\partial B} |u_{\phi}|^2 \varphi_1^2 d\phi \right) = \int_{\partial B} u_{\phi} \cdot u_{\phi, t} \varphi_1^2 d\phi = - \int_{\partial B} \partial_{\phi} (u_{\phi} \varphi_1^2) \cdot u_t d\phi
$$
\n
$$
= \int_{\partial B} \partial_{\phi} (u_{\phi} \varphi_1^2) \cdot d\pi_N(u) u_r d\phi = - \int_{\partial B} (u_{\phi} \cdot u_{r\phi} - u_{\phi} \cdot \partial_{\phi} (\nu(u) \nu(u) \cdot u_r)) \varphi_1^2 d\phi
$$
\n
$$
= -\frac{1}{2} \int_{\partial B} \partial_r (|u_{\phi}|^2) \varphi_1^2 d\phi - \int_{\partial B} u_{\phi} \cdot d\nu(u) u_{\phi} \nu(u) \cdot u_r \varphi_1^2 d\phi.
$$

With  $\Delta |u_{\phi}|^2 = 2|\nabla u_{\phi}|^2$  we obtain

$$
\frac{1}{2} \int_{\partial B} \partial_r (|u_\phi|^2) \varphi_1^2 d\phi = \int_B |\nabla u_\phi|^2 \varphi_1^2 dz + \int_B \nabla |u_\phi|^2 \varphi_1 \nabla \varphi_1 dz,
$$

where

$$
\big|\int_{B}\nabla |u_{\phi}|^2\varphi_1\nabla \varphi_1 dz\big|\leq \frac{1}{4}\int_{B}|\nabla u_{\phi}|^2\varphi_1^2dz+C\int_{B}|u_{\phi}|^2|\nabla \varphi_1|^2dz
$$

by Young's inequality. Finally, we can bound

$$
\int_{\partial B} u_r \cdot \nu(u) u_{\phi} \cdot d\nu(u) u_{\phi} \varphi_1^2 d\phi = \int_B \nabla u \cdot \nabla \big( \nu(u) u_{\phi} \cdot d\nu(u) u_{\phi} \varphi_1^2 \big) dz
$$
\n
$$
\leq C \int_B \left( |\nabla u_{\phi}| |\nabla u| |u_{\phi}| + |\nabla u|^2 |u_{\phi}|^2 \right) \varphi_1^2 dz + C \int_B |\nabla u| |\nabla \varphi_1| |u_{\phi}|^2 \varphi_1 dz
$$
\n
$$
\leq \frac{1}{4} \int_B |\nabla u_{\phi}|^2 \varphi_1^2 dz + C \int_B |\nabla u|^2 |u_{\phi}|^2 \varphi_1^2 dz + C \int_B |\nabla u|^2 |\nabla \varphi_1|^2 dz,
$$

and our claim follows.  $\Box$ 

We need a substitute for the global bound [\(4.3\)](#page-10-2). For this, we note that the equation [\(4.1\)](#page-10-3) also implies the pointwise bound  $|u_{rr}|^2 \leq 2|u_{\phi\phi}|^2/r^4 + 2|u_r|^2/r^2$ ; hence we have

$$
|\nabla^2 u|^2 \leq C(|\nabla u_{\phi}|^2 + 2|\nabla u|^2) \text{ in } B_{R_0}(z_0)
$$

with an absolute constant  $C > 0$ , uniformly in  $z_0 \in \partial B$  and  $0 < R_0 < 1/4$ . By induction then, similarly we have

<span id="page-18-2"></span>(4.9) 
$$
|\nabla^{k+1} u|^2 \le C(|\nabla^k \partial_{\phi} u|^2 + |\nabla^k u|^2) \le C \sum_{j=0}^k |\nabla \partial_{\phi}^j u|^2 \text{ in } B_{R_0}(z_0)
$$

with an absolute constant  $C = C(k) > 0$ , uniformly in  $z_0 \in \partial B$  and  $0 < R_0 < 1/4$ for any  $k \in \mathbb{N}$ .

Likewise, as a substitute for the global non-concentration condition [\(4.4\)](#page-11-0) we now suppose that  $z_0 \in \partial B$  is not a concentration point in the sense that for suitably chosen  $\delta > 0$  to be determined in the sequel and some  $0 < R_0 < 1/4$  as above there holds

<span id="page-18-0"></span>(4.10) 
$$
\sup_{0 < t < T_0} \int_{B_{R_0}(z_0) \cap B} |\nabla u(t)|^2 dz < \delta.
$$

We then obtain the following localized version of Proposition [4.2.](#page-11-2)

<span id="page-18-1"></span>**Proposition 4.8.** There exist constants  $\delta > 0$  and  $C > 0$  independent of  $R_0 > 0$ such that whenever [\(4.10\)](#page-18-0) holds then for any  $T \leq T_0$  we have

$$
\sup_{0 < t < T} \int_{\partial B} |u_{\phi}(t)|^2 \varphi_1^2 d\phi + \int_0^T \int_B |\nabla u_{\phi}|^2 \varphi_1^2 dz dt
$$
  

$$
\leq 2 \int_{\partial B} |u_{0,\phi}|^2 \varphi_1^2 d\phi + CTR_0^{-2}E(u_0).
$$

Proof. With the help of the inequality [\(10.1\)](#page-39-7) in the Appendix we can bound

$$
\int_{B}|\nabla u|^4\varphi_1^2dz\leq C\delta\int_{B_R(z_i)}|\nabla^2 u|^2\varphi_1^2dz+C\delta R_0^{-2}\int_{B_R(z_i)}|\nabla u|^2dz.
$$

Thus, for sufficiently small  $\delta > 0$  our claim follows from Lemma [4.7.](#page-17-0)

The next lemma again prepares for a proposition that later will allow us to obtain higher derivative bounds by induction. Note the differences to Lemma [4.3.](#page-12-0)

<span id="page-18-3"></span>**Lemma 4.9.** For any  $k \geq 2$ , with a constant  $C > 0$  depending only on k and N, for the solution  $u = u(t)$  to [\(1.3\)](#page-1-2), [\(1.4\)](#page-1-3) for any  $0 < t < T_0$  there holds

$$
\frac{d}{dt} (\|\partial_{\phi}^{k} u \varphi_{k}\|_{L^{2}(\partial B)}^{2}) + \|\nabla \partial_{\phi}^{k} u \varphi_{k}\|_{L^{2}(B)}^{2}
$$
\n
$$
\leq C \sum_{1 \leq j_{i} \leq k, \ \sum_{i} j_{i} \leq 2k+2} \|\Pi_{i} \nabla^{j_{i}} u \varphi_{k}^{2}\|_{L^{1}(B)} + C \sum_{1 \leq j_{i} \leq k, \ \sum_{i \geq 0} j_{i} \leq k+1} \|\Pi_{i > 0} \nabla^{j_{i}} u \nabla^{j_{0}} \varphi_{k}\|_{L^{2}(B)}^{2} + C R_{0}^{-2k} E(u_{0}).
$$

*Proof.* Fix  $k \geq 2$ . With  $\Delta |\partial_{\phi}^k u|^2 = 2|\nabla \partial_{\phi}^k u|^2$  we compute

$$
\frac{1}{2} \frac{d}{dt} \left( \| \partial_{\phi}^{k} u \varphi_{k} \|_{L^{2}(\partial B)}^{2} \right) = (-1)^{k} \int_{\partial B} \partial_{\phi}^{k} (\partial_{\phi}^{k} u \varphi_{k}^{2}) \cdot u_{t} d\phi
$$
\n
$$
= (-1)^{k+1} \int_{\partial B} \partial_{\phi}^{k} (\partial_{\phi}^{k} u \varphi_{k}^{2}) \cdot (u_{r} - \nu(u) \nu(u) \cdot u_{r}) d\phi
$$
\n
$$
= -\frac{1}{2} \int_{\partial B} \partial_{r} (|\partial_{\phi}^{k} u|^{2}) \varphi_{k}^{2} d\phi + \int_{\partial B} \partial_{\phi}^{k} u \cdot \partial_{\phi}^{k} (\nu(u) \nu(u) \cdot u_{r}) \varphi_{k}^{2} d\phi.
$$
\n
$$
= -\int_{B} |\nabla \partial_{\phi}^{k} u|^{2} \varphi_{k}^{2} dz - \int_{B} \nabla (|\partial_{\phi}^{k} u|^{2}) \varphi_{k} \nabla \varphi_{k} dz + I,
$$

where we split

$$
I = \int_{\partial B} \partial_{\phi}^{k} u \cdot \partial_{\phi}^{k} (\nu(u) \nu(u) \cdot u_{r})) \varphi_{k}^{2} d\phi = \sum_{j=0}^{k} {k \choose j} I_{j}
$$

with

$$
I_j = (\partial_{\phi}^k u \cdot \partial_{\phi}^j (\nu(u) \nu(u)) \varphi_k^2, \partial_{\phi}^{k-j} u_r)_{L^2(\partial B)}
$$
  
= 
$$
\left( \nabla (\partial_{\phi}^k u \cdot \partial_{\phi}^j (\nu(u) \nu(u)) \varphi_k^2), \nabla \partial_{\phi}^{k-j} u \right)_{L^2(B)}, \ 0 \le j \le k.
$$

For  $1\leq j\leq k$  we bound

$$
\begin{aligned} |I_j| \leq & C \sum_{0 \leq i \leq j} \|\nabla \partial_{\phi}^k u \varphi_k\|_{L^2(B)} \|\partial_{\phi}^{j-i} \nu(u) \partial_{\phi}^i \nu(u) \nabla \partial_{\phi}^{k-j} u \varphi_k\|_{L^2(B)} \\ & + C \sum_{0 \leq i \leq j} \|\partial_{\phi}^k u \cdot \nabla \big(\partial_{\phi}^{j-i} \nu(u) \partial_{\phi}^i \nu(u) \varphi_k^2\big) \cdot \nabla \partial_{\phi}^{k-j} u\|_{L^1(B)} \end{aligned}
$$

By the chain rule then for  $1\leq j\leq k$  we have

$$
|I_j| \leq C \sum_{1 \leq j_i \leq k, \ \sum_{i} j_i = k+1} \|\nabla \partial_{\phi}^k u \varphi_k\|_{L^2(B)} \|\Pi_i \nabla^{j_i} u \varphi_k\|_{L^2(B)} + C \sum_{1 \leq j_i \leq k, \ \sum_{i} j_i = k+2} \|\partial_{\phi}^k u \cdot \Pi_i \nabla^{j_i} u \varphi_k^2\|_{L^1(B)} + C \sum_{1 \leq j_i \leq k, \ \sum_{i} j_i = k+1} \|\partial_{\phi}^k u \cdot \Pi_i \nabla^{j_i} u \varphi_k \nabla \varphi_k\|_{L^1(B)}.
$$

By Cauchy-Schwarz and Young's inequality then we can bound

$$
\sum_{1 \leq j \leq k} |I_j| \leq \frac{1}{4} \|\nabla \partial_{\phi}^k u \varphi_k\|_{L^2(B)}^2 + C \sum_{1 \leq j_i \leq k, \ \Sigma_i j_i = k+1} \|\Pi_i \nabla^{j_i} u \varphi_k\|_{L^2(B)}^2 + C \sum_{1 \leq j_i \leq k, \ \Sigma_i j_i = 2k+2} \|\Pi_i \nabla^{j_i} u \varphi_k^2\|_{L^1(B)} + C \|\partial_{\phi}^k u \nabla \varphi_k\|_{L^2(B)}^2 + \sum_{\substack{1 \leq j_i \leq k, \ \Sigma_i j_i = 2k+2}} \|\Pi_i \nabla^{j_i} u \varphi_k^2\|_{L^1(B)} + C \|\partial_{\phi}^k u \nabla \varphi_k\|_{L^2(B)}^2,
$$

as claimed. Finally, with

$$
\nu(u) \cdot u_{\phi r} = (dist_N(u))_{\phi r} - u_r \cdot d\nu(u)u_{\phi}
$$

as in the proof of Lemma [4.3,](#page-12-0) for  $j = 0$  we can write

$$
\nu(u) \cdot \partial_{\phi}^{k} u_{r} = \partial_{\phi}^{k-1} (\nu(u) \cdot u_{\phi r}) + II = \partial_{\phi}^{k} (dist_{N}(u))_{r} + III,
$$

where the terms in  $II$  and  $III$  involve products of at least two derivatives of orders between 1 and  $k$  of  $u$ . Thus we have

$$
I_0 = (\partial_{\phi}^k u \cdot \nu(u)\varphi_k^2, \nu(u) \cdot \partial_{\phi}^k u_r)_{L^2(\partial B)}
$$
  
= 
$$
(\partial_{\phi}^k u \cdot \nu(u)\varphi_k^2, \partial_{\phi}^k (dist_N(u))_r)_{L^2(\partial B)} + II_0
$$

with a term  $II_0$  that can be dealt with in the same way as the terms  $I_j$ ,  $1 \leq j \leq k$ .

