

# Directional distributions and the half-angle principle

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## Abstract

Angle halving, or alternatively the reverse operation of angle doubling, is a useful tool when studying directional distributions. It is especially useful on the circle where, in particular, it yields an identification between the wrapped Cauchy distribution and the angular central Gaussian distributions, as well as a matching of their parameterizations. The operation of angle halving can be extended to higher dimensions, but its effect on distributions is more complicated than on the circle. In all dimensions angle halving provides a simple way to interpret stereographic projection from the sphere to Euclidean space.

**Key words:** angular central Gaussian distribution, gnomonic projection, Möbius transformation, multivariate  $t$  distribution, stereographic projection, wrapped Cauchy distribution

## 1 Introduction

The wrapped Cauchy (WC) distribution on the circle is a remarkable distribution that appears in a wide variety of seemingly unrelated settings in probability and statistics. The ACG distribution is another important distribution in directional statistics. It was used by Tyler (1987a,b) to construct and study a robust estimator of a covariance matrix, or more generally a scatter matrix, for  $q$ -dimensional multivariate data.

As noted in Kent & Tyler (1988), the ACG distribution in  $q = 2$  dimensions (i.e. on the circle) can be identified with the WC distribution after angle doubling. Equivalently, WC distribution can be identified with the ACG distribution after angle halving. Hence algorithms to estimate the parameters of one distribution can be used with little change to estimate the parameters of the other distribution. Several algorithms to compute the maximum likelihood estimates based on the EM algorithm have been explored in Kent & Tyler (1988) and Kent et al. (1994). See also Arslan et al. (1995) for further discussion.

The current paper extends the analysis as follows:

- to use angle halving on the circle to recast the Möbius transformation in terms of a rescaled linear transformation of the plane, a result which additionally allows us to match the parameterizations of the WC and ACG distributions;
- to extend angle halving to higher dimensions and to show the connection between gnomonic projection and stereographic projection;
- to note that the ACG distribution under gnomonic projection maps to a multivariate Cauchy distribution; and to contrast it with the spherical Cauchy distribution of Kato & McCullagh (2020), which under stereographic projection maps to a multivariate  $t$ -distribution;
- to summarize some further properties of the WC distribution.

To set the scene for the main investigation of the paper, recall some basic properties of the WC and ACG distributions on the circle  $S_1$ . The WC distribution, written  $WC(\lambda)$ , has probability density function (p.d.f.)

$$f_{WC}(\theta; \lambda) = (2\pi)^{-1} \frac{1 - \lambda^2}{1 + \lambda^2 - 2\lambda \cos \theta}, \quad \theta \in S_1. \quad (1)$$

Here  $0 \leq |\lambda| < 1$  is a concentration parameter. The distribution has been centered to have its mode at  $\theta = 0$  if  $\lambda > 0$  and  $\theta = \pi$  if  $\lambda < 0$ ; it reduces to the uniform distribution if  $\lambda = 0$ .

The ACG distribution on  $S_1$ , written  $ACG(b)$ , has probability density function (p.d.f.)

$$\begin{aligned} f_{ACG}(\varphi; b) &= (2\pi)^{-1} b / \{b^2 \cos^2 \varphi + \sin^2 \varphi\} \\ &= \pi^{-1} b / \{b^2(1 + \cos 2\varphi) + (1 - \cos 2\varphi)\} \\ &= \pi^{-1} b / \{(1 + b^2) - (1 - b^2) \cos 2\varphi\}, \quad \varphi \in S_1. \end{aligned} \quad (2)$$

Here  $0 < b < \infty$  is a concentration parameter. The density is antipodally symmetric,  $f(\theta) = f(\theta + \pi)$ . The distribution has been centered to have its modes at  $\theta = 0, \pi$  if  $b < 1$  and  $\theta = \pm\pi/2$  if  $b > 1$ ; it reduces to the uniform distribution if  $b = 1$ .

If

$$b = (1 - \lambda)/(1 + \lambda), \quad (3)$$

it can be checked that (2) is the same as (1) under the angle doubling relation  $\theta = 2\varphi$ . That is, if  $\Phi$  is a random angle following the  $ACG(b)$  distribution and (3) holds, then  $\Theta = 2\Phi$  is a random angle following the  $WC(\lambda)$  distribution. The relation (3) between  $b$  and  $\lambda$  will be assumed throughout the paper.

