Mahonian and Euler-Mahonian statistics for set partitions

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Abstract. A partition of the set $[n] := \{1, 2, \ldots, n\}$ is a collection of disjoint nonempty subsets (or blocks) of $[n]$, whose union is $[n]$. In this paper we consider the following rarely used representation for set partitions: given a partition of $[n]$ with blocks $B_1, B_2, \ldots, B_m$ satisfying $\max B_1 < \max B_2 < \cdots < \max B_m$, we represent it by a word $w = w_1 w_2 \ldots w_n$ such that $i \in B_{w_i}$, $1 \le i \le n$. We prove that the Mahonian statistics INV, MAJ, MAJ_d, r-MAJ, Z, DEN, MAK, MAD are all equidistributed on set partitions via this representation, and that the Euler-Mahonian statistics (des, MAJ), (mstc, INV), (exc, DEN), (des, MAK) are all equidistributed on set partitions via this representation.

Keywords: Set partition, Inversion, Major index, Mahonian statistic, q-Stirling number

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1. Introduction

1.1. Mahonian and Euler-Mahonian statistics

We use the notation $M = \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$ for the multiset M consisting of k_i copies of i, for all $i \in [m] := \{1, 2, \ldots, m\}$. Let $n = k_1 + k_2 + \cdots + k_m$, we write $|M| = n$ and $\langle M \rangle = m$. Throughout this paper, we always assume that $M = \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$ with $k_i \ge 1$ for all $i \in [m]$ and that $|M| = n$. Let $\mathfrak{S}_M$ be the set of multipermutations of multiset M.

Given $w = w_1 w_2 \ldots w_n \in \mathfrak{S}_M$, a pair (i, j) is called an *inversion* of w if $i < j$ and $w_i > w_j$. Let $\text{INV}(w)$ be the number of inversions of w. An index i, $1 \le i \le n-1$, is called a *descent* of w if $w_i > w_{i+1}$. Let $\text{Des}(w)$ be the set of all descents of w, and let $\text{des}(w) := |\text{Des}(w)|$, where $|\cdot|$ indicates cardinality. Define the *major index* of w, denoted $\text{MAJ}(w)$, to be

$$\text{MAJ}(w) = \sum_{i \in \text{Des}(w)} i.$$

Name	Reference	Year
INV	Rodriguez [35]	1839
MAJ	MacMahon [29]	1916
r-MAJ	Rawlings [32] (for permutations)	1981
	Rawlings [33] (for words)	1981
MAJ_d	Kadell [25]	1985
Z	Zeilberger-Bressoud [39]	1985
DEN	Denert [12], Foata-Zeilberger [15] (for permutations)	1990
	Han [22] (for words)	1994
MAK	Foata-Zeilberger [15] (for permutations)	1990
	Clarke-Steingrímsson-Zeng [11] (for words)	1997
MAD	Clarke-Steingrímsson-Zeng [11]	1997
STAT	Babson-Steingrímsson [2] (for permutations)	2000
	Kitaev-Vajnovszki [26] (for words)	2016

Table 1. Mahonian statistics on words.

Name	Reference	Year
(des, MAJ)	MacMahon [29]	1916
(exc, DEN)	Denert [12], Foata-Zeilberger [15] (for permutations)	1990
	Han [22] (for words)	1994
(des, MAK)	Foata-Zeilberger [15] (for permutations)	1990
	Clarke-Steingrímsson-Zeng [11] (for words)	1997
(mstc, INV)	Skandera [37] (for permutations)	2001
	Carnevale [4] (for words)	2017

Table 2. Euler-Mahonian statistics on words.

MacMahon's equidistribution theorem asserts that for any multiset M,

$$\sum_{w\in\mathfrak{S}_M} q^{\text{INV}(w)} = \sum_{w\in\mathfrak{S}_M} q^{\text{MAJ}(w)}.$$

In other words, the statistics INV and MAJ are equidistributed on $\mathfrak{S}_M$. This famous result was obtained by MacMahon [29] in 1916. It was not until 1968 that a famous bijective proof was found by Foata [14].

Any statistic that is equidistributed with des is said to be *Eulerian*, while any statistic equidistributed with MAJ is said to be *Mahonian*. A bivariate statistic that is equidistributed with (des, MAJ) is said to be *Euler-Mahonian*. Following [11], we will write Mahonian statistics with capital letters. Tables 1 and 2 list the most common Mahonian and Euler-Mahonian statistics on words in the literature respectively.

There are many research articles devoted to finding MacMahon type results for other combinatorial objects. For example, see [10] for 01-fillings of moon polyominoes, [18] for standard Young tableaux, [34] for ordered set partitions, [28] for k-Stirling permutations. In this paper we consider set partitions.

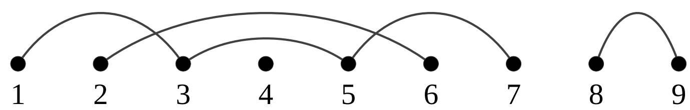


Fig. 1. The standard representation of $\{\{1,3,5,7\},\{2,6\},\{4\},\{8,9\}\}$.

1.2. Set partitions

A *partition of the set* $[n] = \{1, 2, \ldots, n\}$ is a collection of disjoint nonempty subsets (or blocks) of $[n]$, whose union is $[n]$. For example, $\{\{1,3,5,7\},\{2,6\},\{4\},\{8,9\}\}$ is a partition of [9]. We denote by Π_n the set of all partitions of $[n]$, and by $\Pi_{n,m}$ the set of all partitions of $[n]$ with exactly m blocks. There are several well-known representations for set partitions, and each of them has its aim value and its result. In the following, we provide three most common representations.

Given a partition of $[n]$, the graph on the vertex set $[n]$ whose edge set consists of the arcs connecting the elements of each block in numerical order is called the *standard representation*. For example, the standard representation of $\{\{1,3,5,7\},\{2,6\},\{4\},\{8,9\}\}$ has the arc set $\{(1,3),(3,5),(5,7),(2,6),(8,9)\}$; see Fig. 1. Using the standard representation, Chen, Gessel, Yan and Yang [9] introduced a major index statistic and then obtained a MacMahon type result for set partitions.

Given a partition of $[n]$, write it as $B_1/B_2/\cdots/B_m$, where $B_1, B_2, \ldots, B_m$ are the blocks and satisfy the following order property

$$\min B_1 < \min B_2 < \cdots < \min B_m.$$

This representation is called the *block representation*. For example, the block representation of $\{\{4\},\{2,6\},\{8,9\},\{1,3,5,7\}\}$ is $\{1,3,5,7\}/\{2,6\}/\{4\}/\{8,9\}$. From the historical point of view, the set partitions are defined by block representation. Using the block representation, Sagan [36] introduced a major index statistic and then obtained a MacMahon type result for set partitions.

Given a partition w of $[n]$ with blocks $B_1, B_2, \cdots, B_m$, where

$$\min B_1 < \min B_2 < \cdots < \min B_m,$$

we write $w = w_1 w_2 \ldots w_n$, where w_i is the block number in which i appears, that is, $i \in B_{w_i}$. This representation is called the *canonical representation*. For example, the canonical representation of $\{\{1,3,5,7\},\{2,6\},\{4\},\{8,9\}\}$ is 121312144. The canonical representation is one of the most popular representations in the theory of set partitions. Many statistics, especially the pattern-based statistics, on set partitions are defined via canonical representation, see, e.g., the book [30] for more information.

In this paper, we consider the following rarely used representation for set partitions: given a partition w of $[n]$ with blocks $B_1, B_2, \ldots, B_m$, where

$$\max B_1 < \max B_2 < \cdots < \max B_m,$$

we write $w = w_1 w_2 \ldots w_n$, where w_i is the block number in which i appears, that is, $i \in B_{w_i}$. Note that the only difference between this representation and canonical representation is that here we use the ordering according to the *maximal* element of the blocks. This representation was also considered by Johnson [23, 24] and Deodhar-Srinivasan [13]. In this paper we prove that the Mahonian statistics on words in Table 1, except for STAT, are all equidistributed on set partitions via this representation, and that the Euler-Mahonian statistics on words in Table 2 are all equidistributed on set partitions via this representation. For this reason, we call this representation the *Mahonian representation* for set partitions. We will state formally the main results of this paper in the next section, and will prove them in the remaining sections.

2. Main results

2.1. Definitions and the main results

Given $w = w_1 w_2 \ldots w_n \in \mathfrak{S}_M$, define the *tail permutation of* w to be the subword $w_{t_1} w_{t_2} \ldots w_{t_m}$ of w, where $t_1 < t_2 < \ldots < t_m$, so that w_{t_i} is the last (rightmost) occurrence of that letter for any $i \in [m]$. For example, the tail permutation of 331322112441 is 3241. Clearly the tail permutation of $w \in \mathfrak{S}_M$ is a permutation of $[m]$ (recall that we always assume that $\langle M \rangle = m$).

Let $\mathfrak{S}_M^\tau$ be the set of words in $\mathfrak{S}_M$ with tail permutation τ. In particular, let $\mathcal{P}_M$ be the set of words in $\mathfrak{S}_M$ with increasing tail permutation, that is, $\mathcal{P}_M = \mathfrak{S}_M^\tau$, where $\tau = 12\cdots m$. Thus, $\mathcal{P}_M$ is the set of words in $\mathfrak{S}_M$ in which the last occurrences of $1, 2, \ldots, m$ occur in that order. For example, let $M = \{1^2, 2^2\}$, then $\mathfrak{S}_M = \{1122, 1212, 1221, 2112, 2121, 2211\}$, $\mathfrak{S}_M^{21} = \{1221, 2121, 2211\}$, and $\mathcal{P}_M = \mathfrak{S}_M^{12} = \{1122, 1212, 2112\}$.

Let w be a partition of $[n]$ with blocks $B_1, B_2, \ldots, B_m$, where

$$\max B_1 < \max B_2 < \cdots < \max B_m,$$

if $|B_i| = k_i$, $1 \le i \le m$, we say that w is of *type* $(k_1, k_2, \ldots, k_m)$. It is obvious that $\mathcal{P}_M$ is the set of the partitions of $[n]$ of type $(k_1, k_2, \ldots, k_m)$ via Mahonian representation. Throughout this paper we always use the Mahonian representation to represent a set partition, that is, we always think of a set partition as an element in $\mathcal{P}_M$ for some M. Then

$$\Pi_n = \bigcup_{|M|=n} \mathcal{P}_M \quad \text{and} \quad \Pi_{n,m} = \bigcup_{|M|=n, \langle M \rangle = m} \mathcal{P}_M.$$

The main results of this paper can be summarized as the following three theorems.

Theorem 2.1. *Let* $M = \{1^{k_1}, 2^{k_2}, \dots, m^{k_m}\}$ *with* $k_i \geq 1$ *for all* $i \in [m]$*, then*

$$\begin{aligned} \sum_{w\in\mathcal{P}_M} q^{\mathrm{INV}(w)} &= \sum_{w\in\mathcal{P}_M} q^{\mathrm{MAJ}(w)} = \sum_{w\in\mathcal{P}_M} q^{\mathrm{MAJ}_d(w)} = \sum_{w\in\mathcal{P}_M} q^{\mathrm{Z}(w)} \\ &= \sum_{w\in\mathcal{P}_M} q^{r\text{-}\mathrm{MAJ}(w)} = \sum_{w\in\mathcal{P}_M} q^{\mathrm{DEN}(w)} = \sum_{w\in\mathcal{P}_M} q^{\mathrm{MAK}(w)} = \sum_{w\in\mathcal{P}_M} q^{\mathrm{MAD}(w)}. \end{aligned} \tag{2.1}$$

Note that (2.1) shows that the Mahonian statistics on words in Table 1, except for STAT, are all equidistributed on set partitions of given type via Mahonian representation. The following theorem shows that the Euler-Mahonian statistics on words in Table 2 are all equidistributed on set partitions of given type via Mahonian representation.

Theorem 2.2. *Let* $M = \{1^{k_1}, 2^{k_2}, \dots, m^{k_m}\}$ *with* $k_i \geq 1$ *for all* $i \in [m]$*, then*

$$\sum_{w\in\mathcal{P}_M} t^{\mathrm{des}(w)} q^{\mathrm{MAJ}(w)} = \sum_{w\in\mathcal{P}_M} t^{\mathrm{mstc}(w)} q^{\mathrm{INV}(w)} = \sum_{w\in\mathcal{P}_M} t^{\mathrm{exc}(w)} q^{\mathrm{DEN}(w)} = \sum_{w\in\mathcal{P}_M} t^{\mathrm{des}(w)} q^{\mathrm{MAK}(w)}.$$

A permutation $\tau = \tau_1\tau_2\dots\tau_m$ of $[m]$ is said to be *consecutive* if $\{\tau_1, \tau_2, \dots, \tau_i\}$ forms a set of consecutive numbers for all $i \in [m]$. For example, $\tau = 54362718$ is consecutive, whereas $\tau = 54236718$ is not. In particular, the increasing permutation $\tau = 12\dots m$ is consecutive. Throughout this paper we always assume that τ is a consecutive permutation. For the statistics INV, MAJ, MAJ_d, Z, we prove the following more general result.

