RINGS GENERATED BY IDEMPOTENTS AND NILPOTENTS

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Abstract. We present new characterizations of the rings in which every element is the sum of two idempotents and a nilpotent that commute, and the rings in which every element is the sum of two tripotents and a nilpotent that commute. We prove that such rings are completely determined by the additive decompositions of their square elements. These improve the results of Chen and Sheibani[J. Algebra Appl., 16, 1750178(2017)] and Zhou [J. Algebra Appl., 16, 1850009(2017)].

1. Introduction

A ring R is strongly 2-nil-clean ring if every element in R is the sum of two idempotents and a nilpotent that commute. An element $p \in R$ is tripotent if $p^3 = p$. A ring R is Zhou nil-clean ring if every element in R is the sum of two tripotents and a nilpotent that commute. Many elementary properties and structure theorems of such rings were investigated in [3, 5, 6, 7, 10, 12, 13]. In this paper we shall characterize preceding rings by means of the additive decomposition of their square elements. These improve the known results, e.g., [2, Theorem 16], [5, Theorem 2.8] and [13, Theorem 2.11].

In Section 2, we prove that a ring R is strongly 2-nil-clean if and only every square element in R is the sum of two idempotents and a nilpotent that commute if and only if every square element in R is the sum of an idempotent, an involution and a nilpotent that commute.

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In Section 3, we further prove that a ring R is Zhou nil-clean if and only if every square element in R is the sum of two tripotents and a nilpotent that commute if and only if every square element in R is the sum of a tripotent, an involution and a nilpotent that commute.

Throughout, all rings are associative with an identity. We use $N(R)$ to denote the set of all nilpotents in R. $a \in R$ is square if $a = x^2$ for some $x \in R$. $a \equiv b(mod\ N(R))$ means that $a-b \in N(R)$.

2. strongly 2-nil-clean rings

In this section, we establish new characterizations of strongly 2-nil-clean rings by means of their square elements. We begin with

Lemma 2.1. *The following are equivalent for a ring R:*

1. *R is strongly 2-nil-clean.*
2. *For any $a \in R$, $a - a^3 \in N(R)$.*
3. *Every element in R is the sum of a tripotent and a nilpotent that commute.*

Proof. See [5, Theorem 2.3 and Theorem 2.8]. □

Theorem 2.2. *The following are equivalent for a ring R:*

1. *R is strongly 2-nil-clean.*
2. *Every square element in R is the sum of an idempotent and a nilpotent that commute.*
3. *Every square element in R is the sum of two idempotents and a nilpotent that commute.*
4. *Every square element in R is the sum of three idempotents and a nilpotent that commute.*

Proof. (1) $\Rightarrow$ (2) For any $a \in R$, it follows by Lemma 2.1 that $a - a^3 \in N(R)$. Hence $a^2 - (a^2)^2 = a(a - a^3) \in N(R)$. In view of [12, Lemma 3.5], a^2 is the sum of an idempotent and a nilpotent that commute.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) These are trivial.

(4) $\Rightarrow$ (1) Step 1. By hypothesis, there exist idempotents $e, f, g \in R$ and a nilpotent $w \in R$ that commute such that $2^2 = e+f+g+w$.

Hence $4e = e + ef + eg + ew$, and so $3e = ef + eg + ew$. This implies that $3ef = ef + efg + efw$; whence, $2ef = efg + efw$. Therefore

$$2ef = (2ef)^2 - 2ef = (efg + efw)^2 - (efg + efw) \in N(R).$$

Likewise, we have $2eg, 2fg \in N(R)$. Accordingly,

$$\begin{aligned} 12 &= 4^2 - 4 \\ &= (e + f + g + w)^2 - (e + f + g + w) \\ &\equiv 2(ef + eg + fg)(mod\ N(R)), \end{aligned}$$

i.e., $6^2 = 3 \times 12 \in N(R)$. Hence $6 \in N(R)$. Write $2^n \times 3^n = 0$ for some $n \in \mathbb{N}$. Since $2^nR \bigcap 3^nR = 0$, we have $R \cong R_1 \times R_2$, where $R_1 = R/2^nR, R_2 = R/3^nR$.