Using the divergence theorem and integrating by parts we can write the leading term as

$$
\begin{aligned} \hat{I}_0 &:= (\partial_\phi^k u \cdot \nu(u) \varphi_k^2, \partial_\phi^k (dist_N(u))_r)_{L^2(\partial B)} \\ & = \left( \nabla \big( \partial_\phi^k u \cdot \nu(u) \varphi_k^2 \big), \nabla \partial_\phi^k (dist_N(u)) \big)_{L^2(B)} \right. \\ & \qquad \qquad + \left( \partial_\phi^k u \cdot \nu(u) \varphi_k^2, \Delta \partial_\phi^k (dist_N(u)) \right)_{L^2(B)} \\ & = \left( \nabla \big( \partial_\phi^k u \cdot \nu(u) \varphi_k^2 \big), \nabla \partial_\phi^k (dist_N(u)) \big)_{L^2(B)} \right. \\ & \qquad \qquad - \left( \partial_\phi \big( \partial_\phi^k u \cdot \nu(u) \varphi_k^2 \big), \Delta \partial_\phi^{k-1} (dist_N(u)) \right)_{L^2(B)} \end{aligned}
$$

to see that this term may be bounded

$$
|\hat{I}_0| \leq C \| (|\nabla \partial_{\phi}^k u| + |\partial_{\phi}^k u \nabla u|) \varphi_k + |\partial_{\phi}^k u \nabla \varphi_k| \|_{L^2(B)} \| \nabla^{k+1}(dist_N(u)) \varphi_k \|_{L^2(B)}.
$$

But by elliptic regularity we again have

$$
\|\nabla^{k+1}(dist_N(u))\varphi_k\|_{L^2(B)}\leq \|\nabla^{k+1}(dist_N(u)\varphi_k)\|_{L^2(B)} + C \sum_{1 \leq j \leq k+1} \|\nabla^{k+1-j}(dist_N(u))\nabla^j \varphi_k\|_{L^2(B)}\leq C \|\Delta(dist_N(u)\varphi_k)\|_{H^{k-1}(B)} + C \sum_{1 \leq j \leq k+1} \|\nabla^{k+1-j}(dist_N(u))\nabla^j \varphi_k\|_{L^2(B)},
$$

where from [\(3.5\)](#page-7-1) we can bound the first term on the right

$$
\|\Delta(dist_N(u))\varphi_k\|_{H^{k-1}(B)} \leq \sum_{0 \leq j < k} \|\nabla^j (\nabla u \cdot d\nu(u) \nabla u \varphi_k)\|_{L^2(B)}
$$
  

$$
\leq C \sum_{0 \leq j_0 < k, 1 \leq j_i \leq k, \ \Sigma_{i \geq 0} j_i \leq k+1} \|\Pi_i \nabla^{j_i} u \nabla^{j_0} \varphi_k\|_{L^2(B)}.
$$

Moreover, using that  $dist_N(u) = 0$  on  $\partial B$ , with the help of Poincaré's inequality we find the bound

$$
||dist_N(u)\nabla^{k+1}\varphi_k||_{L^2(B)}^2 \leq CR_k^{-2k} ||\nabla dist_N(u))||_{L^2(B_{R_k}(z_0))}^2 \leq CR_0^{-2k}E(u).
$$

The remaining terms for  $1 \leq j \leq k$  can be estimated

$$
\|\nabla^{k+1-j}(\operatorname{dist}_N(u))\nabla^j\varphi_2\|_{L^2(B)} \leq C \sum_{1 \leq j_i \leq k, \ \sum_{ij} = k+1-j} \|\Pi_i\nabla^{j_i}u\nabla^j\varphi_2\|_{L^2(B)}
$$

via the chain rule. Thus, finally, we obtain the bound

$$
\|\nabla^{k+1}(dist_N(u))\varphi_k\|_{L^2(B)}\n\n\leq C\sum_{1\leq j_0,j_i\leq k,\ \Sigma_{i\geq 0}j_i\leq k+1}\|\Pi_{i>0}\nabla^{j_i}u\nabla^{j_0}\varphi_2\|_{L^2(B)}+CR_0^{-2k}E(u_0).
$$

By Cauchy-Schwarz and Young's inequality thus we can bound

$$
|\hat{I}_0| \leq \frac{1}{4} \|\nabla \partial_{\phi}^k u \varphi_k\|_{L^2(B)}^2 + C \|\partial_{\phi}^k u \nabla u \varphi_k\|_{L^2(B)}^2 + C \sum_{1 \leq j_i \leq k, \ \Sigma_i j_i \leq k+1} \|\Pi_{i>0} \nabla^{j_i} u \nabla^{j_0} \varphi_k\|_{L^2(B)}^2 + C R_0^{-2k} E(u_0),
$$

and together with our above estimate for the terms  $I_j, j \geq 1$ , our claim follows.  $\Box$ 

<span id="page-21-0"></span>**Proposition 4.10.** There exists a constant  $\delta > 0$  independent of  $R_0 > 0$  such that whenever [\(4.10\)](#page-18-0) holds then for any  $T \leq T_0$  with a constant  $C_2 = C_2(T, R, u_0) > 0$ bounded by the terms on the right hand side in the statement of Proposition [4.8](#page-18-1) there holds the estimate

$$
\sup_{0 < t < T} \int_{\partial B} |u_{\phi\phi}(t)|^2 \varphi_2^2 d\phi + \int_0^T \int_B |\nabla u_{\phi\phi}|^2 \varphi_2^2 dz dt
$$
  

$$
\leq C_2 \int_{\partial B} |u_{0,\phi\phi}|^2 \varphi_2^2 d\phi + C_2.
$$

*Proof.* For  $k = 2$  with the help of Young's inequality we can bound

$$
J_1 = \sum_{1 \le j_i \le k, \ \sum_{i} j_i \le 2k+2} ||\Pi_i \nabla^{j_i} u \varphi_k^2||_{L^1(B)}
$$
  
\n
$$
\le C ||(\nabla^2 u|^3 + |\nabla^2 u|^2 |\nabla u|^2 + |\nabla^2 u||\nabla u|^4 + |\nabla u|^6 + 1)\varphi_2^2||_{L^1(B)}
$$
  
\n
$$
\le C ||(\nabla^2 u|^3 + |\nabla u|^6 + 1)\varphi_2^2||_{L^1(B)},
$$

and

$$
J_2 = \sum_{1 \le j_0, j_i \le k, \ \Sigma_{i \ge 0} j_i \le k+1} || \Pi_{i > 0} \nabla^{j_i} u \nabla^{j_0} \varphi_2 ||_{L^2(B)}^2
$$
  

$$
\le C || (|\nabla^2 u|^2 + |\nabla u|^4 + 1) |\nabla \varphi_2|^2 + (|\nabla u|^2 + 1) |\nabla^2 \varphi_2|^2 ||_{L^1(B)}.
$$

Observing that  $\varphi_1 = 1$  on the support of  $\varphi_2$ , by [\(10.2\)](#page-39-4) for the first term in  $J_1$ we have

$$
\begin{aligned}\n\|\nabla^2 u|^3 \varphi_2^2 \|_{L^1(B)} &\leq \|\nabla^2 u \varphi_2\|_{L^4(B)}^2 \|\nabla^2 u \varphi_1\|_{L^2(B)} \\
&\leq C \|\nabla^2 u \varphi_2\|_{H^1(B)} \|\nabla^2 u \varphi_2\|_{L^2(B)} \|\nabla^2 u \varphi_1\|_{L^2(B)} \\
&\leq C \|\nabla^3 u \varphi_2\|_{L^2(B)} + \|\nabla^2 u \varphi_1\|_{L^2(B)} \|\nabla^2 u \varphi_2\|_{L^2(B)} \|\nabla^2 u \varphi_1\|_{L^2(B)}.\n\end{aligned}
$$

Moreover, arguing as in [\(10.1\)](#page-39-7) for the function  $|\nabla u|^6 \varphi_2^2$  in place of  $|v|^4 \varphi^2$ , we can bound

$$
\int_{B} |\nabla u|^6 \varphi_2^2 dz \le C \Big( \int_{B} \left( |\nabla^2 u| |\nabla u|^2 \varphi_2 + |\nabla u|^3 |\nabla \varphi_2| \right) dz \Big)^2
$$
  

$$
\le C \Big( \int_{B} |\nabla^2 u|^3 \varphi_2^2 dz \Big)^{2/3} \Big( \int_{B} |\nabla u|^3 \varphi_2^{1/2} dz \Big)^{4/3} + C \Big( \int_{B} |\nabla u|^3 |\nabla \varphi_2| \Big) dz \Big)^2,
$$

where by Hölder's inequality we have

$$
\int_{B} |\nabla u|^3 \varphi_2^{1/2} dz \le \Big( \int_{B} |\nabla u|^6 \varphi_2^2 dz \Big)^{1/4} \Big( \int_{B} |\nabla u|^2 \varphi_1^2 dz \Big)^{3/4}
$$

so that with Young's inequality we obtain

$$
\int_{B} |\nabla u|^6 \varphi_2^2 dz \le C \delta \Big( \int_{B} |\nabla^2 u|^3 \varphi_2^2 dz \Big)^{2/3} \Big( \int_{B} |\nabla u|^6 \varphi_2^2 dz \Big)^{1/3} \n+ C \Big( \int_{B} |\nabla u|^3 |\nabla \varphi_2| \Big) dz \Big)^2 \n\le \frac{1}{2} \int_{B} |\nabla u|^6 \varphi_2^2 dz + C \int_{B} |\nabla^2 u|^3 \varphi_2^2 dz + C \Big( \int_{B} |\nabla u|^3 |\nabla \varphi_2| \Big) dz \Big)^2
$$

With Young's inequality for suitable  $\varepsilon > 0$ , and using [\(4.9\)](#page-18-2), we then can bound

$$
J_1 \leq C \|\n(\nabla^2 u|^3 + 1)\varphi_2^2\|_{L^1(B)} + C \|\nabla u|^3 |\nabla \varphi_2|\|_{L^1(B)}^2 \leq \varepsilon \|\nabla^3 u \varphi_2\|_{L^2(B)}^2 + C(1 + \|\nabla^2 u \varphi_2\|_{L^2(B)}^2) \|\nabla^2 u \varphi_1\|_{L^2(B)}^2 + C \|\n|\nabla u|^3 |\nabla \varphi_2|\|_{L^1(B)}^2
$$
  

$$
\leq \frac{1}{2} \|\nabla \partial^2_\varphi u \varphi_2\|_{L^2(B)}^2 + C(1 + \|\nabla \partial_\varphi u \varphi_2\|_{L^2(B)}^2) \|\nabla \partial_\varphi u \varphi_1\|_{L^2(B)}^2 + C_1,
$$

where we also have estimated

$$
\|\nabla u|^3 |\nabla \varphi_2|\|_{L^1(B)}^2 \le C \|\nabla u \varphi_1\|_{L^4(B)}^4 \|\nabla u \varphi_1\|_{L^2(B)}^2
$$
  

$$
\le C \big( \|\nabla^2 u \varphi_1\|_{L^2(B)}^2 + E(u) \big) \|\nabla u \varphi_1\|_{L^2(B)}^4 \le C \|\nabla \partial_{\phi} u \varphi_1\|_{L^2(B)}^2 + C.
$$

Similarly, with [\(10.2\)](#page-39-4) we have

$$
J_2 \le C \| \nabla^2 u \varphi_1 \|_{L^2(B)}^2 + C.
$$

Thus, from Lemma [4.7](#page-17-0) we obtain

<span id="page-22-0"></span>(4.11) 
$$
\frac{d}{dt} \left( \|\partial_{\phi}^2 u \varphi_2\|_{L^2(\partial B)}^2 \right) + \frac{1}{2} \|\nabla \partial_{\phi}^2 u \varphi_2\|_{L^2(B)}^2 \leq C \left(1 + \|\nabla \partial_{\phi} u \varphi_2\|_{L^2(B)}^2\right) \|\nabla \partial_{\phi} u \varphi_1\|_{L^2(B)}^2 + C.
$$

Denote as  $C_1 = C_1(T, R, u_0) > 0$  a constant bounded by the terms on the right hand side in the statements of Propositions [4.8.](#page-18-1) By elliptic regularity, using that  $|\Delta(u\varphi_2| \leq 2|\nabla u \nabla \varphi_2| + C$  we can bound

$$
\begin{aligned} \|\nabla^2 u\varphi_2\|_{L^2(B)}^2 &\le \|u\varphi_2\|_{H^2(B)}^2 + C\|\nabla u\nabla\varphi_2\|_{L^2(B)}^2 + C\\ &\le C\|u\varphi_2\|_{H^2(\partial B)}^2 + \|\Delta(u\varphi_2)\|_{L^2(B)}^2 + C\|\nabla u\nabla\varphi_2\|_{L^2(B)}^2 + C\\ &\le C\|\partial^2_\varphi u\varphi_2\|_{L^2(\partial B)}^2 + CE(u) + C_1. \end{aligned}
$$

From [\(4.11\)](#page-22-0) we then obtain the differential inequality

$$
\frac{d}{dt}\left(1+\|\partial_{\phi}^2 u\varphi_2\|_{L^2(\partial B)}^2\right) \leq C\left(1+\|\partial_{\phi}^2 u\varphi_2\|_{L^2(\partial B)}^2\right) \|\nabla \partial_{\phi} u\varphi_1\|_{L^2(B)}^2 + C_1;
$$

that is,

$$
\frac{d}{dt}\Big(\log\left(1+\|\partial_{\phi}^2 u\varphi_2\|_{L^2(\partial B)}^2\right)\Big)\leq C\|\nabla\partial_{\phi} u\varphi_1\|_{L^2(B)}^2+C_1,
$$

and the right hand side is integrable in time by Proposition [4.8.](#page-18-1) The claim follows.

 $\Box$ 

We continue by induction.

.

<span id="page-23-0"></span>**Proposition 4.11.** There exists a constant  $\delta > 0$  independent of  $R_0 > 0$  with the following property. Whenever [\(4.10\)](#page-18-0) holds, then for any  $k \geq 3$ , any smooth  $u_0 \in H^{1/2}(S^1;N)$ , and any  $T < T_0$ , there holds

$$
\sup_{0
$$

where we denote as  $C_k = C_k(T, R, u_0) > 0$  a constant bounded by the terms on the right hand side in the statement of the proposition for  $k - 1$ .