Theorem 2.3. *Let* $M = \{1^{k_1}, 2^{k_2}, \dots, m^{k_m}\}$ *with* $k_i \geq 1$ *for all* $i \in [m]$*, and let* τ *be a consecutive permutation of* $[m]$*, then*

$$\sum_{w\in\mathfrak{S}_M^\tau} q^{\mathrm{INV}(w)} = \sum_{w\in\mathfrak{S}_M^\tau} q^{\mathrm{MAJ}(w)} = \sum_{w\in\mathfrak{S}_M^\tau} q^{\mathrm{MAJ}_d(w)} = \sum_{w\in\mathfrak{S}_M^\tau} q^{\mathrm{Z}(w)}. \tag{2.2}$$

Setting $\tau = 12\dots m$ in (2.2) gives the top row of (2.1).

Remark 2.1. Small examples show that none of r-MAJ, DEN, MAK and MAD is equidistributed with INV (or MAJ, MAJ_d, Z) on $\mathfrak{S}_M^\tau$ for consecutive τ in general.

Remark 2.2. Small examples show that any two of (des, MAJ), (mstc, INV), (exc, DEN) and (des, MAK) are *not* equidistributed on $\mathfrak{S}_M^\tau$ for consecutive τ in general.

To conclude this subsection, we give an example. Table 3 gives the distributions of the

$\mathcal{P}_{\{1,1,2,2,3,3\}}$	INV	MAJ	MAJ_2	Z	2-MAJ	DEN	MAK	MAD	STAT
112233	0	0	0	0	0	0	0	0	0
112323	1	4	1	2	1	4	4	1	3
113223	2	3	4	1	2	3	3	2	4
121233	1	2	1	2	1	2	2	1	5
121323	2	6	2	4	2	6	6	2	8
123123	3	3	6	6	5	5	3	4	4
131223	3	2	3	3	4	2	2	3	5
132123	4	5	4	5	3	3	5	3	9
211233	2	1	2	1	2	1	1	2	4
211323	3	5	3	3	3	5	5	3	7
213123	4	4	5	5	6	4	4	6	8
231123	5	2	4	4	5	3	2	5	3
311223	4	1	2	2	3	1	1	4	2
312123	5	4	3	4	4	2	3	5	6
321123	6	3	5	3	4	4	4	4	7

Table 3. The distributions of the statistics INV, MAJ, MAJ_2, Z, 2-MAJ, DEN, MAK, MAD, STAT on $\mathcal{P}_{\{1,1,2,2,3,3\}}$.

statistics INV, MAJ, MAJ_2, Z, 2-MAJ, DEN, MAK, MAD, STAT on $\mathcal{P}_{\{1,1,2,2,3,3\}}$. We see that

$$\sum_{w\in\mathcal{P}_{\{1,1,2,2,3,3\}}} q^{\text{STAT}(w)} = 1+q^2+2q^3+3q^4+2q^5+q^6+2q^7+2q^8+q^9,$$
$$\sum_{w\in\mathcal{P}_{\{1,1,2,2,3,3\}}} q^{S(w)} = 1+2q+3q^2+3q^3+3q^4+2q^5+q^6,$$

where S is any of INV, MAJ, MAJ_2, Z, 2-MAJ, DEN, MAK, MAD.

2.2. q-Stirling numbers of the second kind

A q-analog of a mathematical object is an object depending on the variable q that reduces to the original object when we set $q = 1$. Let

$$[n]_q = 1+q+q^2+\cdots+q^{n-1}, \qquad [n]_q! = [1]_q[2]_q\ldots[n]_q, \qquad \binom{n}{i}_q = \frac{[n]_q!}{[i]_q![n-i]_q!}.$$

Then $[n]_q$, $[n]_q!$ and $\binom{n}{i}_q$ are the q-analogs of n, $n!$ and $\binom{n}{i}$ respectively.

It is well-known that the total number of partitions of $[n]$ is the *Bell number* $B(n)$, and that the number of partitions of $[n]$ with exactly m blocks is the *Stirling number of the second kind* $S(n,m)$. That is,

$$|\Pi_n| = B(n), \qquad |\Pi_{n,m}| = S(n,m).$$

There are two recursions for $S(n,m)$:

$$S(n,m) = S(n-1,m-1) + mS(n-1,m), \quad S(0,m) = \delta_{0,m}, \tag{2.3}$$

and

$$S(n+1,m) = \sum_{i=0}^{n} \binom{n}{i} S(n-i,m-1), \quad S(0,m) = \delta_{0,m}, \tag{2.4}$$

where $\delta_{n,m}$ is the Kronecker delta, defined by $\delta_{n,m} = 1$ if $n = m$, and $\delta_{n,m} = 0$ otherwise. Considering the q-analogs of the above two recursions leads to two different kinds of q-Stirling numbers of the second kind.

Carlitz's q-Stirling numbers of the second kind, denoted $S_q(n,m)$, are defined by the following recursion

$$S_q(n,m) = S_q(n-1,m-1) + [m]_q S_q(n-1,m), \qquad S_q(0,m) = \delta_{0,m},$$

which is a q-analog of (2.3). These polynomials were first studied by Carlitz [5, 6] and then Gould [16]. Milne [31] introduced an inversion statistic, here we denote $\underline{\text{inv}}$, for set partitions via canonical representation, and he proved

$$S_q(n,m) = \sum_{w \in \Pi_{n,m}} q^{\underline{\text{inv}}(w)}.$$

Sagan [36] introduced a major index statistic, here we denote $\underline{\text{maj}}$, for set partitions via block representation, and he proved

$$S_q(n,m) = \sum_{w \in \Pi_{n,m}} q^{\underline{\text{maj}}(w)}.$$

Johnson's q-Stirling numbers of the second kind, denoted $\left\{ {n \atop m} \right\}_q$, are defined by the following recursion

$$\left\{ {n+1 \atop m} \right\}_q = \sum_{i=0}^{n} \binom{n}{i}_q \left\{ {n-i \atop m-1} \right\}_q, \qquad \left\{ {0 \atop m} \right\}_q = \delta_{0,m},$$

which is a q-analog of (2.4). These polynomials were first studied by Johnson [24]. As pointed out by Johnson [24], $\left\{ {n \atop m} \right\}_q$ is different from $S_q(n,m)$. Johnson [24] proved

$$\left\{ {n \atop m} \right\}_q = \sum_{w \in \Pi_{n,m}} q^{\mathrm{INV}(w)},$$

where w uses the Mahonian representation. Combining Johnson's result with Theorem 2.1, we have the following corollary.

Corollary 2.1. *Using the Mahonian representation, we have*

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_q = \sum_{w \in \Pi_{n,m}} q^{S(w)},$$

where S is any of INV, MAJ, MAJ$_d$, r-MAJ, Z, DEN, MAK, MAD.

2.3. Structure of the paper

This paper is organized as follows. In Sections 3, 4 and 5, we establish the equidistributions of INV and MAJ, INV and MAJ$_d$, MAJ and Z on $\mathfrak{S}_M^{\tau}$ for consecutive τ, respectively. In Section 6, we establish the equidistribution of the bi-statistics (mstc, INV) and (des, MAJ) on $\mathcal{P}_M$. In Section 7, we establish the equidistribution of INV and r-MAJ on $\mathcal{P}_M$. In Section 8, we establish the equidistribution of the bi-statistics (des, MAJ) and (exc, DEN) on $\mathcal{P}_M$. In Section 9, we establish the equidistribution of the triple statistics (des, MAK, MAD) and (exc, DEN, INV) on $\mathcal{P}_M$. Combining these results we obtain Theorems 2.1, 2.2 and 2.3.

3. Equidistribution of INV and MAJ on $\mathfrak{S}_M^{\tau}$

The equidistribution of INV and MAJ was proved bijectively for the first time by Foata [14]. We denote this famous bijection by F. The goal of this section is to prove that the classical statistics INV and MAJ are equidistributed on $\mathfrak{S}_M^{\tau}$ for consecutive τ. We will show this by proving that Foata's bijection F preserves the consecutive tail permutation. We first give a brief description of F.

Given a word $w = w_1 w_2 \dots w_n$ and a letter x. We first define an operator J_x on w. If $w_n \le x$, write $w = u_1 b_1 u_2 b_2 \dots u_s b_s$, where each b_i is a letter less than or equal to x, and each u_i is a word (possibly empty), all of whose letters are greater than x. Similarly, if $w_n > x$, write $w = u_1 b_1 u_2 b_2 \dots u_s b_s$, where each b_i is a letter greater than x, and each u_i is a word (possibly empty), all of whose letters less than or equal to x. In each case we define the operator J_x on w to be

$$J_x(w) = b_1 u_1 b_2 u_2 \dots b_s u_s.$$

We call the letters $b_1, b_2, \dots, b_s$ the *jumping letters*, the remaining letters the *fixed letters*. The following property for the operator J_x is crucial

$$\mathrm{INV}(J_x(w)x) - \mathrm{INV}(w) = \begin{cases} n, & \text{if } w_n > x, \\ 0, & \text{if } w_n \le x. \end{cases}$$

Given $w = w_1 w_2 \dots w_n$. Define $\gamma_1 = w_1$, and $\gamma_{i+1} = J_{w_{i+1}}(\gamma_i) w_{i+1}$ for $1 \le i \le n-1$. Finally set $F(w) = \gamma_n$.

Theorem 3.1 (Foata [14]). *$F : \mathfrak{S}_M \to \mathfrak{S}_M$ is a bijection satisfying*

$$\mathrm{MAJ}(w) = \mathrm{INV}(F(w))$$

for all $w \in \mathfrak{S}_M$.

Example 3.1. Let $w = 211323$, it is not hard to see that $\mathrm{MAJ}(w) = 5$. We give the procedure for creating $F(w)$:

$$\begin{aligned}
\gamma_1 &= 2, \\
\gamma_2 &= J_1(2)1 = 21, \\
\gamma_3 &= J_1(21)1 = 121, \\
\gamma_4 &= J_3(121)3 = 1213, \\
\gamma_5 &= J_2(1213)2 = 31212, \\
\gamma_6 &= J_3(31212)3 = 312123 = F(w).
\end{aligned}$$

Note that $\mathrm{INV}(F(w)) = 5$.

The following result shows that Foata's bijection preserves the consecutive tail permutation.

Proposition 3.1. *Let $M = \{1^{k_1}, 2^{k_2}, \dots, m^{k_m}\}$ with $k_i \ge 1$ for all $i \in [m]$, and let τ be a consecutive permutation of $[m]$. Then the set $\mathfrak{S}_M^\tau$ is invariant under Foata's bijection, that is*

$$F(\mathfrak{S}_M^\tau) = \mathfrak{S}_M^\tau.$$

Proof. To prove $F(\mathfrak{S}_M^\tau) = \mathfrak{S}_M^\tau$, it suffices to show that $F(w) \in \mathfrak{S}_M^\tau$ for all $w \in \mathfrak{S}_M^\tau$, because F is a bijection. Suppose that $\tau = \tau_1 \tau_2 \dots \tau_m$.

Given a word with some letters overlined, we call the subword consisting of the overlined letters the *overlined subword*. For example, the overlined subword of $331\overline{3}2211\overline{2}4\overline{4}\overline{1}$ is $\overline{3241}$.

Given $w = w_1 w_2 \dots w_n \in \mathfrak{S}_M^\tau$, we overline the last occurrences of the letters $1, 2, \dots, m$. Thus, the overlined subword of w is $\overline{\tau_1 \tau_2 \dots \tau_m}$ as $w \in \mathfrak{S}_M^\tau$. Note that γ_i is a word with some letters overlined for $1 \le i \le n$, because γ_i is a rearrangement of $w_1 w_2 \dots w_i$. We claim that for any i, $1 \le i \le n$, we have

(i) the overlined subword of γ_i is the same as the overlined subword of $w_1 w_2 \dots w_i$.

(ii) each overlined letter of γ_i is the last occurrence of that letter in γ_i.