Step 2. Let $a \in R$. Then there exist idempotents $e, f, g \in R$ and a nilpotent $w \in R$ that commute such that $(a+1)^2 = e + f + g + v$, and so $a^2 + 2a = e - (1 - f) + g + w = e + h - k + w$, where $h = fg, k = (1 - f)(1 - g)$. We easily check that $h^2 = h, k^2 = k$ and $hk = kh = 0$. Hence, we have

$$\begin{aligned} (a^2 + 2a)^3 &\equiv (e + h + k + 2eh - 2ek)(e + h - k) \\ &\equiv e + h - k + 6eh \\ &\equiv a^2 + 2a(mod\ N(R)). \end{aligned}$$

Therefore $(a^2 + 2a)^3 - (a^2 + 2a) \in N(R)$. Likewise, $(a - 1)^2$ is the sum of three idempotents and a nilpotent that commute. As the preceding discussion, we have $(a^2 - 2a)^3 - (a^2 - 2a) \in N(R)$.

Case 1. $a \in R_1$. Then $2 \in N(R_1)$. Hence, $(a^3 - a)(a^3 + a) = a^6 - a^2 \in N(R_1)$. This implies that $(a^3 - a)^2 \in N(R_1)$, i.e., $a - a^3 \in N(R_1)$.

Case 2. $a \in R_2$. Since $3 \in N(R_2)$, we get

$$\begin{aligned} (a^2 - a)^3 - (a^2 - a) \in N(R_2), \\ (a^2 + a)^3 - (a^2 + a) \in N(R_2). \end{aligned}$$

Moreover, we have

$$\begin{aligned} (a^6 - a^3) - (a^2 - a) \in N(R_2), \\ (a^6 + a^3) - (a^2 + a) \in N(R_2). \end{aligned}$$

Thus, $2a^3 - 2a \in N(R_2)$. Clearly, $2 \in R_2^{-1}$, and then $a - a^3 \in N(R_2)$.

Therefore for any $a \in R$, we have $a - a^3 \in N(R)$. In light of Lemma 2.1, R_2 is strongly 2-nil-clean. □

We note that "three idempotents" in the proceeding theorem can not be replaced by "four idempotents" as the following shows.

Example 2.3. *Let $R = \mathbb{Z}_5$. Then every square element in R is the sum of four idempotents and a nilpotent that commute. But R it is not strongly 2-nil-clean.*

Proof. Obviously, $\{x^2 \mid x \in R\} = \{0, 1, 4\}$. For any $a \in R$, we see that a^2 is the sum of four idempotents that commute. As $2 \neq 2^3$, R is not strongly 2-nil-clean. $\square$

An element $v \in R$ is an involution if $v^2 = 1$. As a consequence of Theorem 2.2, we now derive

Theorem 2.4. *The following are equivalent for a ring R:*

1. *R is strongly 2-nil-clean.*
2. *Every square element in R is the sum of an idempotent, an involution and a nilpotent that commute.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R$. In view of Theorem 2.2, there exist an idempotent $e \in R$ and a nilpotent $w \in R$ that commute such that $a^2 = e + w$. Hence $a^2 = (1 - e) + (2e - 1) + w$ with $(1 - e)^2 = 1$ and $(2e - 1)^2 = 1$. That is, a^2 is the sum of an idempotent, an involution and a nilpotent that commute.

(2) $\Rightarrow$ (1) Write $2^2 = e + v + w$, where $e^2 = e, v^2 = 1$ and $w \in N(R)$ that commute. Then

$$16 \equiv e + 2ev + 1 \equiv (4 - v) + 2(4 - v)v + 1 \equiv 3 + 5v(mod\ N(R)).$$

This implies that $13 \equiv 5v(mod\ N(R))$. Hence $169 \equiv 25v^2 = 25(mod\, N(R))$, and then $2^4 \times 3^2 \in N(R)$. That is, $2 \times 3 = 6 \in N(R)$. Write $2^n \times 3^n = 0$ for some $n \in \mathbb{N}$. Then we have $R \cong R_1 \times R_2$, where $R_1 = R/2^n R$ and $R_2 = R/3^n R$.