The paper is organized as follows. Basic transformations of the circle are defined and examined in Section 2. These transformations are used in Section 3 to obtain the ACG and WC distributions on the circle as transformations of the uniform distribution. The basic transformations are extended to the sphere in Section 4 and interpreted through two projections in Section 5. The transformations are used to obtain the ACG distribution on the sphere (Section 6) and a spherical analog of the WC distribution (Section 7) as transformations of the uniform distribution. Finally, Section 8 summarizes some further derivations and motivations for the WC distribution on the circle.

For some standard background on directional distributions, see, e.g., Mardia & Jupp (2000) and Chikuse (2003). For basic results from multivariate analysis, see,

e.g., Mardia et al. (1979). A fundamental reference is McCullagh (1996), which goes further than the current paper in exploring how the family of WC distributions is closed under the group of Möbius transformations on the unit circle.

## 2 Basic operations on the circle

A point on the circle can be written as an angle  $\varphi$ , where without loss of generality,  $\varphi \in (-\pi, \pi]$ . The point can also be expressed as a unit vector

$$\mathbf{x} = (x_1, x_2)^T = (\cos \varphi, \sin \varphi)^T = \pm(1, r)^T / \sqrt{1 + r^2}, \quad r = \tan \varphi, \quad (4)$$

or as a complex number  $x_1 + ix_2 = C(\mathbf{x})$ . It is convenient to denote the mappings between vector and angular representations by

$$\varphi = \text{Arg}(\mathbf{x}), \quad \mathbf{x} = \text{vec}(\varphi). \quad (5)$$

For later use note that the derivatives of the mappings between  $\varphi$  and  $r = \tan \varphi$  are given by

$$dr/d\varphi = \sec^2 \varphi = 1/\cos^2 \varphi = 1 + r^2, \quad d\varphi/dr = 1/(1 + r^2). \quad (6)$$

Another important representation of an angle, where this time the angle is denoted  $\theta$ , is in terms of the tangent of the half-angle,  $s = \tan(\theta/2)$ . Square both sides and use the double angle formulas to get

$$s^2 = \tan^2(\theta/2) = \frac{\sin^2(\theta/2)}{\cos^2(\theta/2)} = \frac{1 - \cos \theta}{1 + \cos \theta}, \quad (7)$$

which can be inverted to give

$$\cos \theta = \frac{1 - s^2}{1 + s^2},$$

so that  $1 + \cos \theta = 2/(1 + s^2)$ .

Throughout the paper we assume that  $\theta$  and  $\varphi$  are related by the double angle condition,  $\theta = 2\varphi$ , so that  $r = s$ . However, it is helpful to use both notations  $r$  and  $s$  to emphasize that  $r$  is obtained from  $\varphi$  and  $s$  is obtained from  $\theta$ .

Three important mappings from  $S_1$  to itself are as follows.

- (a) *Squaring*, denoted  $S(\mathbf{x})$ . In vector form the transformation is defined by

$$S(\mathbf{x}) = (x_1^2 - x_2^2, 2x_1x_2)^T, \quad \mathbf{x} \in S_1. \quad (8)$$

If  $\mathbf{y} = S(\mathbf{x})$ , then in complex arithmetic  $y_1 + iy_2 = (x_1 + ix_2)^2$ . Further, if  $\varphi = \text{Arg}(\mathbf{x})$  and  $\theta = \text{Arg}(\mathbf{y})$  are the two points in angular coordinates, then  $\theta = 2\varphi$ . Hence squaring is a two-to-one mapping of  $S_1$  to itself.

- (b) The *rescaled diagonal linear transformation*, denoted  $L(\mathbf{x}; b)$ , where  $b > 0$  is a scaling constant. In vector form the transformation is defined by

$$L(\mathbf{x}; b) = (x_1, bx_2)^T / \sqrt{x_1^2 + b^2x_2^2}. \quad (9)$$

That is, the second component of  $\mathbf{x}$  is scaled by a factor  $b$ , and the resulting vector is rescaled to be a unit vector. The rescaled diagonal linear transformation can also be described as follows. If  $\mathbf{z} = L(\mathbf{x}; b)$  then

$$\tan \text{Arg}(\mathbf{z}) = b \tan \text{Arg}(\mathbf{x}). \quad (10)$$

(c) The *diagonal Möbius transformation*, denoted  $M(\mathbf{y}; \lambda)$ . In vector form the transformation is defined for  $\lambda > 0$  by

$$M(\mathbf{y}; \lambda) = (2\lambda + (1 + \lambda^2)y_1, (1 - \lambda^2)y_2)^T / (1 + \lambda^2 + 2\lambda y_1), \quad \mathbf{y} \in S_1. \quad (11)$$