We use induction on i to prove our claim. The initial case of $i = 1$ is obvious. Assume that our claim is true for i and prove it for $i+1$, where $1 \le i \le n-1$. Suppose that the overlined subword of $w_1 w_2 \dots w_i$ is $\overline{\tau_1 \tau_2 \dots \tau_s}$. For $1 \le k \le s$, because $\overline{\tau}_k$, which is the last occurrence of τ_k in w, appears in $w_1 w_2 \dots w_i$, we have that $w_{i+1} \notin \{\tau_1, \tau_2, \dots, \tau_s\}$.

Below we assume that $\gamma_i = c_1 c_2 \dots c_i$. (This will simplify the description of the proof of Proposition 4.1 in the next section although it is not necessary.)

(A) We first prove (i) for $i+1$. By the induction hypothesis, the overlined subword of γ_i is $\overline{\tau_1 \tau_2 \dots \tau_s}$. Since $\{\tau_1, \tau_2, \dots, \tau_s\}$ is a set of consecutive numbers and $w_{i+1} \notin \{\tau_1, \tau_2, \dots, \tau_s\}$, we see that either all of $\tau_1, \tau_2, \dots, \tau_s$ are greater than w_{i+1}, or all of them are smaller than w_{i+1}. It follows that either all of the overlined letters of γ_i are greater than w_{i+1}, or all of them are smaller than w_{i+1}. Then either all of the overlined letters of $c_1 c_2 \dots c_i$ are jumping letters, or all of them are fixed letters. Combining this with the definition of the operator $J_{w_{i+1}}$, we can see that the operator $J_{w_{i+1}}$ dose not change the overlined subword of $c_1 c_2 \dots c_i$. Then by the definition of γ_{i+1}, we have that the overlined subword of γ_{i+1} is the same as overlined subword of $w_1 w_2 \dots w_i w_{i+1}$, this completes the proof of (i).

(B) We now prove (ii) for $i+1$. For $1 \le k \le s$, by the induction hypothesis we see that $\overline{\tau}_k$ is the last occurrence of the letter τ_k in γ_i. It is not hard to see that either all the occurrences of τ_k in $c_1 c_2 \dots c_i$ are jumping letters, or all of them are fixed letters. Combining this with the fact that $w_{i+1} \notin \{\tau_1, \tau_2, \dots, \tau_s\}$, we obtain that $\overline{\tau}_k$ is the last occurrence of the letter τ_k in γ_{i+1}, $1 \le k \le s$. If w_{i+1} is not overlined, we complete the proof of (ii) for $i+1$. If w_{i+1} is overlined, it must be the last occurrence of that letter in γ_{i+1}, we also complete the proof of (ii) for $i+1$.

By (i) of our claim, the overlined subword of $\gamma_n = F(w)$ is $\overline{\tau_1 \tau_2 \dots \tau_m}$. By (ii) of our claim, the tail permutation of $F(w)$ is $\tau_1 \tau_2 \dots \tau_m$. So $F(w) \in \mathfrak{S}_M^\tau$, completing the proof. ∎

Remark 3.1. It is not hard to prove that if τ is a consecutive permutation of $[m]$, then $F(\tau) = \tau$. (This can also be deduced by a theorem of Björner and Wachs [3, Theorem 4.2]. Also see [8].) Proposition 3.1 can be viewed a generalization of this result to words, because Proposition 3.1 gives the above result when we set $M = \{1, 2, \dots, m\}$.

By Theorem 3.1 and Proposition 3.1 we obtain the equidistribution of INV and MAJ on $\mathfrak{S}_M^\tau$ for consecutive τ, and we achieve the goal of this section.

4. Equidistribution of INV and MAJ_d on $\mathfrak{S}_M^\tau$

Given $w = w_1w_2 \dots w_n \in \mathfrak{S}_M$, let d be a positive integer, define

$$\text{MAJ}_d(w) = \text{inv}_d(w) + \sum_{w_i > w_{i+d}} i,$$

where

$$\text{inv}_d(w) = |\{(i,j) : i < j < i+d,\ w_i > w_j\}|.$$

It is not hard to see that $\text{MAJ}_1 = \text{MAJ}$ and $\text{MAJ}_n = \text{INV}$, thus the family of MAJ_d interpolates between MAJ and INV. The statistic MAJ_d was introduced by Kadell [25], who gave a bijective proof that this statistic is Mahonian. Kadell's bijection takes INV to MAJ_d, with the extreme case taking INV to MAJ corresponding precisely to the inverse of Foata's bijection. Assaf [1] exhibited a different family of bijections, taking MAJ_{d-1} to MAJ_d. In his Ph.D. thesis [27], Liang gave a bijection F_d that takes MAJ_d to INV (the author independently found this bijection, and later discovered the Ph.D. thesis of Liang [27]). The bijection F_d is a natural extension of Foata's bijection and is the inverse of Kadell's bijection.

We now describe the bijection F_d. Given $w = w_1w_2 \dots w_n \in \mathfrak{S}_M$. We will define words $\gamma_1, \gamma_2, \dots, \gamma_n$, where γ_i is a rearrangement of $w_1w_2 \dots w_i$. First define $\gamma_i = w_1w_2 \dots w_i$ for $1 \le i \le d$. Assume that $\gamma_i = c_1c_2 \dots c_i$ has been defined for some $d \le i \le n-1$. Then define

$$\gamma_{i+1} = J_{w_{i+1}}(c_1c_2 \dots c_{i-d+1})c_{i-d+2}c_{i-d+3} \dots c_i w_{i+1}. \tag{4.5}$$

Finally, set $F_d(w) = \gamma_n$. When $d = 1$, we have $\gamma_{i+1} = J_{w_{i+1}}(\gamma_i)w_{i+1}$. Thus F_d reduces to Foata's bijection F when $d = 1$. We have the following theorem.

Theorem 4.1 (Liang [27])**.** *$F_d : \mathfrak{S}_M \to \mathfrak{S}_M$ is a bijection satisfying*

$$\text{MAJ}_d(w) = \text{INV}(F_d(w))$$

for all $w \in \mathfrak{S}_M$.

Example 4.1. Let $w = 213123$ and $d = 2$, it is not hard to see that $\text{MAJ}_2(w) = 5$. We give the procedure for creating $F_2(w)$:

$$\begin{aligned}\gamma_1 &= 2,\\ \gamma_2 &= 21,\\ \gamma_3 &= J_3(2)13 = 213,\\ \gamma_4 &= J_1(21)31 = 1231,\\ \gamma_5 &= J_2(123)12 = 31212,\\ \gamma_6 &= J_3(3121)23 = 312123 = F_2(w).\end{aligned}$$

Note that $\text{INV}(F_2(w)) = 5$.

The following proposition shows that F_d preserves the consecutive tail permutation, which is a generalization of Proposition 3.1.

Proposition 4.1. *Let $M = \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$ with $k_i \geq 1$ for all $i \in [m]$, and let τ be a consecutive permutation of $[m]$, then $\mathfrak{S}_M^\tau$ is invariant under F_d, that is,*

$$F_d(\mathfrak{S}_M^\tau) = \mathfrak{S}_M^\tau.$$

Replacing each $c_1c_2\ldots c_i$ by $c_1c_2\ldots c_{i+d-1}$ in the paragraphs (A) and (B) of the proof of Proposition 3.1 gives the proof of Proposition 4.1.

By Theorem 4.1 and Proposition 4.1, we obtain the equidistribution of INV and MAJ_d on $\mathfrak{S}_M^\tau$ for consecutive τ, and we achieve the goal of this section.

5. Equidistribution of MAJ and Z on $\mathfrak{S}_M^\tau$

Given a word $w = w_1w_2\ldots w_n$, the *z-index* of w, denoted by $\text{Z}(w)$, is defined by

$$\text{Z}(w) = \sum_{i<j} \text{MAJ}(w_{ij}),$$

where w_{ij} is a word obtained from w by deleting all elements except i and j. For example, let $w = 312432314$,

$$w_{12} = 1221, \quad w_{13} = 31331, \quad w_{14} = 1414, \quad w_{23} = 32323, \quad w_{24} = 2424, \quad w_{34} = 34334,$$

then

$$\text{Z}(w) = \sum_{i<j} \text{MAJ}(w_{ij}) = 18.$$

Zeilberger and Bressoud [39] proved Z is Mahonian by induction. Greene [17] presented a combinatorial proof. Han [21] gave another combinatorial proof by exhibiting a Foata-style bijection, which we will denote as H_{Z}. The goal of this section is to establish the equidistribution of the statistics MAJ and Z on $\mathfrak{S}_M^\tau$ for consecutive τ by proving that Han's bijection H_{Z} preserves the consecutive tail permutation.

Before stating the bijection H_{Z}, we need some notions, see [21]. Recall that, throughout this paper we let $M = \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$ with $k_i \geq 1$ for all i. In this section, we let m be a fixed number and let $\mathbf{m} = (k_1, k_2, \ldots, k_m) := \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$ with $k_i \geq 0$ for all i. That is, for the multiset M we assume that $k_i \geq 1$ and for the multiset $\mathbf{m}$ we assume that $k_i \geq 0$.

Let $w = x_1 x_2 \ldots x_n \in \mathfrak{S}_{\mathbf{m}}$ and let x be a positive integer, define

$$C^x(w) = y_1 y_2 \ldots y_n, \qquad \text{where } \; y_i = C^x(x_i) := \begin{cases} x_i - x, & \text{if } x_i > x, \\ x_i - x + m, & \text{if } x_i \leq x, \end{cases}$$

and

$$C_x(w) = z_1 z_2 \ldots z_n, \qquad \text{where } \; z_i = C_x(x_i) := \begin{cases} x_i, & \text{if } x_i < x, \\ x_i - 1, & \text{if } x_i > x, \\ m, & \text{if } x_i = x. \end{cases}$$

Note that $C^x(w) \in \mathfrak{S}_{\mathbf{m}^x}$ and $C_x(w) \in \mathfrak{S}_{\mathbf{m}_x}$, where

$$\mathbf{m}^x = (k_{x+1}, k_{x+2}, \ldots, k_m, k_1, k_2, \ldots, k_{x-1}, k_x), \tag{5.6}$$

$$\mathbf{m}_x = (k_1, k_2, \ldots, k_{x-1}, k_{x+1}, k_{x+2}, \ldots, k_m, k_x). \tag{5.7}$$

Let us recall the construction of a bijection $\theta_{\mathbf{m},\mathbf{m}'} : \mathfrak{S}_{\mathbf{m}} \to \mathfrak{S}_{\mathbf{m}'}$, fixing the statistic MAJ. It is enough to give this construction when $\mathbf{m}$ and $\mathbf{m}'$ differ only by two consecutive letters, say i and $i+1$, that is the bijection

$$\theta_i : \mathfrak{S}_{\mathbf{m}} \to \mathfrak{S}_{\mathbf{m}'}, \text{ where } \mathbf{m} = (k_1, \ldots, k_i, k_{i+1}, \ldots, k_m), \mathbf{m}' = (k_1, \ldots, k_{i+1}, k_i, \ldots, k_m).$$

We do it as follows: let $w \in \mathfrak{S}_{\mathbf{m}}$. We replace all the $(i+1)i$ factors of this word with a special letter "$\sim$". In the word thus obtained, the maximum factors containing the two letters i and $i+1$ have the form $i^a(i+1)^b$ $(a \geq 0, b \geq 0)$. We then change these factors to $i^b(i+1)^a$ and replace each "$\sim$" by $(i+1)i$, to obtain the word $\theta_i(w) \in \mathfrak{S}_{\mathbf{m}'}$. For example, let

$w = 1112111222215622$ and $i = 1$, we have

$$\begin{aligned} w &= 111\ 21\ 11222\ 21\ 5622 \\ &\mapsto 111 \sim\ 11222 \sim\ 5622 \\ &\mapsto 222 \sim\ 11122 \sim\ 5611 \\ &\mapsto 222\ 21\ 11122\ 21\ 5611 \ = \ \theta_1(w). \end{aligned}$$

Note that $\mathrm{Des}(w) = \mathrm{Des}(\theta_i(w))$ and thus θ_i fixes the statistic MAJ.

The bijection H_{Z} is defined, for any word $w \in \mathfrak{S}_{\mathbf{m}}$ and any letter x, by the following composition:

$$H_{\mathrm{Z}}(wx) = \left(C_x^{-1} \circ H_{\mathrm{Z}} \circ \theta_{\mathbf{m}^x, \mathbf{m}_x} \circ C^x(w)\right) x.$$

Theorem 5.1 (Han [21]). *H_{Z} is a bijection satisfying*

$$\mathrm{MAJ}(w) = \mathrm{Z}(H_{\mathrm{Z}}(w))$$

for any $w \in \mathfrak{S}_{\mathbf{m}}$.