*Proof.* By Proposition [4.10](#page-21-0) the claimed result holds true for  $k = 2$ . Suppose the claim holds true for some  $k_0 \geq 2$  and let  $k = k_0 + 1$ . Note that by elliptic regularity, as in the proof of Proposition [4.10](#page-21-0) we can bound

$$
\|\nabla^{k}u\varphi_{k}\|_{L^{2}(B)}^{2} \leq \|u\varphi_{k}\|_{H^{k}(B)}^{2} + C \sum_{j  
\n
$$
\leq C \|u\varphi_{k}\|_{H^{k}(\partial B)}^{2} + C \|\Delta(u\varphi_{k})\|_{H^{k-2}(B)}^{2} + C \sum_{j  
\n
$$
\leq C \|\partial_{\phi}^{k}u\varphi_{k}\|_{L^{2}(\partial B)}^{2} + C \sum_{j
$$
$$
$$

By induction hypothesis and Sobolev's embedding  $H^2(B) \hookrightarrow W^{1,4} \cap C^0(\overline{B})$  for  $0 \leq t < T$  we then have the uniform bounds

$$
\|\nabla^{k_0} u\varphi_{k_0}\|_{L^2(B)}^2 + \|\nabla^{k_0-1} u\varphi_{k_0}\|_{L^4(B)}^2 + \sum_{j=1}^{k_0-2} \|\nabla^j u\varphi_{k_0}\|_{L^\infty(B)}^2 \leq C_k,
$$

and it follows that

$$
\|\nabla^k u\varphi_k\|_{L^2(B)}^2 + \|\nabla^{k_0} u\varphi_k\|_{L^4(B)}^2 + \|\nabla^{k_0-1} u\varphi_k\|_{L^\infty(B)}^2 \le C \|\partial_\phi^k u\varphi_k\|_{L^2(\partial B)}^2 + C_k.
$$

Again let

$$
J_1 := \sum_{1 \leq j_i \leq k, \, \Sigma_i j_i = 2k+2} \|\Pi_i \nabla^{j_i} u \varphi_k^2\|_{L^1(B)} \\
\leq \|(|\nabla^k u|^2 (|\nabla^2 u| + |\nabla u|^2) + |\nabla^k u||\nabla^{k_0} u||\nabla^3 u| + \dots + |\nabla u|^{2k+2} \varphi_k^2\|_{L^1(B)}.
$$

and set

$$
J_2 = \sum_{1 \leq j_0, j_i \leq k, \ \Sigma_{i \geq 0} j_i \leq k+1} \|\Pi_{i > 0} \nabla^{j_i} u \nabla^{j_0} \varphi_2\|_{L^2(B)}^2.
$$

Suppose  $k_0 = 2$ . Recalling that  $\varphi_k = \varphi_k \varphi_{k_0}$ , we can bound the listed terms  $\|\nabla^3 u|^2 (|\nabla^2 u| + |\nabla u|^2) \varphi_3^2\|_{L^1(B)}$ 

$$
\leq \|\nabla^3 u\varphi_3\|_{L^4(B)}^2 (\|\nabla^2 u\varphi_2\|_{L^2(B)} + \|\nabla u\varphi_2\|_{L^4(B)}^2)
$$
  
\n
$$
\leq C_3 \|\nabla \partial^3_\varphi u\varphi_3\|_{L^2(B)} \|\nabla^3 u\varphi_3\|_{L^2(B)} + C_3 \|\nabla^3 u\varphi_2\|_{L^2(B)}^2 + C_3
$$
  
\n
$$
\leq C_3 \|\nabla \partial^3_\varphi u\varphi_3\|_{L^2(B)} \|\partial^3_\varphi u\varphi_3\|_{L^2(\partial B)} + C_3 \|\nabla \partial^2_\varphi u\varphi_2\|_{L^2(B)}^2 + C_3
$$

$$
\leq \varepsilon \|\nabla \partial^3_\phi u \varphi_3\|^2_{L^2(B)} + C_3 \|\partial^3_\phi u \varphi_3\|^2_{L^2(\partial B)} + C_3 \|\nabla \partial^2_\phi u \varphi_2\|^2_{L^2(B)} + C_3,
$$

and

$$
\|\nabla u|^8 \varphi_3^2\|_{L^1(B)} \le \|\nabla u\varphi_3\|_{L^\infty(B)}^2 \|\nabla u\varphi_2\|_{L^6(B)}^6
$$
  

$$
\le C_3 \|\partial_\phi^3 u\varphi_3\|_{L^2(\partial B)}^2 + C_3,
$$

respectively. Here we also have used [\(10.1\)](#page-39-7), [\(10.2\)](#page-39-4) to bound

$$
\|\nabla u\varphi_{2}\|_{L^{6}(B)}^{3} \leq \|\nabla(|\nabla u|^{3}\varphi_{2}^{3})\|_{L^{1}(B)}\n\leq C\|(|\nabla^{2}u|\varphi_{2} + |\nabla u||\nabla\varphi_{2}|)|\nabla u|^{2}\varphi_{2}^{2}\|_{L^{1}(B)}\n\leq C(\|\nabla^{2}u\varphi_{2}\|_{L^{2}(B)} + \|\nabla u\nabla\varphi_{2}\|_{L^{2}(B)})\|\nabla u\varphi_{2}\|_{L^{4}(B)}^{2}\n\leq C(\|\nabla^{2}u\varphi_{2}\|_{L^{2}(B)} + \|\nabla u\nabla\varphi_{2}\|_{L^{2}(B)})^{2}\|\nabla u\varphi_{2}\|_{L^{2}(B)} \leq C_{3}.
$$

Similarly, we can bound the remaining terms and the terms in  $J_2$  to obtain

$$
\frac{d}{dt} \left( \|\partial_{\phi}^{3} u\varphi_{3}\|_{L^{2}(\partial B)}^{2} \right) + \frac{1}{2} \|\nabla \partial_{\phi}^{3} u\varphi_{3}\|_{L^{2}(B)}^{2}
$$
\n
$$
\leq C_{3} (1 + \|\partial_{\phi}^{3} u\varphi_{3}\|_{L^{2}(\partial B)}^{2}) (1 + \|\nabla \partial_{\phi}^{2} u\varphi_{2}\|_{L^{2}(B)}^{2}) + C_{3}
$$

from Lemma [4.9](#page-18-3) and then

$$
\frac{d}{dt}\Big(\log\big(1+\|\partial_{\phi}^3 u\varphi_2\|_{L^2(\partial B)}^2\big)\Big)\leq C_3(1+\|\nabla\partial_{\phi}^2 u\varphi_2\|_{L^2(B)}^2),
$$

where the right hand side is integrable in time by Proposition [4.10.](#page-21-0) The claim for  $k = 3$  thus follows.

<span id="page-24-0"></span>For  $k \geq 4$  the analysis is similar (but simpler) and may be left to the reader.  $\Box$ 

## <span id="page-24-1"></span>5. Local existence

In order to show local existence we approximate the flow equation [\(1.3\)](#page-1-2) by the equation

(5.1) 
$$
u_t = -(\varepsilon + d\pi_N(u))u_r \text{ on } \partial B.
$$

where  $\varepsilon > 0$  and where we smoothly extend the nearest-neighbor projection  $\pi_N$ , originally defined only in the  $\rho$ -neighborhood  $N_\rho$  of N, to the whole ambient  $\mathbb{R}^n$ . Our aim then is to show that for given smooth initial data  $u_0$  the evolution problem  $(5.1), (1.4)$  $(5.1), (1.4)$  $(5.1), (1.4)$  admits a smooth solution  $u_{\varepsilon}$  which remains uniformly smoothly bounded on a uniform time interval as  $\varepsilon \downarrow 0$ . Fixing some  $0 < \varepsilon < 1/2$ , we show existence for the problem [\(5.1\)](#page-24-1) with data [\(1.4\)](#page-1-3) by means of a fixed-point argument.

To set up the argument, fix smooth initial data  $u_0: S^1 \to N$  with harmonic extension  $u_0 \in C^\infty(\bar{B}; \mathbb{R}^n)$  and some  $k \geq 2$ . For suitable  $T > 0$  to be determined let

$$
X = L^{\infty}([0, T]; H^{k+1}(B; \mathbb{R}^n)) \cap H^1(S^1 \times [0, T]; \mathbb{R}^n)
$$

and set

$$
V = \{ v \in X; \ v(0) = u_0, \ \Delta v(t) = 0 \text{ in } B \text{ for } 0 \le t \le T, \|v\|_X^2 = \sup_{0 \le t \le T} \|v(t)\|_{H^{k+1}(B)}^2 + \int_0^T \int_{S^1} |v_t|^2 d\phi dt \le 4R_0^2 \},
$$

where  $R_0 = ||u_0||_{H^{k+1}(B)}$ . We endow the space V with the metric derived from the semi-norm

$$
|v|_X^2 = \sup_{0 \le t \le T} \|\nabla v(t)\|_{L^2(B)}^2 + \int_0^T \int_{S^1} |v_t|^2 d\phi dt.
$$

Note that this metric is positive definite on  $V$  in view of the initial condition that we impose.

**Lemma 5.1.**  $V$  is a complete metric space.

*Proof.* Let  $(v_m)_{m\in\mathbb{N}} \subset V$  with  $|v_l - v_m|_X \to 0$   $(l, m \to \infty)$ . By the theorem of Banach-Alaoglu a subsequence  $v_m \to v$  weakly- $*$  in  $L^{\infty}([0,T]; H^{k+1}(B))$  with  $v_{m,t} \to v_t$  weakly in  $L^2([0,T] \times S^1)$ , and by weak lower semi-continuity of the norm there holds

$$
||v||_X^2 \le \limsup_{m \to \infty} ||v_m||_X^2 \le 4R_0^2.
$$

Moreover, we have  $\Delta v(t) = 0$  for all  $0 \le t \le T$  and  $v(0) = u_0$  by compactness of the trace operator  $H^1(S^1 \times [0,T]) \ni u \mapsto u(0) \in L^2(S^1)$ . Hence  $v \in V$ .

Moreover, we have

<span id="page-25-0"></span>
$$
|v_l - v|_X \le \limsup_{m \to \infty} |v_l - v_m|_X \to 0 \text{ as } l \to \infty.
$$

 $\Box$ 

<span id="page-25-2"></span>**Lemma 5.2.** There is  $T_2 > 0$  such that for any  $T \leq T_2$ , any  $v \in V$  there is a solution  $u = \Phi(v) \in V$  of the equation

(5.2) 
$$
u_t = -(\varepsilon + d\pi_N(v))u_r \text{ on } \partial B \times [0, T_2],
$$

satisfying  $(1.4)$ .

*Proof.* For  $v \in V$  we construct a solution  $u = \Phi(v) \in X$  of [\(5.2\)](#page-25-0) via Galerkin approximation. For this let  $(\varphi_l)_{l \in \mathbb{N}_0}$  be Steklov eigenfunctions of the Laplacian, satisfying

$$
\Delta \varphi_l = 0 \text{ in } B
$$

with boundary condition

$$
\partial_r \varphi_l = \lambda_l \varphi_l
$$
 on  $\partial B$ ,  $l \in \mathbb{N}_0$ .

Note that the Steklov eigenvalues are given by  $\lambda_0 = 0$  and  $\lambda_{2l-1} = \lambda_{2l} = l, l \in \mathbb{N}$ . In fact, we may choose  $\varphi_0 \equiv 1/\sqrt{2\pi}$  and

(5.3) 
$$
\varphi_{2l-1}(re^{i\theta}) = \frac{1}{\sqrt{\pi}}r^{l}sin(l\theta), \ \varphi_{2l}(re^{i\theta}) = \frac{1}{\sqrt{\pi}}r^{l}cos(l\theta), \ l \in \mathbb{N}.
$$

to obtain an orthonormal basis for  $L^2(S^1)$  consisting of these functions. Given  $m \in \mathbb{N}$  then let  $u^{(m)}(t, z) = \sum_{l=0}^{m} a_l^{(m)}$  $\binom{m}{l}(t)\varphi_l(z)$  solve the system of equations

<span id="page-25-1"></span>(5.4)  

$$
\partial_t a_l^{(m)} = (\varphi_l, u_t^{(m)})_{L^2(S^1)} = -(\varphi_l, (\varepsilon + d\pi_N(v))u_r^{(m)})_{L^2(S^1)}
$$

$$
= -\sum_{j=0}^m a_j^{(m)} \lambda_j (\varphi_l, (\varepsilon + d\pi_N(v))\varphi_j)_{L^2(S^1)}, 0 \le l \le m.
$$

Since for any  $m \in \mathbb{N}$  the coefficients  $\lambda_j(\varphi_l, (\varepsilon + d\pi_N(v))\varphi_j)_{L^2(S^1)}$  of this system are uniformly bounded for any  $v \in V$ , for any  $m \in \mathbb{N}$  there exists a unique global solution  $a^{(m)} = (a^{(m)}_l)$  $\binom{m}{l}$ <sub>0≤l≤m</sub> of [\(5.4\)](#page-25-1) with initial data  $a_l^{(m)}$  $\binom{m}{l}(0) = a_{l0} = (u_0, \varphi_l)_{L^2(S^1)},$  $0 \leq l \leq m$ .

Note that for any  $m \in \mathbb{N}$  and any  $j \in \mathbb{N}_0$  the function

$$
\partial_{\phi}^{2j}(ru_r^{(m)}) \in span\{\varphi_l; \ 0 \le l \le m\},\
$$

and  $\partial_{\phi}^{2j}u^{(m)}$  is harmonic. In particular, for  $j=0$  we obtain

$$
\frac{1}{2} \frac{d}{dt} \left( \|\nabla u^{(m)}\|_{L^2(B)}^2 \right) = \int_B \nabla u^{(m)} \nabla u_t^{(m)} dz = (u_r^{(m)}, u_t^{(m)})_{L^2(S^1)}
$$
\n
$$
= -(u_r^{(m)}, (\varepsilon + d\pi_N(v)) u_r^{(m)})_{L^2(S^1)}
$$
\n
$$
= -\varepsilon \|u_r^{(m)}\|_{L^2(S^1)}^2 - \|d\pi_N(v) u_r^{(m)}\|_{L^2(S^1)}^2
$$
\n
$$
\leq -\frac{1}{2} \|u_t^{(m)}\|_{L^2(S^1)}^2 \leq 0,
$$

and we find the uniform  $H^1$ -bound

<span id="page-26-1"></span>
$$
\sup_{t\geq 0} \|\nabla u^{(m)}(t)\|_{L^2(B)}^2 + \varepsilon \|u_r^{(m)}\|_{L^2([0,\infty[\times S^1)}^2 + \|u_t^{(m)}\|_{L^2([0,\infty[\times S^1)}^2)
$$
  

$$
\leq 2\|\nabla u^{(m)}(0)\|_{L^2(B)}^2 \leq 2\|\nabla u_0\|_{L^2(B)}^2 \leq 2R_0^2.
$$

Moreover, for  $j = k \in \mathbb{N}$  as in the definition of X upon integrating by parts we find

<span id="page-26-0"></span>
$$
\frac{1}{2} \frac{d}{dt} \left( \|\nabla \partial_{\phi}^{k} u^{(m)}\|_{L^{2}(B)}^{2} \right) = (-1)^{k} \int_{B} \nabla \partial_{\phi}^{2k} u^{(m)} \nabla u_{t}^{(m)} dz
$$
\n
$$
(5.7)
$$
\n
$$
= (-1)^{k} (\partial_{\phi}^{2k} u_{r}^{(m)}, u_{t}^{(m)})_{L^{2}(S^{1})}
$$
\n
$$
= (-1)^{k+1} (\partial_{\phi}^{2k} u_{r}^{(m)}, (\varepsilon + d\pi_{N}(v)) u_{r}^{(m)})_{L^{2}(S^{1})}
$$
\n
$$
= -\varepsilon ||\partial_{\phi}^{k} u_{r}^{(m)}||_{L^{2}(S^{1})}^{2} - ||d\pi_{N}(v)\partial_{\phi}^{k} u_{r}^{(m)}||_{L^{2}(S^{1})}^{2} + I,
$$

where  $I = \sum_{j=1}^{k} \binom{k}{j} I_j$  with

$$
I_j = -(\partial_{\phi}^k u_r^{(m)}, \partial_{\phi}^j (d\pi_N(v)) \partial_{\phi}^{k-j} u_r^{(m)})_{L^2(S^1)}
$$

similar to the proof of Lemma [4.3.](#page-12-0) However, now we simply bound

$$
|I_j|\leq C\sum_{\Sigma_i j_i=j}\|\partial_{\phi}^k u^{(m)}_r\|_{L^2(S^1)}\|\Pi_i \partial_{\phi}^{j_i}v \partial_{\phi}^{k-j} u^{(m)}_r\|_{L^2(S^1)},\ 1\leq j\leq k.
$$

Note that by compactness of Sobolev's embedding  $H^1(S^1) \hookrightarrow L^{\infty}(S^1)$  and Ehrlich's lemma for any number  $1 \leq j \leq k$ , any  $\delta > 0$  we can bound

$$
\begin{aligned} \|\partial_{\phi}^{k-j}u_{r}^{(m)}\|_{L^{\infty}(S^{1})} &\leq \delta \|\partial_{\phi}^{k-j+1}u_{r}^{(m)}\|_{L^{2}(S^{1})}+C(\delta)\|\partial_{\phi}^{k-j}u_{r}^{(m)}\|_{L^{2}(S^{1})}\\ &\leq 2\delta \|\partial_{\phi}^{k}u_{r}^{(m)}\|_{L^{2}(S^{1})}+C(\delta)\|u_{r}^{(m)}\|_{L^{2}(S^{1})}.\end{aligned}
$$

On the other hand, for any  $v \in V$  by the trace theorem we have

$$
\|\partial_{\phi}^k v\|_{L^2(S^1)} \le C \|\partial_{\phi}^k v\|_{H^1(B)} \le C \|v\|_{H^{k+1}(B)} \le CR_0
$$

and we therefore also can bound

$$
\|\partial_{\phi}^{j}v\|_{L^{\infty}(S^{1})} \leq C\|\partial_{\phi}^{k}v\|_{L^{2}(S^{1})} + \|\partial_{\phi}^{j}v\|_{L^{2}(S^{1})} \leq C\|v\|_{H^{k+1}(B)} \leq CR_{0}.
$$

for any  $1 \leq j < k$ .