If  $\mathbf{w} = M(\mathbf{y}; \lambda)$  where  $\text{Arg}(\mathbf{y}) = \theta$  and  $\text{Arg}(\mathbf{w}) = \eta$ , then  $\theta$  and  $\eta$  are related by

$$\tan \eta/2 = b \tan \theta/2, \quad (12)$$

where  $b$  and  $\lambda$  are related by (3). That is, the Möbius transformation is the same as the rescaled diagonal linear transformation after the angles  $\theta$  and  $\eta$  are divided by 2. The Möbius transformation is most commonly defined using complex arithmetic,

$$C(M(\mathbf{y}; \lambda)) = \frac{y_1 + iy_2 + \lambda}{\lambda(y_1 + iy_2) + 1}, \quad \mathbf{y} \in S_1 \quad (13)$$

where for our purposes here,  $0 < \lambda < 1$  is restricted to being real.

These transformations can be combined to give the following result, which it is helpful to call the *fundamental diagonal Möbius identity*:

$$M(S(\mathbf{x}); \lambda) = S(L(\mathbf{x}; b)), \quad \mathbf{x} \in S_1, \quad (14)$$

where  $b$  and  $\lambda$  are related by (3). That is, a rescaled diagonal linear transformation followed by squaring is the same as squaring followed by a diagonal Möbius transformation.

The identity in (14) has been stated for diagonal case. However, it is possible to construct a more general version by allowing rotations before and after the relevant transformation. Let

$$\mathbf{R}_\alpha = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \quad (15)$$

denote a  $2 \times 2$  rotation matrix by an angle  $\alpha$ . Also, recall that any  $2 \times 2$  matrix  $\mathbf{B}$  with positive determinant can be writing using the singular value decomposition as

$$\mathbf{B} = c\mathbf{R}_\alpha \text{diag}(1, b)\mathbf{R}_\beta^T$$

where  $c > 0$  and  $b > 0$ . Note that  $\mathbf{R}_\alpha^T \text{vec}(\varphi) = \text{vec}(\varphi - \alpha)$  and  $S(\mathbf{R}_\alpha^T \mathbf{x}) = \mathbf{R}_\alpha^{2T} S(\mathbf{x}) = \text{vec}(2(\theta - \alpha))$ .

Define more general versions of the rescaled diagonal linear and Möbius transformations by

$$\begin{aligned} L(\mathbf{x}; \mathbf{B}) &= \mathbf{B}\mathbf{x} / \|\mathbf{B}\mathbf{x}\| = \mathbf{R}_\alpha L(\mathbf{R}_\beta^T \mathbf{x}; b), \\ M(\mathbf{x}; \lambda, \exp(2i\alpha), \exp(2i\beta)) &= \mathbf{R}_\alpha^2 M(\mathbf{R}_\beta^{2T} \mathbf{x}; \lambda), \end{aligned} \quad (16)$$

where  $\|\mathbf{x}\|^2 = \mathbf{x}^T \mathbf{x}$ . In complex notation, the Möbius transformation becomes

$$M(\mathbf{y}; \lambda, \exp(2i\alpha), \exp(2i\beta)) = \exp(2i(\alpha - \beta)) \frac{y_1 + iy_2 + \lambda \exp(2i\beta)}{\lambda \exp(-2i\beta)(y_1 + iy_2) + 1}.$$

Note the  $L$  now depends on the matrix  $\mathbf{B}$  and  $M$  now depends on a real number and two complex numbers. The more general version of the fundamental Möbius identity becomes

$$M(S(\mathbf{x}); \lambda, \exp(2i\alpha), \exp(2i\beta)) = S(L(\mathbf{x}; \mathbf{B})). \quad (17)$$

### 3 Transformations of distributions on the circle

Let  $\Phi^*$  follow a uniform distribution on the circle, with density  $f(\varphi^*) = 1/(2\pi)$ ,  $-\pi < \varphi^* < \pi$ . Let  $R^* = \tan \Phi^*$  and  $\mathbf{X}^* = \text{vec}(\Phi^*)$  denote the corresponding tangent of the angle and the Euclidean coordinates. Consider the rescaled diagonal linear transformation  $\mathbf{X} = L(\mathbf{X}^*; b)$ , where  $b > 0$ , and let  $\Phi = \text{Arg}(\mathbf{X})$  and  $R = \tan(\Phi)$  denote the corresponding angular and tangent values.