The main result of this section is the following proposition, which implies the equidistribution of the statistics MAJ and Z on $\mathfrak{S}_M^\tau$ for consecutive τ.

Proposition 5.1. *Let $M = \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$ with $k_i \geq 1$ for all $i \in [m]$, and let τ be a consecutive permutation of $[m]$, the set $\mathfrak{S}_M^\tau$ is invariant under Han's bijection H_{Z}, that is*

$$H_{\mathrm{Z}}(\mathfrak{S}_M^\tau) = \mathfrak{S}_M^\tau.$$

In the rest of this section, we prove this proposition. As we will see, this is a somewhat more difficult task. We first give some notations and lemmas.

For any $j \geq 0$, we denote

$$\theta_{j!} = \theta_j \circ \cdots \circ \theta_2 \circ \theta_1 \circ \theta_0, \text{ where } \theta_0 = \mathrm{id}.$$

Let S be a set, we denote $S-1 := \{s-1 : s \in S\}$. In general, we denote $S-i := \{s-i : s \in S\}$.

Lemma 5.1. *Given $w = w_1 w_2 \ldots w_n \in \mathfrak{S}_{\mathbf{m}}$, assume that $\theta_{(m-2)!}(w) = c_1 c_2 \ldots c_n$. Let $i \in [n]$, if $w_i = m$, then $c_i = m$; if $w_i < m$, then $c_i \in R_i - 1$, where $R_i = \{w_{i-1}, w_i, \ldots, w_n, m\}$, and we assume that $w_0 = m$.*

Proof. Let

$$\theta_{j!}(w) = w_1^{(j)} w_2^{(j)} \ldots w_n^{(j)},$$

for $0 \le j \le m-2$. So $w_i = w_i^{(0)}$ and $c_i = w_i^{(m-2)}$, $1 \le i \le n$. Let $w_0^{(j)} = w_{n+1}^{(j)} = m$ for all j. Given an index $i \in [n]$, we assume that $w_i = a$. Obviously,

$$w_i^{(0)} = w_i^{(1)} = w_i^{(2)} = \cdots = w_i^{(a-2)} = a.$$

Then when $w_i = a = m$, we have $c_i = w_i^{(m-2)} = m$. Below we assume that $w_i = a < m$. Then $a-1 \le m-2$. Consider the word $\theta_{(a-1)!}(w)$. If $a > 1$ and $w_i^{(a-1)} = a-1$, then

$$w_i^{(a)} = w_i^{(a+1)} = \cdots = w_i^{(m-2)} = a-1.$$

So $c_i = w_i^{(m-2)} = a-1 = w_i - 1 \in R_i - 1$. Below we assume that $w_i^{(a-1)} \ne a-1$, so $w_i^{(a-1)} = a$ (including the case of $a = 1$). Assume that $w_{i-1}^{(a-1)} = p$, and that

$$w_i^{(a-1)} = w_{i+1}^{(a-1)} = \cdots = w_{j-1}^{(a-1)} = a, \text{ and } w_j^{(a-1)} = q \ne a,$$

where $1 \le i < j \le n+1$. The above assumption means that

$$w_{i-1}^{(a-1)} w_i^{(a-1)} w_{i+1}^{(a-1)} \ldots w_{j-1}^{(a-1)} w_j^{(a-1)} = paa\ldots aq, \; a \ne q.$$

We distinguish two cases.

Case 1: $p \le a$. We further distinguish two subcases.

Subcase 1.1: $a > q$. It is not hard to see that

$$w_i^{(a)} = a+1, \; w_i^{(a+1)} = a+2, \ldots, \; w_i^{(m-2)} = m-1.$$

Then $c_i = w_i^{(m-2)} = m-1 \in R_i - 1$.

Subcase 1.2: $a < q$. Since $w_j^{(a-1)} = q > a$, we have $w_j = q$. It is not hard to see that

$$\begin{aligned}
w_i^{(a)} w_{i+1}^{(a)} \ldots w_{j-1}^{(a)} w_j^{(a)} &= (a+1)(a+1)\ldots(a+1)q, \\
w_i^{(a+1)} w_{i+1}^{(a+1)} \ldots w_{j-1}^{(a+1)} w_j^{(a+1)} &= (a+2)(a+2)\ldots(a+2)q, \\
&\vdots \\
w_i^{(q-2)} w_{i+1}^{(q-2)} \ldots w_{j-1}^{(q-2)} w_j^{(q-2)} &= (q-1)(q-1)\ldots(q-1)q.
\end{aligned}$$

If $q = m$, then $c_i = w_i^{(m-2)} = w_i^{(q-2)} = m-1 \in R_i - 1$. If $q < m$, then $q-1 \le m-2$. Since $w_i^{(q-2)} = q-1$, then $w_i^{(q-1)}$ is either $q-1$ or q. If $w_i^{(q-1)} = q-1$, then

$$w_i^{(q)} = w_i^{(q+1)} = \cdots = w_i^{(m-2)} = q-1.$$

We have $c_i = w_i^{(m-2)} = q-1 = w_j - 1 \in R_i - 1$. If $w_i^{(q-1)} = q$. Clearly,

$$w_i^{(q-1)} w_{i+1}^{(q-1)} \dots w_{j-1}^{(q-1)} w_j^{(q-1)} = qq \dots q.$$

We now assume that

$$w_{i-1}^{(q-1)} w_i^{(q-1)} w_{i+1}^{(q-1)} \dots w_k^{(q-1)} = sqq \dots qr, \ q \neq r,$$

where $1 \leq i < j < k \leq n+1$. As $p \leq a$, i.e., $w_{i-1}^{(a-1)} \leq w_i^{(a-1)}$, and θ_l fixes the decent set for all l, we have $s \leq q$. Then we return to **Case 1** and we can give the proof inductively.

Case 2: $p > a$. We further consider two subcases.

Subcase 2.1: $a > q$. Since $w_{i-1}^{(a-1)} = p > a$, we see that $w_{i-1} = p$. It is clear that

$$w_i^{(a)} = a+1, \quad w_i^{(a+1)} = a+2, \quad \dots, \quad w_i^{(p-2)} = w_i^{(p-1)} = \dots = w_i^{(m-2)} = p-1.$$

Then $c_i = w_i^{(m-2)} = p - 1 = w_{i-1} - 1 \in R_i - 1$.

Subcase 2.2: $a < q$. Combining the arguments of **Subcase 1.2** and **Subcase 2.1** gives the proof.

We complete the proof. ∎

Given a word $w = w_1 w_2 \dots w_n$ and a set A, if $w \cap A \neq \emptyset$, that is $\{w_1, w_2, \dots, w_n\} \cap A \neq \emptyset$, we denote by $\mathrm{Last}_A(w)$ the rightmost letter in w that belongs to A. Obvious that $\mathrm{Last}_A(w) \in A$. For example, let $A = \{2, 3, 5\}$, then $\mathrm{Last}_A(123453441) = 3$ and $\mathrm{Last}_A(123435441) = 5$.

Lemma 5.2. *Given a set A with $1 < \min(A) \leq \max(A) < m$. Let $w \in \mathfrak{S}_{\mathbf{m}}$, if $w \cap A \neq \emptyset$, then*

$$\mathrm{Last}_{A-1}\left(\theta_{(m-2)!}(w)\right) = \mathrm{Last}_A(w) - 1.$$

Proof. Assume that $w = w_1 w_2 \dots w_n$ and $\theta_{(m-2)!}(w) = c_1 c_2 \dots c_n$. Let $B = [m+1] - A$, clearly $(B-1) \cap (A-1) = \emptyset$. Since $\max(A) < m$, we have $m, m+1 \in B$. Let w_p be the rightmost letter in w that belongs to A. Assume that $w_p = a$. Since $a \in A$, we have $2 \leq a \leq m-1$. It is clear that $\mathrm{Last}_A(w) = w_p = a \in A$ and $w_{p+1}, w_{p+2}, \dots, w_n \in B$. Then

$$E := \{w_{p+1}, w_{p+2}, \dots, w_n, m, m+1\} \subseteq B.$$

Given an index i with $p+2 \leq i \leq n$, by Lemma 5.1 we have

$$c_i \in \{w_{i-1} - 1, w_i - 1, \dots, w_n - 1, m-1, m\} \subseteq E - 1 \subseteq B - 1. \tag{5.8}$$

For $i = p+1$, by Lemma 5.1 we have

$$c_{p+1} \in \{w_p - 1\} \cup (E-1) \subseteq \{a-1\} \cup (B-1). \tag{5.9}$$

Since $(B-1) \cap (A-1) = \emptyset$, by (5.9) and (5.8) we have

$$c_{p+1} = a-1 \in A-1 \text{ or } c_{p+1} \notin A-1, \tag{5.10}$$

$$c_{p+2}, c_{p+3}, \ldots, c_n \notin A-1. \tag{5.11}$$

Assume that $\theta_{(a-2)!}(w) = e = e_1 e_2 \ldots e_n$ (note that $a \geq 2$). Clearly $e_p = w_p = a$. We now apply the operator θ_{a-1} to e. If e_p becomes $a-1$, since $a-1$ is unchanged after we apply the operator $\theta_{m-2} \circ \cdots \circ \theta_{a+1} \circ \theta_a$, then $c_p = a-1$, combining this with (5.10) and (5.11) we get

$$\text{Last}_{A-1}(c_1 c_2 \ldots c_n) = a-1 = w_p - 1 = \text{Last}_A(w) - 1,$$

we complete the proof for this case. Below we assume that e_p is unchanged after we apply the operator θ_{a-1} to e. We consider two cases.

Case 1: $e_p e_{p+1} = a(a-1)$. It is clear that e_{p+1} is unchanged after we apply the operator $\theta_{m-2} \circ \cdots \circ \theta_a \circ \theta_{a-1}$ to e. Therefore, $c_{p+1} = a-1$. Combining this with (5.11), we have

$$\text{Last}_{A-1}(c_1 c_2 \ldots c_n) = c_{p+1} = a-1 = w_p - 1 = \text{Last}_A(w) - 1.$$

Case 2: $e_{p+1} \neq a-1$. Let $\theta_{a-1}(e) = \theta_{(a-1)!}(w) = h = h_1 h_2 \ldots h_n$. By our assumption that e_p is unchanged after we apply the operator θ_{a-1} to e, we have $h_p = e_p = a$. Because $h_p = e_p = a$, there must be an index q with $q < p$ such that $e_q e_{q+1} \ldots e_p = (a-1) \ldots (a-1) a \ldots a$, and $h_q h_{q+1} \ldots h_p$ also has the form $(a-1) \ldots (a-1) a \ldots a$. Assume that $h_s = a-1$ and $h_{s+1} = \cdots = h_p = a$, where $q \leq s < p$. Since $a-1$ is unchanged after we apply the operator $\theta_{m-2} \circ \cdots \circ \theta_{a+1} \circ \theta_a$ to h, then $c_s = h_s = a-1$. Because $h = \theta_{(a-1)!}(w)$ and $h_s h_{s+1} \ldots h_p = (a-1) a \ldots a$, we see that

$$F := \{w_s, w_{s+1}, \ldots, w_p\} \subseteq \{1, 2, \ldots, a\}. \tag{5.12}$$

Note that applying the operator $\theta_{m-2} \circ \cdots \circ \theta_{a+1} \circ \theta_a$ to h will not decrease any letter a in h. Since $h_{s+1} = h_{s+2} = \cdots = h_p = a$, we have

$$c_{s+1}, c_{s+2}, \ldots, c_p \geq a. \tag{5.13}$$

Given an index i with $s+1 \leq i \leq p$, by Lemma 5.1, we have

$$c_i \in \{w_{i-1} - 1, w_i - 1, \ldots, w_n - 1, m-1, m\} \subseteq (F-1) \cup (E-1).$$

By (5.12) and (5.13) we have $c_i \notin F-1$, then $c_i \in E-1 \subseteq B-1$, where $s+1 \le i \le p$. Since $(B-1)\cap(A-1)=\emptyset$, we have

$$c_{s+1}, c_{s+2}, \ldots, c_p \notin A-1. \tag{5.14}$$

Combining (5.14), (5.10), (5.11) and the fact that $c_s = a-1 \in A-1$, we get

$$\mathrm{Last}_{A-1}\left(c_1c_2\ldots c_n\right) = a-1 = w_p - 1 = \mathrm{Last}_A(w)-1.$$