Thus, for sufficiently small  $\delta > 0$  with a constant  $C > 0$  depending on  $\varepsilon > 0$  and  $R_0$  there holds

$$
|I| \le \varepsilon/2 \|\partial_{\phi}^k u_r^{(m)}\|_{L^2(S^1)}^2 + C \|u_r^{(m)}\|_{L^2(S^1)}^2
$$

and from  $(5.7)$  with the help of  $(3.7)$  we obtain the inequality

$$
\frac{d}{dt} \left( \|\nabla \partial_{\phi}^{k} u^{(m)}\|_{L^{2}(B)}^{2} \right) \leq C \|u_{r}^{(m)}\|_{L^{2}(S^{1})}^{2} = C \|u_{\phi}^{(m)}\|_{L^{2}(S^{1})}^{2} \leq C \|u_{\phi}^{(m)}\|_{H^{1}(B)}^{2}
$$
\n
$$
\leq C \|\nabla \partial_{\phi}^{k} u^{(m)}\|_{L^{2}(B)}^{2} + C \|\nabla u^{(m)}\|_{L^{2}(B)}^{2} \leq C(1 + \|\nabla \partial_{\phi}^{k} u^{(m)}\|_{L^{2}(B)}^{2}),
$$

where we recall  $(5.6)$  for the last conclusion.

It follows that for suitably small  $T > 0$  there holds  $||u_m^{(m)}||_X^2 \leq 4R_0^2$  for all  $m \in \mathbb{N}$ . Thus, there is a sequence  $m \to \infty$  such that  $u^{(m)} \to u$  weakly- $*$  in  $L^{\infty}([0,T]; H^{k+1}(B))$  with  $u_t^{(m)} \to u_t$  weakly in  $L^2([0,T] \times S^1)$ , where  $u =: \Phi(v) \in V$ solves equation [\(5.2\)](#page-25-0).  $\Box$ 

**Lemma 5.3.** There is  $T > 0$  such that for  $v_1, v_2 \in V$  there holds

$$
|\Phi(v_1) - \Phi(v_2)|_X \le \frac{1}{2}|v_1 - v_2|_X.
$$

*Proof.* Let  $T_2 > 0$  be as determined in Lemma [5.2](#page-25-2) and fix some  $0 < T \leq T_2$ . For  $v_1, v_2 \in V$  then we have  $u_i =: \Phi(v_i) \in V, i = 1, 2$ . Set  $w = u_1 - u_2, v = v_1 - v_2$ , and compute

(5.8) 
$$
w_t = -(\varepsilon + d\pi_N(v_1))w_r - (d\pi_N(v_1) - d\pi_N(v_2))u_{2,r} \text{ on } \partial B = S^1.
$$

Multiplying with  $w_r$  and integrating we obtain

<span id="page-27-0"></span>
$$
\frac{1}{2}\frac{d}{dt}\left(\|\nabla w\|_{L^2(B)}^2\right) = \int_B \nabla w \nabla w_t \, dx = (w_r, w_t)_{L^2(S^1)} = -\varepsilon \|w_r\|_{L^2(S^1)}^2
$$

$$
-\|d\pi_N(v_1)w_r\|_{L^2(S^1)}^2 - (w_r, (d\pi_N(v_1) - d\pi_N(v_2)u_{2,r})_{L^2(S^1)},
$$

where with  $||u_{2,r}||_{L^{\infty}(S^1)} \leq C||u_2||_{H^3(B)} \leq CR_0$  we can bound

$$
|(w_r, (d\pi_N(v_1) - d\pi_N(v_2))u_{2,r})_{L^2(S^1)}| \leq C ||w_r||_{L^2(S^1)} ||v||_{L^2(S^1)} ||u_{2,r}||_{L^{\infty}(S^1)}
$$
  

$$
\leq C ||w_r||_{L^2(S^1)} ||v||_{L^2(S^1)} \leq \frac{\varepsilon}{2} ||w_r||_{L^2(S^1)}^2 + C ||v||_{L^2(S^1)}^2.
$$

Thus, with a constant  $C = C(\varepsilon) > 0$  we find

<span id="page-27-1"></span>(5.9) 
$$
\frac{d}{dt} \|\nabla w\|_{L^2(B)}^2 + \varepsilon \|w_r\|_{L^2(S^1)}^2 \leq C \|v\|_{L^2(S^1)}^2.
$$

Similarly, from [\(5.8\)](#page-27-0) we can bound

(5.10) 
$$
||w_t||_{L^2(S^1)}^2 \leq C||w_r||_{L^2(S^1)}^2 + C||v||_{L^2(S^1)}^2
$$

Integrating over  $0 \le t \le T$  and observing that we have

<span id="page-27-2"></span>
$$
\sup_{0\leq t\leq T}||v(t)||_{L^{2}(S^{1})}^{2} \leq \left(\int_{0}^{T}||v_{t}(t)||_{L^{2}(S^{1})}dt\right)^{2} \leq T\int_{0}^{T}||v_{t}(t)||_{L^{2}(S^{1})}^{2}dt,
$$

.

from [\(5.9\)](#page-27-1) we first obtain

$$
\sup_{0\leq t\leq T} \|\nabla w(t)\|_{L^2(B)}^2 + \varepsilon \|w_r\|_{L^2([0,T]\times S^1)}^2 \leq CT \sup_{0\leq t\leq T} \|v(t)\|_{L^2(S^1)}^2 \leq CT^2|v|_X^2,
$$

which we may use together with [\(5.10\)](#page-27-2) to bound

$$
|w|_X^2 = \sup_{0 \le t \le T} \|\nabla w(t)\|_{L^2(B)}^2 + \|w_t\|_{L^2([0,T] \times S^1)}^2 \le CT^2 |v|_X^2.
$$

For sufficiently small  $T > 0$  then our claim follows.

$$
^{28}
$$

Thus, by Banach's fixed point theorem, for any  $\varepsilon > 0$ , any smooth  $u_0 \in$  $H^{1/2}(S^1;N)$  there exists  $T > 0$  and a solution  $u = u(t) \in V$  of the initial value problem [\(5.1\)](#page-24-1), [\(1.4\)](#page-1-3). We now show that the number  $T > 0$  may be chosen uniformly as  $\varepsilon \downarrow 0$ . Indeed, we have the following result.

<span id="page-28-0"></span>**Lemma 5.4.** There exists a constant  $C > 0$  such that for any  $k \geq 2$ , any smooth  $u_0 \in H^{1/2}(S^1;N)$ , and any  $0 < \varepsilon \leq 1/2$  for the solution u to  $(5.1)$  with  $u(0) = u_0$ there holds

$$
\frac{d}{dt} \left( \|\nabla \partial_{\phi}^{k} u\|_{L^{2}(B)}^{2} \right) \leq C(1 + \|\nabla u\|_{L^{2}(B)}^{2} + \|\nabla \partial_{\phi}^{k} u\|_{L^{2}(B)}^{k+3}.
$$

*Proof.* Similar to the proof of Lemma [5.2,](#page-25-2) for given  $2 \leq k \in \mathbb{N}$  we compute

$$
\frac{1}{2} \frac{d}{dt} (\|\nabla \partial_{\phi}^{k} u\|_{L^{2}(B)}^{2}) = (-1)^{k} \int_{B} \nabla \partial_{\phi}^{2k} u \nabla u_{t} dx
$$
\n(5.11)\n
$$
= (-1)^{k} (\partial_{\phi}^{2k} u_{r}, u_{t})_{L^{2}(S^{1})} = (-1)^{k+1} (\partial_{\phi}^{2k} u_{r}, (\varepsilon + d\pi_{N}(u)) u_{r})_{L^{2}(S^{1})}
$$
\n
$$
\leq -\|d\pi_{N}(u)\partial_{\phi}^{k} u_{r}\|_{L^{2}(S^{1})}^{2} - I,
$$

where we now drop the term  $\varepsilon \|\partial_{\phi}^k u_r\|_{L^2(S^1)}^2$  from [\(5.7\)](#page-26-0). Again we split  $I = \sum_{j=1}^k {k \choose j} I_j$ with

$$
I_j = (\partial_{\phi}^k u_r, \partial_{\phi}^j (d\pi_N(u)) \partial_{\phi}^{k-j} u_r)_{L^2(S^1)}
$$
  
= 
$$
(\nabla \partial_{\phi}^k u, \nabla (\partial_{\phi}^j (d\pi_N(u)) \partial_{\phi}^{k-j} u_r))_{L^2(B)},
$$

but now we bound these terms as in the proof of Lemma [4.3](#page-12-0) via

$$
|I_j| \leq C \|\nabla \partial_{\phi}^k u\|_{L^2(B)} \left( \|\nabla \partial_{\phi}^j (d\pi_N(u)) \partial_{\phi}^{k-j} u_r\|_{L^2(B)} + \|\partial_{\phi}^j (d\pi_N(u)) \nabla \partial_{\phi}^{k-j} u_r\|_{L^2(B)} \right)
$$
  

$$
\leq C \sum_{1 \leq j_i \leq k+1, \, \Sigma_i j_i = k+2} \|\nabla \partial_{\phi}^k u\|_{L^2(B)} \|\Pi_i \nabla^{j_i} u\|_{L^2(B)}.
$$

Using that for any  $k \geq 2$  by Sobolev's embedding  $H^2(B) \hookrightarrow W^{1,4} \cap C^0(\overline{B})$  be can bound

$$
\sum_{1 \le j_i \le k+1, \, \Sigma_i j_i = k+2} \|\Pi_i \nabla^{j_i} u\|_{L^2(B)} \le C(1 + \|\nabla u\|_{L^2(B)} + \|\nabla^{k+1} u\|_{L^2(B)})^{k+2},
$$

and also using [\(4.3\)](#page-10-2), we obtain the claim.

$$
\qquad \qquad \Box
$$

We now are able to conclude.

<span id="page-28-1"></span>**Proposition 5.5.** For any  $k \geq 2$ , any smooth  $u_0 \in H^{1/2}(S^1; N)$  there exists  $T > 0$ and a solution  $u \in V$  to [\(1.3\)](#page-1-2) with initial data  $u(0) = u_0$ .

*Proof.* In view of Lemma [5.4,](#page-28-0) there exists a uniform number  $T > 0$  such that, with V as defined above, for any  $0 < \varepsilon \leq 1/2$  for there exists a solution  $u_{\varepsilon} \in V$ to [\(5.1\)](#page-24-1). By definition of V, as  $\varepsilon \downarrow 0$  suitably, we have  $u_{\varepsilon} \rightarrow u$  weakly- $*$  in  $L^{\infty}([0,T];H^{k+1}(B))\cap H^1(S^1\times[0,T]).$  But this suffices to pass to the limit  $\varepsilon\downarrow 0$ in [\(5.1\)](#page-24-1), and  $u \in V$  solves [\(1.3\)](#page-1-2) with  $u(0) = u_0$ . □

*Proof of Theorem 1.1.i*). By Proposition [5.5](#page-28-1) for any smooth  $u_0 \in H^{1/2}(S^1; N)$  and any  $k \geq 2$  there exists  $T > 0$  and a solution  $u \in V$  of [\(1.3\)](#page-1-2), [\(1.4\)](#page-1-3) for  $0 < t < T$ . Alternatingly employing Propositions [4.11](#page-23-0) and [4.6,](#page-16-1) we then obtain smoothness of u for  $0 < t \leq T$ , including the final time T. (This argument later appears in more detail in Section [6](#page-29-0) after Lemma [6.2.](#page-29-1)) Iterating, the solution  $u$  may be extended smoothly until some maximal time  $T_0$  where condition [\(4.4\)](#page-11-0) ceases to hold.

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<span id="page-29-0"></span>Uniqueness (even within a much larger class of competing functions) is established in Section [7.](#page-31-0)

## 6. Weak solutions

Given  $u_0 \in H^{1/2}(S^1; N)$ , there are smooth functions  $u_{0k} \in H^{1/2}(S^1; N)$  with  $u_{0k} \to u_0$  in  $H^1(B)$  as  $k \to \infty$ . Indeed, similar to an argument of Schoen-Uhlenbeck [\[39\]](#page-41-13), Theorem 3.1, with a standard mollifying sequence  $(\rho_k)_{k\in\mathbb{N}}$  for the mollified functions  $v_{0k} := u_0 * \rho_k$  we have  $dist_N(v_{0k}) \to 0$  uniformly, and  $u_{0k} := \pi_N(v_{0k}) \to$  $u_0 \in H^{1/2}(S^1;N)$  as  $k \to \infty$ .

