# THE ZARISKI-LIPMAN CONJECTURE FOR TORIC VARIETIES

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ABSTRACT. We give a short proof of the Zariski–Lipman conjecture for toric varieties : any complex toric variety with locally free tangent sheaf is smooth.

#### 1. INTRODUCTION

The Zariski–Lipman conjecture states that an algebraic variety X over  $\mathbb{C}$  with locally free tangent sheaf is necessarily smooth. This conjecture has been proved under various additional assumptions, see for example [8] in the N-graded case, [9] for complete intersections, [4] for log canonical varieties and [1,2,5] for surfaces. In this note, we give a direct and simple proof for toric varieties, that is for irreducible varieties endowed with an algebraic effective action of a complex torus with a dense open orbit. Recall that the tangent sheaf  $\mathcal{T}_X$  of an algebraic variety X is the dual of its sheaf of Kähler differentials, that is  $\mathcal{T}_X = \operatorname{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ .

**Theorem 1.1.** Let X be a toric variety over  $\mathbb{C}$ . Assume that the tangent sheaf of X is locally free. Then X is smooth.

As we will see in Remark 1.2 and Remark 1.3, Theorem 1.1 can be obtained as a corollary of the results from [6] or from [11]. Nevertheless, the proof presented here is fairly simple and relies purely on methods from toric geometry.

Remark 1.2. If (X, 0) is a germ of a rational singularity over  $\mathbb{C}$ , then (X, 0) is Cohen-Macaulay. If moreover (X, 0) has a locally free tangent sheaf, its canonical divisor is Cartier, and thus (X, 0) is Gorenstein. As noticed in [5], rational Gorenstein singularities are canonical, a class of singularities for which the Zariski-Lipman conjecture has been proved in [4, 6]. Normal toric singularities being rational [3, Theorem 11.4.2], Theorem 1.1 is a corollary of the results in [6].

*Remark* 1.3. Locally, a normal toric variety with no torus factor is a categorical quotient of a smooth affine toric variety by a reductive group action [3, Theorem 5.1.11]. From [11, Corollary 5.11], this implies that the Nakai conjecture holds true for normal toric varieties. As the Nakai conjecture implies the Zariski-Lipman conjecture [14, Proposition 2], Theorem 1.1 follows from [11]. The author would like to thank Thierry Levasseur for pointing to him this reference.

*Remark* 1.4. The Zariski–Lipman conjecture is actually stated over any field of characteristic zero. As noticed in [8], with no loss of generality one may assume the field to be algebraically closed. In this note, we will work over the complex numbers, but Theorem 1.1 and its proof hold over any algebraically closed field of characteristic zero.

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### 2. Klyachko's description of toric reflexive sheaves

Let X be a toric variety of dimension n over  $\mathbb{C}$ , with torus  $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ , for N the rank n lattice of its one-parameter subgroups. Denote by  $M = \operatorname{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$  its character lattice. Let  $\mathcal{T}_X = \operatorname{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$  be the tangent sheaf of X, that is the dual of its sheaf of Kähler differentials (see e.g. [3, Section 8.0]). According to [12, Theorem 3], if  $\mathcal{T}_X$  is locally free, then X is normal, which we will assume from now on. Then, X is the toric variety associated to a fan  $\Sigma$  of strongly convex rational polyhedral cones in  $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$  [3, Chapter 3]. In particular, X is covered by the  $T_N$ -invariant affine varieties  $U_{\sigma} = \operatorname{Spec}(\mathbb{C}[M \cap \sigma^{\vee}])$ , for  $\sigma \in \Sigma$ .

Recall that a coherent sheaf  $\mathcal{F}$  on X is called  $T_N$ -equivariant if there is an isomorphism  $\varphi : \alpha^* \mathcal{F} \to \pi_2^* \mathcal{F}$  satisfying some cocycle condition (see e.g. [13, Section 5] or [7]) where  $\alpha : T_N \times X \to X$ ,  $\pi_1 : T_N \times X \to T_N$  and  $\pi_2 : T_N \times X \to X$  stand for respectively the  $T_N$ -action, the projection on  $T_N$  and the projection on X. The  $T_N$ -action on X induces naturally an equivariant structure on its sheaf of Kähler differentials given by the composition :