The inverse transformation between  $\mathbf{X}$  and  $\mathbf{X}^*$  is  $\mathbf{X}^* = L(\mathbf{X}; 1/b)$ . Then the p.d.f. of  $\Phi$  is given by

$$\begin{aligned} \frac{1}{2\pi} \frac{d\varphi^*}{d\varphi} &= \frac{1}{2\pi} \frac{d\varphi^*}{dr^*} \frac{dr^*}{dr} \frac{dr}{d\varphi} \\ &= \frac{1}{2\pi} \frac{1}{1+r^{*2}} b^{-1} (1+r^2) \\ &= \frac{1}{2\pi b} \frac{\cos^2 \varphi}{\cos^2 \varphi + b^{-2} \sin^2 \varphi} \frac{1}{\cos^2 \varphi} \\ &= \frac{b}{2\pi} \frac{1}{b^2 \cos^2 \varphi + \sin^2 \varphi} = f_{\text{ACG}}(\varphi; b), \end{aligned} \quad (18)$$

where we have used the fact that  $r^{*2} = b^{-2} r^2 = b^{-2} \sin^2 \varphi / \cos^2 \varphi$ , and  $1/(1+r^2) = \cos^2 \varphi$ . In other words  $\Phi$  follows the  $\text{ACG}(b)$  distribution.

If  $\Phi^*$  follows a uniform distribution, then so does  $\Theta^* = 2\Phi^*$ . Hence

$$\Theta = \text{Arg}(M(\Theta^*, \lambda)) = 2\Phi = 2\text{Arg}(L(\Phi^*, b))$$

has p.d.f. (18) as a function of  $\varphi$  (the factor  $1/2$  from the Jacobian  $d\varphi^*/d\theta^*$  cancels the factor 2 which arises since the mapping from  $\varphi^*$  to  $\theta^*$  is two-to-one). After writing the p.d.f. in terms of  $\theta$ , the wrapped Cauchy density  $f_{\text{WC}}(\theta; \lambda)$  in (1) is obtained, where  $\lambda$  is related to  $b$  by (3).

In particular, if  $0 < \lambda < 1$ , i.e.  $0 < b < 1$ , the diagonal Möbius mapping  $\mathbf{Y} = M(\mathbf{Y}^*, \lambda)$  pulls probability mass towards the direction  $\theta = 0$ ; similarly the rescaled diagonal linear mapping  $\mathbf{X} = L(\mathbf{X}^*; b)$  pulls probability mass towards the directions  $\varphi = 0$  and  $\pi$ . Hence the WC distribution for  $\mathbf{Y}$  has a mode in the zero direction and the ACG distribution for  $\mathbf{X}$  has its modes in the directions 0 and  $\pi$ .

In summary, both the ACG and WC distributions can be obtained from suitable transformations of the uniform distribution. For simplicity, attention has been focused on the centered distributions in this section, but rotations of the modal direction can be easily included.

### 4 Basic operations on the sphere

In higher dimensions, more notation is needed. For  $q \geq 2$ , let  $S_{q-1} = \{\mathbf{x} \in \mathbb{R}^q : \mathbf{x}^T \mathbf{x} = 1\}$  denote the unit sphere in  $\mathbb{R}^q$  with surface area

$$\pi_q = 2\pi^{q/2} / \Gamma(q/2). \quad (19)$$

A point  $\mathbf{x} \in S_{q-1}$  can be written in the polar form about the north pole  $\mathbf{e}_1 = (1, 0, \dots, 0)^T$  as

$$\mathbf{x} = \pm \begin{bmatrix} \cos \varphi \\ \sin \varphi \mathbf{u} \end{bmatrix}, \quad 0 \leq \varphi \leq \pi, \quad (20)$$

where  $\mathbf{u}$  is a unit  $(q-1)$ -dimensional vector. If  $q = 2$  then  $u = \pm 1$  is just a scalar.