Let  $u_k$  be the corresponding solutions of [\(1.4\)](#page-1-3) with initial data  $u_k(0) = u_{0k}$ , defined on a maximal time interval  $[0, T_k], k \in \mathbb{N}$ . We claim that each function  $u_k$ can be smoothly extended to a uniform time interval  $[0, T]$  for some  $T > 0$ . To see this, we first establish the following non-concentration result.

<span id="page-29-2"></span>**Lemma 6.1.** For any  $\delta > 0$  there exists a number  $R > 0$  and a time  $T_0 > 0$  such that

$$
\sup_{z_0 \in B, 0 < t < T_0} \int_{B_R(z_0) \cap B} |\nabla u_k(t)|^2 \, dz < \delta \quad \text{for all } k \in \mathbb{N}.
$$

*Proof.* Given  $\delta > 0$ , by absolute continuity of the Lebesgue integral and  $H^1$ convergence  $u_{0k} \to u_0$   $(k \to \infty)$  we can find  $R > 0$  such that

$$
\sup_{z_0 \in B} \int_{B_{2R}(z_0) \cap B} |\nabla u_{0k}|^2 dz < \delta \text{ for all } k \in \mathbb{N}.
$$

Choosing  $T_0 = \delta R$ , by Lemma [2.2](#page-6-2) then we have

$$
\sup_{z_0 \in B, 0 < t < T_0} \int_{B_R(z_0) \cap B} |\nabla u_k(t)|^2 \, dz < 4\delta + C\delta E(u_{k0}) < L\delta
$$

with a uniform constant  $L > 0$  for all  $k \in \mathbb{N}$ . The claim follows, if we replace  $\delta$  with  $\delta/L$ . with  $\delta/L$ .

In view of Proposition [3.3,](#page-8-4) from Lemma [6.1](#page-29-2) and Lemma [2.1](#page-6-1) we obtain the following bound for  $u_k$  in  $H^1(S^1)$ .

<span id="page-29-1"></span>**Lemma 6.2.** There exist a time  $T_0 > 0$  and constants  $C > 0$ ,  $C_0 = C_0(E(u_0)) > 0$ such that

$$
\int_0^{T_0} \int_{S^1} |\partial_{\phi} u_k(t)|^2 d\phi dt \le CE(u_{k0}) \le C_0 \text{ for all } k \in \mathbb{N}.
$$

From Lemma [6.2](#page-29-1) we obtain locally in time uniform smooth bounds for  $(u_k)$  for  $t > 0$  by iteratively applying our previous regularity results. More precisely, Fatou's lemma and Lemma [6.2](#page-29-1) first yield the bound

$$
\int_0^{T_0} \liminf_{k \to \infty} \big( \int_{S^1} |\partial_{\phi} u_k(t)|^2 d\phi \big) dt \le C_0.
$$

Thus for almost every  $0 < t_0 < T_0$  there holds

$$
\liminf_{k \to \infty} \int_{S^1} |\partial_{\phi} u_k(t_0)|^2 d\phi < \infty.
$$

For any such  $0 < t_0 < T_0$ , if  $\delta > 0$  is sufficiently small, from Proposition [4.2](#page-11-2) with another appeal to Fatou's lemma we may conclude

$$
\int_{t_0}^{T_0} \liminf_{k \to \infty} \int_B |\nabla \partial_{\phi} u_k|^2 dz dt \le \liminf_{k \to \infty} \int_{t_0}^{T_0} \int_B |\nabla \partial_{\phi} u_k|^2 dz dt \le C_1
$$

for some  $C_1 > 0$ , so that now we even have

$$
\liminf_{k \to \infty} \int_B |\nabla \partial_{\phi} u_k(t_1)|^2 dz < \infty.
$$

for almost every  $t_0 < t_1 < T_0$ . Hence we may next invoke Proposition [4.4](#page-13-0) and [\(4.2\)](#page-10-1) to obtain the bound

$$
\liminf_{k \to \infty} \int_{t_1}^{T_0} \int_{\partial B} |\nabla \partial_{\phi} u_k|^2 dz dt < \infty
$$

for any such  $t_0 < t_1 < T_0$ , and Fatou's lemma gives that

$$
\liminf_{k \to \infty} \int_{\partial B} |\nabla \partial_{\phi} u_k(t_2)|^2 d\phi < \infty
$$

for almost every  $t_1 < t_2 < T_0$ . Now Proposition [4.10](#page-21-0) may be applied with  $\varphi_0 = 1$ , and we obtain

$$
\liminf_{k \to \infty} \int_{t_2}^{T_0} \int_B |\nabla \partial_{\phi}^2 u_k|^2 dz dt < \infty
$$

for any such  $t_1 < t_2 < T_0$ . Another application of Fatou's lemma gives

$$
\liminf_{k \to \infty} \int_B |\nabla \partial_{\phi}^2 u_k(t_3)|^2 dz < \infty
$$

for almost every  $t_2 < t_3 < T_0$ , and Proposition [4.5](#page-14-1) yields

$$
\liminf_{k \to \infty} \int_{t_3}^{T_0} \int_{\partial B} |\nabla \partial_{\phi}^2 u_k|^2 d\phi dt < \infty
$$

for any such  $t_2 < t_3 < T_0$ . We may then iterate, using [\(3.7\)](#page-8-1) and alternatingly employing Propositions [4.11](#page-23-0) and [4.6](#page-16-1) for  $3 \leq k \in \mathbb{N}$ , to find a subsequence  $(u_k)$ satisfying uniform smooth bounds on  $[t_0, T_0]$  for any  $t_0 > 0$ . Passing to the limit  $k \to \infty$  for this subsequence we obtain a weak solution to [\(1.3\)](#page-1-2), [\(1.4\)](#page-1-3) of energy-class in the following sense.

<span id="page-30-0"></span>**Definition 6.3.** A function  $u \in H^1([0, T_0] \times S^1; N) \cap L^{\infty}([0, T_0]; H^{1/2}(S^1; N))$  is a weak solution of  $(1.3)$ ,  $(1.4)$  of energy-class, if  $(1.3)$  is satisfied in the weak sense, that is, if there holds

<span id="page-30-1"></span>(6.1) 
$$
\int_0^{T_0} \int_{\partial B} (u_t + d\pi_N(u)u_r) \cdot \varphi d\phi dt
$$

$$
= \int_0^{T_0} \int_{\partial B} u_t \cdot \varphi d\phi dt + \int_0^{T_0} \int_B \nabla u \cdot \nabla (d\pi_N(u)\varphi) dz dt = 0
$$

for all  $\varphi \in C_c^{\infty}(S^1 \times ]0, T_0[)$ , and if there holds the energy inequality

<span id="page-30-2"></span>(6.2) 
$$
E(u(T)) + \int_0^T \int_{\partial B} |u_t|^2 d\phi dt \le E(u_0)
$$

for any  $0 < T < T_0$ , with the initial data  $u_0 \in H^{1/2}(S^1; N)$  being attained in the sense of traces.

We then may summarize our results, as follows.

<span id="page-31-1"></span>**Proposition 6.4.** For any  $u_0 \in H^{1/2}(S^1; N)$  there exists  $T_0 > 0$  and a weak solution u to [\(1.3\)](#page-1-2), [\(1.4\)](#page-1-3) on [0, T<sub>0</sub>] of energy-class, which is smooth for  $t > 0$ .

*Proof.* For any open  $U \subset S^1 \times ]0, T_0[$  we have uniform smooth bounds for  $u_k$  on  $U$ ; thus a suitable sub-sequence  $u_k \to u$  smoothly locally as  $k \to \infty$ . The equation  $(6.1)$  follows from the corresponding identites for  $u_k$ .

Moreover,  $(6.2)$  follows from the energy identity, Lemma [2.1,](#page-6-1) for  $u_k$  in view of  $H^1$ -convergence  $u_{0k} \to u_0$  as well as weak lower semi-continuity of the energy and of the  $L^2$ -norm.

Finally, with error  $o(1) \to 0$  as  $k \to \infty$  for  $0 < t < T_0$  we can estimate

$$
||u(t) - u_0||_{L^2(\partial B)}^2 \le ||u_k(t) - u_{0k}||_{L^2(\partial B)}^2 + o(1)
$$
  
\n
$$
\le \left(\int_0^t ||\partial_t u_k(t')||_{L^2(\partial B)} dt'\right)^2 + o(1) \le t \int_0^t ||\partial_t u_k(t')||_{L^2(\partial B)}^2 dt' + o(1)
$$
  
\n
$$
\le t E(u_0) + o(1) \to 0 \text{ as } t \downarrow 0,
$$

and  $u(t) \to u_0$  weakly in  $H^{1/2}(S^1; N) \cap H^1(B; \mathbb{R}^n)$  as  $t \downarrow 0$ . In fact, by [\(6.2\)](#page-30-2) we then even have strong convergence.

# 7. Uniqueness

<span id="page-31-0"></span>With the help of the tools developed in Section [3](#page-7-2) we can show uniqueness of partially regular weak energy-class solutions as in Proposition [6.4.](#page-31-1)

<span id="page-31-5"></span>**Theorem 7.1.** Let  $u_0 \in H^{1/2}(S^1; N)$ . Suppose u and v both are weak energy-class solutions of [\(1.3\)](#page-1-2), [\(1.4\)](#page-1-3) on [0, T<sub>0</sub>] for some  $T_0 > 0$  with initial data  $u_0$ , and suppose that u and v are smooth for  $t > 0$ . Then  $u = v$ .

*Proof.* Using the identity [\(3.2\)](#page-7-0) for u and v, respectively, for the function  $w = u - v$ for almost every  $0 < t < T_0$  we have

<span id="page-31-3"></span>(7.1) 
$$
\begin{aligned} \partial_t w + \partial_r w &= \nu(u)\partial_r(dist_N(u)) - \nu(v)\partial_r(dist_N(v)) \\ &= (\nu(u) - \nu(v))\partial_r(dist_N(u)) + \nu(v)\partial_r(dist_N(u) - dist_N(v)) \end{aligned}
$$

on  $\partial B = S^1$ . From equation [\(3.5\)](#page-7-1), moreover, we obtain

<span id="page-31-2"></span>(7.2) 
$$
|\Delta(dist_N(u) - dist_N(v))| = |\nabla u \cdot d\nu(u)\nabla u - \nabla v \cdot d\nu(v)\nabla v| \leq C(|w||\nabla u|^2 + (|\nabla u| + |\nabla v|)|\nabla w|) \text{ in } B.
$$

Observing that

$$
|dist_N(u) - dist_N(v)| \le C|w|,
$$

upon multiplying [\(7.2\)](#page-31-2) with the function  $(distr_N(u) - dist_N(v)) \in H_0^1(B)$ , integrating by parts, and using Young's inequality, for any  $\varepsilon > 0$  we obtain

<span id="page-31-4"></span>
$$
\|\nabla (dist_N(u) - dist_N(v))\|_{L^2(B)}^2
$$
  
(7.3)  

$$
\leq C \int_B (|w|^2 |\nabla u|^2 + (|\nabla u| + |\nabla v|) |\nabla w||w|) dz
$$
  

$$
\leq \varepsilon \|\nabla w\|_{L^2(B)}^2 + C(\varepsilon) \|w\|_{L^4(B)}^2 (\|\nabla u\|_{L^4(B)}^2 + \|\nabla v\|_{L^4(B)}^2).
$$

On the other hand, for any  $0 < t_0 < T \leq T_0$ , multiplying the equation [\(7.1\)](#page-31-3) with w and integrating by parts on  $S^1 \times [t_0, T]$ , upon letting  $t_0 \downarrow 0$  we find

$$
\sup_{0  
=  $C \int_{0}^{T} \int_{\partial B} w(\nu(u) - \nu(v)) \partial_{r} (dist_{N}(u)) d\phi dt$   
+  $C \int_{0}^{T} \int_{\partial B} w \nu(v) \partial_{r} (dist_{N}(u) - dist_{N}(v)) d\phi dt =: C \int_{0}^{T} (I + II) dt.$
$$

We first estimate the term

$$
I = I(t) = \int_{\partial B} w(\nu(u) - \nu(v)) \partial_r (dist_N(u)) d\phi
$$
  
= 
$$
\int_B \nabla (w(\nu(u) - \nu(v))) \nabla (dist_N(u)) dz
$$
  
+ 
$$
\int_B w(\nu(u) - \nu(v)) \Delta (dist_N(u)) dz.
$$

Using

$$
|\nabla \big(w(\nu(u) - \nu(v))\big)| \le C|\nabla w||w| + |w\big((d\nu(u) - d\nu(v))\nabla u + d\nu(v)\nabla w\big)|
$$
  

$$
\le C(|\nabla w||w| + |w|^2|\nabla u|)
$$

we can bound

$$
\begin{aligned} & \|\int_{B} \nabla \big(w(\nu(u) - \nu(v))\big) \nabla (dist_N(u))dz| \le C \int_{B} |(\nabla w||w| + |w|^2 |\nabla u|) |\nabla u| dz \\ &\le \varepsilon \|\nabla w\|_{L^2(B)}^2 + C(\varepsilon) \|w\|_{L^4(B)}^2 \|\nabla u\|_{L^4(B)}^2 \end{aligned}
$$

for each  $t$ . Also using  $(3.5)$ , we can moreover estimate

$$
\left| \int_{B} w(\nu(u) - \nu(v)) \Delta(dist_N(u)) \, dz \right| \leq C \|w\|_{L^4(B)}^2 \|\nabla u\|_{L^4(B)}^2
$$

for almost every  $0 < t < T$  to obtain

$$
|I| \leq \varepsilon ||\nabla w||_{L^{2}(B)}^{2} + C(\varepsilon)||w||_{L^{4}(B)}^{2} ||\nabla u||_{L^{4}(B)}^{2}.
$$

Similarly, we estimate the term

$$
II = II(t) = \int_{\partial B} w \,\nu(v)\partial_r((dist_N(u) - dist_N(v))\,d\phi
$$

$$
= \int_B \nabla(w\nu(v))\nabla(dist_N(u) - dist_N(v))\,dz
$$

$$
+ \int_B w \,\nu(v)\Delta(dist_N(u) - dist_N(v))\,dz.
$$

Noting that with [\(7.3\)](#page-31-4) we can bound

$$
\begin{aligned} &\|\int_{B}\nabla(w\nu(v))\nabla(dist_N(u)-dist_N(v))dz|\\ &\leq C(\|\nabla w\|_{L^2(B)}+\|w\nabla v\|_{L^2(B)})\|\nabla(dist_N(u)-dist_N(v))\|_{L^2(B)}\\ &\leq \varepsilon\|\nabla w\|_{L^2(B)}^2+C(\varepsilon)\|w\|_{L^4(B)}^2(\|\nabla u\|_{L^4(B)}^2+\|\nabla v\|_{L^4(B)}^2) \end{aligned}
$$

and that with [\(7.2\)](#page-31-2) we have

$$
\begin{aligned} \|\int_{B} w\nu(v)\Delta(dist_N(u) - dist_N(v))\,dz| \\ &\leq C \int_{B} (|w|^2 |\nabla u|^2 + |w||\nabla w| (|\nabla u| + |\nabla v|))\,dz \\ &\leq \varepsilon \|\nabla w\|_{L^2(B)}^2 + C(\varepsilon) \|w\|_{L^4(B)}^2 (\|\nabla u\|_{L^4(B)}^2 + \|\nabla v\|_{L^4(B)}^2) \end{aligned}
$$

we find the estimate

$$
|II| \leq \varepsilon ||\nabla w||_{L^{2}(B)}^{2} + C(\varepsilon)||w||_{L^{4}(B)}^{2} (||\nabla u||_{L^{4}(B)}^{2} + ||\nabla v||_{L^{4}(B)}^{2})
$$

for almost every  $0 < t < T$ .