Step 1. Let $a \in R_1$. Then there exist $e, v, w \in R_1$ such that $a^2 = e + v + w$, $e^2 = e, v^2 = 1$ and $w \in N(R_1)$ that commute. Clearly, $2 \in N(R_1)$. Then we have

$$\begin{aligned} a^8 &\equiv (e + v)^4 \equiv e + 1(\text{ mod } N(R_1)), \\ a^{12} &= a^8 a^4 \equiv (e + 1)(e + 1) \equiv e + 1(\text{ mod } N(R_1)). \end{aligned}$$

Hence $a^8(1 - a^2)(1 + a^2) = a^8(1 - a^4) = a^8 - a^{12} \in N(R_1)$, and so $a^6(a - a^3)^2 = a^8(1 - a^2)^2 \in N(R_1)$. This implies that $(a - a^3)^8 =$

$a^6(a-a^3)^2(1-a^2)^6 \in N(R_1)$, i.e., $a-a^3 \in N(R_1)$. According to Lemma 2.1, R_1 is strongly 2-nil-clean.

Step 2. Let $a \in R_2$. Then there exist $e, v, w \in R_2$ such that $a^2 = e+v+w$, $e^2 = e, v^2 = 1$ and $w \in N(R_2)$ that commute. Since $3 \in N(R_2)$, we have $a^6 \equiv (e+v)^3 \equiv e+v \equiv a^2 (mod\ N(R_2))$. Hence $a^2 - a^6 \in N(R_2)$. Clearly, $2 \in R_2^{-1}$. Set $p = \frac{a^4+a^2}{2}, q = \frac{a^4-a^2}{2}$. We compute that

$$p^2 - p = \frac{1}{4}(a^8 + 2a^6 - a^4 - 2a^2) = \frac{1}{4}(a^2+2)(a^6-a^2),$$
$$q^2 - q = \frac{1}{4}(a^8 - 2a^6 - a^4 + 2a^2) = \frac{1}{4}(a^2-2)(a^6-a^2).$$

Hence $p^2 - p, q^2 - q \in N(R_2)$. In light of [12, Lemma 3.5], we can find two idempotents $g, h \in \mathbb{Z}[a]$ such that $p-g, q-h \in N(R_2)$. Hence $a^2 = g - h + w$ for some $w \in \mathbb{Z}[a]$. As $3 \in N(R_2)$, we see that $a^2 = g + h + h + (w - 3h)$ with $w - 3h \in N(R_2)$. Therefore a^2 is the sum of three idempotents and a nilpotent that commute. According to Theorem 2.2, R_2 is strongly 2-nil-clean.

Therefore $R \cong R_1 \times R_2$ is strongly 2-nil-clean, as asserted. □

3. Zhou nil-clean rings

The aim of this section is to further characterize Zhou nil-clean rings by means of the additive decompositions of their square elements. An element $p \in R$ is 5-potent if $p^5 = p$. We have

Lemma 3.1. *The following are equivalent are equivalent for a ring R:*

1. *R is Zhou nil-clean.*
2. *For any $a \in R$, $a - a^5 \in N(R)$.*
3. *Every element in R is the sum of a 5-potent and a nilpotent that commute.*

Proof. See [2, Theorem 19] and [13, Theorem 2.11]. □

Lemma 3.2. *A ring R is Zhou nil-clean if and only if $7 \in R^{-1}$ and every square element in R is the sum of four idempotents and a nilpotent that commute.*

Proof. $\Longrightarrow$ In view of Lemma 3.1, $30 = 2^5 - 2 \in N(R)$. Since $(7, 30) = 1$, we can find some $k, l \in \mathbb{N}$ such that $7k + 30l = 1$, and so $7 \in R^{-1}$. By virtue of [13, Theorem 2.11], $R \cong A \times B \times C$, where $A = 0$ or $A/J(A)$ is Boolean with $J(A)$ nil, $B = 0$ or $B/J(B)$ is a subdirect product of $\mathbb{Z}_3$'s with $J(B)$ nil; $C = 0$ or $C/J(C)$ is a subdirect product of $\mathbb{Z}_5$'s with $J(C)$ nil. In light of [2, Theorem 15], every element in R is the sum of four idempotents and a nilpotent that commute.

