

Heat coefficients for magnetic Laplacians on the complex projective space $\mathbf{P}^n(\mathbb{C})$

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Abstract

Denoting by Δ_ν the Fubini-Study Laplacian perturbed by a uniform magnetic field strength proportional to ν , this operator has a discrete spectrum consisting on eigenvalues β_m , $m \in \mathbb{Z}_+$, when acting on bounded functions of the complex projective n -space. For the corresponding eigenspaces, we give a new proof for their reproducing kernels by using Zarembo's expansion directly. These kernels are then used to obtain an integral representation for the heat kernel of Δ_ν . Using a suitable polynomial decomposition of the multiplicity of each β_m , we write down a trace formula for the heat operator associated with Δ_ν in terms of Jacobi's theta functions and their higher order derivatives. Doing so enables us to establish the asymptotics of this trace as $t \searrow 0^+$ by giving the corresponding heat coefficients in terms of Bernoulli numbers and polynomials. The obtained results can be exploited in the analysis of the spectral zeta function associated with Δ_ν .

1 Introduction

Let Δ denote the Laplace-Beltrami operator on a Riemannian manifold (\mathcal{M}, g) of dimension d , given by

$$\Delta = -\frac{1}{\sqrt{\det g}} \sum_{i,j} \partial_i \left(g^{ij} \sqrt{\det g} \partial_j \right). \quad (1.1)$$

The heat kernel is a function $H(x, y, t) \in \mathcal{C}^\infty(\mathcal{M} \times \mathcal{M} \times \mathbb{R}_+)$ that solves the problem

$$(\partial_t + \Delta)H = 0 \quad \text{and} \quad \lim_{t \rightarrow 0} \int_{\mathcal{M}} H(x, y, t) f(y) dy = f(x) \quad (1.2)$$

for any smooth function f of compact support. For \mathcal{M} compact, there exists a complete orthonormal basis $\{\phi_k\}$ in $L^2(\mathcal{M})$, consisting of eigenfunctions of Δ , associated with eigenvalues $0 \leq \lambda_0 \leq \lambda_1, \dots$, such that

$$H(x, y, t) = \sum_{k=0}^{+\infty} e^{-\lambda_k t} \phi_k(x) \phi_k(y) \quad (1.3)$$

with uniformly convergence for any fixed $t > 0$, see [44] for the general theory. The asymptotic expansion $H(x, y, t)$ had be studied by Minakshisudaran and Pleijel [1] and one has [45]:

$$H(x, x, t) \sim \frac{1}{(2\sqrt{\pi t})^d} (1 + a_1(x)t + a_2(x)t^2 + \dots), \quad t \searrow 0^+ \quad (1.4)$$

which, after being integrated with respect to the volume form, gives the trace formula

$$\sum_{k=0}^{+\infty} e^{-t\lambda_k} \sim \frac{1}{(2\sqrt{\pi t})^d} (a_0 + a_1 t + a_2 t^2 + \dots), \quad t \searrow 0^+. \quad (1.5)$$

The numbers a_k in (1.5) are called *heat coefficients* and may be expressed in terms of geometrical invariants of \mathcal{M} . Indeed, a_0 is the volume of \mathcal{M} , and for $d = 2$, a_1 is proportional to the Euler characteristic of \mathcal{M} . In general, $\{a_k\}$ depend on the curvature tensor R and its successive covariant derivatives meaning

that it is not easy to determine them. Indeed, few of these coefficients have been calculated for a general manifold [46] while in the case of compact symmetric spaces of rank one they can be found in [41].

For $\mathcal{M} = \mathbf{P}^n(\mathbb{C})$ the complex projective n -space which is a real manifold of dimension $2n$ representing the prototype of rank-one complex Riemannian symmetric space of compact-type, the computation of the heat coefficients was discussed in [12] through the spectral zeta functions and via the resolvent of the Laplacian operator in [6]. Other methods can be found in [7] and for further discussion on these coefficients we refer to [8].

In [13] the author has examined the role played by the coefficients $a_k^{(n)} \equiv a_k(\mathbf{P}^n(\mathbb{C}))$ in describing a new class of heat coefficients and then introduced the associated zeta function. He also proved that for the Fubini-Study Laplacian

$$\Delta_0 := 4(1 + \langle z, z \rangle) \sum_{i,j=1}^n (\delta_{ij} + z_i \bar{z}_j) \frac{\partial^2}{\partial z_i \partial \bar{z}_j}, \quad (1.6)$$

the expansion of the trace of the heat operator

$$\mathrm{Tr}(e^{-t\Delta_0}) \sim \frac{1}{(4\pi t)^n} \sum_{k=0}^{+\infty} a_k^{(n)} t^k, \quad t \searrow 0^+, \quad (1.7)$$

can be expressed purely in terms of Jacobi theta functions $\vartheta_2(t)$ and $\vartheta_3(t)$ and their higher order derivatives, depending on the cases n odd and n even respectively. A similar discussion involving these special functions already appeared in [36] where the authors have established an integral representation for the heat kernel

$$H_0(x, y, t) = \frac{e^{n^2 t}}{2^{n-2} \pi^{n+1}} \int_{\rho}^{\frac{\pi}{2}} \frac{-d(\cos u)}{\sqrt{\cos^2 \rho - \cos^2 u}} \left(-\frac{1}{\sin u} \frac{d}{du} \right)^n [\Theta_{n+1}(t, u)] \quad (1.8)$$

associated with Δ_0 , where $\rho = d_{FS}(x, y)$ denotes the Fubini-Study distance (given by (2.4) below) and

$$\Theta_{n+1}(t, u) := \sum_{j=0}^{+\infty} e^{-4t(j+\frac{n}{2})^2} \cos(2j+n)u.$$

They also suggested tackling the heat trace asymptotics problem by exploiting (1.8).

In the present paper, we deal with similar questions in the context of the complex projective n -space and for the following perturbed form of Δ_0 :

$$\Delta_\nu := 4(1 + \langle z, z \rangle) \left(\sum_{i,j=1}^n (\delta_{ij} + z_i \bar{z}_j) \frac{\partial^2}{\partial z_i \partial \bar{z}_j} - \nu \sum_{j=1}^n \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right) - \nu^2 \right) + 4\nu^2 \quad (1.9)$$

called magnetic Laplacian (with $2\nu \in \mathbb{Z}_+$) as introduced in [31], which can also be viewed as the Bochner Laplacian on powers of the Hopf line bundle and of its conjugate [3, 17]. When acting on the space of bounded functions, Δ_ν admits a discrete spectrum consisting on eigenvalues called spherical Landau levels (ν is proportional to the magnetic field strength). For the corresponding eigenspaces \mathcal{A}_m^ν , $m \in \mathbb{Z}_+$, the authors [53] have established formulae for the associated Berezin transforms as functions of Δ_0 .

For these spaces \mathcal{A}_m^ν , we present a new proof for their reproducing kernels using Zarembo's expansion directly. These reproducing kernels are then used to establish an integral representation for the heat kernel of Δ_ν , extending the one in (1.8) for $\nu \neq 0$. To obtain the heat coefficients in the asymptotic expansion of the trace of the heat operator $\exp(\frac{1}{4}t\Delta_\nu)$, we first use a suitable polynomial decomposition of dimensions of eigenspaces \mathcal{A}_m^ν to write down a trace formula for this operator. Next, we express this trace in terms of Jacobi's theta functions and their higher order derivatives, and then take the asymptotics of these special functions as $t \searrow 0^+$. Doing so enables us, after straightforward calculations, to find out precise formulae for the heat coefficients in terms of Bernoulli numbers and polynomials.

The paper is organized as follows. In Section 2, we recall the geometrical construction of magnetic Laplacians Δ_ν on the complex projective n -space and we illustrate this construction in the case $n = 1$ by explaining the role of Dirac monopoles. In Section 3, we summarize some notations about spherical

harmonics that help us recalling some results on eigenspaces of Δ_ν . For these spaces, we give in Section 4 a new proof for their reproducing kernels. In Section 5, we recall the Cauchy problem for the heat equation associated with Δ_ν . In Section 6 we establish an integral representation for the corresponding heat kernel. In Section 7, we write down an asymptotic expansion of the trace of the heat semigroup $\exp(\frac{1}{4}t\Delta_\nu)$ where the heat coefficients are obtained explicitly. In Section 8, we exhibit these coefficients for specific values of $n \in \{1, 2, 3, 4\}$. In appendix A, we list the definitions of some needed special functions and orthogonal polynomials.

2 Magnetic Laplacians Δ_ν

We here recall from [31] the construction of magnetic Laplacians Δ_ν on the complex projective n -space, $n \geq 1$. Let \mathbb{S}^{2n+1} be the $(2n+1)$ -dimensional unit sphere of \mathbb{C}^{n+1} . Then, the unit circle $U(1) \equiv \mathbb{S}^1$ acts freely on \mathbb{S}^{2n+1} and define the complex projective space by $\mathbf{P}^n(\mathbb{C}) = \mathbb{S}^1 \backslash \mathbb{S}^{2n+1}$ to be the set of all complex lines of \mathbb{C}^{n+1} . Indeed, the Hopf fibration $\mathbb{S}^1 \rightarrow \mathbb{S}^{2n+1} \rightarrow \mathbf{P}^n(\mathbb{C})$ defines a principal $U(1)$ -bundle on $\mathbf{P}^n(\mathbb{C})$ whose associated complex line is $\mathcal{L} = \{(l, z) \in \mathbf{P}^n(\mathbb{C}) \times \mathbb{C}^{n+1}, z \in l\}$. It is endowed with the Fubini-Study metric ds_{FS}^2 which reads in a standard charte

$$\mathbb{C}^n \equiv \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1}; z_{n+1} = 1\}$$

or local coordinates as

$$ds_{FS}^2 := \sum_{i,j=1}^n ((1 + \langle z, z \rangle) \delta_{ij} - z_i \bar{z}_j) dz_i \otimes d\bar{z}_j, \quad \langle z, z \rangle = |z|^2, \quad (2.1)$$

where $g_{ij}(z) = (1 + \langle z, z \rangle)^{-2} ((1 + \langle z, z \rangle) \delta_{ij} - z_i \bar{z}_j)$. The complex projective n -space equipped with this metric is a Kählerian compacte manifold of complex dimension n . The associated Laplace-Beltrami operator is given by

$$\sum_{i,j=1}^n g^{ij}(z) \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \quad (2.2)$$

where $(g^{ij}(z))$ is the inverse of the matrix $(g_{ij}(z))$. In local coordinates $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, it reads

$$\Delta_0 = 4(1 + \langle z, z \rangle) \sum_{i,j=1}^n (\delta_{ij} + z_i \bar{z}_j) \frac{\partial^2}{\partial z_i \partial \bar{z}_j}. \quad (2.3)$$

The Fubini-Study distance is defined by

$$\cos^2 d_{FS}(z, w) = \frac{|1 + \langle z, w \rangle|^2}{(1 + \langle z, z \rangle)(1 + \langle w, w \rangle)}. \quad (2.4)$$

Let $\nabla = d + \partial \log(1 + \langle z, z \rangle)$ be the unique hermitian connection associated with ds_{FS}^2 on \mathcal{L} . Now, for a positive integer ν let $\mathcal{L}^\nu := (\mathcal{L}^*)^{\otimes \nu} \otimes (\mathcal{L}^*)^{\otimes \nu}$ the complex line bundle on $\mathbf{P}^n(\mathbb{C})$. Then, the corresponding hermitian connection reads

$$\nabla_\nu = d + \nu (\partial - \bar{\partial}) \partial \log(1 + \langle z, z \rangle). \quad (2.5)$$

Next, consider the operator

$$\Delta_\nu := -(\nabla_\nu)^* \nabla_\nu \quad (2.6)$$

acting on the space of smooth sections $\Gamma_{n,\nu}^\infty := C^\infty(\mathbf{P}^n(\mathbb{C}), \mathcal{L}^\nu)$, which is also known as the Bochner Laplacian on hermitian line bundles parametrized by the magnetic field strength ν . Precisely, in the local coordinates the operator Δ_ν takes the form

$$\Delta_\nu = 4(1 + \langle z, z \rangle) \left(\sum_{i,j=1}^n (\delta_{ij} + z_i \bar{z}_j) \frac{\partial^2}{\partial z_i \partial \bar{z}_j} - \nu \sum_{j=1}^n \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right) - \nu^2 \right) + 4\nu^2 \quad (2.7)$$

which, according to [31], will be called a magnetic Laplacians on $\mathbf{P}^n(\mathbb{C})$. The dependence of this operator on n is omitted.