We complete the proof. ∎

Comparing equations (5.6) and (5.7), we see that

$$\theta_{\mathbf{m}^x,\mathbf{m}_x} = \theta^{m-x}_{(m-2)!} := \underbrace{\theta_{(m-2)!}\circ\theta_{(m-2)!}\circ\cdots\circ\theta_{(m-2)!}}_{m-x}.$$

We define the map $\phi_x := \theta_{\mathbf{m}^x,\mathbf{m}_x}\circ C^x = \theta^{m-x}_{(m-2)!}\circ C^x$. Then

$$H_{\mathrm{Z}}(wx) = \left(C_x^{-1}\circ H_{\mathrm{Z}}\circ\phi_x(w)\right)x.$$

Lemma 5.3. *Given a set A and a number x with $1 \le \min(A) \le \max(A) < x \le m$. Let $w \in \mathfrak{S}_{\mathbf{m}}$, if $w \cap A \ne \emptyset$, then*

$$\mathrm{Last}_A(\phi_x(w)) = \mathrm{Last}_A(w).$$

Proof. Let $m-x=d$. Since $\max(A)<x$, by the definition of C^x we see that

$$\mathrm{Last}_{A+d}(C^x(w)) = \mathrm{Last}_A(w)+d. \tag{5.15}$$

Since $\max(A)<x$, we have $\max(A)+d<m$. Thus, $1<\min(A+j)\le\max(A+j)<m$ for any $1\le j\le d$. By Lemma 5.2, we have

$$\begin{aligned}
\mathrm{Last}_A\left(\theta^d_{(m-2)!}\circ C^x(w)\right) &= \mathrm{Last}_{A+1}\left(\theta^{d-1}_{(m-2)!}\circ C^x(w)\right)-1\\
&= \mathrm{Last}_{A+2}\left(\theta^{d-2}_{(m-2)!}\circ C^x(w)\right)-2\\
&\ \ \vdots\\
&= \mathrm{Last}_{A+d-1}\left(\theta^{1}_{(m-2)!}\circ C^x(w)\right)-(d-1)\\
&= \mathrm{Last}_{A+d}\left(C^x(w)\right)-d.
\end{aligned}$$

Combining this with (5.15) yields

$$\mathrm{Last}_A\left(\phi_x(w)\right) = \mathrm{Last}_A(w),$$

completing the proof. ∎

Proof of Proposition 5.1. We prove the following stronger property. Given $w \in \mathfrak{S}_{\mathbf{m}}$. For any l with $1 \le l \le m$, assume that $\tau = \tau_1\tau_2 \dots \tau_l$ is a consecutive permutation of $[l]$. Denote

$$A_r = \{\tau_1, \tau_2, \dots, \tau_r\},\ 1 \le r \le l.$$

If

$$\mathrm{Last}_{A_r}(w) = \tau_r, \text{ for any } 1 \le r \le l, \tag{5.16}$$

then

$$\mathrm{Last}_{A_r}(H_{\mathrm{z}}(w)) = \tau_r, \text{ for any } 1 \le r \le l. \tag{5.17}$$

To see the above property implies Proposition 5.1, note that taking $k_i \ge 1$ for $1 \le i \le m$ in $\mathbf{m} = (k_1, k_2, \dots, k_m)$, we have $\mathbf{m} = M$. Then taking $l = m$, (5.16) means that $w \in \mathfrak{S}_M^\tau$, and (5.17) means that $H_{\mathrm{z}}(w) \in \mathfrak{S}_M^\tau$. Thus, the above property implies Proposition 5.1.

Below we use induction on n, which is the length of w, to prove the above property. The initial case of $n = 1$ is obvious. Assume the above property is true for $n-1$ and prove it for n. Let $w = w_1w_2 \dots w_n$ and $w' = w_1w_2 \dots w_{n-1}$. By definition we see that

$$H_{\mathrm{z}}(w) = \left(C_{w_n}^{-1} \circ H_{\mathrm{z}} \circ \phi_{w_n}(w')\right) w_n. \tag{5.18}$$

We first consider the case that $w_n \in A_l$. Then $\mathrm{Last}_{A_l}(w) = w_n$. By (5.16) we see that $w_n = \tau_l$. Since τ is a consecutive permutation of $[l]$, then $\tau_l = l$ or $\tau_l = 1$. We distinguish two cases.

Case 1: $\tau_l = l$. From (5.18) we have

$$H_{\mathrm{z}}(w) = \left(C_l^{-1} \circ H_{\mathrm{z}} \circ \phi_l(w')\right) l. \tag{5.19}$$

Given r with $1 \le r \le l-1$. Obviously, $1 \le \min(A_r) \le \max(A_r) < l \le m$. Since $w \cap A_r \neq \emptyset$ and $l \notin A_r$, then $w' \cap A_r \neq \emptyset$. By Lemma 5.3, we get

$$\mathrm{Last}_{A_r}(\phi_l(w')) = \mathrm{Last}_{A_r}(w') = \mathrm{Last}_{A_r}(w) = \tau_r, \text{ for } 1 \le r \le l-1. \tag{5.20}$$

Note that $\phi_l(w')$ is of length $n-1$, we apply the induction hypothesis to $\phi_l(w')$, with taking $l' = l-1$ and $\tau' = \tau_1\tau_2 \dots \tau_{l-1}$. Clearly, τ' is a consecutive permutation of $[l-1]$ as $\tau_l = l$. By (5.20), we see that $\phi_l(w')$ and τ' satisfy the condition (5.16), then

$$\mathrm{Last}_{A_r}(H_{\mathrm{z}} \circ \phi_l(w')) = \tau_r, \text{ for } 1 \le r \le l-1. \tag{5.21}$$

Since $\max(A_r) < l$, we see that

$$\mathrm{Last}_{A_r}\left(C_l^{-1} \circ H_{\mathrm{z}} \circ \phi_l(w')\right) = \mathrm{Last}_{A_r}\left(H_{\mathrm{z}} \circ \phi_l(w')\right) = \tau_r, \text{ for } 1 \le r \le l-1. \tag{5.22}$$

Combining (5.22) with (5.19) yields

$$\mathrm{Last}_{A_r}\left(H_{\mathrm{z}}(w)\right) = \tau_r, \text{ for } 1 \le r \le l-1.$$

Since $A_l = \{1, 2, \ldots, l\}$, by (5.19) we have

$$\mathrm{Last}_{A_l}\left(H_{\mathrm{z}}(w)\right) = l = \tau_l.$$

Thus, (5.17) holds for $1 \le r \le l$, and we complete the proof for this case.

Case 2: $\tau_l = 1$. In this case we have

$$H_{\mathrm{z}}(w) = \left(C_1^{-1} \circ H_{\mathrm{z}} \circ \phi_1(w')\right) 1. \tag{5.23}$$

Clearly $\theta_{\mathbf{m}^1, \mathbf{m}_1} = \mathrm{id}$, then $\phi_1(w') = C^1(w') = d_1 d_2 \ldots d_{n-1}$, where

$$d_i = \begin{cases} w_i - 1, & \text{if } w_i > 1, \\ m, & \text{if } w_i = 1. \end{cases} \tag{5.24}$$

Let $\tau' = \tau_1' \tau_2' \ldots \tau_{l-1}'$, where $\tau_i' = \tau_i - 1$. Since $\tau = \tau_1 \tau_2 \ldots \tau_l$ is a consecutive permutation of $[l]$ and $\tau_l = 1$, we see that τ' is a consecutive permutation of $[l-1]$. For $1 \le r \le l-1$, let

$$A_r' = \{\tau_1', \tau_2', \ldots, \tau_r'\} = A_r - 1.$$

Note that $\min(A_r) > 1$ for $1 \le r \le l-1$, by (5.24) we see that

$$\mathrm{Last}_{A_r'}\left(\phi_1(w')\right) = \mathrm{Last}_{A_r}\left(w'\right) - 1 = \tau_r - 1 = \tau_r', \text{ for } 1 \le r \le l-1. \tag{5.25}$$

Applying the induction hypothesis to $\phi_1(w')$, then

$$\mathrm{Last}_{A_r'}\left(H_{\mathrm{z}} \circ \phi_1(w')\right) = \tau_r', \text{ for } 1 \le r \le l-1. \tag{5.26}$$

It is not hard to see that

$$\mathrm{Last}_{A_r}\left(C_1^{-1} \circ H_{\mathrm{z}} \circ \phi_1(w')\right) = \mathrm{Last}_{A_r'}\left(H_{\mathrm{z}} \circ \phi_1(w')\right) + 1. \tag{5.27}$$

Then we obtain

$$\mathrm{Last}_{A_r}\left(C_1^{-1} \circ H_{\mathrm{z}} \circ \phi_1(w')\right) = \tau_r' + 1 = \tau_r, \text{ for } 1 \le r \le l-1. \tag{5.28}$$

Note that $1 \notin A_r$ for $1 \le r \le l-1$, combining (5.28) with (5.23), we have

$$\mathrm{Last}_{A_r}\left(H_{\mathrm{z}}(w)\right) = \tau_r, \text{ for } 1 \le r \le l-1.$$

Clearly, $A_l = \{1, 2, \ldots, l\}$, by (5.23) we have

$$\mathrm{Last}_{A_l}\left(H_{\mathrm{z}}(w)\right) = 1 = \tau_l.$$

Thus, (5.17) holds for $1 \le r \le l$, completing the proof for this case.

We now consider the case that $w_n \notin A_l$. Since $A_l = [l]$ and $w_n \notin A_l$, we have $l < w_n \le m$. Then for any r, $1 \le r \le l$, we have $1 \le \min(A_r) \le \max(A_r) < w_n \le m$. The proof for this case is very similar to that of **Case 1**, and we omit it. ∎

6. Equidistribution of (mstc,INV) and (des,MAJ) on $\mathcal{P}_M$

The permutation statistic stc was introduced by Skandera [37], he proved that (stc, INV) is Euler-Mahonian. Carnevale [4] extended the statistic stc to words, which he called mstc, and proved that (mstc, INV) is Euler-Mahonian, that is (mstc, INV) and (des, MAJ) are equidistributed on words. In this section, we prove that (mstc, INV) and (des, MAJ) are equidistributed when restricted to $\mathcal{P}_M$. Here we give an equivalent definition of the statistic mstc.

First, we define the map std : $\mathfrak{S}_M \to \mathfrak{S}_n$. Given $w = w_1 w_2 \ldots w_n \in \mathfrak{S}_M$, define std$(w)$ to be the permutation $\pi_1\pi_2\ldots\pi_n \in \mathfrak{S}_n$ such that $\pi_i < \pi_j$ if and only if either $w_i < w_j$ or $w_i = w_j$ with $i < j$. For example, std(32112133) = 64125378. Clearly, $\mathrm{INV}(\mathrm{std}(w)) = \mathrm{INV}(w)$.

Second, we define the map $I : \mathfrak{S}_n \to \mathbf{I}_n$, where $\mathbf{I}_n := \{(c_1, c_2, \ldots, c_n) : 0 \le c_i \le i-1\}$. Given $\pi \in \mathfrak{S}_n$, for $i \in [n]$, let c_i be the number of letters j to the right of the letter i in π such that $j < i$. Then define

$$I(\pi) = (c_1, c_2, \ldots, c_n).$$

Clearly, $\mathrm{INV}(\pi) = \sum_{i=1}^{n} c_i$. For example, let $\pi = 64125378$, then $I(\pi) = (0, 0, 0, 3, 1, 5, 0, 0)$ and $\mathrm{INV}(\pi) = 3 + 1 + 5 = 9$.

Third, we give the definition of the statistic eul on $\mathbf{I}_n$, see [19]. Let $c = (c_1, c_2, \ldots, c_n) \in \mathbf{I}_n$. If $n = 1$, define $\mathrm{eul}(c) = 0$; if $n \ge 2$, let $c' = (c_1, c_2, \ldots, c_{n-1})$, then define

$$\mathrm{eul}(c) = \begin{cases} \mathrm{eul}(c'), & \text{if } c_n \le \mathrm{eul}(c'), \\ \mathrm{eul}(c') + 1, & \text{if } c_n > \mathrm{eul}(c'). \end{cases}$$

Finally, for $w \in \mathfrak{S}_M$, define

$$\operatorname{mstc}(w) = \operatorname{eul} \circ I \circ \operatorname{std}(w). \tag{6.29}$$

Remark 6.1. Let $\mathbf{i}(\pi) = \pi^{-1}$. Originally, Carnevale [4] defined the statistic mstc to be

$$\operatorname{mstc}(w) = \operatorname{stc} \circ \mathbf{i} \circ \operatorname{std}(w). \tag{6.30}$$

In [28] (at the end of Section 7), the author proved that for any permutation π,

$$\operatorname{eul} \circ I(\pi) = \operatorname{stc} \circ \mathbf{i}(\pi).$$

Then

$$\operatorname{eul} \circ I \circ \operatorname{std}(w) = \operatorname{stc} \circ \mathbf{i} \circ \operatorname{std}(w),$$

which shows the equivalence of (6.29) and (6.30).