$$\alpha^*\Omega^1_X \xrightarrow{d\alpha} \Omega^1_{T_N \times X} \xrightarrow{\simeq} \pi_1^*\Omega^1_{T_N} \oplus \pi_2^*\Omega^1_X \xrightarrow{\operatorname{pr}_2} \pi_2^*\Omega^1_X$$

where  $pr_2$  is the projection on the second factor [7, Section 2]. By duality,  $\mathcal{T}_X$  is  $T_N$ -equivariant, and, being dual to the coherent sheaf  $\Omega^1_X$ , it is reflexive. Klyachko proved that equivariant reflexive sheaves on toric varieties are described by families of filtrations [10] (see also [13]). Let us recall briefly this description for the tangent sheaf. As  $\mathcal{T}_X$  is reflexive, its sections extend over codimension 2 subvarieties. Denote by  $\Sigma(l)$  the set of *l*-dimensional cones in  $\Sigma$ , and consider

$$X_0 = \bigcup_{\sigma \in \Sigma(0) \cup \Sigma(1)} U_{\sigma}.$$

By the orbit-cone correspondence [3, Section 3.2],  $X_0$  is the complement of  $T_N$ -orbits of co-dimension greater or equal to 2, and thus  $\mathcal{T}_X = \iota_* \mathcal{T}_{X_0}$ , where  $\iota : X_0 \to X$  is the inclusion. By equivariance,  $\mathcal{T}_{X_0}$  is entirely characterised by the sections  $\Gamma(U_\sigma, \mathcal{T}_X)$ , for  $\sigma \in \Sigma(0) \cup \Sigma(1)$ . If  $\sigma = \{0\}$ ,  $U_{\{0\}} = T_N$ , and

$$\Gamma(U_{\{0\}}, \mathfrak{T}_X) = N_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}[M],$$

for  $N_{\mathbb{C}} := N \otimes_{\mathbb{Z}} \mathbb{C}$  is the Lie algebra of  $T_N$ . Then, if  $\rho \in \Sigma$  is a ray (i.e. a onedimensional cone),  $\Gamma(U_{\rho}, \mathfrak{T}_X)$  is graded by  $M/(M \cap \rho^{\perp}) \simeq \mathbb{Z}$ . As  $T_N$  is a dense open subset of  $U_{\rho}$ , the restriction map  $\Gamma(U_{\rho}, \mathfrak{T}_X) \to \Gamma(U_{\{0\}}, \mathfrak{T}_X)$  is injective and induces a decreasing  $\mathbb{Z}$ -filtration

$$\ldots \subset E^{\rho}(i) \subset E^{\rho}(i-1) \subset \ldots \subset N_{\mathbb{C}}$$

such that one has

$$\Gamma(U_{\rho}, \mathfrak{T}_X) = \bigoplus_{m \in M} E^{\rho}(-\langle m, u_{\rho} \rangle) \otimes \chi^m,$$

where we denote by  $u_{\rho}$  the primitive generator of  $\rho$  and  $\langle \cdot, \cdot \rangle$  the duality pairing. Explicitly, the family of filtrations  $(E^{\rho}(\bullet))_{\rho \in \Sigma(1)}$  for  $\mathcal{T}_X$  is given by [10, Example 2.3(5) on page 350]:

$$E^{\rho}(i) = \begin{cases} N_{\mathbb{C}} & \text{if} \quad i \leq 0\\ \mathbb{C} \cdot u_{\rho} & \text{if} \quad i = 1\\ \{0\} & \text{if} \quad i \geq 2. \end{cases}$$

Finally, by Klyachko's compatibility condition [10, Theorem 2.2.1] (see [13, Section 5] for a detailed treatment on normal toric varieties),  $\mathcal{T}_X$  is locally free if and only if the family of filtrations  $(E^{\rho}(\bullet))_{\rho \in \Sigma(1)}$  satisfies that for each  $\sigma \in \Sigma$ , there exists a decomposition

$$N_{\mathbb{C}} = \bigoplus_{[m] \in M/(M \cap \sigma^{\perp})} E^{\sigma}_{[m]}$$

such that for each ray  $\rho \subset \sigma$ :

$$E^{\rho}(i) = \bigoplus_{\langle m, u_{\rho} \rangle \ge i} E^{\sigma}_{[m]}.$$