Using the polar representation (20), the surface measure on  $S_{q-1}$ , written  $[d\mathbf{x}]$ , say, can be written recursively as

$$[d\mathbf{x}] = \sin^{q-2} \varphi d\varphi [d\mathbf{u}]. \quad (21)$$

When  $q = 2$ , the formula simplifies to  $[d\mathbf{x}] = d\varphi$ . However, note (2) used a slightly different convention for  $\varphi$ ; the scalar  $u = \pm 1$  was not present and the angle  $\varphi$  was allowed to range through the whole circle,  $-\pi < \varphi \leq \pi$ .

For all dimensions  $q \geq 2$ , changing  $\varphi$  to  $\pi - \varphi$  and  $\mathbf{u}$  to  $-\mathbf{u}$  changes  $\mathbf{x}$  to  $-\mathbf{x}$ . Hence when studying antipodally symmetric p.d.f.s, it is sufficient to restrict  $\varphi$  to the range  $0 \leq \varphi < \pi/2$ .

Let  $\mathbf{y}$  be another point in  $S_{q-1}$  with polar representation

$$\mathbf{y} = \begin{bmatrix} \cos \theta \\ \sin \theta \mathbf{u} \end{bmatrix}. \quad (22)$$

If  $\mathbf{u}$  is the same as in (20) and  $\theta = 2\varphi$ , then  $\mathbf{y}$  can be said to be obtained from  $\mathbf{x}$  by *doubling the angle*, where ‘‘angle’’ here means the colatitude  $\varphi$ . In dimensions  $q > 2$  the concept of doubling the angle is less general than the squaring operation on the circle ( $q = 2$ ) given in (8). In particular, when  $q > 2$  the operation of doubling the angle depends on the choice of north pole.

For use below, consider the following linear function of a  $q$ -dimensional unit vector  $\mathbf{y}$ ,

$$P(\mathbf{y}) = 1 + \lambda^2 - 2\lambda \mathbf{y}^T \boldsymbol{\mu}_0, \quad (23)$$

and partition the unit vector  $\boldsymbol{\mu}_0 = (\mu_1, \boldsymbol{\mu}_2^T)^T$  in terms of a scalar and a  $(q-1)$ -vector. Using (20) and (22),  $P(\mathbf{y})$  can be rewritten as a quadratic function of  $\mathbf{x}$  as follows,

$$\begin{aligned} P(\mathbf{y}) &= 1 + \lambda^2 - 2\lambda \mathbf{y}^T \boldsymbol{\mu}_0 \\ &= (1 + \lambda^2) - 2\lambda \mu_1 \cos \theta - 2\lambda (\boldsymbol{\mu}_2^T \mathbf{u}) \sin \theta \\ &= (1 + \lambda^2)(\cos^2 \varphi + \sin^2 \varphi) - 2\lambda \mu_1 (\cos^2 \varphi - \sin^2 \varphi) - 4\lambda (\boldsymbol{\mu}_2^T \mathbf{u}) \sin \varphi \cos \varphi \\ &= (1 + \lambda^2)(x_1^2 + \mathbf{x}_2^T \mathbf{x}_2) - 2\lambda \mu_1 (x_1^2 - \mathbf{x}_2^T \mathbf{x}_2) - 4\lambda (\boldsymbol{\mu}_2^T \mathbf{x}_2) x_1 \\ &= (1 + \lambda^2 - 2\lambda \mu_1) x_1^2 + (1 + \lambda^2 + 2\lambda \mu_1) \mathbf{x}_2^T \mathbf{x}_2 - 4\lambda (\boldsymbol{\mu}_2^T \mathbf{x}_2) x_1 \\ &= Q(\mathbf{x}), \text{ say,} \end{aligned} \quad (24)$$

a homogeneous quadratic form  $\mathbf{x}^T \mathbf{A} \mathbf{x}$  with matrix

$$\mathbf{A} = \begin{bmatrix} 1 + \lambda^2 - 2\lambda \mu_1 & -2\lambda \boldsymbol{\mu}_2^T \\ -2\lambda \boldsymbol{\mu}_2 & (1 + \lambda^2 + 2\lambda \mu_1) I_{q-1} \end{bmatrix} \quad (25)$$

Since  $\boldsymbol{\mu}_0^T \boldsymbol{\mu}_0 = 1$ , and  $|\lambda| < 1$ , it can be checked that  $\mathbf{A}$  is positive definite.