We now review the bijection Ψ on permutations that takes INV to MAJ, which is essentially due to Carlitz [7]. Also see [34]. See [28] for a k-extension of this bijection.

We will recursively define the bijection Ψ. Given a permutation $\pi \in \mathfrak{S}_n$, assume that $I(\pi) = (c_1, c_2, \ldots, c_n)$. Let $\pi' \in \mathfrak{S}_{n-1}$ be the permutation obtained from π by deleting the letter n. Assume that $\Psi(\pi') \in \mathfrak{S}_{n-1}$ has been defined, we will define $\Psi(\pi)$ from $\Psi(\pi')$ by inserting the letter n. There are n positions where we can insert n in $\Psi(\pi')$ to obtain a permutation $\Psi(\pi)$. We first label the positions of $\Psi(\pi')$ according to the following rules:

(a) Label the position after $\Psi(\pi')$ with 0.

(b) Label the positions following the descents of $\Psi(\pi')$ from right to left with $1, 2, \ldots, \operatorname{des}(\Psi(\pi'))$.

(c) Label the remaining positions from left to right with $\operatorname{des}(\Psi(\pi')) + 1, \ldots, n-1$.

Finally, we insert the letter n into the position labeled by c_n, let $\Psi(\pi)$ be the resulting permutation.

For example, if $\Psi(\pi') = 64125378$, we can write the labeling of the positions of $\Psi(\pi')$ as subscripts to get

$${}_4 6_3 4_2 1_5 2_6 5_1 3_7 7_8 8_0.$$

If $c_9 = 6$, then $\Psi(\pi) = 641295378$.

Given a multiset $M = \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$ with $|M| = n$, we extend Ψ to $\mathfrak{S}_M$, which we denote Ψ_M. First, we define a map $\operatorname{istd}_M : \mathfrak{S}_n \to \mathfrak{S}_M$. Given $\pi \in \mathfrak{S}_n$, $\operatorname{istd}_M(\pi) \in \mathfrak{S}_M$ is obtained from π by replacing the letters $1, 2, \ldots, k_1$ with 1, replacing the letters $k_1 + 1, k_1 + 2, \ldots, k_1 + k_2$ with 2, ..., replacing the letters $1 + \sum_{i=1}^{m-1} k_i, \ldots, \sum_{i=1}^{m} k_i$ with m. Clearly for any $w \in \mathfrak{S}_M$, we have

$$\mathrm{istd}_M \circ \mathrm{std}(w) = w.$$

For $w \in \mathfrak{S}_M$, we define

$$\Psi_M(w) = \mathrm{istd}_M \circ \Psi \circ \mathrm{std}(w). \tag{6.31}$$

See [38] (Section 2.2) for another equivalent description of Ψ_M.

Example 6.1. Let $M = \{1^2, 2^2, 3^2\}$ and let $w = 213123 \in \mathfrak{S}_M$. So $\pi = \mathrm{std}(w) = 315246$. Clearly, $I(\pi) = (c_1, c_2, c_3, c_4, c_5, c_6) = (0, 0, 2, 0, 2, 0)$. We give the procedure for creating $\Psi(\pi)$:

$$_1 1_0 \xrightarrow{c_2=0} {}_1 1_2 2_0 \xrightarrow{c_3=2} {}_2 1_3 3_1 2_0 \xrightarrow{c_4=0} {}_2 1_3 3_1 2_4 4_0 \xrightarrow{c_5=2} {}_3 5_2 1_4 3_1 2_5 4_0 \xrightarrow{c_6=0} 513246 = \Psi(\pi).$$

Then $\Psi_M(w) = \mathrm{istd}_M \circ \Psi(\pi) = \mathrm{istd}_M(513246) = 312123$.

Our next goal is to prove the following property of Ψ_M.

Proposition 6.1. *Let $M = \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$ with $k_i \geq 1$ for all $i \in [m]$. For any $w \in \mathfrak{S}_M$, we have*

$$(\mathrm{mstc}, \mathrm{INV})w = (\mathrm{des}, \mathrm{MAJ})\Psi_M(w).$$

To prove Proposition 6.1, we need some lemmas.

Lemma 6.1. *Given $\pi \in \mathfrak{S}_n$, let $I(\pi) = (c_1, c_2, \ldots, c_n)$. For any $1 \leq i \leq n-1$, if $c_i \geq c_{i+1}$, then in permutation $\Psi(\pi)$, the letter $i+1$ is not immediately followed by the letter i.*

Proof. Let $\pi_{(i)}$ be the subpermutation of π by deleting the letters $i+1, i+2, \ldots, n$, and let $\gamma_i = \Psi(\pi_{(i)})$. Clearly $\gamma_n = \Psi(\pi)$. Consider γ_{i-1}, assume that we have labeled the positions of γ_{i-1} by $0, 1, \ldots, i-1$. Assume that $\mathrm{des}(\gamma_{i-1}) = d$. We now insert the letters i and $i+1$ (in order) into γ_{i-1} to obtain γ_i and γ_{i+1}. If $c_i \leq d$, then $c_{i+1} \leq c_i \leq d$, by rules (a) and (b) we see that the letter $i+1$ is on the right of the letter i in γ_{i+1}, thus, the letter $i+1$ is not immediately followed by i in γ_{i+1}, as well as in γ_n. If $c_i > d$ and the letter $i+1$ is immediately followed by i in γ_{i+1}, by rule (c), it is not hard to see that $c_{i+1} = c_i + 1$, contradicting the assumption that $c_i \geq c_{i+1}$, and we complete the proof. ∎

By a similar argument, we can obtain the following lemma, and we omit the proof of it.

Lemma 6.2. *Given $\pi \in \mathfrak{S}_n$, let $I(\pi) = (c_1, c_2, \ldots, c_n)$. For any $1 \leq i \leq n-1$, and $1 \leq s \leq n-i$, if $c_i \geq c_{i+1} \geq \cdots \geq c_{i+s}$, then in permutation $\Psi(\pi)$, the letter $i+s$ is not immediately followed by the letter i.*

Lemma 6.3. *Let $M = \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$ with $k_i \geq 1$ for all $i \in [m]$. Let $w \in \mathfrak{S}_M$, then*

$$\mathrm{Des}(\mathrm{istd}_M \circ \Psi \circ \mathrm{std}(w)) = \mathrm{Des}(\Psi \circ \mathrm{std}(w)).$$

Proof. Assume that $\pi = \mathrm{std}(w) \in \mathfrak{S}_n$. Given $i \in [m]$, assume that $w_{i_1} = w_{i_2} = \cdots = w_{i_{k_i}} = i$, where $i_1 < i_2 < \cdots < i_{k_i}$. Let $a = \sum_{j=1}^{i-1} k_j$, then $\pi_{i_1} = a+1, \pi_{i_2} = a+2, \ldots, \pi_{i_{k_i}} = a + k_i$. Assume that $I(\pi) = (c_1, c_2, \ldots, c_n)$. It is not hard to see that

$$c_{a+1} \geq c_{a+2} \geq \cdots \geq c_{a+k_i}.$$

To obtain $\mathrm{istd}_M \circ \Psi(\pi)$, we need replace the letters $a+1, a+2, \ldots, a+k_i$ in $\Psi(\pi)$ with i. By Lemma 6.2, we see that in $\Psi(\pi)$, the letter $a + s_2$ is not immediately followed by the letter $a + s_1$, for any $1 \leq s_1 < s_2 \leq k_i$. Thus, the replacement fixes the descent set, that is,

$$\mathrm{Des}(\mathrm{istd}_M \circ \Psi(\pi)) = \mathrm{Des}(\Psi(\pi)),$$

which completes the proof. ∎

Lemma 6.4. *For any permutation π, we have*

$$(\mathrm{eul} \circ I, \mathrm{INV})\pi = (\mathrm{des}, \mathrm{MAJ})\Psi(\pi).$$

The proof of the lemma can be found in the proof of Theorem 7.1 in [28] (the case $k = 1$).

Proof of Proposition 6.1. Given $w \in \mathfrak{S}_M$, we have

$$\begin{aligned} \mathrm{mstc}(w) &= \mathrm{eul} \circ I \circ \mathrm{std}(w) && \text{(by (6.29))} \\ &= \mathrm{des}(\Psi \circ \mathrm{std}(w)) && \text{(by Lemma 6.4)} \\ &= \mathrm{des}(\mathrm{istd}_M \circ \Psi \circ \mathrm{std}(w)) && \text{(by Lemma 6.3)} \\ &= \mathrm{des}(\Psi_M(w)) && \text{(by (6.31))} \end{aligned}$$

and

$$\begin{aligned} \mathrm{INV}(w) &= \mathrm{INV}(\mathrm{std}(w)) \\ &= \mathrm{MAJ}(\Psi \circ \mathrm{std}(w)) && \text{(by Lemma 6.4)} \\ &= \mathrm{MAJ}(\mathrm{istd}_M \circ \Psi \circ \mathrm{std}(w)) && \text{(by Lemma 6.3)} \\ &= \mathrm{MAJ}(\Psi_M(w)), && \text{(by (6.31))} \end{aligned}$$

completing the proof. ∎

The following proposition shows that Ψ_M preserves the increasing tail permutation.

Proposition 6.2. *Let $M = \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$ with $k_i \geq 1$ for all $i \in [m]$, the set $\mathcal{P}_M$ is invariant under Ψ_M, that is*

$$\Psi_M(\mathcal{P}_M) = \mathcal{P}_M.$$

Proof. Let $w = w_1 w_2 \ldots w_n \in \mathcal{P}_M$, assume that w_{s_j} is the last occurrence of j, $1 \leq j \leq m$.

Then $w_{s_1}w_{s_2}\dots w_{s_m} = 12\dots m$, where $s_1 < s_2 < \cdots < s_m$. Let $\pi = \pi_1\pi_2\dots\pi_n = \mathrm{std}(w)$. We denote $a_j = \sum_{l=1}^{j} k_l$. Then $\pi_{s_1} = a_1, \pi_{s_2} = a_2, \dots, \pi_{s_m} = a_m$. Let $I(\pi) = (c_1, c_2, \dots, c_n)$. It is not hard to see that $c_{a_1} = c_{a_2} = \cdots = c_{a_m} = 0$. By rule (a) of the construction of $\Psi(\pi)$, we see that in $\Psi(\pi)$, all the letters to the right of a_j are larger than a_j, for $1 \le j \le m$. This implies that $\mathrm{istd}_M(\Psi(\pi)) \in \mathcal{P}_M$, that is, $\Psi_M(w) \in \mathcal{P}_M$, completing the proof. ∎

Combining Proposition 6.1 with Proposition 6.2 gives the equidistribution of the bi-statistics (mstc, INV) and (des, MAJ) on $\mathcal{P}_M$.

7. Equidistribution of INV and r-MAJ on $\mathcal{P}_M$

The r-major index, denoted r-MAJ, was introduced by Rawlings [32] for permutations, and extended to words in [33].

First, define the *r-descent set*, denoted r-Des, and *r-inversion set*, denoted r-Inv, for word $w = w_1w_2\dots w_n$ as follows:

$$r\text{-Des}(w) = \{i \in \{1, 2, \dots, n-1\} : w_i \ge w_{i+1} + r\},$$
$$r\text{-Inv}(w) = \{(i, j) : 1 \le i < j \le n, w_i - r < w_j < w_i\}.$$

Now, r-MAJ is defined by

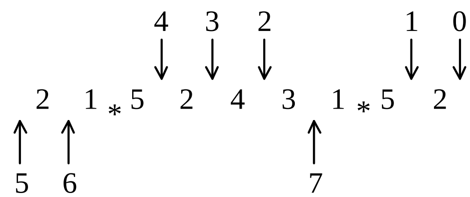
$$r\text{-MAJ}(w) = |r\text{-Inv}(w)| + \sum_{i \in r\text{-Des}(w)} i.$$

It is clear that r-MAJ reduces to MAJ when $r = 1$ and to INV when $r \ge n$, thus the family of r-MAJ interpolates between MAJ and INV.