But Sobolev's embedding  $H^{1/2}(B) \hookrightarrow L^4(B)$  and Fourier expansion give the bound

$$
||w||_{L^{4}(B)}^{2} \leq C||w||_{H^{1/2}(B)}^{2} \leq C||w||_{L^{2}(\partial B)}^{2}
$$

and similar bounds for  $\nabla u$  as well as  $\nabla v$ . Moreover, since by the energy inequality  $(6.2)$  we have  $u(t), v(t) \rightarrow u_0$  strongly in  $H^1(B)$  as  $t \downarrow 0$ , there exist a radius  $0 < R \leq 1/2$  and a time  $0 < T < T_0$  such that condition [\(3.13\)](#page-10-4) in Proposition [3.3](#page-8-4) holds true on  $[0, T]$  for both u and v, allowing to bound

$$
\int_0^T \|\nabla u(t)\|_{L^4(B)}^2 dt \le C \int_0^T \|\nabla u(t)\|_{L^2(\partial B)}^2 dt \le C \int_0^T \|\partial_\phi u(t)\|_{L^2(\partial B)}^2 dt
$$
  

$$
\le C \int_0^T \int_{\partial B} |u_t|^2 d\phi dt + C(R)TE(u_0) \le C(R)(1+T_0)E(u_0)
$$

with the help of [\(3.7\)](#page-8-1), and similarly for  $|\nabla v|$ . Choosing  $\varepsilon = 1/4$ , for sufficiently small  $0 < T < T_0$  by absolute continuity of the integral we thus can estimate

$$
\sup_{0 < t < T} \|w(t)\|_{L^2(\partial B)}^2 + \int_0^T \int_B |\nabla w|^2 dz dt
$$
\n
$$
\leq \frac{1}{2} \|\nabla w\|_{L^2(B \times [0,T])}^2 + C \sup_{0 < t < T} \|w(t)\|_{L^2(\partial B)}^2 \int_0^T (\|\nabla u\|_{L^4(B)}^2 + \|\nabla v\|_{L^4(B)}^2) dt
$$
\n
$$
\leq \frac{1}{2} \Big( \sup_{0 < t < T} \|w(t)\|_{L^2(\partial B)}^2 + \int_0^T \int_B |\nabla w|^2 dz dt \Big),
$$

and it follows that  $w = 0$ , as claimed.

Proof of Theorem [1.2.](#page-2-0) Existence for short time and uniqueness of a partially reg-ular weak solution to [\(1.3\)](#page-1-2), [\(1.4\)](#page-1-3) for given data  $u_0 \in H^{1/2}(S^1;N)$  follow from Proposition [6.4](#page-31-1) and Theorem [7.1,](#page-31-5) respectively. Since by Proposition [6.4](#page-31-1) our weak solution is smooth for  $t > 0$ , the remaining assertions follow from Theorem [1.1.](#page-2-2)

Note that at any blow-up time  $T_{i-1}$ ,  $i \geq 1$ , of the flow as in Theorem [1.1.](#page-2-2)ii) there exists a unique weak limit  $u_i = \lim_{t \uparrow T_{i-1}} u(t) \in H^{1/2}(S^1; N)$ , and we may uniquely continue the flow using Proposition [6.4.](#page-31-1)  $\Box$ 

### 8. Blow-up

<span id="page-33-0"></span>Preparing for the proof of part ii) of Theorem [1.1](#page-2-2) suppose now that for the solution constructed in part i) of that theorem there holds  $T_0 < \infty$ . Then, as we shall see in more detail below, by the results in Section [4](#page-10-0) condition [\(4.4\)](#page-11-0) must be

$$
\bot
$$

violated for  $T = T_0$  and there exist  $\delta > 0$  and points  $z_k \in B$  as well as radii  $r_k \downarrow 0$ as  $k \to \infty$  such that for suitable  $t_k \uparrow T_0$  there holds

$$
\int_{B_{r_k}(z_k)\cap B} |\nabla u(t_k)|^2 dz = \sup_{z_0 \in B, t \le t_k} \int_{B_{r_k}(z_0)\cap B} |\nabla u(t)|^2 dz = \delta.
$$

We may later choose a smaller constant  $\delta > 0$ , if necessary. Moreover, for later use from now on we consider local concentrations in the sense that for some  $z_0 \in B$ and some fixed radius  $r_0 > 0$  for a sequence of points  $z_k \in B$  with  $z_k \to z_0$  and radii  $r_k \downarrow 0$  for suitable  $t_k \uparrow T_0$  as  $k \to \infty$  there holds

$$
\int_{B_{r_k}(z_k)\cap B} |\nabla u(t_k)|^2 dz = \sup_{z' \in B_{r_0}(z_0), t \le t_k} \int_{B_{r_k}(z')\cap B} |\nabla u(t)|^2 dz = \delta.
$$

Scale

$$
u_k(z,t) = u(z_k + r_k z, t_k + r_k t)
$$

for

$$
z \in \Omega_k = \{z; \ z_k + r_k z \in B\}, \ t \in I_k = \{t; \ 0 \le t_k + tr_k < T_0\}.
$$

Note that then there holds

<span id="page-34-0"></span>(8.1) 
$$
\int_{B_1(0)\cap\Omega_k} |\nabla u_k(0)|^2 dz = \sup_{z_k + r_k z' \in B_{r_0}(z_0), -t_k/r_k \le t < 0} \int_{B_1(z')\cap\Omega_k} |\nabla u_k(t)|^2 dz = \delta.
$$

Passing to a sub-sequence we may assume that the domains  $\Omega_k$  exhaust a limit domain  $\Omega_{\infty} \subset \mathbb{R}^2$ , which either is the whole space  $\mathbb{R}^2$  or a half-space H.

<span id="page-34-1"></span>By the energy inequality Lemma [2.1](#page-6-1) for  $t \in I_k$  there holds

(8.2) 
$$
\int_{\Omega_k} |\nabla u_k(t)|^2 dz = \int_B |\nabla u(t_k + r_k t)|^2 dz \leq 2E(u_0),
$$

and for any  $t_0 < 0$  and sufficiently large  $k \in \mathbb{N}$  we have

<span id="page-34-2"></span>(8.3) 
$$
\int_{t_0}^0 \int_{\partial \Omega_k} |\partial_t u_k|^2 ds dt = \int_{t_0}^0 \int_{\partial \Omega_k} |d\pi_N(u_k) \partial_{\nu_k} u_k|^2 ds dt
$$

$$
= \int_{t_k + r_k t_0}^{t_k} \int_{\partial B} |u_t|^2 d\phi dt \le \int_{t_k + r_k t_0}^{T_0} \int_{\partial B} |u_t|^2 d\phi dt \to 0
$$

as  $k \to \infty$ , where ds is the element of length and where  $\nu_k$  is the outward unit normal along  $\partial \Omega_k$ . Expressing the harmonic functions  $\partial_t u_k(t)$  in Fourier series for each  $t < 0$ , it then also follows that  $\partial_t u_k \to 0$  locally in  $L^2$  on  $\Omega_{\infty} \times ]-\infty,0[$ . Finally, again using the fact that  $u_k(t)$  for each t is harmonic, by the maximum principle we have the uniform bound  $|u_k| \leq \sup_{p \in \Gamma} |p|$  as well as uniform smooth bounds locally away from the boundary of  $\Omega_{\infty}$ .

Hence we may assume that as  $k \to \infty$  we have  $u_k \to u_\infty$  weakly locally in  $H^1$ on  $\Omega_{\infty} \times ]-\infty, 0[$ , where  $u_{\infty}(z, t) = u_{\infty}(z)$  is independent of time, harmonic, and bounded. Moreover, we have smooth convergence away from  $\partial\Omega_{\infty}$ . Thus, if we assume that  $\Omega_{\infty} = \mathbb{R}^2$  by [\(8.1\)](#page-34-0) it follows that

$$
\int_{B_1(0)} |\nabla u_\infty|^2 dz = \delta.
$$

But any function  $v: \mathbb{R}^2 \to \mathbb{R}$  which is bounded and harmonic must be constant, which rules out this possibility. Hence  $\Omega_{\infty}$  can only be a half-space.

After a suitable rotation of the domain B and shift of coordinates in  $\mathbb{R}^2 \cong \mathbb{C}$  we may then assume that  $z_k = (0, -y_k)$  with  $1 - y_k \le Mr_k$  for some  $M \in \mathbb{N}$  and that  $\Omega_{\infty} = \{(x, y); y > y_0\}$  for some  $y_0$ . Finally, replacing  $r_k > 0$  with  $(M + 1)r_k$  and  $z_k$  with  $z_k = (0, -1)$ , if necessary, we may assume that  $\Omega_k \subset \mathbb{R}^2_+ = \{(x, y); y > 0\}$ is the ball of radius  $1/r_k$  around the point  $(0, 1/r_k)$  with  $0 \in \partial \Omega_k$ , while from [\(8.1\)](#page-34-0) with a uniform number  $L \in \mathbb{N}$  we have

<span id="page-35-0"></span>
$$
(8.4) \ L \int_{B_1(0) \cap \Omega_k} |\nabla u_k(0)|^2 dz \ge L\delta \ge \sup_{|z'| \le r_0/r_k, -t_k/r_k \le t < 0} \int_{B_1(z') \cap \Omega_k} |\nabla u_k(t)|^2 dz
$$

for any  $k \in \mathbb{N}$ . Let  $\Phi_k: \mathbb{R}_+^2 \to \Omega_k$  be the conformal maps given by

$$
\Phi_k(z) = \frac{2z}{2 - ir_k z}, \ z \in \mathbb{R}^2_+, \ k \in \mathbb{N},
$$

with  $\Phi_k \to id$  locally uniformly on  $\mathbb{R}^2 \cong \mathbb{C}$  as  $k \to \infty$ .

Let  $v_k = u_k \circ \Phi_k$ ,  $k \in \mathbb{N}$ . By conformal invariance of the Dirichlet energy, from  $(8.2)$  for any t we have

(8.5) 
$$
\int_{\mathbb{R}^2_+} |\nabla v_k(t)|^2 dz = \int_{\Omega_k} |\nabla u_k(t)|^2 dz \leq 2E(u_0),
$$

and by [\(8.4\)](#page-35-0) with a uniform number  $L_1 \in \mathbb{N}$  there holds

<span id="page-35-1"></span>
$$
(8.6) \qquad L_1 \int_{B_2^+(0)} |\nabla v_k(0)|^2 dz \ge L_1 \delta \ge \sup_{|z'| \le r_0/r_k, -t_k/r_k \le t < 0} \int_{B_1^+(z')} |\nabla v_k(t)|^2 dz,
$$

where  $B_r^+(z) = B_r(z) \cap \mathbb{R}^2_+$  for any  $r > 0$  and any  $z = (x, y) \in \mathbb{R}^2$ . Moreover, from [\(8.3\)](#page-34-2) for any  $t_0 < 0$  and any  $R > 0$  for the integral over  $]-R, R[\times{0} \subset \partial \mathbb{R}^2_+$  we obtain

(8.7) 
$$
\int_{t_0}^0 \int_{-R}^R |\partial_t v_k|^2 dx dt
$$
  
 
$$
\leq C \int_{t_0}^0 \int_{-R}^R |d\pi_N(v_k) \partial_y v_k|^2 dx dt \to 0 \text{ as } k \to \infty,
$$

and  $\partial_t v_k \to 0$  locally in  $L^2$  on  $\mathbb{R}^2_+ \times ]-\infty,0[$ . In addition, from our choice of  $(u_k)$ it follows that  $v_k \to v_\infty$  weakly locally in  $H^1$  on  $\mathbb{R}^2_+ \times ]-\infty, 0[$  as  $k \to \infty$ , where  $v_{\infty}(z, t) =: w_{\infty}(z)$  is harmonic and bounded.

For a suitable sequence of times  $t_0 < s_k < 0$ , we then also have locally weak convergence  $w_k := v_k(s_k) \to w_\infty$  in  $H^1$  on  $\mathbb{R}^2_+$  and, in addition,

<span id="page-35-2"></span>(8.8) 
$$
d\pi_N(w_k)\partial_y w_k \to 0 \text{ in } L^2_{loc}(\partial \mathbb{R}^2_+) \text{ as } k \to \infty.
$$

Thus, for sufficiently small  $\delta > 0$  by Proposition [3.3,](#page-8-4) applied to the functions  $w_k \circ \Psi$ , where  $\Psi: B \to \mathbb{R}^2_+$  is a suitable conformal map, we also have uniform local L<sup>2</sup>-bounds for  $\partial_x w_k$  on  $\partial \mathbb{R}^2_+$ , and we may assume that  $w_k \to w_\infty$  locally uniformly and weakly locally in  $H^1$  on  $\partial \mathbb{R}^2_+$  as  $k \to \infty$ . Since  $w_k$  is harmonic, we then also have locally strong  $H^1$ -convergence  $w_k \to w_\infty$  on  $\mathbb{R}^2_+$ .