$\Longleftarrow$ Step 1. By hypothesis, there exists idempotents $e, f, g, h \in R$ and a nilpotent $w \in R$ that commute such that $3^2 = e+f+g+h+w$. Hence $7 = \alpha + \beta + w$, where $\alpha = e - (1 - f), \beta = g - (1 - h)$. Since $e, 1-f, g, 1-h$ are idempotents, we easily check that $\alpha^3 = \alpha, \beta^3 = \beta$ and $\alpha\beta = \beta\alpha$. Then $48 \times 7 = 7^3 - 7 \equiv 3\alpha\beta(\alpha + \beta)(mod\ N(R))$, hence, $48(\alpha+\beta) \equiv 3\alpha\beta(\alpha+\beta)(mod\ N(R))$. Multiplying both sides by $\alpha\beta$ gives

$$48\alpha\beta(\alpha + \beta) \equiv 3\alpha\beta(\alpha + \beta)\alpha\beta \equiv 3\alpha\beta(\alpha + \beta)(mod\ N(R)),$$

and so

$$\begin{aligned} 48 \times 7 \times 15 &\equiv 15[3\alpha\beta(\alpha + \beta)] \\ &= 45\alpha\beta(\alpha + \beta) \\ &\equiv 0(mod\ N(R)). \end{aligned}$$

Hence, $2^4 \times 3^2 \times 5 \times 7 \in N(R)$. It follows from $7 \in R^{-1}$ that $2^4 \times 3^2 \times 5 \in N(R)$, and so $2 \times 3 \times 5 \in N(R)$. Write $2^n \times 3^n \times 5^n = 0$ for some $n \in \mathbb{N}$. Then $R \cong R_1 \times R_2 \times R_3$, where $R_1 = R/2^nR, R_2 = R/3^nR, R_3 = R/5^nR$.

Step 2. By hypothesis, there exist idempotents $e, f, g, h \in R$ and a nilpotent $w \in R$ that commute such that $(a+2)^2 = e+f+g+h+w$, and so $a^2+4a+2 = e-(1-f)+g-(1-h)+w$. Set $p = e-(1-f)$ and $q = g-(1-h)$. Then $a^2+4a+2 = p+q(mod\ N(R)), p^3 = p, q^3 = q$ and $pq = qp$.

Case 1. $a \in R_1$. Then $2 \in N(R_1)$. We have

$$\begin{aligned} a^8 &\equiv (p + q)^4 \\ &\equiv p^4 + q^4 \\ &\equiv p^2 + q^2 \\ &\equiv a^4(mod\ N(R_1)), \end{aligned}$$

Hence $a^3(a - a^5) \in N(R_1)$, and so $(a - a^5)^4 = a^3(a - a^5)(1 - a^4)^3 \in N(R_1)$. Therefore $a - a^5 \in N(R_1)$.

Case 2. $a \in R_2$. Then $3 \in N(R_2)$. We have

$$\begin{aligned}(a^2+4a+2)^3 &\equiv (p+q)^3 \\ &\equiv p^3+q^3 \\ &\equiv p+q \\ &\equiv a^2+4a+2 (mod\ N(R_2)),\end{aligned}$$

Likewise, we have $(a^2-4a+2)^3-(a^2-4a+2) \in N(R_2)$. So we get

$$\begin{aligned}(a^2+2)^3+(4a)^3-(a^2+4a+2) \in N(R_2),\\ (a^2+2)^3-(4a)^3-(a^2-4a+2) \in N(R_2).\end{aligned}$$

Hence $2(4a)^3-8a \in N(R_2)$, and so $2^3(16a^3-a) \equiv 2^3(a^3-a) \in N(R_2)$. Accordingly, $a^3-a \in N(R_2)$, and so $a-a^5=(1+a^2)(a-a^3) \in N(R_2)$.