Rawlings [32] showed that r-MAJ is equidistributed with INV on permutations by constructing a bijection on permutations that takes INV to r-MAJ, which is a generalization of Carlitz's bijection which we have given in the previous section. In [33], Rawlings extended his bijection to words and then proved that INV and r-MAJ are equidistributed on $\mathfrak{S}_M$. Here we use a different way to describe Rawlings' bijection R on words given in [33] that takes INV to r-MAJ.

Given $w \in \mathfrak{S}_M$, let $w' \in \mathfrak{S}_{M'}$, where $M' = \{1^{k_1}, 2^{k_2}, \dots, (m-1)^{k_{m-1}}\}$, be the word obtained from w by deleting all of the occurrences of m. Consider the jth m in w from left to right, let $u_j(m)$ be the letters in w that are on the right of this m and that are smaller than m. Clearly, $u_1(m) \ge u_2(m) \ge \cdots \ge u_{k_m}(m)$. For example, $w = 2152431552$, $w' = 2124312$, and $u_1(5) = 5$, $u_2(5) = 1$, $u_3(5) = 1$. We will recursively define Rawlings' bijection R. Assume that $R(w')$ has been defined, we will define $R(w) \in \mathfrak{S}_M$ from $R(w')$ by inserting k_m m's. Assume that we have inserted $(j-1)$ m's, we are going to insert the jth m. First, we star the positions before each m. Then, we label the positions that are not occupied by any star according to the following rules. Using the labels $0, 1, 2, \dots$ in order, first read from right to left and label those positions that will *not* result in the creation of a new r-descent. Then, reading back from left to right the positions that *will* create a new r-descent are labeled. Finally, we insert the jth m into the position labeled by $u_j(m)$.

For example, assume that $r = 3$, and we will insert the third 5 into 215243152, then the labels in the top and bottom rows of



respectively indicate the positions that will not and that will result in a new 3-descent. If $u_3(5) = 1$, we obtain 2152431552.

With the above notations, we now consider $w \in \mathcal{P}_M$. It is not hard to see that $w' \in \mathcal{P}_{M'}$ and $u_{k_m}(m) = 0$. By the procedure of creating $R(w)$ from $R(w')$, we see that the rightmost letter of $R(w)$ is m. Then an inductive proof shows that $R(w) \in \mathcal{P}_M$. This is the content of the following proposition, which implies the equidistribution of INV and r-MAJ on $\mathcal{P}_M$.

Proposition 7.1. *Let $M = \{1^{k_1}, 2^{k_2}, \dots, m^{k_m}\}$ with $k_i \geq 1$ for all $i \in [m]$, the set $\mathcal{P}_M$ is invariant under Rawlings' bijection, that is,*

$$R(\mathcal{P}_M) = \mathcal{P}_M.$$

8. Equidistribution of (des,MAJ) and (exc,DEN) on $\mathcal{P}_M$

Denert's permutation statistic, DEN, was introduced by Denert in [12], and she conjectured that (exc, DEN) is Euler-Mahonian. This conjecture was first proved by Foata and Zeilberger [15], Han [19, 20] gave two bijective proofs. Han [22] extended DEN to words, and proved that (exc, DEN) is Euler-Mahonian by giving a bijection on words that takes (des, MAJ) to (exc, DEN). We denote Han's bijection given in [22] by H_{DEN}. The goal of this section is to establish the equidistribution of the bi-statistics (des, MAJ) and (exc, DEN) on $\mathcal{P}_M$ by proving that H_{DEN} preserves the increasing tail permutation.

Let $w = w_1 w_2 \dots w_n \in \mathfrak{S}_M$, the two-line notation of w is written as

$$w = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ w_1 & w_2 & w_3 & \dots & w_n \end{pmatrix},$$

where $\overline{w} := a_1a_2\ldots a_n$ is the nondecreasing rearrangement of $w = w_1w_2\ldots w_n$, this notation will be adhered to throughout; that is, if $w = w_1w_2\ldots w_n$ is a word, $\overline{w}$ has the above meaning.

Let $w = w_1w_2\ldots w_n \in \mathfrak{S}_M$, with $\overline{w} = a_1a_2\ldots a_n$. An *excedance* in w is a triple (i, a_i, w_i) such that $w_i > a_i$. Here i is called the *excedance place*, a_i is called the *excedance bottom*, w_i is called the *excedance top*. The number of excedances of w is denoted by $\mathrm{exc}(w)$. Let $\mathrm{Exc}\ w$ be the subword consisting of all the excedance tops of w, in the order induced by w. Let $\mathrm{Nexc}\ w$ be the subword consisting of those letters of w that are not excedance tops. For example, if $w = 121442314$, then $\mathrm{Exc}\ w = 244$ and $\mathrm{Nexc}\ w = 112314$. Let $\mathrm{imv}(w)$ be the number of weak inversions of w, i.e.,

$$\mathrm{imv}(w) = |\{(i,j) : i < j, w_i \geq w_j\}|.$$

Denert's statistic of w, $\mathrm{DEN}(w)$, is defined by

$$\mathrm{DEN}(w) = \sum_i \{i : w_i > a_i\} + \mathrm{imv}(\mathrm{Exc}\ w) + \mathrm{INV}(\mathrm{Nexc}\ w).$$

For example, let

$$w = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 5 \\ 5 & 3 & 1 & 1 & 2 & 4 & 4 & 3 & 2 & 3 \end{pmatrix},$$

then $\mathrm{Exc}\ w = 5344$ and $\mathrm{Nexc}\ w = 112323$, and

$$\mathrm{DEN}(w) = (1+2+6+7) + \mathrm{imv}(5344) + \mathrm{INV}(112323) = 16 + 4 + 1 = 21.$$

Before stating Han's bijection H_{DEN}, we need some notions. A *biword* is an ordered pair of words of the same length, written as

$$w = \begin{pmatrix} x_1 & x_2 & x_3 & \ldots & x_n \\ w_1 & w_2 & w_3 & \ldots & w_n \end{pmatrix}.$$

In particular, the two-line notation of a word is a biword satisfying $x_1x_2\ldots x_n$ is the nondecreasing rearrangement of $w_1w_2\ldots w_n$.

Definition 8.1. A biword $w = \begin{pmatrix} x_1 & x_2 & x_3 & \ldots & x_n \\ w_1 & w_2 & w_3 & \ldots & w_n \end{pmatrix}$ is called a *dominated cycle* if $n = 1$ and $w_1 = x_n$, or $n > 1$, $w_1 = x_n$, $w_i = x_{i-1}$ and $w_1 > w_i$ for all $2 \leq i \leq n$.

Definition 8.2. Let $\mathbb{P}$ be the set of positive integers. Given $x, y \in \mathbb{P}$, the cyclic interval $\rrbracket x, y\rrbracket$ is defined by

$$\rrbracket x, y\rrbracket = \begin{cases} \{z \in \mathbb{P} : x < z \leq y\}, & \text{if } x \leq y; \\ \{z \in \mathbb{P} : x < z \text{ or } z \leq y\} & \text{if } x > y. \end{cases}$$

Definition 8.3. For $1 \le i \le n-1$, define the operator T_i on biword $w = \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ w_1 & w_2 & w_3 & \dots & w_n \end{pmatrix}$ to be

$$T_i(w) = \begin{pmatrix} x_1 & x_2 & \dots & x_{i-1} \\ w_1 & w_2 & \dots & w_{i-1} \end{pmatrix} T \begin{pmatrix} x_i & x_{i+1} \\ w_i & w_{i+1} \end{pmatrix} \begin{pmatrix} x_{i+2} & \dots & x_n \\ w_{i+2} & \dots & w_n \end{pmatrix},$$

where

$$T \begin{pmatrix} x & y \\ \alpha & \beta \end{pmatrix} = \begin{cases} \begin{pmatrix} y & x \\ \beta & \alpha \end{pmatrix}, & \text{if exactly one of } \alpha \text{ and } \beta \text{ lies in } \rrbracket x, y \rrbracket; \\ \begin{pmatrix} y & x \\ \alpha & \beta \end{pmatrix}, & \text{otherwise.} \end{cases}$$

Han's bijection H_{DEN} is devised to decompose a word into dominated cycles. Below is a description of Han's bijection.

Let $w = w_1 w_2 \dots w_n \in \mathfrak{S}_M$, we write it in the two-line notation

$$w = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ w_1 & w_2 & w_3 & \dots & w_n \end{pmatrix}.$$

If $n = 1$, then w itself is a dominated cycle. We assume that $n \ge 2$. If $w_n = a_n$, then set

$$w' = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} \\ w_1 & w_2 & w_3 & \dots & w_{n-1} \end{pmatrix}, \quad u = \begin{pmatrix} a_n \\ w_n \end{pmatrix}.$$

If $w_n \ne a_n$, let i_1 be the largest index such that $a_{i_1} = w_n$, and set

$$w^{(1)} = T_{n-2} \circ T_{n-1} \circ \dots \circ T_{i_1}(w) = \begin{pmatrix} a_1^{(1)} & a_2^{(1)} & \dots & a_{n-2}^{(1)} & w_n & a_n \\ w_1^{(1)} & w_2^{(1)} & \dots & w_{n-2}^{(1)} & w_{n-1}^{(1)} & w_n \end{pmatrix}.$$

If $w_{n-1}^{(1)} = a_n$, then set

$$w' = \begin{pmatrix} a_1^{(1)} & a_2^{(1)} & \dots & a_{n-2}^{(1)} \\ w_1^{(1)} & w_2^{(1)} & \dots & w_{n-2}^{(1)} \end{pmatrix}, \quad u = \begin{pmatrix} w_n & a_n \\ a_n & w_n \end{pmatrix}.$$

If $w_{n-1}^{(1)} \ne a_n$, let i_2 be the largest index such that $a_{i_2}^{(1)} = w_{n-1}^{(1)}$, and set

$$w^{(2)} = T_{n-3} \circ T_{n-1} \circ \dots \circ T_{i_2}(w^{(1)}) = \begin{pmatrix} a_1^{(2)} & a_2^{(2)} & \dots & a_{n-3}^{(2)} & w_{n-1}^{(1)} & w_n & a_n \\ w_1^{(2)} & w_2^{(2)} & \dots & w_{n-3}^{(2)} & w_{n-2}^{(2)} & w_{n-1}^{(1)} & w_n \end{pmatrix}.$$

Similarly, we can repeat the above process by considering whether $w_{n-2}^{(2)}$ is equal to a_n. So we can obtain a sequence of words $w^{(1)}, w^{(2)}, \ldots, w^{(t)}$ such that

$$w^{(t)} = \begin{pmatrix} a_1^{(t)} & a_2^{(t)} & \ldots & a_{n-t-1}^{(t)} & w_{n-t+1}^{(t-1)} & \ldots & w_n & a_n \\ w_1^{(t)} & w_2^{(t)} & \ldots & w_{n-t-1}^{(t)} & w_{n-t}^{(t)} & \ldots & w_{n-1}^{(1)} & w_n \end{pmatrix},$$

where $w_{n-t}^{(t)} = a_n$ and $w_{n-t+1}^{(t-1)} \neq a_n, \ldots, w_{n-1}^{(1)} \neq a_n, w_n \neq a_n$. Set

$$w' = \begin{pmatrix} a_1^{(t)} & a_2^{(t)} & \ldots & a_{n-t-1}^{(t)} \\ w_1^{(t)} & w_2^{(t)} & \ldots & w_{n-t-1}^{(t)} \end{pmatrix}, \quad u = \begin{pmatrix} w_{n-t+1}^{(t-1)} & w_{n-t+2}^{(t-2)} & \ldots & w_n & a_n \\ a_n & w_{n-t+1}^{(t-1)} & \ldots & w_{n-1}^{(1)} & w_n \end{pmatrix},$$

Note that $a_n \geq a_i$ for $1 \leq i \leq n$, and $w_{n-t+1}^{(t-1)} \neq a_n, \ldots, w_{n-1}^{(1)} \neq a_n, w_n \neq a_n$, then we see that u is a dominated cycle.

It is not hard to see that w' is the two-line notation of the word $w_1^{(t)} w_2^{(t)} \ldots w_{n-t-1}^{(t)}$. Appealing to induction, assume that w' admits a decomposition $u_1, u_2, \ldots, u_s$ into dominated cycles, then the decomposition of w is defined to be $u_1, u_2, \ldots, u_s, u$, and $H_{\text{DEN}}(w)$ is obtained by concatenating the bottom rows of these cycles.

Theorem 8.1 (Han [22]). *$H_{\text{DEN}} : \mathfrak{S}_M \to \mathfrak{S}_M$ is a bijection satisfying*

$$(\text{exc}, \text{DEN})\, w = (\text{des}, \text{MAJ})\, H_{\text{DEN}}(w)$$

for all $w \in \mathfrak{S}_M$.