2.1 Example

On elements of $\Gamma_{1,\nu}^\infty$, which are smooth sections of the $U(1)$ -bundle with the first Chern class ν acts the Hamiltonian H_ν of the Dirac (point) monopole in \mathbb{R}^3 with magnetic charge ν (in suitable units). Indeed, eigenfunctions of this monopole have been identified as *sections* by Wu and Yang [20] and are known as *monopole harmonics*. Their explicit expression in the coordinate z are given below by (2.9). For more information on Dirac monopoles see [21]. The restriction $2\nu \in \{1, 2, 3, \dots\}$ results from Dirac's quantization condition for monopole charges, which requires that the total flux of the magnetic field across a closed surface must be an integer multiple of a universal constant. It can also be understood in the context of cohomology groups for hermitian line bundles [22] or as the Weil-Souriau-Kostant quantization condition [16]. In the stereographic coordinate $z \in \mathbb{C} \cup \{\infty\} \equiv \mathbb{S}^2 \equiv \mathbf{P}^1(\mathbb{C})$ (and suitable units) this Hamiltonian reads ([19], p.598) :

$$L_\nu = -(1 + |z|^2) \left((1 + |z|^2) \frac{\partial^2}{\partial z \partial \bar{z}} + \nu \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right) - \nu^2 \right) - \nu^2 \equiv -\frac{1}{4} \Delta_\nu. \quad (2.8)$$

The stereographic projection bridges between the monopole system and the Landau system which describes spinless charged particles in perpendicular homogeneous magnetic fields ([4], p.240). Precisely, to determine eigenstates of the monopole Hamiltonian (2.8) one proceeds exactly as for the Landau Hamiltonian of the Euclidean setting. This leads, for each fixed integer $m \in \mathbb{Z}_+$, to a finite dimensional L^2 eigenspace whose orthonormal basis vectors $\{\Phi_k^{\nu,m}\}$, $-m \leq k \leq 2\nu + m$, are given by [24] :

$$\Phi_k^{\nu,m}(z) := \sqrt{\frac{(2\nu + 2m + 1)(2\nu + m)!m!}{(m+k)!(2\nu + m - k)!}} (1 + z\bar{z})^{-\nu} z^k P_m^{(k, 2\nu - k)} \left(\frac{1 - z\bar{z}}{1 + z\bar{z}} \right), \quad (2.9)$$

$P_m^{(\alpha,\beta)}(\cdot)$ being the Jacobi polynomial [2]. These eigenfunctions (2.9) are associated with the eigenvalue

$$\tau_m := (2m + 1)\nu + m(m + 1) \quad (2.10)$$

called a spherical Landau level.

3 Spaces of bounded eigenfunctions of Δ_ν

In order to summarize some needed results [31] about eigenspaces of Δ_ν , we first need to fix some notations.

3.1 Spherical harmonics

Let $\mathcal{P}(\mathbb{C}^n)$ denote the space of polynomials in the independent variable z and \bar{z} of \mathbb{C}^n . Elements of this space can be written in the form

$$u(z, \bar{z}) = \sum_{|\alpha| \leq k} \sum_{|\beta| \leq l} c_{\alpha,\beta} z^\alpha \bar{z}^\beta, \quad c_{\alpha,\beta} \in \mathbb{C}, \quad \alpha, \beta \in \mathbb{Z}_+^n, \quad (3.1)$$

for non-negative integers k and l , where standard multi-index is used.

The subspace of $\mathcal{P}(\mathbb{C}^n)$ composed of polynomials that are homogeneous of degree p in z and of degree q in \bar{z} will be denoted by $\mathcal{P}_{p,q}(\mathbb{C}^n)$. The dimension of $\mathcal{P}_{p,q}(\mathbb{C}^n)$ is given by

$$\delta(n, p, q) = \binom{p+n-1}{p-1} \binom{q+n-1}{q-1} \quad (3.2)$$

The subspace of $\mathcal{P}_{p,q}(\mathbb{C}^n)$ composed of harmonic elements, that is, elements in the kernel of the complex Laplacian

$$\sum_{j=1}^n \frac{\partial^2}{\partial z_j \partial \bar{z}_j} \quad (3.3)$$

will be denoted by $\mathfrak{H}_{p,q}(\mathbb{C}^n)$.

The set of restrictions of elements of $\mathfrak{H}_{p,q}(\mathbb{C}^n)$ to the unit sphere $\mathbb{S}^{2n-1} = \{\zeta \in \mathbb{C}^n, \langle \zeta, \zeta \rangle = 1\}$, denoted by $\mathcal{H}(p, q)$, is called the space of complex spherical harmonics of degree p in z and degree q in \bar{z} . Note

that $\mathcal{H}(p, 0)$ consists of holomorphic polynomials, and $\mathcal{H}(0, q)$ consists of polynomials whose complex conjugates are holomorphic.

The dimensions of spaces $\mathcal{H}(p, q)$, denoted by $d(n, p, q)$, are given by

$$d(n, p, q) = \delta(n, p, q) - \delta(n, p-1, q-1), \quad p, q \neq 0, \quad (3.4)$$

$$d(n, p, 0) = \delta(n, p, 0) \quad \text{and} \quad d(n, 0, q) = \delta(n, 0, q). \quad (3.5)$$

For $n = 1$, $d(n, p, 0) = d(n, 0, q) = 1$, but $\mathcal{H}(p, q) = \{0\}$ if both $p > 0$ and $q > 0$. It is a standard fact that $\mathcal{H}(p, q)$ are pairwise orthogonal in $L^2(\mathbb{S}^{2n-1}, d\omega)$ where $d\omega$ is the uniform measure on the sphere.

3.2 Bounded eigenfunctions of Δ_ν

For $\lambda \in \mathbb{C}$, we set $\Lambda_{n,\nu}(\lambda) := n^2 - \lambda^2 + 4\nu^2$ and consider the equation

$$\Delta_\nu F(z) = \Lambda_{n,\nu}(\lambda) F(z), \quad (3.6)$$

where F is a bounded function on \mathbb{C}^n . Define the eigenspace

$$\mathcal{A}_m^\nu := \{F \in L^\infty(\mathbb{C}^n), \Delta_\nu F = \Lambda_{n,\nu}(\lambda) F\} \subset L^2(\mathbb{C}^n, d\mu_n), \quad (3.7)$$

where

$$d\mu_n(z) := (1 + \langle z, z \rangle)^{-(n+1)} d\mu(z), \quad (3.8)$$

$d\mu(z)$ being the Lebesgue measure on \mathbb{C}^n . The eigenspace $\mathcal{A}_m^\nu = \{0\}$ if

$$\lambda \notin \left\{ \xi \in \mathbb{C}, \frac{1}{2}(n \pm \xi) + \nu \in \mathbb{Z}_- \right\} \cup \left\{ \xi \in \mathbb{C}, \frac{1}{2}(n \pm \xi) - \nu \in \mathbb{Z}_- \right\}. \quad (3.9)$$

otherwise it is not trivial if and only if λ has the form $\lambda = \pm(2(m + \nu) + n)$ for some $m \in \mathbb{Z}_+$. Note that when $n = 1$ and $\lambda = \pm(2(m + \nu) + 1)$ the above parametrization of the eigenvalue of Δ_ν gives that $\frac{-1}{4}\Lambda_{1,\nu}(\lambda) = \tau_m$ (where τ_m was given by (2.10)) as expected from the example in Subsection 2.1. For $n \geq 1$ and under the condition $\lambda = \pm(2(m + \nu) + n)$ one gets the eigenvalue

$$\beta_m := \Lambda_{n,\nu}(\pm(2(m + \nu) + n)) = -4(m + \nu)(m + \nu + n) + 4\nu^2 \quad (3.10)$$

and any function F in \mathcal{A}_m^ν admits the expansion

$$F(r\omega) = \frac{1}{(1+r^2)^{(m+\nu)}} \sum_{\substack{0 \leq p \leq m \\ 0 \leq q \leq m+2\nu}} r^{p+q} {}_2F_1 \left(\begin{matrix} p-m, q-m-2\nu \\ n+p+q \end{matrix} \middle| -r^2 \right) \sum_{j=1}^{d(n,p,q)} a_j^{\nu,p,q} h_{p,q}^j(\omega, \bar{\omega}), \quad (3.11)$$

where $r > 0$, $\omega \in \mathbb{S}^{2n-1}$, $a_j^{\nu,p,q}$ are constant complex numbers, ${}_2F_1$ is the Gauss hypergeometric function (see Appendix A) and $\{h_{p,q}^j\}_{j=0}^{d(n,p,q)}$ is an orthonormal basis of $\mathcal{H}(p, q)$. Note that F satisfies the growth condition

$$\lim_{r \rightarrow \infty} F(r\omega) = \sum_{0 \leq p \leq m} \frac{\Gamma(m-p+1)\Gamma(n+2p+2\nu)}{(-1)^{p-m}\Gamma(m+n+p+2\nu)} \sum_{j=1}^{d(n,p,p+2\nu)} a_j^{\nu,p} h_{p,p+2\nu}^j(\omega, \bar{\omega}) \quad (3.12)$$

where we wrote $a_j^{\nu,p} = a_j^{\nu,p,p+2\nu}$. In particular, \mathcal{A}_m^ν is finite-dimensional and has the following orthonormal basis [31, p.150]:

$$\Phi_j^{p,q,m}(z) = \gamma_{p,q}^{n,\nu,m} (1 + \langle z, z \rangle)^{-m-\nu} {}_2F_1 \left(\begin{matrix} p-m, q-m-2\nu \\ n+p+q \end{matrix} \middle| -\langle z, z \rangle \right) h_{p,q}^j(z, \bar{z}) \quad (3.13)$$

where

$$\gamma_{p,q}^{n,\nu,m} = \sqrt{\frac{2(2m+2\nu+n)\Gamma(m+q+n)\Gamma(m+2\nu+p+n)}{\Gamma^2(n+p+q)\Gamma(m-p+1)\Gamma(m+2\nu-q+1)}}.$$

for varying $0 \leq p \leq m$, $0 \leq q \leq m+2\nu$ and $j = 1, \dots, d(n; p, q)$. Here $\gamma_{p,q}^{n,\nu,m} = \|\Phi_j^{p,q,m}(z)\|^{-1}$ where the norm is taken in $L^2(\mathbb{C}^n, d\mu_n(z))$. The reproducing kernel of \mathcal{A}_m^ν is given by [31, p.148]:

$$K_{\nu,m}(z, w) := \frac{(2m+2\nu+n)\Gamma(m+n+2\nu)}{\pi^n \Gamma(m+2\nu+1)} \left(\frac{(1 + \langle w, z \rangle)^2}{(1 + |z|^2)(1 + |w|^2)} \right)^\nu P_m^{(n-1, 2\nu)}(\cos 2d_{FS}(z, w)), \quad (3.14)$$

where $P_m^{(n-1, 2\nu)}(\cdot)$ is the Jacobi polynomial of parameters $n-1$ and 2ν (see (A.9)).