Case 3. $a \in R_3$. Then $5 \in N(R_3)$. We have

$$\begin{aligned}(a^2+4a+2)^5 &\equiv (p+q)^5 \\ &\equiv p^5+q^5 \\ &\equiv p+q \\ &\equiv a^2+4a+2 (mod\ N(R_3)),\end{aligned}$$

Likewise, we get $(a^2-4a+2)^5-(a^2-4a+2) \in N(R_3)$. Then

$$\begin{aligned}(a^2+2)^5+(4a)^5-(a^2+4a+2) \in N(R_3),\\ (a^2+2)^5-(4a)^5-(a^2-4a+2) \in N(R_3).\end{aligned}$$

Hence $2(4a)^5-8a \in N(R_3)$, and so $2^3(256a^5-a)=2^3(a^5-a) \in N(R)$. Accordingly, $a-a^5 \in N(R_3)$.

Therefore $a-a^5 \in N(R)$ for all $a \in R$. This completes the proof by Lemma 3.1. □

We are now ready to prove the following.

Theorem 3.3. *The following are equivalent for a ring R:*

(1) *R is Zhou nil-clean.*
(2) *Every square element in R is the sum of a tripotent and a nilpotent that commute.*
(3) *Every square element in R is the sum of two tripotents and a nilpotent that commute.*

Proof. (1) ⇒ (2) In view of [13, Theorem 2.11], $R \cong R_1 \times R_2$, where R_1 is strongly 2-nil-clean with $2 \in N(R_1)$ and R_2 is Zhou nil-clean with $3 \times 5 \in N(R_2)$.

Let $a \in R_1$. Then a^2 is the sum of an idempotent and a nilpotent that commute.

Let $a \in R_2$. In view of [13, Theorem 2.11], $a - a^5 \in N(R_2)$. Hence, $a^2 - (a^2)^3 = a(a - a^5) \in N(R_2)$. Since $2 \in R_2^{-1}$, it follows by [13, Lemma 2.6] that there exists a tripotent $p \in \mathbb{Z}[a]$ such that $a^2 - p \in N(R_2)$.

Therefore every square element in R is the sum of a tripotent and a nilpotent that commute, as required.

(2) ⇒ (3) This is trivial.

(3) ⇒ (1) By hypothesis, there exist tripotents $e, f \in R$ a nilpotent $w \in R$ that commute such that $2^2 = e + f + w$. We check that

$$15(e + f) \equiv 4(4^2 - 1) = 4^3 - 4 \equiv 3ef(e + f)(mod\ N(R)).$$

Multiplying both sides by ef gives

$$15ef(e + f) \equiv 3ef(e + f)(mod\ N(R)).$$

This implies that

$$2^4 \times 3 \times 5 = 4 \times (4^3 - 4) \equiv 12ef(e + f) \equiv 0(mod\ N(R)).$$

It follows that $2 \times 3 \times 5 \in N(R)$. Write $2^n \times 3^n \times 5^n = 0$ for some $n \in \mathbb{N}$. Then

$$R \cong R_1 \times R_2 \times R_3, R_1 = R/2^nR, R_2 = R/3^nR, R_3 = R/5^nR.$$

Case 1. $a \in R_1$. Then $2 \in N(R_1)$. By hypothesis, there exist tripotents $p, q \in R_1$ and a nilpotent $w \in R_1$ that commute such that $a^2 = p + q + w$. Hence we have

$$\begin{aligned} (a^2)^4 &\equiv (p + q)^4 \\ &\equiv p^4 + q^4 \\ &\equiv p^2 + q^2 \\ &\equiv a^4(mod\ N(R_2)), \end{aligned}$$

Hence $a^3(a-a^5) \in N(R_1)$; whence, $(a-a^5)^4 = a^3(a-a^5)(1-a^4)^3 \in N(R_1)$. This implies that $a - a^5 \in N(R_1)$.

Case 2. $a \in R_2$. Then $3 \in N(R_2)$. By hypothesis, there exist tripotents $p, q \in R_2$ and a nilpotent $w \in R_2$ that commute such that $a^2 = p + q + w$. Hence we have

$$\begin{aligned} (a^2)^3 &\equiv (p+q)^3 \\ &\equiv p^3 + q^3 \\ &\equiv p + q \\ &\equiv a^2 (mod\ N(R_2)), \end{aligned}$$

This implies that $a(a - a^5) \in N(R_2)$, and so $(a - a^5)^2 = a(a - a^5)(1 - a^4) \in N(R_2)$. Accordingly, $a - a^5 \in N(R_2)$.