Example 8.1. Let $w = 124324$, then

$$\begin{pmatrix} 1 & 2 & 2 & 3 & 4 & 4 \\ 1 & 2 & 4 & 3 & 2 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & \mathbf{2} & 3 & 4 \\ 1 & 2 & 4 & 3 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \xrightarrow{T_3} \begin{pmatrix} 1 & 2 & 3 & 2 & 4 \\ 1 & 2 & 3 & 4 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$
$$\longrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$

So $H_{\text{DEN}}(w) = 123424$. We see that $(\text{exc}, \text{DEN})w = (\text{des}, \text{MAJ})H_{\text{DEN}}(w) = (1, 4)$.

The following proposition shows that Han's bijection H_{DEN} preserves the increasing tail permutation, which implies the equidistribution of the bi-statistics (des, MAJ) and (exc, DEN) on $\mathcal{P}_M$.

Proposition 8.1. *Let $M = \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$ with $k_i \geq 1$ for all $i \in [m]$, the set $\mathcal{P}_M$ is invariant under Han's bijection H_{DEN}, that is,*

$$H_{\text{DEN}}(\mathcal{P}_M) = \mathcal{P}_M.$$

Proof. Given $w \in \mathcal{P}_M$, assume that the operations that we used to create $H_{\text{DEN}}(w)$ are T_{k_1},

$T_{k_2}, \ldots, T_{k_s}$ in order. Let

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \overline{w} \\ w \end{pmatrix} \text{ and } \begin{pmatrix} \alpha_{i+1} \\ \beta_{i+1} \end{pmatrix} = T_{k_i} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \text{ for } 1 \le i \le s.$$

Consider the sequence

$$\beta_1, \beta_2, \beta_3, \ldots, \beta_{s+1},$$

where $\beta_1 = w$, $\beta_{s+1} = H_{\text{DEN}}(w)$. It is clear that for $1 \le i \le s$, either $\beta_{i+1} = \beta_i$, or β_{i+1} is obtained from β_i by interchanging two consecutive entries. Note that the tail permutation of β_1 is $123\ldots m$. If all the tail permutations of $\beta_1, \beta_2, \beta_3, \ldots, \beta_{s+1}$ are $123\ldots m$, our proof is completed. Now we assume that all the tail permutations of $\beta_1, \beta_2, \ldots, \beta_t$ are $123\ldots m$, and the tail permutation of β_{t+1} is not $123\ldots m$. Assume that $\beta_t = b_1 b_2 \ldots b_n$ and $\beta_{t+1} = b_1 b_2 \ldots b_{c-1} b_{c+1} b_c b_{c+2} \ldots b_n$. Because the tail permutations of β_t and β_{t+1} are different and the tail permutation of β_t is $123\ldots m$, we see that the tail permutation of β_{t+1} has the form $12\ldots(i-1)(i+1)i(i+2)\ldots m$. Thus, in β_t, b_c is the last occurrence of the letter i and b_{c+1} is the last occurrence of the letter $i+1$. Since the tail permutation of β_t is $123\ldots m$ and b_c is the last occurrence of the letter i, we have $\min\{b_{c+1}, b_{c+2}, \ldots, b_n\} > i$. Note that

$$\begin{pmatrix} \alpha_{t+1} \\ \beta_{t+1} \end{pmatrix} = T_{k_t} \begin{pmatrix} \alpha_t \\ \beta_t \end{pmatrix} = \begin{pmatrix} * & * & \ldots & * \\ b_1 & b_2 & \ldots & b_{c-1} \end{pmatrix} T \begin{pmatrix} x & y \\ i & i+1 \end{pmatrix} \begin{pmatrix} * & \ldots & * \\ b_{c+2} & \ldots & b_n \end{pmatrix}.$$

By the process of creating $H_{\text{DEN}}(w)$, we can see the following two facts: (i) $x \le y$; (ii) $x \in \{b_{c+2}, b_{c+3}, \ldots, b_n\}$. From fact (i) we see that

$$\rrbracket x, y \rrbracket = \{x+1, x+2, \ldots, y\}.$$

By fact (ii) we have

$$x \ge \min\{b_{c+2}, b_{c+3}, \ldots, b_n\} > i.$$

Thus, $i < i+1 < x+1$. Then both i and $i+1$ are not in the set $\rrbracket x, y \rrbracket$. By the definition of the operator T, we have $\beta_{t+1} = \beta_t$, which is a contradiction and the proof is completed. ∎

9. Equidistribution of (des,MAK,MAD) and (exc,DEN,INV) on $\mathcal{P}_M$

The Mahonian permutation statistic MAK was introduced by Foata and Zeilberger [15]. Clarke, Steingrímsson and Zeng [11] extended it to words. In the same paper, Clarke, Steingrímsson and Zeng introduced a new Mahonian statistic MAD on words, and they proved that the triple statistics (des, MAK, MAD) and (exc, DEN, INV) are equidistributed on words by exhibiting a bijection Φ on words that takes (des, MAK, MAD) to (exc, DEN, INV). The goal of this section is to establish the equidistribution of (des, MAK, MAD) and (exc, DEN, INV) on $\mathcal{P}_M$ by proving that Φ preserves the increasing tail permutation.

Let $w = w_1w_2\dots w_n \in \mathfrak{S}_M$, the *height* $h(a)$ of a letter a in w is one more than the number of letters in w that are strictly smaller than a. The *value* of the ith letter in w, denoted by v_i, is defined by

$$v_i = h(w_i) + l(i),$$

where $l(i)$ is the number of letters in w that are to the left of w_i and equal to w_i. For example, given $w = 21144231$, then $\overline{w} = 11122344$, so the heights of $1, 2, 3, 4$ are, respectively, $1, 4, 6, 7$. The values of the letters of w are given by $4, 1, 2, 7, 8, 5, 6, 3$, in the order in which they appear in w. It is not hard to see that $v_1v_2\dots v_n = \mathrm{std}(w)$.

Let $w = w_1w_2\dots w_n \in \mathfrak{S}_M$, recall that a decent of w is an index i such that $w_i > w_{i+1}$, we call w_i a *descent top*, and w_{i+1} a *descent bottom*. The *descent tops sum* of w, denoted by $\mathrm{Dtop}(w)$, is the sum of the heights of the descent tops of w. The *descent bottoms sum* of a word w, denoted by $\mathrm{Dbot}(w)$, is the sum of the values of the descent bottoms of w. The *descent difference* of w is

$$\mathrm{Ddif}(w) = \mathrm{Dtop}(w) - \mathrm{Dbot}(w).$$

Given a word $w = w_1w_2\dots w_n$, we separate w into its *descent blocks* by putting in dashes between w_i and w_{i+1} whenever $w_i \le w_{i+1}$. A maximal contiguous subword of w which lies between two dashes is a *descent block*. A descent block is an *outsider* if it has only one letter; otherwise, it is a *proper* descent block. The leftmost letter of a proper descent block is its *closer* and the rightmost letter is its *opener*. Let B be a proper descent block of the word w and let $C(B)$ and $O(B)$ be the closer and opener of B, respectively. Let a be a letter of w, we say that a is *embraced* by B if $C(B) \ge a > O(B)$.

The *right embracing numbers* of a word $w = w_1w_2\dots w_n$ are the numbers $e_1, e_2, \dots, e_n$ where e_i is the number of descent blocks in w that are strictly to the right of w_i and that embrace w_i. The *right embracing sum* of w, denoted by $\mathrm{Res}(w)$, is defined by

$$\mathrm{Res}(w) = e_1 + e_2 + \cdots + e_n.$$

Definition 9.1.

$$\mathrm{MAK}(w) = \mathrm{Dbot}(w) + \mathrm{Res}(w),$$
$$\mathrm{MAD}(w) = \mathrm{Ddif}(w) + \mathrm{Res}(w).$$

We now give an overview of the bijection Φ, see [11]. Given $w = w_1w_2\dots w_n \in \mathfrak{S}_M$, let $\pi = \mathrm{std}(w)$. For permutation π, we first construct two biwords $\begin{pmatrix} f \\ f' \end{pmatrix}$ and $\begin{pmatrix} g \\ g' \end{pmatrix}$, and then form the biword $\begin{pmatrix} f & g \\ f' & g' \end{pmatrix}$ by concatenating f and g, and f' and g', respectively. The word f is defined as the subword of descent bottoms in π, ordered increasingly. The word g is defined as the subword of non-descent bottoms in π, also ordered increasingly. The word f' is the subword of descent tops in π, ordered so that for any letter x in f', there are exactly d letters in f' that are on the left of x and that are greater than x, where d is the embracing number of the letter x in π. The word g' is the subword of non-descent tops in π, ordered so that for any letter x in g', there are exactly d letters in g' that are on the right of x and that are smaller than x, where d is the embracing number of the letter x in π. Rearranging the columns of $\begin{pmatrix} f & g \\ f' & g' \end{pmatrix}$, so that the top row is in increasing order, then let π' be the bottom row of the rearranged biword. We point out that f and f' are the excedance bottoms and excedance tops in π', respectively. Finally, we let $\Phi(w) = \mathrm{istd}_M(\pi')$.

Example 9.1. Consider the word

$$w = 1\ 3\ 2\ 1\ 3\ 2\ 2\ 3.$$

Then

$$\pi = \mathrm{std}(w) = 1 - 6\ 3\ 2 - 7\ 4 - 5 - 8.$$

It is not hard to see that

$$\begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 7 & 6 \end{pmatrix}, \qquad \begin{pmatrix} g \\ g' \end{pmatrix} = \begin{pmatrix} 1 & 5 & 6 & 7 & 8 \\ 1 & 2 & 4 & 5 & 8 \end{pmatrix}.$$

Then

$$\begin{pmatrix} f & g \\ f' & g' \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 & 1 & 5 & 6 & 7 & 8 \\ 3 & 7 & 6 & 1 & 2 & 4 & 5 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 7 & 6 & 2 & 4 & 5 & 8 \end{pmatrix},$$

and

$$\pi' = 1\ 3\ 7\ 6\ 2\ 4\ 5\ 8.$$

Thus,

$$\Phi(w) = 1\ 2\ 3\ 3\ 1\ 2\ 2\ 3.$$

Theorem 9.1 (Clarke-Steingrímsson-Zeng [11])**.** $\Phi : \mathfrak{S}_M \to \mathfrak{S}_M$ *is a bijection satisfying*

$$(\mathrm{des}, \mathrm{MAK}, \mathrm{MAD})\, w = (\mathrm{exc}, \mathrm{DEN}, \mathrm{INV})\, \Phi(w)$$

for all $w \in \mathfrak{S}_M$.

The following proposition shows that Φ preserves the increasing tail permutation, it implies that the triple statistics (des, MAK, MAD) and (exc, DEN, INV) are equidistributed on $\mathcal{P}_M$.

Proposition 9.1. *Let $M = \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$ with $k_i \geq 1$ for all $i \in [m]$, the set $\mathcal{P}_M$ is invariant under Φ, that is,*

$$\Phi(\mathcal{P}_M) = \mathcal{P}_M.$$

Proof. Let $w = w_1 w_2 \ldots w_n \in \mathcal{P}_M$. For any given $s \in \{2, 3, \ldots, m\}$, assume that $w_p = s$ is the last occurrence of the letter s in w. Let $c = \sum_{j=1}^{s} k_j$. Let $\pi = \pi_1 \pi_2 \ldots \pi_n = \mathrm{std}(w)$, then $\pi_p = \sum_{j=1}^{s} k_j = c$. We claim that in permutation π', for any b with $b < c$, the letter b is on the left of the letter c. Note that our claim implies the proposition. Below we prove our claim. Because the tail permutation of w is $12 \ldots m$ and $w_p = s$ is the last occurrence of the letter s in w, we have $w_p < w_j$ for $j > p$. Then $c = \pi_p < \pi_j$ for $j > p$. It follows that in permutation π the embracing number of the letter c is 0 and that c is a non-descent top. By the definition of π', we see that $c \in g'$, and there is no letter in g' that is smaller than c and that is on the right of c. If $b \in g'$, it must be on the left of c in g' since $b < c$. In this case we see that b is on the left of c in π'. If $b \in f'$, then it is an excedance top in π'. Assume that $\pi'_j = b$, then $j < b$. Assume that $\pi'_k = c$, since $c \in g'$ is not an excedance top, we have $c \leq k$. Therefore, $j < b < c \leq k$. So in this case we also have that $b = \pi'_j$ is on the left of $c = \pi'_k$ in π' as $j < k$. Then our claim is true and we complete the proof. ∎

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