To see that  $w_{\infty}$  is non-constant, let  $\varphi_k = \varphi_{z_0, 4r_k}, k \in \mathbb{N}$ . Integrating the identity  $(2.1)$  from the proof of Lemma [2.2](#page-6-2) in time, with error  $o(1) \rightarrow 0$  and suitable numbers

 $\varepsilon_k \downarrow 0$  as  $k \to \infty$  in view of [\(8.3\)](#page-34-2) we find

<span id="page-36-0"></span>
$$
\frac{1}{2} \Big| \int_{B} |\nabla u(t_k)|^2 \varphi_k^2 dz - \int_{B} |\nabla u(t_k + r_k s_k)|^2 \varphi_k^2 dz \Big|
$$
\n
$$
\leq \int_{t_k + r_k s_k}^{t_k} \int_{\partial B} |u_t|^2 \varphi_k^2 d\phi dt + 2 \int_{t_k + r_k s_k}^{t_k} \int_{B} |u_t \nabla u \varphi_k \nabla \varphi_k| dz dt
$$
\n
$$
\leq o(1) + 8\varepsilon_k r_k \int_{t_k + r_k s_k}^{t_k} \int_{B} |\nabla u|^2 |\nabla \varphi_k|^2 dz dt
$$
\n
$$
+ (8\varepsilon_k r_k)^{-1} \int_{t_k + r_k s_k}^{t_k} \int_{B} |u_t|^2 \varphi_k^2 dz dt.
$$

With the help of [\(2.2\)](#page-7-6) and [\(8.3\)](#page-34-2) for suitable  $\varepsilon_k \downarrow 0$  we can bound

$$
(8\varepsilon_k r_k)^{-1} \int_{t_k + r_k s_k}^{t_k} \int_B |u_t|^2 \varphi_k^2 dz \, dt \le C \varepsilon_k^{-1} \int_{t_k + r_k s_k}^{t_k} \int_{\partial B} |u_t|^2 dz \, dt \to 0.
$$

Since for any choice  $t_0 < s_k < 0$  we also can estimate

$$
8\varepsilon_k r_k \int_{t_k + r_k s_k}^{t_k} \int_B |\nabla u|^2 |\nabla \varphi_k|^2 dz dt \leq C\varepsilon_k |t_0| E(u_0) \to 0,
$$

from [\(8.9\)](#page-36-0) and [\(8.6\)](#page-35-1) it follows that with error  $o(1) \rightarrow 0$  as  $k \rightarrow \infty$  we have

<span id="page-36-2"></span>
$$
L_1 \int_{B_4^+(0)} |\nabla w_k|^2 dz + o(1) = L_1 \int_{B_4^+(0)} |\nabla v_k(s_k)|^2 dz + o(1)
$$
  
(8.10)  

$$
\geq L_1 \int_B |\nabla u(t_k + r_k s_k)|^2 \varphi_k^2 dz + o(1) \geq L_1 \int_B |\nabla u(t_k)|^2 \varphi_k^2 dz
$$
  

$$
\geq L_1 \int_{B_2^+(0)} |\nabla v_k(0)|^2 dz \geq L_1 \delta.
$$

Finally, in view of locally uniform convergence  $w_k \to w_\infty$  and weak local  $L^2$ convergence of the traces  $\nabla w_k \to \nabla w_\infty$  on  $\partial \mathbb{R}^2_+$ , we may pass to the limit  $k \to \infty$ in [\(8.8\)](#page-35-2) to conclude that

(8.11) 
$$
d\pi_N(w_\infty)\partial_y w_\infty = 0 \text{ on } \partial \mathbb{R}^2_+.
$$

Since  $w_{\infty}$  is harmonic, the Hopf differential

<span id="page-36-1"></span>
$$
f = |\partial_x w_{\infty}|^2 - |\partial_y w_{\infty}|^2 - 2i\partial_x w_{\infty} \cdot \partial_y w_{\infty}
$$

defines a holomorphic function  $f \in L^1(\mathbb{R}^2_+,\mathbb{C})$ . Moreover,  $w_\infty \in H^{3/2}_{loc}(\mathbb{R}^2_+)$  with trace  $\nabla w_{\infty} \in L^2_{loc}(\partial \mathbb{R}^2_+)$ ; thus also the trace of f is well-defined on  $\partial \mathbb{R}^2_+$ . By [\(8.11\)](#page-36-1) now the trace of f is real-valued; thus  $f \equiv c$  for some constant  $c \in \mathbb{R}$ . But  $\nabla w_\infty \in L^2(\mathbb{R}^2_+)$ ; hence  $f \in L^1(\mathbb{R}^2_+)$ . It follows that  $c = 0$ , and  $w_\infty$  is conformal.

With stereographic projection  $\Phi: B \to \mathbb{R}^2_+$  from a point  $z_0 \in \partial B$  define the map  $\bar{u} = w_{\infty} \circ \Phi \in H^{1/2}(S^1; N)$ . By conformal invariance,  $\bar{u}$  again is harmonic with finite Dirichlet integral and satisfies [\(1.6\)](#page-2-1) on  $\partial B \setminus \{z_0\}$ ; since the point  $\{z_0\}$  has vanishing  $H^1$ -capacity,  $\bar{u}$  then is stationary in the sense of [\[21\]](#page-40-16). Moreover,  $\bar{u}$  is conformal. For such mappings, smooth regularity on  $B$  was shown by Grüter-Hildebrandt-Nitsche [\[21\]](#page-40-16); thus condition [\(1.6\)](#page-2-1) holds everywhere on  $\partial B$  in the pointwise sense, and  $\bar{u}$  parametrizes a minimal surface of finite area supported by N which meets N orthogonally along its boundary.

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*Proof of Theorem [1.1.](#page-2-2)ii*). For given smooth data  $u_0 \in H^{1/2}(S^1; N)$  let u be the unique solution to  $(1.3)$ ,  $(1.4)$  guaranteed by part i) of the theorem, and suppose that the maximal time of existence  $T_0 < \infty$ . Then condition [\(4.4\)](#page-11-0) must fail as  $t \uparrow T_0$ ; else from Propositions [4.11](#page-23-0) and [4.6](#page-16-1) we obtain smooth bounds for  $u(t)$  as  $t \uparrow T_0$  and there exists a smooth trace  $u_1 = \lim_{t \uparrow T_0} u(t)$ . But by the first part of the theorem there is a smooth solution to the initial value problem for [\(1.3\)](#page-1-2) with initial data  $u_1$  at time  $T_0$ , and this solution extends the original solution u to an interval  $[0, T_1]$  for some  $T_1 > T_0$ , contradicting maximality of  $T_0$ .

Let  $z^{(i)} \in B$ ,  $1 \leq i \leq i_0$ , such that for some number  $\delta > 0$  and suitable  $t_k^{(i)}$  $\binom{n}{k}$   $\uparrow$   $T_0$ ,  $z_k^{(i)} \to z^{(i)}, r_k^{(i)} \to 0$  as  $k \to \infty$  there holds

$$
\liminf_{k \to \infty} \int_{B_{r_k^{(i)}}(z_k^{(i)}) \cap B} |\nabla u(t_k^{(i)})|^2 dz \ge \delta.
$$

By the argument following [\(8.9\)](#page-36-0) thus for a suitable sequence of radii  $0 < r_k^{(0)} \to 0$ such that  $r_k^{(i)}$  $\binom{(i)}{k}$   $\binom{(0)}{k}$   $\to 0$  as well as  $(T_0 - t_k^{(i)})$  $\binom{(i)}{k}/r_k^{(0)} \to 0$  then with error  $o(1) \to 0$  as  $k\to\infty$  there holds

$$
\int_{B_{r_k^{(0)}}(z^{(i)})\cap B} |\nabla u(t)|^2 dz + o(1) \ge \int_{B_{r_k^{(i)}}(z_k^{(i)})\cap B} |\nabla u(t_k^{(i)})|^2 dz \ge \delta.
$$

for all  $T_0 - r_k^{(0)} < t < T_0$ , uniformly in  $1 \le i \le i_0$ . For sufficiently large  $k \in \mathbb{N}$ such that  $r_k^{(0)} < \inf_{i \le j} |z^{(i)} - z^{(j)}|/4$  it follows that  $i_0 \le E(u_0)/\delta$ , and we may fix  $r_0 > 0$  and redefine  $t_k^{(i)}$  $\binom{i}{k},\ r_k^{(i)}$  $\mathbf{z}_k^{(i)}$ , and  $\mathbf{z}_k^{(i)}$  $\binom{n}{k}$ , if necessary, such that for each  $1 \leq i \leq i_0$ there holds

$$
\int_{B_{r_k^{(i)}}(z_k^{(i)})\cap B} |\nabla u(t_k^{(i)})|^2 dz=\sup_{z'\in B_{r_0}(z^{(i)}),\, 0< t\leq t_k^{(i)}}\int_{B_{r_k^{(i)}}(z')\cap B} |\nabla u(t)|^2 dz=\delta.
$$

Moreover, we may assume that  $\delta < \delta_0$ , as defined in Proposition [3.1.](#page-7-5) The char-acterization of the concentration points as in Theorem [1.2.](#page-2-0)ii) via solutions  $\bar{u}^{(i)}$  of [\(1.6\)](#page-2-1) then follows from our above analysis.

In addition, Corollary [3.2](#page-8-5) yields the uniform lower bound

$$
\lim_{r_0 \downarrow 0} \liminf_{t \uparrow T} \int_{B_{r_0}(z^{(i)}) \cap B} |\nabla u(t)|^2 dz \ge 2E(\bar{u}^{(i)}) \ge 2\delta_0^2
$$

for the concentration energy quanta, which gives the claimed upper bound for the total number of concentration points.

<span id="page-37-0"></span>Finally, with the help of Proposition [4.11](#page-23-0) we can smoothly extend the solution u to  $B \setminus \{z^{(1)}, \ldots, z^{(i_0)}\}$  at time  $t = T_0$ .

# 9. ASYMPTOTICS

Suppose next that the solution u to [\(1.3\)](#page-1-2), [\(1.4\)](#page-1-3) exists for all time  $0 < t < \infty$ . Then u either concentrates for suitable  $t_k \uparrow \infty$  in the sense that condition [\(4.4\)](#page-11-0) does not hold true uniformly in time, or u satisfies uniform smooth bounds, as shown in Section [4.](#page-10-0)

In the latter case, the claim made in Theorem [1.1.](#page-2-2)iii) easily follows.

**Proposition 9.1.** Suppose that for any  $\delta > 0$  there exists  $R > 0$  such that con-dition [\(4.4\)](#page-11-0) holds true for all  $0 < t < \infty$ . Then there exists a smooth solution

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 $u_{\infty} \in H^{1/2}(S^1;N)$  of  $(1.6)$  such that  $u(t) \to u_{\infty}$  smoothly as  $t \to \infty$  suitably, and  $u_{\infty}$  parametrizes a minimal surface of finite area supported by N which meets N orthogonally along its boundary.

*Proof.* For sufficiently small  $\delta > 0$ , for any  $j \in \mathbb{N}$  by iterative reference to Propositions [4.2,](#page-11-2) [4.4](#page-13-0)- [4.6,](#page-16-1) and [4.10,](#page-21-0) [4.11,](#page-23-0) respectively, as in Section [6](#page-29-0) we can find constants  $C_j > 0$  such that  $||u(t)||_{H^j(B)} \leq C_j$  for all  $t > 1$ , Moreover, by the energy inequality Lemma [2.1](#page-6-1) for a suitable sequence  $t_k \to \infty$  there holds  $u_t(t_k) \to 0$  in  $L^2(\partial B)$  as  $k \to \infty$ . Then for any  $j \in \mathbb{N}$  a subsequence  $u(t_k) \to u_\infty$  in  $H^j(B)$ , and a diagonal subsequence converges smoothly, where  $u_{\infty}$  solves [\(1.6\)](#page-2-1). By the argument after  $(8.11)$  in Section [8](#page-33-0) then  $u_{\infty}$  is conformal and  $u_{\infty}$  parametrizes a minimal surface with free boundary on N which meets N orthogonally along its boundary.  $\square$ 

In the remaining case that for some  $\delta > 0$  condition [\(4.4\)](#page-11-0) fails to hold, there exists a sequence  $t_k \uparrow \infty$  and points  $z^{(1)}, \ldots, z^{(i_0)}$  such that for sequences  $z_k^{(i)} \to z^{(i)}$ , radii  $r_k^{(i)} \to 0$  as  $k \to \infty$  there holds

$$
\liminf_{k\to\infty}\int_{B_{r_k^{(i)}}(z_k^{(i)})\cap B}|\nabla u(t_k)|^2dz\geq \delta,~1\leq i\leq i_0.
$$

By Lemma [2.1](#page-6-1) there holds the a-priori bound  $i_0 \leq E(u_0)/\delta$  for the number of concentration points. By the argument leading to [\(8.10\)](#page-36-2) then for a suitable number  $0 < r_0 \le \inf_{i < j} |z^{(i)} - z^{(j)}|/4$  with error  $o(1) \to 0$  as  $k \to \infty$  and with some constant  $L \in \mathbb{N}$  for all  $1 \leq i \leq i_0$  there holds

$$
\begin{split} L & \int_{B_{2r_k^{(i)}}(z_k^{(i)}) \cap B} |\nabla u(t_k)|^2 dz + o(1) \\ &\geq \sup_{z_0 \in B_{r_0}(z_k^{(i)}), \, t_k - r_0 \leq t \leq t_k} \int_{B_{r_k^{(i)}}(z_0) \cap B} |\nabla u(t)|^2 dz \geq \delta. \end{split}
$$

Fixing any index  $1 \leq i \leq i_0$  and renaming  $z_k^{(i)} = z_k$ ,  $r_k^{(i)} = r_k$ , we then scale

$$
u_k(z,t) = u(z_k + r_k z, t_k + r_k t), \ z \in \Omega_k = \{z; z_k + r_k z \in B\}, \ -t_k/r_k \le t \le 0,
$$

as before and observe that for any  $t_0 < 0$  there holds

(9.1) 
$$
\int_{t_0}^0 \int_{\partial \Omega_k} |\partial_t u_k|^2 ds dt = \int_{t_0}^0 \int_{\partial \Omega_k} |d\pi_N(u_k) \partial_{\nu_k} u_k|^2 ds dt
$$

$$
= \int_{t_k + r_k t_0}^{t_k} \int_{\partial B} |u_t|^2 d\phi dt \le \int_{t_k + r_k t_0}^{\infty} \int_{\partial B} |u_t|^2 d\phi dt \to 0
$$

as  $k \to \infty$ , where  $\nu_k$  is the outward unit normal along  $\partial \Omega_k$ . Just as in Section [8](#page-33-0) for suitable  $t_0 < s_k < 0$  we then obtain local uniform and  $H^1$ -convergence of a subsequence of the conformally rescaled maps  $w_k = u_k(s_k) \circ \Phi_k \in H^1_{loc}(\mathbb{R}^2_+)$  to a smooth, harmonic and conformal limit  $w_{\infty}$  with finite energy and continuously mapping  $\partial \mathbb{R}^2_+$  to N, inducing a solution  $\bar{u}_{\infty} = w_{\infty} \circ \Phi \in H^{1/2}(S^1; N)$  of [\(1.6\)](#page-2-1) corresponding to a minimal surface with free boundary on  $N$ . This ends the proof of Theorem [1.1.](#page-2-2)iii)

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#### 10. Appendix

In this section, for the convenience of the reader we derive two interpolation inequalities that play a crucial role in our arguments.