Case 3. $a \in R_3$. Then $5 \in N(R_3)$. By hypothesis, there exist tripotents $p, q \in R_3$ and a nilpotent $w \in R_3$ that commute such that $(a+2)^2 = p + q + w$. Hence we get

$$\begin{aligned} (a^2 + 4a + 4)^5 &\equiv (p+q)^5 \\ &\equiv p^5 + q^5 \\ &\equiv p + q \\ &\equiv a^2 + 4a + 4 (mod\ N(R_3)), \end{aligned}$$

Likewise, we have $(a^2 - 4a + 4)^5 \equiv a^2 - 4a + 4(mod\ N(R_3))$. We derive that

$$(a^2 - a - 1)^5 - (a^2 - a - 1) \in N(R_3),$$
$$(a^2 + a - 1)^5 - (a^2 + a - 1) \in N(R_3).$$

This implies that

$$(a^2 - 1)^5 - a^5 - (a^2 - 1 - a) \in N(R_3),$$
$$(a^2 - 1)^5 + a^5 - (a^2 - 1 + a) \in N(R_3).$$

Thus we have $2a^5 - 2a \in N(R_3)$. Clearly, $2 \in R_3^{-1}$, and therefore $a - a^5 \in N(R_3)$.

According to Lemma 3.1, $R \cong R_1 \times R_2 \times R_3$ is Zhou nil-clean. □

As a consequence, we now derive:

Corollary 3.4. *The following are equivalent for a ring R:*

(1) *R is Zhou nil-clean.*
(2) *Every square element in R is the sum of a tripotent, an involution and a nilpotent that commute.*

Proof. (1) ⇒ (2) Clearly, $2 \times 15 = 30 \in N(R)$. Then we can find some $n \in \mathbb{N}$ such that

$$R \cong R_1 \times R_2, R_1 = R/2^n R, R_2 = R/15^n R.$$

Case 1. Let $a \in R_1$. Clearly, R_1 is strongly 2-nil-clean. By virtue of Theorem 2.4, a^2 is the sum of an idempotent, an involution u and a nilpotent that commute.

Case 2. Let $a \in R_2$. In view of Theorem 3.3, there exist a tripotent $p \in R$ and a nilpotent $w \in R$ that commute such that $a^2 = p + w$. Clearly, $2 \in R_2^{-1}$. Let $e = \frac{p^2+p}{2}$ and $f = \frac{p^2-p}{2}$. Then we directly verify that

$$e^2 - e = \frac{1}{4}(p^4 + 2p^3 - p^2 - 2p) = \frac{1}{4}(p+2)(p^3 - p).$$

Hence $e^2 = e$. Likewise, $f^2 = f$. Therefore $a^2 = e - f + w = (1-e) - f + (2e-1) + w$. We check that $[(1-e)-f]^3 = (1-e) - f$ and $(2e-1)^2 = 1$, as desired.

(2) ⇒ (1) Since every involution is a tripotent, we complete the proof by Theorem 3.3. □

Example 3.5. *Let $R = \{a, b, c, d\}$ be the ring defined by the following operations:*

$+$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

$\times$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	d	b
d	a	d	b	c

Then x^3 is an idempotent for all $x \in R$, but R is not Zhou nil-clean.

Proof. For all $x \in R$, we check that $x = x^4$, and so $(x^3)^2 = x^3$. That is, $x^3 \in R$ is an idempotent. Since $c - c^5 = b$ is not nilpotent, R is not Zhou nil-clean. □

Example 3.6. *Let $R = \{\begin{pmatrix} x & y \\ y & x+y \end{pmatrix} \mid x, y \in \mathbb{Z}_3\}$. Then R is a finite field with 9 elements. Let $a \in R$. Then $a = a^9$, and so $a^2 = (a^2)^5$, i.e., a^2 is 5-potent. Therefore every square element in R can be written as the sum of a 5-potent and a nilpotent that commute.*

Choose $c = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. *Then* $c - c^5 = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}^{-1}$. *Hence* $c - c^5$ *is not nilpotent. According to* [13, Theorem 2.11], R *is not Zhou nil-clean.*

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