4 A new proof for reproducing kernels of \mathcal{A}_m^ν

In order to obtain the reproducing kernel (3.14), the authors [31] have used the Zaremba expansion ([40]) of the reproducing kernel to compute $K_{\nu,m}(z, 0)$ as well as to establish a two-point invariance-type property of this kernel, involving the transitive action of the group $SU(n+1)$ on $\mathbf{P}^n(\mathbb{C})$. Here, we shall retrieve the same result by using the Zaremba expansion directly.

Lemma 1 *The functions (3.13) satisfy*

$$\begin{aligned} \sum_{p=0}^m \sum_{q=0}^{m+2\nu} \sum_{j=1}^{d(n,p,q)} \Phi_{m,p,q}^j(z) \overline{\Phi_{m,p,q}^j(w)} &= \frac{(2m+2\nu+n)\Gamma(m+2\nu+n)}{\pi^n \Gamma(m+2\nu+1)} \left(\frac{(1+\langle w, z \rangle)^2}{(1+|z|^2)(1+|w|^2)} \right)^\nu \\ &\times P_m^{(n-1, 2\nu)} \left(\frac{2|1+\langle z, w \rangle|^2}{(1+|z|^2)(1+|w|^2)} - 1 \right), \end{aligned} \quad (4.1)$$

for every $z, w \in \mathbb{C}^n$.

Proof. Denoting the sum in the l.h.s of (4.1) by $K_{\nu,m}(z, w)$ and replacing the orthonormal basis (3.13) of \mathcal{A}_m^ν by their expressions, we obtain

$$\begin{aligned} K_{\nu,m}(z, w) &= ((1+\langle z, z \rangle)(1+\langle w, w \rangle))^{-\nu-m} \sum_{\substack{1 \leq j \leq d(n,p,q) \\ 0 \leq p \leq m, 0 \leq q \leq m+2\nu}} (\gamma_{p,q}^{n,\nu,m})^2 h_{p,q}^j(z, \bar{z}) \overline{h_{p,q}^j(w, \bar{w})} \\ &\times {}_2F_1 \left(\begin{matrix} p-m, q-m-2\nu \\ n+p+q \end{matrix} \middle| -\langle z, z \rangle \right) {}_2F_1 \left(\begin{matrix} p-m, q-m-2\nu \\ n+p+q \end{matrix} \middle| -\langle w, w \rangle \right) \end{aligned} \quad (4.2)$$

which can also be written as

$$\begin{aligned} K_{\nu,m}(z, w) &= ((1+\langle z, z \rangle)(1+\langle w, w \rangle))^{-\nu-m} \sum_{p=0}^m \sum_{q=0}^{m+2\nu} (\gamma_{p,q}^{n,\nu,m})^2 \mathfrak{S}_{n;p,q}^{z,w} \\ &\times {}_2F_1 \left(\begin{matrix} p-m, q-m-2\nu \\ n+p+q \end{matrix} \middle| -\langle z, z \rangle \right) {}_2F_1 \left(\begin{matrix} p-m, q-m-2\nu \\ n+p+q \end{matrix} \middle| -\langle w, w \rangle \right) \end{aligned} \quad (4.3)$$

where

$$\begin{aligned} \mathfrak{S}_{n;p,q}^{z,w} &:= \sum_{1 \leq j \leq d(n,p,q)} h_{p,q}^j(z, \bar{z}) \overline{h_{p,q}^j(w, \bar{w})} \\ &= (|z||w|)^{p+q} \sum_{1 \leq j \leq d(n,p,q)} h_{p,q}^j \left(\frac{z}{|z|}, \frac{\bar{z}}{|z|} \right) \overline{h_{p,q}^j \left(\frac{w}{|w|}, \frac{\bar{w}}{|w|} \right)}. \end{aligned} \quad (4.4)$$

As usual, in the spherical harmonics setting, this last sum can be computed via the Koornwinder's formula [38] as

$$\mathfrak{S}_{n;p,q}^{z,w} = \frac{\Gamma(n)d(n,p,q)}{2\pi^n} (|z||w|)^{p+q} \left| \left\langle \frac{z}{|z|}, \frac{w}{|w|} \right\rangle \right|^{p-q} e^{i(p-q) \arg \left(\left\langle \frac{z}{|z|}, \frac{w}{|w|} \right\rangle \right)} \frac{P_{\min(p,q)}^{(n-2, |p-q|)} \left(2 \left| \left\langle \frac{z}{|z|}, \frac{w}{|w|} \right\rangle \right|^2 - 1 \right)}{P_{\min(p,q)}^{(n-2, |p-q|)}(1)}.$$

Now, using the notation $R_{p,q}^\gamma(\xi)$ in (A.14), $\mathfrak{S}_{n;p,q}^{z,w}$ also reads

$$\mathfrak{S}_{n;p,q}^{z,w} = (2\pi^n)^{-1} \Gamma(n) d(n,p,q) (|z||w|)^{p+q} R_{p,q}^{n-2} \left(\left\langle \frac{z}{|z|}, \frac{w}{|w|} \right\rangle \right). \quad (4.5)$$

On the other hand, we also need to write the terminating Gauss hypergeometric functions ${}_2F_1$ in (4.3) in terms of Zernike polynomials (A.14). For that, we use (A.15) to obtain

$${}_2F_1 \left(\begin{matrix} p-m, q-m-2\nu \\ n+p+q \end{matrix} \middle| -|z|^2 \right) = (1+|z|^2)^{m+\nu-\frac{p+q}{2}} R_{m-p, m-q+2\nu}^{n+p+q-1} \left((1+|z|^2)^{-\frac{1}{2}} \right). \quad (4.6)$$

Now, by replacing (4.5) and (4.6) in (4.3), the sum (4.2) takes the form

$$K_{\nu,m}(z,w) = \sum_{p=0}^m \sum_{q=0}^{m+2\nu} \left[(2\pi^n)^{-1} \Gamma(n) (\gamma_{p,q}^{n,\nu,m})^2 \right] d(n,p,q) \left(\frac{(|z||w|)^2}{(1+|z|^2)(1+|w|^2)} \right)^{\frac{1}{2}(p+q)} \quad (4.7)$$

$$\times R_{m-p,m-q+2\nu}^{n+p+q-1} \left((1+|z|^2)^{-\frac{1}{2}} \right) R_{m-p,m-q+2\nu}^{n+p+q-1} \left((1+|w|^2)^{-\frac{1}{2}} \right) R_{p,q}^{n-2} \left(\left\langle \frac{z}{|z|}, \frac{w}{|w|} \right\rangle \right)$$

with

$$(2\pi^n)^{-1} \Gamma(n) (\gamma_{p,q}^{n,\nu,m})^2 = \beta_{\nu,n,m} \frac{(-1)^{p+q} (-m)_p (-m-2\nu)_q (n+m+2\nu)_p (n+m)_q}{(n)_{p+q} (n)_{p+q}}, \quad (4.8)$$

where the constant

$$\beta_{\nu,n,m} = \frac{(2m+2\nu+n)\Gamma(n+m)\Gamma(n+m+2\nu)}{\pi^n \Gamma(n) m! (m+2\nu)!} \quad (4.9)$$

was obtained by using identities (A.3) and (A.5). Therefore, (4.7) becomes

$$K_{\nu,m}(z,w) = \beta_{\nu,n,m} \sum_{\substack{0 \leq p \leq m \\ 0 \leq q \leq m+2\nu}} \frac{(-1)^{p+q} (-m)_p (-m-2\nu)_q (n+m+2\nu)_p (n+m)_q}{(d(n,p,q))^{-1} (n)_{p+q} (n)_{p+q}} \left(\frac{|z|^2 |w|^2}{(1+|z|^2)(1+|w|^2)} \right)^{\frac{p+q}{2}}$$

$$\times R_{m-p,m-q+2\nu}^{n+p+q-1} \left((1+|z|^2)^{-\frac{1}{2}} \right) R_{m-p,m-q+2\nu}^{n+p+q-1} \left((1+|w|^2)^{-\frac{1}{2}} \right) R_{p,q}^{n-2} \left(\left\langle \frac{z}{|z|}, \frac{w}{|w|} \right\rangle \right).$$

We now are in position to apply the addition formula for Zernike polynomials due to Šapiro [30] and Koornwinder [29]:

$$R_{k,l}^\gamma \left(z_1 \bar{z}_2 + (1-z_1 \bar{z}_1)^{\frac{1}{2}} (1-z_2 \bar{z}_2)^{\frac{1}{2}} y \right) = \sum_{p=0}^k \sum_{q=0}^l \tau_{p,q,\gamma-1} \frac{(-1)^{p+q} (-k)_p (-l)_q (l+\gamma+1)_p (k+\gamma+1)_q}{(\gamma+1)_{p+q} (\gamma+1)_{p+q}}$$

$$\times [(1-z_1 \bar{z}_1)(1-z_2 \bar{z}_2)]^{\frac{p+q}{2}} R_{k-p,l-q}^{\gamma+p+q}(z_1) \overline{R_{k-p,l-q}^{\gamma+p+q}(z_2)} R_{p,q}^{\gamma-1}(y),$$

where

$$\tau_{p,q,\gamma} := \frac{(p+q+\gamma+1)(\gamma+1)_p (\gamma+1)_q}{(\gamma+1)p!q!}. \quad (4.10)$$

Indeed, we specialize this formula for $z_1 = (1+|z|^2)^{-\frac{1}{2}}$, $z_2 = (1+|w|^2)^{-\frac{1}{2}}$, $y = \left\langle \frac{z}{|z|}, \frac{w}{|w|} \right\rangle$, $k = m$, $l = m+2\nu$, and $\gamma = n-1$ to obtain, after replacement, the expression

$$K_{\nu,m}(z,w) = \beta_{\nu,n,m} R_{m,m+2\nu}^{n-1} \left(\frac{1 + \langle z, w \rangle}{(1+|z|^2)^{\frac{1}{2}} (1+|w|^2)^{\frac{1}{2}}} \right). \quad (4.11)$$

Using again (A.14), equation (4.11) can be rewritten as:

$$K_{\nu,m}(z,w) = \frac{(2m+2\nu+n)\Gamma(m+2\nu+n)}{\pi^n \Gamma(m+2\nu+1)} \left(\frac{(1+\langle w, z \rangle)^2}{(1+|z|^2)(1+|w|^2)} \right)^\nu P_m^{(n-1,2\nu)} \left(\frac{2|1+\langle z, w \rangle|^2}{(1+|z|^2)(1+|w|^2)} - 1 \right). \quad (4.12)$$

Recalling (2.4), we may also write (4.12) in the form (4.1). This ends the proof. ■

Remark 2 For $\nu = 0$, the Fubini-Study Laplacian $\Delta_0 \equiv \Delta_{FS}$ has a discrete spectral decomposition with eigenvalues $\beta_m = (-4m(m+n))_{m \geq 0}$. Besides, each eigenspace is finite-dimensional and has a orthonormal basis given by homogeneous spherical harmonics of degree zero. Setting $\nu = 0$ in (3.14), then the kernel of the orthogonal projection from $L^2(\mathbb{C}^n, d\mu_n)$ onto the m -th eigenspace of Δ_0 reduces to

$$K_{0,m}(z,w) = \pi^{-n} (2m+n) \frac{(m+n-1)!}{m!} P_m^{(n-1,0)}(\cos 2d_{FS}(z,w)), \quad (4.13)$$

in agreement with the result of Koornwinder [14, Theorem 3.8].