Let  $v \in H^1(B)$ , and let  $\varphi_{z_i,r}$  as above such that the collection of balls  $B_r(z_i)$ ,  $1 \leq i \leq i_0$  covers B with at most L balls  $B_{2r}(z_i)$  overlapping at any  $z \in B$ , with  $L \in \mathbb{N}$  independent of  $r > 0$ . We may assume  $r < 1/8$  so that for any  $1 \leq i \leq i_0$ there is a pair of orthogonal vectors  $e_{1,i}$ ,  $e_{2,i}$  such that for any  $z \in B_r(z_i)$  there holds  $z + s e_{1,i} + t e_{2,i} \in B$  for any  $0 \leq s, t \leq 2r$ . After a rotation of coordinates, we may assume that  $e_{1,i} = (1,0), e_{2,i} = (0,1)$  are the standard basis vectors. Writing  $\varphi$  for  $\varphi_{z_i,r}$  for any  $z = (x, y) \in B_r(z_i)$ , by arguing as Ladyzhenskaya [\[26\]](#page-40-17), using that

$$
(v^{2}\varphi)(x+2r, y) = 0 = (v^{2}\varphi)(x, y+2r),
$$

then we can estimate

<span id="page-39-7"></span>
$$
v^{4}(z) = |(v^{2}\varphi)(z)|^{2} \le \int_{0}^{2r} |\partial_{x}(v^{2}\varphi)(x+s,y)|ds \cdot \int_{0}^{2r} |\partial_{y}(v^{2}\varphi)(x,y+t)|dt
$$
  
(10.1)  

$$
\le \int_{\{s;(s,y)\in B\}} |\partial_{x}(v^{2}\varphi)(s,y)|ds \cdot \int_{\{t;(x,t)\in B\}} |\partial_{y}(v^{2}\varphi)(x,t)|dt,
$$

and with the help of Fubini's theorem we find

$$
\int_{B_r(z_i)} |v|^4 dz \leq \int_B |v|^4 \varphi^2 dz \leq \int_{-\infty}^{\infty} \left( \int_{\{x;(x,y)\in B\}} |(v^2 \varphi)(x,y)|^2 dx \right) dy
$$
\n
$$
\leq \int_{-\infty}^{\infty} \int_{\{s;(s,y)\in B\}} |\partial_x (v^2 \varphi)(s,y)| ds dy \cdot \int_{-\infty}^{\infty} \int_{\{t;(x,t)\in B\}} |\partial_y (v^2 \varphi)(x,t)| dt dx
$$
\n
$$
\leq \left( \int_B |\nabla (v^2 \varphi)| dz \right)^2 \leq \left( \int_B (2|\nabla v||v\varphi| + v^2 |\nabla \varphi|) dz \right)^2
$$
\n
$$
\leq C \left( \int_{B_{2r}(z_i)} |\nabla v|^2 dz + r^{-2} \int_{B_{2r}(z_i)} v^2 dz \right) \int_{B_{2r}(z_i)} v^2 dz.
$$

Fixing  $r = 1/5$  and summing over  $1 \leq i \leq i_0$  with an absolute constant  $C > 0$  we obtain the bound

<span id="page-39-4"></span>(10.2) 
$$
||v||_{L^4(B)}^4 \leq C||v||_{H^1(B)}^2||v||_{L^2(B)}^2
$$

for any  $v \in H^1(B)$ .

### **REFERENCES**

- <span id="page-39-5"></span>[1] Brézis, H.; Gallouet, T.: *Nonlinear Schrödinger evolution equations*, Nonlinear Anal. 4 (1980), no. 4, 677-681.
- <span id="page-39-6"></span><span id="page-39-2"></span>[2] Brendle, Simon: *Curvature flows on surfaces with boundary*, Math. Ann. 324 (2002), no. 3, 491-519.
- [3] Brézis, Haim; Wainger, Stephen: *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Comm. Partial Differential Equations 5 (1980), no. 7, 773-789.
- <span id="page-39-0"></span>[4] Caffarelli, Luis; Silvestre, Luis: *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations 32 (2007), no. 7-9, 1245-1260.
- <span id="page-39-1"></span>[5] Chang, Kung-Ching; Ding, Wei Yue; Ye, Rugang: *Finite-time blow-up of the heat flow of harmonic maps from surfaces*, J. Differential Geom. 36 (1992), no. 2, 507-515.
- <span id="page-39-3"></span>[6] Chang, Kung-Ching; Liu, Jia-Quan: *Boundary flow for the minimal surfaces in*  $\mathbb{R}^n$  *with Plateau boundary condition*, Proc. Roy. Soc. Edinburgh Sect. A 135 (2005), no. 3, 537-562.
- [7] Chang, Kung-Ching; Liu, Jia-Quan: *An evolution of minimal surfaces with Plateau condition*, Calc. Var. Partial Differential Equations 19 (2004), no. 2, 117-163.

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- [8] Chang, Kung-Ching; Liu, Jia-Quan: *Heat flow for the minimal surface with Plateau boundary condition*, Acta Math. Sin. (Engl. Ser.) 19 (2003), no. 1, 1-28.
- [9] Chang, Kung-Ching; Liu, Jia-Quan: *Another approach to the heat flow for Plateau problem*, J. Differential Equations 189 (2003), no. 1, 46-70.
- [10] Courant, Richard: *Plateau's problem and Dirichlet's principle*, Ann. of Math. (2) 38 (1937), no. 3, 679-724.
- [11] Da Lio, Francesca: *Compactness and bubble analysis for* 1/2*-harmonic maps*, Ann. Inst. H. Poincaré Anal. Non Linéaire 32 (2015), no. 1, 201-224.
- [12] Da Lio, Francesca; Martinazzi, Luca; Rivi`ere, Tristan: *Blow-up analysis of a nonlocal Liouville-type equation*, Anal. PDE 8 (2015), no. 7, 1757-1805.
- [13] Da Lio, Francesca; Pigati, Alessandro: *Free boundary minimal surfaces: a nonlocal approach*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 20 (2020), no. 2, 437-489.
- [14] Da Lio, Francesca; Rivière, Tristan: *Three-term commutator estimates and the regularity of* 1/2*-harmonic maps into spheres*, Anal. PDE 4 (2011), no. 1, 149-190.
- [15] Da Lio, Francesca; Schikorra, Armin: *On regularity theory for* n/p*-harmonic maps into manifolds*, Nonlinear Anal. 165 (2017), 182-197.
- <span id="page-40-8"></span>[16] Douglas, Jesse: *Solution of the problem of Plateau*, Trans. Amer. Math. Soc. 33 (1931), no. 1, 263-321.
- <span id="page-40-14"></span>[17] Eells, James, Jr.; Sampson, J. H.: *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86 (1964), 109-160.
- <span id="page-40-2"></span>[18] Freire, Alexandre: *Uniqueness for the harmonic map flow from surfaces to general targets*, Comment. Math. Helv. 70 (1995), no. 2, 310-338.
- <span id="page-40-3"></span>[19] Freire, Alexandre: *Correction to: "Uniqueness for the harmonic map flow from surfaces to general targets"*, Comment. Math. Helv. 71 (1996), no. 2, 330-337.
- <span id="page-40-6"></span>[20] Gehrig, Manuela Iris: *Prescribed curvature on the boundary of the disc*, Dissertation, ETH Zurich, 2020, https://doi.org/10.3929/ethz-b-000445412.
- <span id="page-40-16"></span>[21] Grüter, Michael; Hildebrandt, Stefan; Nitsche, Johannes C. C.: *On the boundary behavior of minimal surfaces with a free boundary which are not minima of the area*, Manuscripta Math. 35 (1981), no. 3, 387-410.
- <span id="page-40-7"></span>[22] Hyder, Ali; Segatti, Antonio; Sire, Yannick; Wang, Changyou: *Partial regularity of the heat flow of half-harmonic maps and applications to harmonic maps with free boundary*, [arXiv:2111.14171.](http://arxiv.org/abs/2111.14171)
- <span id="page-40-12"></span>[23] Imbusch, Cordula; Struwe, Michael: *Variational principles for minimal surfaces*, Topics in nonlinear analysis, 477-498, Progr. Nonlinear Differential Equations Appl., 35, Birkhäuser, Basel, 1999.
- [24] Jost, J¨urgen: *Univalency of harmonic mappings between surfaces*, J. Reine Angew. Math. 324 (1981), 141-153.
- <span id="page-40-13"></span>[25] Jost, J.; Struwe, M.: *Morse-Conley theory for minimal surfaces of varying topological type*, Invent. Math. 102 (1990), no. 3, 465-499.
- <span id="page-40-17"></span>[26] Ladyzhenskaya, O. A.: *The mathematical theory of viscous incompressible flow*, Second English edition, revised and enlarged Translated from the Russian by Richard A. Silverman and John Chu, Mathematics and its Applications, Vol. 2, Gordon and Breach, Science Publishers, New York-London-Paris, 1969.
- <span id="page-40-4"></span>[27] Lenzmann, Enno; Schikorra, Armin: *Sharp commutator estimates via harmonic extensions*, Nonlinear Anal. 193 (2020), 111375, 37 pp.
- <span id="page-40-5"></span>[28] Mazowiecka, Katarzyna; Schikorra, Armin: *Fractional div-curl quantities and applications to nonlocal geometric equations*, J. Funct. Anal. 275 (2018), no. 1, 1-44.
- <span id="page-40-0"></span>[29] Millot, Vincent; Sire, Yannick: *On a fractional Ginzburg-Landau equation and* 1/2*-harmonic maps into spheres*, Arch. Ration. Mech. Anal. 215 (2015), no. 1, 125-210.
- <span id="page-40-11"></span>[30] Morse, Marston: *Functional topology and abstract variational theory*, Ann. of Math. (2) 38 (1937), no. 2, 386-449.
- <span id="page-40-10"></span>[31] Morse, Marston; Tompkins, C.: *The existence of minimal surfaces of general critical types*, Ann. of Math. (2) 40 (1939), no. 2, 443-472.
- <span id="page-40-9"></span><span id="page-40-1"></span>[32] Moser, Roger: *Intrinsic semiharmonic maps*, J. Geom. Anal. 21 (2011), no. 3, 588-598.
- <span id="page-40-15"></span>[33] Radó, Tibor: *On Plateau's problem*, Ann. of Math. (2) 31 (1930), no. 3, 457-469.
- [34] Rivi`ere, Tristan: *Le flot des applications faiblement harmoniques en dimension deux*, published in "Applications harmoniques entre variétés", Thèse de l'université Paris 6, 1993.

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- [35] Rupflin, Melanie: *Teichm¨uller harmonic map flow from cylinders*, Math. Ann. 368 (2017), no. 3-4, 1227-1276.
- <span id="page-41-11"></span>[36] Rupflin, Melanie; Schrecker, Matthew R. I.: *Analysis of boundary bubbles for almost minimal cylinders*, Calc. Var. Partial Differential Equations 57 (2018), no. 5, Paper No. 121.
- <span id="page-41-12"></span>[37] Rupflin, Melanie; Topping, Peter M.: *Global weak solutions of the Teichmüller harmonic map flow into general targets*, Anal. PDE 12 (2019), no. 3, 815-842.
- [38] Schikorra, Armin: *Regularity of* n/2*-harmonic maps into spheres*, J. Differential Equations 252 (2012), no. 2, 1862-1911.
- <span id="page-41-13"></span>[39] Schoen, Richard; Uhlenbeck, Karen: *A regularity theory for harmonic maps*, J. Differential Geometry 17 (1982), no. 2, 307-335.
- [40] Schoen, Richard; Yau, Shing Tung: *On univalent harmonic maps between surfaces*, Invent. Math. 44 (1978), no. 3, 265-278.
- <span id="page-41-7"></span>[41] Struwe, Michael: *On a critical point theory for minimal surfaces spanning a wire in*  $\mathbb{R}^n$ , J. Reine Angew. Math. 349 (1984), 1-23.
- <span id="page-41-3"></span>[42] Struwe, Michael: *On the evolution of harmonic mappings of Riemannian surfaces*, Comment. Math. Helv. 60 (1985), no. 4, 558-581.
- <span id="page-41-9"></span>[43] Struwe, Michael: *A Morse theory for annulus-type minimal surfaces*, J. Reine Angew. Math. 368 (1986), 1-27.
- <span id="page-41-8"></span>[44] Struwe, Michael: *Plateau's problem and the calculus of variations*, Mathematical Notes, 35. Princeton University Press, Princeton, NJ, 1988.
- <span id="page-41-4"></span>[45] Topping, Peter: *Reverse bubbling and nonuniqueness in the harmonic map flow*, Int. Math. Res. Not. 2002, no. 10, 505-520.
- <span id="page-41-6"></span>[46] Tromba, Anthony Joseph: *Degree theory on oriented infinite-dimensional varieties and the Morse number of minimal surfaces spanning a curve in*  $\mathbb{R}^n$ . *I,*  $n \geq 4$ , Trans. Amer. Math. Soc. 290 (1985), no. 1, 385-413.
- <span id="page-41-5"></span>[47] Tromba, Anthony Joseph: *Degree theory on oriented infinite-dimensional varieties and the Morse number of minimal surfaces spanning a curve in*  $\mathbb{R}^n$ . *II,*  $n = 3$ , Manuscripta Math. 48 (1984), no. 1-3, 139-161.
- <span id="page-41-0"></span>[48] Wettstein, Jerome: *Uniqueness and Regularity of the Fractional Harmonic Gradient Flow in*  $S^{n-1}$ , Nonlinear Analysis 214 (2022).
- <span id="page-41-1"></span>[49] Wettstein, Jerome: *Existence, Uniqueness and Regularity of the Fractional Harmonic Gradient Flow in General Target Manifolds*, arxiv 2109.11458.
- <span id="page-41-2"></span>[50] Wettstein, Jerome: *Half-Harmonic Gradient Flow: Aspects of a Non-Local Geometric PDE*, [arXiv:2112.08846.](http://arxiv.org/abs/2112.08846)

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