## 3. Proof of Theorem 1.1

Assume from now on that  $\mathcal{T}_X$  is locally free. We want to show that X is smooth. As this is a local condition, we might as well assume X affine, so that  $X = U_{\sigma}$  for some strongly convex rational polyhedral cone  $\sigma \subset N_{\mathbb{R}}$ . With no loss of generality, we can also assume that X has no torus factor, so that  $\{u_{\rho}, \rho \in \sigma(1)\}$  spans  $N_{\mathbb{R}}$  [3, Proposition 3.3.9]. Then, X is smooth if and only if  $\sigma$  is smooth [3, Theorem 1.3.12], which, by definition, is equivalent to the fact that  $\{u_{\rho}, \rho \in \sigma(1)\}$  is a Z-basis for N.

As  $\{u_{\rho}, \rho \in \sigma(1)\}$  spans  $N_{\mathbb{R}}, \sigma^{\perp} = \{0\}$ . Then by Klyachko's compatibility condition, we can find a decomposition

$$N_{\mathbb{C}} = \bigoplus_{m \in M} E_m^{\sigma}$$

with, for  $\rho \in \sigma(1)$ , and  $i \in \{2, 1, 0\}$ ,

(1) 
$$E^{\rho}(i) = \bigoplus_{\langle m, u_{\rho} \rangle \ge i} E_{m}^{\sigma}$$

First, from  $E^{\rho}(2) = \{0\}$ , we deduce that  $E_m^{\sigma} \neq \{0\}$  only if for all  $\rho$ ,  $\langle m, u_{\rho} \rangle \leq 1$ . Secondly, taking i = 1 in (1),

$$\mathbb{C} \cdot u_{\rho} = \bigoplus_{\langle m, u_{\rho} \rangle \ge 1} E_m^{\sigma}.$$

Thus, for each  $\rho \in \sigma(1)$  we can find  $m_{\rho} \in M$  with  $\langle m_{\rho}, u_{\rho} \rangle = 1$  and such that  $\mathbb{C} \cdot u_{\rho} = E^{\sigma}_{m_{\rho}}$ . Note that  $\sigma$  contains no line by strong convexity, so if  $\rho \neq \rho'$ , then  $u_{\rho} \notin \mathbb{Z} \cdot u_{\rho'}$  and  $\mathbb{C} \cdot u_{\rho} \neq \mathbb{C} \cdot u_{\rho'}$ . As  $\{u_{\rho}, \rho \in \sigma(1)\}$  spans  $N_{\mathbb{C}}$  over  $\mathbb{C}$ ,

$$N_{\mathbb{C}} = \sum_{\rho \in \sigma(1)} \mathbb{C} \cdot u_{\rho} = \bigoplus_{\rho \in \sigma(1)} E^{\sigma}_{m_{\rho}} \subset \bigoplus_{m \in M} E^{\sigma}_{m} = N_{\mathbb{C}}.$$

Then,

$$N_{\mathbb{C}} = \bigoplus_{\rho \in \sigma(1)} E^{\sigma}_{m_{\rho}}$$

and  $\sigma(1)$  contains exactly *n* elements. Let  $\rho, \rho' \in \sigma(1)$  be two distinct rays. Necessarily.

$$\mathbb{C} \cdot u_{\rho} \cap \mathbb{C} \cdot u_{\rho'} = \{0\}$$

and by (1) with i = 1,  $m_{\rho}$  must satisfy  $\langle m_{\rho}, u_{\rho'} \rangle \leq 0$ .

Last, taking now i = 0 in (1), we deduce from  $E^{\rho}(0) = N_{\mathbb{C}}$  that  $\langle m_{\rho}, u_{\rho'} \rangle = 0$ . To conclude, for all  $\rho, \rho' \in \sigma(1)$ ,

$$\langle m_{\rho}, u_{\rho'} \rangle = \begin{cases} 1 & \text{if } \rho = \rho' \\ 0 & \text{if } \rho \neq \rho' \end{cases}$$

Hence, each element  $u \in N$  can be uniquely written

$$u = \sum_{\rho \in \sigma(1)} \langle m_{\rho}, u \rangle \, u_{\rho}$$

with  $\langle m_{\rho}, u \rangle \in \mathbb{Z}$  for each  $\rho \in \sigma(1)$ . Thus,  $\{u_{\rho}, \rho \in \sigma(1)\}$  is a basis of N, which ends the proof of Theorem 1.1.

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