5 The heat equation associated with Δ_ν

We here consider the Cauchy problem

$$\partial_t \varphi(t, z) = \Delta_\nu \varphi(t, z), \quad t > 0, \quad z \in \mathbb{C}^n, \quad (5.1)$$

with the initial condition $\varphi(0, z) = g(z)$, $g \in \mathcal{E}^{\nu, \infty} := \left(\bigoplus_{m=0}^{+\infty} \mathcal{A}_m^\nu \right) \cap L^\infty(\mathbb{C}^n)$. The solution is given by ([52]):

$$\varphi(t, z) = \int_{\mathbb{C}^n} H_\nu(t, z, w) g(w) d\mu_n(w) \quad (5.2)$$

where

$$H_\nu(t, z, w) = N_t^{(n, \nu)}(z, w) \sum_{m=0}^{+\infty} (2m + 2\nu + n) \frac{\Gamma(m + 2\nu + n)}{\Gamma(m + 2\nu + 1)} e^{-4t(m + \nu + \frac{n}{2})^2} P_m^{(n-1, 2\nu)}(\cos 2d_{FS}(z, w)) \quad (5.3)$$

is the heat kernel associated with Δ_ν with the prefactor

$$N_t^{(n, \nu)}(z, w) := \pi^{-n} \left(\frac{(1 + \langle w, z \rangle)^2}{(1 + |z|^2)(1 + |w|^2)} \right)^\nu e^{4t(\nu^2 + \frac{n^2}{4})}. \quad (5.4)$$

Precisely, the following facts have been proved [52]:

- (i) The series (5.3) defining $H_\nu(t, z, w)$ is normally convergent. Further, for every $k \in \mathbb{Z}_+$ and every multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$ we have that

$$\sum_{m=0}^{+\infty} \sup_{(z, w) \in \mathbb{C}^n \times \mathbb{C}^n} |\partial_t^k D_{z, w}^{\alpha, \beta} [e^{\beta m t} K_{\nu, m}(z, w)]| \leq C(t) < +\infty \quad (5.5)$$

- (ii) The function $]0, +\infty[\times \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$, $(t, z, w) \mapsto H_\nu(t, z, w)$ is C^∞ and satisfies the equation (5.1).
- (iii) To see that the function $\varphi(t, z)$ in (5.2) is a solution of the equation (5.1), one may write it as

$$\varphi(t, z) = \sum_{m=0}^{+\infty} e^{\beta m t} \int_{\mathbb{C}^n} K_{\nu, m}(z, w) g(w) d\mu_n(w). \quad (5.6)$$

Recalling that $g \in \mathcal{E}^{\nu, \infty}$, it decomposes as

$$g(z) = \sum_{m=0}^{+\infty} g_m(z), \quad g_m \in \mathcal{A}_m^\nu. \quad (5.7)$$

Moreover, the component functions $\{g_m\}$ satisfy

$$\int_{\mathbb{C}^n} K_{\nu, m}(z, w) g(w) d\mu_n(w) = g_m(z). \quad (5.8)$$

Since the spaces \mathcal{A}_m^ν are pairwise orthogonal in $L^2(\mathbb{C}^n, d\mu_n(z))$, it follows that

$$\varphi(t, z) = \sum_{m=0}^{+\infty} e^{\beta m t} g_m(z) \quad (5.9)$$

from which one can prove that

$$\lim_{t \rightarrow 0} \|\varphi(t, \cdot) - g(\cdot)\|_{L^2(\mathbb{C}^n, d\mu_n(z))} = 0. \quad (5.10)$$

Finally, from (5.9) and (5.7) it is clear that $\varphi(t, z)$ obey the initial condition $\varphi(0, z) = g(z)$.

Remark 3 For $\nu = 0$, the spectral theorem implies that for any suitable function U , the operator $U(\Delta_0)$ is an integral operator whose kernel is given by

$$\sum_{m=0}^{+\infty} U(-4m(m+n))K_{0,m}(z, w). \quad (5.11)$$

In particular, one can write the kernel of the heat semigroup $\exp(t\Delta_0)$ through the series expansion

$$H_0(t; z, w) = \pi^{-n} \sum_{m \geq 0} (2m+n) \frac{(m+n-1)!}{m!} e^{-4m(m+n)t} P_m^{(n-1,0)}(\cos 2d_{FS}(z, w)), \quad (5.12)$$

see [36, p.873].

6 An integral representation for $H_\nu(t, z, w)$

In this section, we extend the formula (1.8) with respect to the parameter ν as follows.

Theorem 4 Let $n \geq 1$, $2\nu \in \mathbb{Z}_+$ and $\rho = d_{FS}(z, w)$. Then,

$$\begin{aligned} H_\nu(t, z, w) &= \frac{e^{4t(\nu^2 + \frac{n^2}{4})}}{2^{n+2\nu-1} \pi^n \Gamma(2\nu + \frac{1}{2})} \left(\frac{(1 + \langle z, z \rangle)(1 + \langle w, w \rangle)}{(1 + \langle z, w \rangle)^2} \right)^\nu \\ &\quad \times \int_\rho^{\frac{\pi}{2}} \frac{-d(\cos u)}{(\cos^2 \rho - \cos^2 u)^{\frac{1}{2}-2\nu}} \left(-\frac{1}{\sin u} \frac{d}{du} \right)^{n+2\nu} [\Theta_{n+1,\nu}(t, u)] \end{aligned} \quad (6.1)$$

where

$$\Theta_{n+1,\nu}(t, u) := \sum_{m=0}^{\infty} e^{-4t(m+\nu+\frac{n}{2})^2} \cos(2m+2\nu+n)u.$$

Proof. We first write the heat kernel in terms of the reproducing kernel (4.1) as follows

$$\begin{aligned} H_\nu(t, z, w) &= \sum_{m=0}^{+\infty} e^{\beta m t} K_{\nu,m}(z, w) \\ &= \sum_{m=0}^{+\infty} e^{\beta m t} \frac{(2m+2\nu+n)\Gamma(m+2\nu+n)}{\pi^n \Gamma(m+2\nu+1)} \left(\frac{(1 + \langle w, z \rangle)^2}{(1 + |z|^2)(1 + |w|^2)} \right)^\nu \\ &\quad \times P_m^{(n-1,2\nu)} \left(\frac{2|1 + \langle z, w \rangle|^2}{(1 + |z|^2)(1 + |w|^2)} - 1 \right). \end{aligned} \quad (6.2)$$

Next, we will use the integral representation for Jacobi polynomials in terms of Gegenbauer polynomials given in [37, p.194] followed by the symmetry relation $P_k^{(\alpha,\beta)}(-x) = (-1)^k P_k^{(\beta,\alpha)}(x)$,

$$P_k^{(\alpha,\beta)}(2t^2 - 1) = \frac{2\Gamma(\alpha + \beta + 1)\Gamma(k + \beta + 1)}{\Gamma(\beta + \frac{1}{2})\Gamma(k + \alpha + \beta + 1)} \int_0^1 (1 - u^2)^{\beta - \frac{1}{2}} C_{2k}^{\alpha + \beta + 1}(tu) du, \quad (6.3)$$

$Re\alpha > -\frac{1}{2}$, $Re\beta > -\frac{1}{2}$ and $|t| < 1$. Precisely, we set $t = \cos \rho$ and $u = \cos \psi / \cos \rho$, then we specialize $\alpha = n - 1$, $\beta = 2\nu$ and $k = m$ in (6.3) to get that

$$P_m^{(n-1,2\nu)}(\cos 2\rho) = \frac{2\Gamma(n+2\nu)\Gamma(m+2\nu+1)}{\Gamma(2\nu+\frac{1}{2})\Gamma(m+2\nu+n)} \frac{1}{\cos^{4\nu} \rho} \int_\rho^{\frac{\pi}{2}} \frac{\sin \psi}{(\cos^2 \rho - \cos^2 \psi)^{\frac{1}{2}-2\nu}} C_{2m}^{n+2\nu}(\cos \psi) d\psi. \quad (6.4)$$

By substituting the above representation in (6.2), we obtain, after some calculations, the following

$$\begin{aligned} H_\nu(t, z, w) &= \frac{2\Gamma(n+2\nu)e^{4t(\nu^2 + \frac{n^2}{4})}}{\Gamma(2\nu + \frac{1}{2}) \cos^{4\nu} \rho} D_\nu(z, w) \\ &\quad \times \int_\rho^{\frac{\pi}{2}} \frac{\sin \psi}{(\cos^2 \theta - \cos^2 \psi)^{\frac{1}{2}-2\nu}} \left(\sum_{m=0}^{+\infty} (2m+2\nu+n) C_{2m}^{n+2\nu}(\cos \psi) e^{-4t(m+\nu+\frac{n}{2})^2} \right) d\psi, \end{aligned} \quad (6.5)$$

where

$$D_\nu(z, w) := \left(\frac{(1 + \langle w, z \rangle)^2 \cos^{-4} \rho}{(1 + |z|^2)(1 + |w|^2)} \right)^\nu.$$

Next, we use the formula [36, p.876]:

$$\left(-\frac{d}{\sin u} \right)^l \left(\frac{\sin(k+l+1)u}{\sin u} \right) = 2^l l! C_k^{l+1}(\cos u), \quad (6.6)$$

for $l = n + 2\nu$, $k = 2m$, $u = \psi$ to rewrite the series occurring in (6.5) as

$$\frac{2^{1-n-2\nu}}{(n+2\nu-1)!} \left(-\frac{d}{\sin \psi} \right)^{n+2\nu} \left(\sum_{m=0}^{+\infty} \frac{(2m+2\nu+n) \sin(2m+2\nu+n)\psi}{\sin \psi} e^{-4t(m+\nu+\frac{\pi}{2})^2} \right). \quad (6.7)$$

But since

$$\frac{(2m+2\nu+n) \sin(2m+2\nu+n)\psi}{\sin \psi} = -\frac{d}{\sin \psi} \cos(2m+2\nu+n)\psi. \quad (6.8)$$

Equation (6.7) may be presented as

$$\frac{2^{1-n-2\nu}}{(n+2\nu-1)!} \left(-\frac{d}{\sin \psi} \right)^{n+2\nu} [\Theta_{n+1,\nu}(t, \psi)] \quad (6.9)$$

where

$$\Theta_{n+1,\nu}(t, \psi) = \sum_{m=0}^{+\infty} e^{-4t(m+\nu+\frac{\pi}{2})^2} \cos(2m+2\nu+n)\psi. \quad (6.10)$$

Finally, by substituting (6.9) into (6.5) we arrive at the announced result (6.1). ■

7 Heat coefficients for Δ_ν

Here our goal is to establish an asymptotic expansion for the trace of $\exp(\frac{1}{4}t\Delta_\nu)$. For that, we start from the following spectral series representation of this trace

$$Tr \left(\exp \left(\frac{1}{4}t\Delta_\nu \right) \right) = e^{\left(\frac{n^2}{4} + \nu^2\right)t} \sum_{m=0}^{+\infty} (\dim_{\mathbb{C}} \mathcal{A}_m^\nu) e^{-(m+\frac{n}{2}+\nu)^2 t}, \quad (7.1)$$

where the dimension of \mathcal{A}_m^ν can be computed via the diagonal of the corresponding reproducing kernel as

$$\dim_{\mathbb{C}} \mathcal{A}_m^\nu := \int_{\mathbb{C}^n} K_{\nu,m}(z, z) d\mu_n(z) = \frac{(2m+n+2\nu)\Gamma(m+n)\Gamma(m+n+2\nu)}{n(\Gamma(n))^2 \Gamma(m+1)\Gamma(m+2\nu+1)}. \quad (7.2)$$

Theorem 5 *The heat trace formula (7.1) satisfies the asymptotic expansion*

$$Tr \left(\exp \left(\frac{1}{4}t\Delta_\nu \right) \right) \simeq \frac{1}{(4\pi t)^n} \sum_{j=0}^{+\infty} b_j^{(\nu,n)} t^j, \quad t \searrow 0^+ \quad (7.3)$$

with the heat coefficients given by

$$b_j^{(\nu,n)} = \frac{(4\pi)^n}{n!} \sum_{i=0}^j \frac{\left(\frac{n^2}{4} + \nu^2\right)^{j-i}}{(j-i)!} c_i^{(\nu,n)},$$

(i) case n odd

$$c_i^{(\nu,n)} = \begin{cases} \frac{(n-i-1)!}{(n-1)!} \gamma_{n-i-1}^{(\nu,n)}, & 0 \leq i \leq n-1 \\ \sum_{p=0}^{n-1} \frac{(-1)^{i-n+1} c_p^{(\nu,n)}}{(i-n)!(i-p)p!} B_{2(i-p)} \left(\nu + \frac{1}{2} \right), & i \geq n \end{cases} \quad (7.4)$$

(ii) case n even

$$c_i^{(\nu, n)} = \begin{cases} 1, & i = 0 \\ \tau_{n-1-i}^{(\nu, n)} \frac{(n-i-1)!}{(n-1)!}, & 1 \leq i \leq n-1 \\ \sum_{p=0}^{n-1} \frac{(-1)^{i-n} c_p^{(\nu, n)}}{(i-n)!(i-p)(n-p-1)!} [2B_{2(i-p)} - B_{2(i-p)}(\nu)], & i \geq n \end{cases} \quad (7.5)$$

where B_d and $B_d(\cdot)$ denote Bernoulli numbers and polynomials respectively (see Appendix A). The coefficients $\tau_i^{(\nu, n)}$ and $\gamma_i^{(\nu, n)}$ are defined by (7.9) and (7.40) respectively.

Proof. We may rewrite (7.2) as

$$\dim_{\mathbb{C}} \mathcal{A}_m^\nu = \frac{2m+n+2\nu}{n!(n-1)!} \prod_{j=1}^{n-1} (m+j)(m+2\nu+j). \quad (7.6)$$

Setting $r = r(m) = m + \nu + \frac{n}{2}$, the product in (7.6) also reads

$$\wp = \prod_{j=1}^{n-1} \left(r - \frac{n}{2} - \nu + j \right) \left(r - \frac{n}{2} + \nu + j \right). \quad (7.7)$$

We treat the cases of n odd and n even separately as they correspond to different choices of theta functions.

(i) **Case odd** $n \geq 0$. The product (7.7) can also be presented as

$$\wp = \prod_{l=\frac{1}{2}+\nu}^{\frac{n}{2}+\nu-1} (r^2 - l^2) \prod_{s=\frac{1}{2}-\nu}^{\frac{n}{2}-\nu-1} (r^2 - s^2) \quad (7.8)$$

and further it can be decomposed as

$$\wp = \sum_{p=0}^{n-1} \gamma_p^{(\nu, n)} r^{2p}. \quad (7.9)$$

where the numbers $\gamma_0^{(\nu, n)}, \dots, \gamma_{n-1}^{(\nu, n)}$ can be computed explicitly (see Sect. 8). By using (7.9), the multiplicity in (7.6) takes the form

$$\dim_{\mathbb{C}} \mathcal{A}_m^\nu = \frac{2}{n!(n-1)!} \sum_{p=0}^{n-1} \gamma_p^{(\nu, n)} \left(m + \nu + \frac{n}{2} \right)^{2p+1} \quad (7.10)$$

which we may use to rewrite the trace formula (7.1) as

$$\begin{aligned} Tr \left(e^{\frac{1}{4}t\Delta_\nu} \right) &= e^{\left(\frac{n^2}{4}+\nu^2\right)t} \sum_{m=0}^{+\infty} \left[\frac{2}{n!(n-1)!} \sum_{p=0}^{n-1} \gamma_p^{(\nu, n)} \left(m + \nu + \frac{n}{2} \right)^{2p+1} \right] e^{-(m+\frac{n}{2}+\nu)^2t} \\ &= \frac{e^{\left(\frac{n^2}{4}+\nu^2\right)t}}{n!(n-1)!} \mathcal{S} \end{aligned} \quad (7.11)$$

with

$$\mathcal{S} := \sum_{p=0}^{n-1} \gamma_p^{(\nu, n)} \left(2 \sum_{m=0}^{+\infty} \left(m + \nu + \frac{n}{2} \right)^{2p+1} e^{-(m+\frac{n}{2}+\nu)^2t} \right). \quad (7.12)$$

The latter sum can be cast into two parts as

$$\mathcal{S} = 2 \sum_{\mu=\nu+\frac{1}{2}}^{+\infty} \mu e^{-\mu^2t} \sum_{p=0}^{n-1} \gamma_p^{(\nu, n)} \mu^{2p} - 2 \sum_{\mu=\nu+\frac{1}{2}}^{\nu+\frac{n}{2}-1} \mu e^{-\mu^2t} \sum_{p=0}^{n-1} \gamma_p^{(\nu, n)} \mu^{2p} \quad (7.13)$$

Recalling (7.8)-(7.9), we see that for μ ranging over the set $\{\nu + \frac{1}{2}, \nu + \frac{3}{2}, \dots, \nu + \frac{n}{2} - 1\}$ the sum

$$\sum_{p=0}^{n-1} \gamma_p^{(\nu, n)} \mu^{2p} = 0 \quad (7.14)$$

and therefore the sum (7.13) reduces to

$$\mathcal{S} = 2 \sum_{p=0}^{n-1} \gamma_p^{(\nu, n)} \mathcal{R}_p^{(\nu)}(t) \quad (7.15)$$

where

$$\mathcal{R}_p^{(\nu)}(t) = \sum_{\mu=\nu+\frac{1}{2}}^{+\infty} \mu^{2p+1} e^{-\mu^2 t} \quad (7.16)$$

By setting $\mu = j + \frac{1}{2}$, we successively obtain

$$\begin{aligned} \mathcal{R}_p^{(\nu)}(t) &= \sum_{j=\nu}^{+\infty} \left(j + \frac{1}{2}\right)^{2p+1} e^{-(j+\frac{1}{2})^2 t} \\ &= \sum_{j=0}^{+\infty} \left(j + \frac{1}{2}\right)^{2p+1} e^{-(j+\frac{1}{2})^2 t} - \sum_{j=0}^{\nu-1} \left(j + \frac{1}{2}\right)^{2p+1} e^{-(j+\frac{1}{2})^2 t} \\ &= \frac{1}{2} \left(-\frac{d}{dt}\right)^p \left[\sum_{j=0}^{+\infty} (2j+1) e^{-(j+\frac{1}{2})^2 t} \right] - \sum_{j=0}^{\nu-1} \left(j + \frac{1}{2}\right)^{2p+1} e^{-(j+\frac{1}{2})^2 t} \\ &= \frac{1}{2} \left(-\frac{d}{dt}\right)^p [\vartheta_2(t)] - \sum_{j=0}^{\nu-1} \left(j + \frac{1}{2}\right)^{2p+1} e^{-(j+\frac{1}{2})^2 t} \end{aligned} \quad (7.17)$$

where $\vartheta_2(t)$ denotes the Jacobi's theta function given by [39, p.280]:

$$\vartheta_2(t) = \sum_{j=0}^{+\infty} (2j+1) e^{-(j+\frac{1}{2})^2 t} \quad (7.18)$$

while the finite sum in (7.17) may be expressed by the series

$$\sum_{\ell=0}^{+\infty} \sigma_p^{(\nu)}(\ell) t^\ell \quad (7.19)$$

with coefficients

$$\sigma_p^{(\nu)}(\ell) = \frac{(-1)^\ell}{\ell!} \sum_{j=0}^{\nu-1} \left(j + \frac{1}{2}\right)^{2(p+\ell)+1}. \quad (7.20)$$

Now, we use the formula [43, p.597]:

$$\sum_{k=0}^m (k+a)^q = \frac{1}{q+1} [B_{q+1}(m+1+a) - B_{q+1}(a)], \quad q = 1, 2, \dots, \quad (7.21)$$

where $B_n(z)$ are the Bernoulli polynomials, for $m = \nu - 1$, $q = 2(\ell + p) + 1$, $a = \frac{1}{2}$, to get

$$\sigma_p^{(\nu)}(\ell) = \frac{(-1)^\ell}{2(p+\ell+1)\ell!} \left[B_{2(p+\ell+1)}\left(\nu + \frac{1}{2}\right) - B_{2(p+\ell+1)}\left(\frac{1}{2}\right) \right]. \quad (7.22)$$

Summarizing the above calculations, then the sum in (7.15) also reads

$$\mathcal{S} = \sum_{p=0}^{n-1} \gamma_p^{(\nu, n)} \left[\left(-\frac{d}{dt}\right)^p \vartheta_2(t) + \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell}{(p+\ell+1)\ell!} \left[B_{2(p+\ell+1)}\left(\frac{1}{2}\right) - B_{2(p+\ell+1)}\left(\nu + \frac{1}{2}\right) \right] t^\ell \right] \quad (7.23)$$

Applying now the asymptotic of the higher order derivative of $\vartheta_2(t)$ as $t \searrow 0^+$ ([39, p.281]) :

$$\left(\frac{d}{dt}\right)^p \vartheta_2(t) \simeq \frac{(-1)^p p!}{t^{1+p}} + \sum_{s=0}^{+\infty} \frac{B_{s+p} t^s}{s!} \quad (7.24)$$

in terms of Bernoulli numbers (B_d) defined by the recursion relation [13, p.8]:

$$B_d = \frac{(-1)^d}{d+1} (1 - 2^{-2d-1}) B_{2d+2}, \quad d = 0, 1, 2, \dots \quad (7.25)$$

By combining (7.25) and the formula [43, p.777]

$$B_d \left(\frac{1}{2}\right) = -(1 - 2^{1-d}) B_d, \quad (7.26)$$

we deduce that

$$B_{2(d+1)} \left(\frac{1}{2}\right) = (-1)^{d+1} (d+1) B_d. \quad (7.27)$$

For $d = p + s$ (7.27) reads

$$B_{2(p+s+1)} \left(\frac{1}{2}\right) = (-1)^{p+s+1} (p+s+1) B_{p+s}. \quad (7.28)$$

Substituting (7.24) and (7.28) into (7.23), we get after simplifications

$$\mathcal{S} \simeq \sum_{p=0}^{n-1} \gamma_p^{(\nu, n)} \left[p! t^{-1-p} + \sum_{s=0}^{+\infty} \frac{(-1)^{s+1} B_{2(p+s+1)} \left(\nu + \frac{1}{2}\right)}{(p+s+1)s!} t^s \right]. \quad (7.29)$$

Therefore, the heat trace asymptotic formula becomes

$$\begin{aligned} Tr \left(e^{\frac{1}{4} t \Delta_\nu} \right) &\simeq \frac{e^{\left(\frac{n^2}{4} + \nu^2\right)t}}{n!(n-1)!} \sum_{p=0}^{n-1} \gamma_p^{(\nu, n)} \left[\frac{p!}{t^{1+p}} + \sum_{s=0}^{+\infty} \frac{(-1)^{s+1} B_{2(p+s+1)} \left(\nu + \frac{1}{2}\right)}{(p+s+1)s!} t^s \right] \\ &= \frac{e^{\left(\frac{n^2}{4} + \nu^2\right)t}}{n!(n-1)!} \left[\sum_{p=0}^{n-1} \gamma_p^{(\nu, n)} \frac{p!}{t^{1+p}} + \sum_{p=0}^{n-1} \gamma_p^{(\nu, n)} \sum_{s=0}^{+\infty} \frac{(-1)^{s+1} B_{2(p+s+1)} \left(\nu + \frac{1}{2}\right)}{(p+s+1)s!} t^s \right] \\ &= \frac{\omega_n e^{\left(\frac{n^2}{4} + \nu^2\right)t}}{(4\pi t)^n} \left[1 + \sum_{p=0}^{n-2} \frac{\gamma_p^{(\nu, n)} p!}{(n-1)!} t^{n-p-1} + \sum_{p=0}^{n-1} \frac{\gamma_p^{(\nu, n)}}{(n-1)!} \sum_{s=p}^{+\infty} \frac{(-1)^{s-p+1} B_{2(s+1)} \left(\nu + \frac{1}{2}\right)}{(s+1)(s-p)!} t^{n+s-p} \right] \end{aligned} \quad (7.30)$$

The first sum in (7.30) can also be written as

$$\sum_{d=0}^{n-1} \frac{\gamma_{n-1-d}^{(\nu, n)}}{(n-1)!} (n-1-d)! t^d \quad (7.31)$$

while the second one also reads

$$\sum_{d=n}^{+\infty} \left(\sum_{p=0}^{n-1} \frac{(-1)^{d-n+1} \gamma_p^{(\nu, n)} B_{2(d-n+p+1)} \left(\nu + \frac{1}{2}\right)}{(n-1)!(d-n+p+1)(d-n)!} \right) t^d. \quad (7.32)$$

By introducing

$$\Omega_p^{(\nu)}(k) := \frac{(-1)^{k+1} B_{2(k+p+1)} \left(\nu + \frac{1}{2}\right)}{(k+p+1)k!}, \quad k = 0, 1, 2, \dots, \quad (7.33)$$

we may rewrite (7.30) as

$$Tr \left(e^{\frac{1}{4} t \Delta_\nu} \right) \simeq \frac{\omega_n e^{\left(\frac{n^2}{4} + \nu^2\right)t}}{(4\pi t)^n} \left[1 + \sum_{d=1}^{n-1} \frac{\gamma_{n-d-1}^{(\nu, n)} (n-d-1)!}{(n-1)!} t^d + \sum_{d=n}^{+\infty} \left(\sum_{p=0}^{n-1} \frac{\gamma_p^{(\nu, n)}}{(n-1)!} \Omega_p^{(\nu)}(d-n) \right) t^d \right] \quad (7.34)$$

$$= \frac{\omega_n}{(4\pi t)^n} \left[\sum_{d=0}^{+\infty} c_d^{(\nu, n)} t^d \right] \left[\sum_{i=0}^{+\infty} \frac{1}{i!} \left(\frac{n^2}{4} + \nu^2 \right)^i \right] \quad (7.35)$$

where the coefficients $c_d^{(\nu, n)}$ are given by

$$c_d^{(\nu, n)} = \begin{cases} \frac{(n-d-1)!}{(n-1)!} \gamma_{n-d-1}^{(\nu, n)}, & 0 \leq d \leq n-1 \\ \sum_{p=0}^{n-1} \frac{\gamma_p^{(\nu, n)}}{(n-1)!} \Omega_p^{(\nu)}(d-n), & d \geq n \end{cases} \quad (7.36)$$

Since $p \in \{0, \dots, n-1\}$ in the second expression of (7.36), we may write

$$\gamma_p^{(\nu, n)} = \frac{(n-1)!}{p!} c_{n-p-1}^{(\nu, n)}, \quad 0 \leq p \leq n-1 \quad (7.37)$$

and then, the expression of $c_d^{(\nu, n)}$ can be simplified as follows

$$c_d^{(\nu, n)} = \begin{cases} \frac{(n-d-1)!}{(n-1)!} \gamma_{n-d-1}^{(\nu, n)}, & 0 \leq d \leq n-1 \\ \sum_{p=0}^{n-1} \frac{c_{n-p-1}^{(\nu, n)}}{p!} \Omega_p^{(\nu)}(d-n), & d \geq n. \end{cases} \quad (7.38)$$

This means that the coefficients $c_d^{(\nu, n)}$ satisfy a n -terms recurrence relation when $d \geq n$.

(i) **Case even $n \geq 2$.** In this case, the product (7.7) can also be presented as

$$\wp' = \prod_{l=\nu}^{\frac{n}{2}+\nu-1} (r^2 - l^2) \prod_{s=1-\nu}^{\frac{n}{2}-\nu-1} (r^2 - s^2) \quad (\text{product omitted for } n=2) \quad (7.39)$$

and further it can be decomposed as

$$\wp' = \sum_{p=0}^{n-1} \tau_p^{(\nu, n)} r^{2p}. \quad (7.40)$$

where the numbers $\tau_0^{(\nu, n)}, \dots, \tau_{n-1}^{(\nu, n)}$ can be computed explicitly (see Sect.8). By using (7.40), the multiplicity in (7.6) takes the form

$$\dim_{\mathbb{C}} \mathcal{A}_m^\nu = \frac{2}{n!(n-1)!} \sum_{p=0}^{n-1} \tau_p^{(\nu, n)} \left(m + \nu + \frac{n}{2} \right)^{2p+1} \quad (7.41)$$

We now may insert the r.h.s of (7.40) into the trace formula (7.1) to get

$$\begin{aligned} \text{Tr} \left(e^{\frac{1}{4}t\Delta_\nu} \right) &= e^{\left(\frac{n^2}{4} + \nu^2\right)t} \sum_{m=0}^{+\infty} \left[\frac{2}{n!(n-1)!} \sum_{p=0}^{n-1} \tau_p^{(\nu, n)} \left(m + \nu + \frac{n}{2} \right)^{2p+1} \right] e^{-(m+\nu+\frac{n}{2})^2 t} \\ &= \frac{2e^{\left(\frac{n^2}{4} + \nu^2\right)t}}{n!(n-1)!} \sum_{\mu=\nu+\frac{n}{2}}^{+\infty} e^{-\mu^2 t} \sum_{p=0}^{n-1} \tau_p^{(\nu, n)} \mu^{2p+1} \\ &= \frac{e^{\left(\frac{n^2}{4} + \nu^2\right)t}}{n!(n-1)!} \left[2 \sum_{\mu=1}^{+\infty} e^{-\mu^2 t} \sum_{p=0}^{n-1} \tau_p^{(\nu, n)} \mu^{2p+1} - 2 \sum_{\mu=1}^{\nu+\frac{n}{2}-1} e^{-\mu^2 t} \sum_{p=0}^{n-1} \tau_p^{(\nu, n)} \mu^{2p+1} \right] \\ &= \frac{e^{\left(\frac{n^2}{4} + \nu^2\right)t}}{n!(n-1)!} [S_1 - S_2]. \end{aligned} \quad (7.42)$$

Now, recall the Jacobi's theta function $([\cdot], p.)$:

$$\vartheta_3(t) := 2 \sum_{l=1}^{+\infty} l e^{-l^2 t} \quad (7.43)$$

Then one can see that the p derivative of $\vartheta_3(t)$ satisfies

$$(-1)^p \left(\frac{d}{dt} \right)^p [\vartheta_3(t)] = 2 \sum_{l=1}^{+\infty} l^{2p+1} e^{-l^2 t} \quad (7.44)$$

Therefore,

$$S_1 = \sum_{p=0}^{n-1} \tau_p^{(\nu, n)} (-1)^p \left(\frac{d}{dt} \right)^p [\vartheta_3(t)]. \quad (7.45)$$

From (7.39) and (7.40) we see that for μ ranging over the set $\{\nu, \nu+1, \dots, \nu + \frac{n}{2} - 1\}$ the sum

$$\sum_{p=0}^{n-1} \tau_p^{(\nu, n)} \mu^{2p} = 0, \quad (7.46)$$

and therefore S_2 reduces to

$$S_2 = 2 \sum_{\mu=1}^{\nu-1} e^{-\mu^2 t} \sum_{p=0}^{n-1} \tau_p^{(\nu, n)} \mu^{2p+1}. \quad (7.47)$$

The above sum can be expressed as follows

$$\begin{aligned} S_2 &= 2 \sum_{p=0}^{n-1} \tau_p^{(\nu, n)} \sum_{\mu=1}^{\nu-1} e^{-\mu^2 t} \mu^{2p+1} \\ &= 2 \sum_{p=0}^{n-1} \tau_p^{(\nu, n)} \sum_{\ell=0}^{+\infty} \varrho_p^{(\nu)}(\ell) t^\ell \end{aligned} \quad (7.48)$$

with coefficients

$$\varrho_p^{(\nu)}(\ell) = \frac{(-1)^\ell}{\ell!} \sum_{\mu=1}^{\nu-1} \mu^{2(p+\ell)+1}. \quad (7.49)$$

Now, we use the formula [43, p.596]:

$$\sum_{k=1}^m k^q = \frac{1}{q+1} [B_{q+1}(m+1) - B_{q+1}], \quad q = 1, 2, \dots, \quad (7.50)$$

for $m = \nu - 1$, $q = 2(\ell + p) + 1$, to get

$$\varrho_p^{(\nu)}(\ell) = \frac{(-1)^\ell}{2(p+\ell+1)\ell!} [B_{2(p+\ell+1)}(\nu) - B_{2(p+\ell+1)}]. \quad (7.51)$$

Therefore, the trace formula becomes

$$\mathcal{T} := \text{Tr} \left(e^{\frac{1}{4}t\Delta_\nu} \right) = \frac{e^{\left(\frac{n^2}{4} + \nu^2\right)t}}{n!(n-1)!} \sum_{p=0}^{n-1} \tau_p^{(\nu, n)} \left[(-1)^p \left(\frac{d}{dt} \right)^p [\vartheta_3(t)] - \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell [B_{2(p+\ell+1)}(\nu) - B_{2(p+\ell+1)}]}{(p+\ell+1)\ell!} t^\ell \right] \quad (7.52)$$

Applying now the asymptotic of the higher order derivative of $\vartheta_3(t)$ as $t \searrow 0^+$ [42, p.11] :

$$\left(\frac{d}{dt} \right)^\ell \vartheta_3(t) \simeq \frac{(-1)^\ell \ell!}{t^{1+\ell}} + \sum_{j=\ell}^{+\infty} \frac{(-1)^j B_{2j+2}}{(j+1)(j-\ell)!} t^{j-\ell} \quad (7.53)$$

where (B_d) are Bernoulli numbers in (7.25), substituting (7.53) into (7.52), we get

$$\begin{aligned}
\mathcal{T} &\simeq \frac{e^{\left(\frac{n^2}{4} + \nu^2\right)t}}{n!(n-1)!} \sum_{p=0}^{n-1} \tau_p^{(\nu, n)} \left[\frac{p!}{t^{1+p}} + (-1)^p \sum_{j=p}^{+\infty} \frac{(-1)^j B_{2j+2}}{(j+1)(j-p)!} t^{j-p} - \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell [B_{2(p+\ell+1)}(\nu) - B_{2(p+\ell+1)}]}{(p+\ell+1)\ell!} t^\ell \right] \\
&= \frac{e^{\left(\frac{n^2}{4} + \nu^2\right)t}}{n!(n-1)!} \sum_{p=0}^{n-1} \tau_p^{(\nu, n)} \left[\frac{p!}{t^{1+p}} + \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell B_{2(\ell+p+1)}}{(\ell+p+1)\ell!} t^\ell - \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell [B_{2(p+\ell+1)}(\nu) - B_{2(p+\ell+1)}]}{(p+\ell+1)\ell!} t^\ell \right] \\
&= \frac{e^{\left(\frac{n^2}{4} + \nu^2\right)t}}{n!(n-1)!} \sum_{p=0}^{n-1} \tau_p^{(\nu, n)} \left[\frac{p!}{t^{1+p}} + \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell [2B_{2(p+\ell+1)} - B_{2(p+\ell+1)}(\nu)]}{(p+\ell+1)\ell!} t^\ell \right]
\end{aligned}$$

We denote $Vol(\mathbf{P}^n(\mathbb{C})) = \omega_n = (4\pi)^n/n!$ and we use the fact that $\tau_{n-1}^{(\nu, n)} = 1$, to write

$$\begin{aligned}
\mathcal{T} &\simeq \frac{\omega_n e^{\left(\frac{n^2}{4} + \nu^2\right)t}}{(4\pi t)^n} \sum_{p=0}^{n-1} \frac{\tau_p^{(\nu, n)}}{(n-1)!} \left[p! t^{n-p-1} + \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell}{(p+\ell+1)\ell!} [2B_{2(p+\ell+1)} - B_{2(p+\ell+1)}(\nu)] t^{n+\ell} \right] \\
&= \frac{\omega_n e^{\left(\frac{n^2}{4} + \nu^2\right)t}}{(4\pi t)^n} \left[1 + \sum_{p=0}^{n-2} \frac{\tau_p^{(\nu, n)} p!}{(n-1)!} t^{n-p-1} + \sum_{p=0}^{n-1} \frac{\tau_p^{(\nu, n)}}{(n-1)!} \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell}{(p+\ell+1)\ell!} [2B_{2(p+\ell+1)} - B_{2(p+\ell+1)}(\nu)] t^{n+\ell} \right] \\
&= \frac{\omega_n e^{\left(\frac{n^2}{4} + \nu^2\right)t}}{(4\pi t)^n} [1 + T_1 + T_2]
\end{aligned}$$

For T_1 , we have

$$T_1 = \sum_{p=0}^{n-2} \frac{\tau_p^{(\nu, n)} p!}{(n-1)!} t^{n-p-1} = \sum_{d=1}^{n-1} \frac{\tau_{n-d-1}^{(\nu, n)} (n-d-1)!}{(n-1)!} t^d. \quad (7.54)$$

For T_2 , we have

$$\begin{aligned}
S_2 &= \sum_{p=0}^{n-1} \frac{\tau_p^{(\nu, n)}}{(n-1)!} \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell}{(p+\ell+1)\ell!} [2B_{2(p+\ell+1)} - B_{2(p+\ell+1)}(\nu)] t^{n+\ell} \\
&= \sum_{p=0}^{n-1} \frac{\tau_p^{(\nu, n)}}{(n-1)!} \sum_{d=n}^{+\infty} \frac{(-1)^{d-n}}{(p+d-n+1)(d-n)!} [2B_{2(p+d-n+1)} - B_{2(p+d-n+1)}(\nu)] t^d \\
&= \sum_{d=n}^{+\infty} \left(\sum_{p=0}^{n-1} \frac{(-1)^{d-n} \tau_p^{(\nu, n)} [2B_{2(p+d-n+1)} - B_{2(p+d-n+1)}(\nu)]}{(n-1)!(p+d-n+1)(d-n)!} \right) t^d
\end{aligned} \quad (7.55)$$

Therefore

$$\mathcal{T} \simeq \frac{\omega_n e^{\left(\frac{n^2}{4} + \nu^2\right)t}}{(4\pi t)^n} \left[1 + \sum_{d=1}^{n-1} \frac{\tau_{n-d-1}^{(\nu, n)} (n-d-1)!}{(n-1)!} t^d + \sum_{d=n}^{+\infty} \left(\sum_{p=0}^{n-1} \frac{(-1)^{d-n} \tau_p^{(\nu, n)} [2B_{2(p+d-n+1)} - B_{2(p+d-n+1)}(\nu)]}{(n-1)!(p+d-n+1)(d-n)!} \right) t^d \right]$$

which can also be written as

$$\frac{\omega_n}{(4\pi t)^n} e^{\left(\frac{n^2}{4} + \nu^2\right)t} \sum_{i=0}^{+\infty} c_i^{(\nu, n)} t^i \quad (7.56)$$

where

$$c_i^{(\nu, n)} = \begin{cases} 1, & i = 0 \\ \tau_{n-1-i}^{(\nu, n)} \frac{(n-i-1)!}{(n-1)!}, & 1 \leq i \leq n-1 \\ \sum_{p=0}^{n-1} \frac{(-1)^{i-n} \tau_p^{(\nu, n)} [2B_{2(p+i-n+1)} - B_{2(p+i-n+1)}(\nu)]}{(n-1)!(p+i-n+1)(i-n)!}, & i \geq n \end{cases} \quad (7.57)$$

Since in the last expression of $c_i^{(\nu, n)}$ where $i \geq n$, we have $p \in \{0, \dots, n-1\}$ we may write

$$\gamma_p^{(\nu, n)} = \frac{(n-1)!}{p!} c_{n-p-1}^{(\nu, n)}, \quad 0 \leq p \leq n-1 \quad (7.58)$$

and then, the expression of $c_i^{(\nu, n)}$ simplifies as follows

$$c_i^{(\nu, n)} = \begin{cases} 1, & i = 0 \\ \tau_{n-1-i}^{(\nu, n)} \frac{(n-i-1)!}{(n-1)!}, & 1 \leq i \leq n-1 \\ \sum_{p=0}^{n-1} \frac{(-1)^{i-n} c_{n-p-1}^{(\nu, n)} [2B_{2(p+i-n+1)} - B_{2(p+i-n+1)}(\nu)]}{p!(p+i-n+1)(i-n)!}, & i \geq n. \end{cases} \quad (7.59)$$

Therefore, we may rewrite (7.56) as

$$Tr \left(e^{\frac{1}{4}t\Delta\nu} \right) \simeq \frac{\omega_n}{(4\pi t)^n} \left[\sum_{k=0}^{+\infty} c_k^{(\nu, n)} t^k \right] \left[\sum_{j=0}^{+\infty} \frac{1}{j!} \left(\frac{n^2}{4} + \nu^2 \right)^j \right]. \quad (7.60)$$

Finally, in both cases the r.h.s of (7.35) and (7.60) may also be seen as a Cauchy product of two power series as

$$Tr \left(e^{\frac{1}{4}t\Delta\nu} \right) \simeq \frac{1}{(4\pi t)^n} \sum_{d=0}^{+\infty} \left[\omega_n \sum_{i=0}^d \frac{(n^2 + 4\nu^2)^{d-i}}{4^{d-i} (d-i)!} c_i^{(\nu, n)} \right] t^d.$$

This ends the proof of the theorem. ■

8 The coefficient $b_j^{(\nu, n)}$ for $n = 1, 2, 3, 4$

In this section, we exhibit the heat coefficients $b_j^{(\nu, n)}$ for the special cases $n = 1, 2, 3, 4$. For that, we first compute the coefficients $\gamma_i^{(\nu, n)}$ and $\tau_i^{(\nu, n)}$.

$$\begin{aligned} \gamma_0^{(\nu, 1)} &= 1 \\ \tau_0^{(\nu, 2)} &= -\nu^2, \quad \tau_1^{(\nu, 2)} = 1 \\ \gamma_0^{(\nu, 3)} &= -2 \left(\frac{1}{4} + \nu^2 \right), \quad \gamma_1^{(\nu, 3)}, \quad \gamma_2^{(\nu, 3)} = 1 \\ \tau_0^{(\nu, 4)} &= \nu(\nu^2 - 1), \quad \tau_1^{(\nu, 4)} = -\nu^2 + 2\nu + 1, \quad \tau_2^{(\nu, 4)} = -\nu - 2, \quad \tau_3^{(\nu, 4)} = 1. \end{aligned}$$

- For $n = 1$, we have that

$$b_j^{(\nu, 1)} = 4\pi \left[\frac{\left(\frac{1}{4} + \nu^2 \right)^j}{j!} + \sum_{i=1}^j \frac{(-1)^i \left(\frac{1}{4} + \nu^2 \right)^{j-i}}{(j-i)! i!} B_{2i}(\nu + 1/2) \right] \quad (8.1)$$

If $\nu = 0$, then (8.3) reduces to

$$b_j^{(0, 1)} = 4\pi \sum_{i=0}^j \frac{\left(\frac{1}{4} \right)^{j-i} u_i^1}{(j-i)!}$$

with

$$u_0^1 = 1, \quad u_i^1 = \frac{B_{i-1}}{(i-1)!}, \quad i \geq 1 \quad (8.2)$$

in agreement with the results obtained in [42, p.8].

- For $n = 2$, we have that

$$b_j^{(\nu, 2)} = 8\pi^2 \left[\frac{(1 + \nu^2)^j}{j!} - \frac{\nu^2 (1 + \nu^2)^{j-1}}{(j-1)!} + \sum_{i=2}^j \frac{(1 + \nu^2)^{j-i}}{(j-i)!} c_i^{(\nu, 2)} \right] \quad (8.3)$$

If $\nu = 0$, then (8.3) reduces to

$$b_j^{(0, 2)} = 8\pi^2 \sum_{i=0}^j \frac{\left(\frac{1}{4} \right)^{j-i} u_i^2}{(j-i)!}$$

with

$$u_0^2 = 1, \quad u_1^2 = 0, \quad u_i^2 = \frac{(-1)^i B_{2i}}{i(i-2)!}, \quad i \geq 2 \quad (8.4)$$

in agreement with the results obtained in [42, p.12].

- For $n = 3$, the heat coefficients read

$$b_j^{(\nu,3)} = \frac{(4\pi)^3}{3!} \sum_{i=0}^j \frac{\left(\frac{9}{4} + \nu^2\right)^{j-i}}{(j-i)!} c_i^{(\nu,3)}, \quad (8.5)$$

where

$$c_0^{(\nu,3)} = 1, \quad c_1^{(\nu,3)} = -\left(\frac{1}{4} + \nu^2\right), \quad c_2^{(\nu,3)} = \frac{1}{2} \left(\frac{1}{4} - \nu^2\right)^2, \\ c_i^{(\nu,3)} = \sum_{p=0}^2 \frac{(-1)^{i-2} c_{2-p}^{(\nu,3)} B_{2(p+i-2)}(\nu + 1/2)}{(i+p-2)p!(i-3)!}, \quad i \geq 3$$

If $\nu = 0$, then (8.5) becomes

$$b_j^{(0,3)} = \frac{(4\pi)^3}{3!} \sum_{i=0}^j \frac{\left(\frac{9}{4}\right)^{j-i}}{(j-i)!} u_i^3,$$

with

$$u_0^3 = 1, \quad u_1^3 = -\frac{1}{4}, \quad u_2^3 = \frac{1}{32}, \\ u_i^3 = \sum_{p=0}^2 \frac{(-1)^p u_{2-p}^3 B_{i+p-3}}{p!(i-3)!} = \frac{1}{2(i-3)!} \left[B_{i-1} + \frac{1}{2} B_{i-2} + \frac{1}{16} B_{i-3} \right], \quad i \geq 3$$

as expected for $n = 3$ (see [42, p.9]).

- For $n = 4$, the heat coefficients read

$$b_j^{(\nu,4)} = \frac{(4\pi)^4}{4!} \sum_{i=0}^j \frac{(1 + \nu^2)^{j-i}}{(j-i)!} c_i^{(\nu,4)}, \quad (8.6)$$

where

$$c_0^{(\nu,4)} = 1, \quad c_1^{(\nu,4)} = -\frac{1}{3}(\nu + 2), \quad c_2^{(\nu,4)} = \frac{1}{6}(-\nu^2 + 2\nu + 1), \quad c_3^{(\nu,4)} = \frac{\nu}{6}(\nu^2 - 1), \\ c_i^{(\nu,4)} = \sum_{p=0}^3 \frac{(-1)^{i-4} c_{3-p}^{(\nu,4)}}{(i+p-3)p!(i-4)!} [2B_{2(p+i-3)} - B_{2(p+i-3)}(\nu)], \quad i \geq 4$$

If $\nu = 0$, then (8.6) becomes

$$b_j^{(0,4)} = \frac{(4\pi)^4}{4!} \sum_{i=0}^j \frac{u_i^4}{(j-i)!},$$

with

$$u_0^4 = 1, \quad u_1^4 = -\frac{2}{3}, \quad u_2^4 = \frac{1}{6}, \quad u_3^4 = 0, \\ u_i^4 = \sum_{p=0}^3 \frac{(-1)^{i-4} u_{3-p}^4}{(i+p-3)p!(i-4)!} B_{2(p+i-3)}, \quad i \geq 4$$

as expected for $n = 4$ (see [42, p.13]).

A Appendix A

Here we list the basic notations and definitions of some special functions and orthogonal polynomials we have used in this paper. For more details on the theory of these functions we refer to [47, 50, 48, 49, 51].

1. For $a \in \mathbb{C}$, the shifted factorial or Pochhammer symbol is defined by

$$(a)_k = a(a+1)\dots(a+k-1), \quad k \in \mathbb{N}, \quad (\text{A.1})$$

where by convention $(a)_0 = 1$. When $a = -n$ with $n \in \mathbb{N}^* = \mathbb{N} - \{0\}$,

$$(-n)_k = \begin{cases} \frac{(-1)^k n!}{(n-k)!}, & 0 \leq k \leq n, \\ 0, & k > n. \end{cases} \quad (\text{A.2})$$

2. For $a \in \mathbb{C}$ and $k \in \mathbb{N}$, the binomial coefficient is defined by

$$\binom{\alpha}{s} := \alpha(\alpha-1)\dots(\alpha-s+1)/s! \text{ if } s \in \mathbb{Z}_+ \setminus \{0\} \text{ and } \binom{\alpha}{0} := 1, \text{ for all } \alpha \in \mathbb{R}$$

$$\binom{a}{k} = \frac{a(a-1)\dots(a-k+1)}{k!} = \frac{(-1)^k (-a)_k}{k!}. \quad (\text{A.3})$$

3. For $\xi \in \mathbb{C}$, the gamma function is defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt, \quad \text{Re } \xi > 0. \quad (\text{A.4})$$

Note that $\Gamma(n+1) = n!$ if $n \in \mathbb{N}$. The shifted factorial or Pochhammer symbol is defined by

$$(a)_k = a(a+1)\dots(a+k-1), \quad (\text{A.5})$$

and $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$ if $a \in \mathbb{C} \setminus \mathbb{Z}_-$.

4. For $a, b \in \mathbb{C}$ such that $\text{Re } a, \text{Re } b > 0$, the beta function is defined by

$$\mathcal{B}(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad (\text{A.6})$$

5. For $a_1, \dots, a_p \in \mathbb{C}$ and $c_1, \dots, c_q \in \mathbb{C} \setminus \mathbb{Z}_-$, the generalized hypergeometric function is defined by the series

$${}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ c_1, \dots, c_q \end{matrix} \middle| \xi \right) = \sum_{k=0}^{+\infty} \frac{(a_1)_k \dots (a_p)_k}{(c_1)_k \dots (c_q)_k} \frac{\xi^k}{k!}, \quad (\text{A.7})$$

which terminates whenever at least one of the p parameters a_i equals $-1, -2, -3, \dots$. It converges for $|\xi| < \infty$ if $p \leq q$ or for $|\xi| < 1$ if $p = q + 1$ and it diverges for all $\xi \neq 0$ if $p > q + 1$. Special cases of this function are the Gauss hypergeometric function ${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| \xi \right)$, the confluent hypergeometric function ${}_1F_1 \left(\begin{matrix} a \\ c \end{matrix} \middle| \xi \right)$ and the binomial series ${}_1F_0 \left(\begin{matrix} a \\ - \end{matrix} \middle| \xi \right)$. The later one reduces to $(1-\xi)^{-a}$ if $|\xi| < 1$. is called the modified Bessel function of the first kind and order a .

6. The Pfaff transformation Pfaff-Kummer transformation [32, p.33(19)]:

$${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| \xi \right) = (1-\xi)^{-a} {}_2F_1 \left(\begin{matrix} a, c-b \\ c \end{matrix} \middle| \frac{\xi}{1-\xi} \right), \quad |\arg(1-\xi)| < \pi, \quad (\text{A.8})$$

7. The Jacobi polynomial of parameters a and b is defined by

$$P_k^{(a,b)}(x) = 2^{-k} \sum_{j=0}^k \binom{k+a}{j} \binom{k+b}{k-j} (x+1)^j (x-1)^{k-j}, \quad a, b > -1. \quad (\text{A.9})$$

In particular

$$P_k^{(a,b)}(1) = \frac{(a+1)_k}{k!}. \quad (\text{A.10})$$

The Jacobi polynomials can be expressed in terms of terminating ${}_2F_1$ -series as

$$P_k^{(a,b)}(x) = \frac{\Gamma(b+k+1)}{k!\Gamma(b+1)} \left(\frac{x-1}{2}\right)^k {}_2F_1\left(\begin{matrix} -k, -a-k \\ b+1 \end{matrix} \middle| \frac{x+1}{x-1}\right), \quad (\text{A.11})$$

or

$${}_2F_1\left(\begin{matrix} -k, b \\ c \end{matrix} \middle| \xi\right) = \frac{k!}{(c)_k} P_k^{(c-1, b-c-k)}(1-2\xi). \quad (\text{A.12})$$

8. The normalized Jacobi polynomials are defined by

$$R_k^{(\alpha,\beta)}(u) := \frac{P_k^{(\alpha,\beta)}(u)}{P_k^{\alpha,\beta}(1)}.$$

By (A.8) and (A.12), we get

$$R_k^{(\alpha,\beta)}(u) = \left(\frac{1+u}{2}\right)^k {}_2F_1\left(\begin{matrix} -k, -k-\beta \\ \alpha+1 \end{matrix} \middle| \frac{1-u}{1+u}\right). \quad (\text{A.13})$$

9. The disk polynomials that were first studied by Zernike and Brinkman [34], [35] are given by

$$R_{p,q}^\gamma(\xi) := |\xi|^{p-q} e^{i(p-q)\arg \xi} R_{\min(p,q)}^{(\gamma, |p-q|)}(2|\xi|^2 - 1). \quad (\text{A.14})$$

From (A.14) and (A.13) by using the relation $\max(s, t) = |s - t| + \min(s, t)$, we can check easily that

$${}_2F_1\left(\begin{matrix} -s, -t \\ \gamma+1 \end{matrix} \middle| y\right) = (1-y)^{\frac{s+t}{2}} R_{s,t}^\gamma\left((1-y)^{-\frac{1}{2}}\right). \quad (\text{A.15})$$

10. Bernoulli number B_d is defined as the coefficient of $x^d/d!$ in the series expansion of $x(e^x - 1)^{-1}$, and the Bernoulli polynomials $B_d(z)$ are defined by $B_d(z) = \sum_{k=0}^d \binom{d}{k} B_k z^{d-k}$.

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