## 5 Projections from the sphere to Euclidean space

In this section we look at two standard tangent projections from the sphere to the Euclidean space. It is convenient to set up the definitions and notation for all dimensions  $q \geq 2$ . We can then specialize to the case  $q = 2$  and describe how the projections are connected to the transformations of Section 3.

The first is *gnomonic projection*, taking the open hemisphere  $H_{q-1} = \{\mathbf{x} \in S_{q-1} : x_1 > 0\}$  to  $\mathbb{R}^{q-1}$ . If  $\mathbf{x}$  is a unit  $q$ -vector in the open hemisphere, it can be written in the form (20) where  $0 \leq \varphi < \pi/2$  and  $\mathbf{u}$  is a unit  $(q-1)$ -vector. As in (4), let  $r = \tan \varphi$ . Then the gnomonic projection is defined by

$$\mathbf{v} = r \mathbf{u} = \frac{\sin \varphi}{\cos \varphi} \mathbf{u} = \frac{\sin \varphi}{x_1} \mathbf{u}. \quad (26)$$

The second is stereographic projection, taking the sphere  $S_{q-1}$ , minus the point at  $-\mathbf{e}_1$ , to  $\mathbb{R}^{q-1}$ . If  $\mathbf{y} \in S_{q-1}$  is a unit vector other than  $-\mathbf{e}_1$ , write it in the form (22), where  $-\pi < \theta < \pi$ . As in (7), let  $s = \tan(\theta/2)$ . Then the *stereographic projection* of  $\mathbf{y}$  is defined by

$$\mathbf{w} = s \mathbf{u} = \frac{\sin(\theta/2)}{\cos(\theta/2)} \mathbf{u} = \frac{\sin \theta}{1 + y_1} \mathbf{u} \quad (27)$$

since  $\sin \theta = 2 \sin(\theta/2) \cos(\theta/2)$  and  $1 + y_1 = 1 + \cos \theta = 2 \cos^2(\theta/2)$ .

If  $\mathbf{y}$  is obtained from  $\mathbf{x}$  by angle doubling, then the two projections are identical. That is, if  $\theta = 2\varphi$ , then  $r = s$  and  $\mathbf{v} = \mathbf{w}$ . However, the mapping of the uniform measure on the sphere to Euclidean space is different for the two projections. For gnomonic projection, the polar coordinate representation  $\mathbf{v} = r \mathbf{u}$  states that  $r$  is the radial part of  $\mathbf{v}$  so that Lebesgue measure in the tangent space  $\mathbb{R}^{q-1}$  is related to the uniform measure on the sphere by

$$\begin{aligned} d\mathbf{v} &= r^{q-2} dr [d\mathbf{u}] \\ &= (\sin \varphi / \cos \varphi)^{q-2} (dr/d\varphi) d\varphi [d\mathbf{u}] \\ &= \cos^{-q} \varphi \{\sin^{q-2} \varphi d\varphi [d\mathbf{u}]\} \\ &= \cos^{-q} \varphi [d\mathbf{x}], \end{aligned} \quad (28)$$

using (21) and  $dr/d\varphi = \sec^2 \varphi$ . On the other hand, for stereographic projection, the polar coordinate representation  $\mathbf{w} = s \mathbf{u}$  implies

$$\begin{aligned} d\mathbf{w} &= s^{q-2} ds [d\mathbf{u}] \\ &= \{\sin(\theta/2) / \cos(\theta/2)\}^{q-2} (ds/d\theta) d\theta [d\mathbf{u}] \\ &= \frac{1}{2} \{\sin(\theta/2) / \cos(\theta/2)\}^{q-2} \{\cos(\theta/2)\}^{-2} \sin^{-(q-2)} \theta \{\sin^{q-2} \theta d\theta [d\mathbf{u}]\} \\ &= \left(\frac{1}{2}\right)^{q-1} \cos^{-2(q-1)}(\theta/2) [d\mathbf{y}] \end{aligned} \quad (29)$$

since  $ds/d\theta = (1/2) \sec^2(\theta/2)$  and  $\sin \theta = 2 \sin(\theta/2) \cos(\theta/2)$ . Except on the circle  $q = 2$ , the two differentials involve different powers of  $\cos(\theta/2) = \cos \varphi$ .

## Two projections

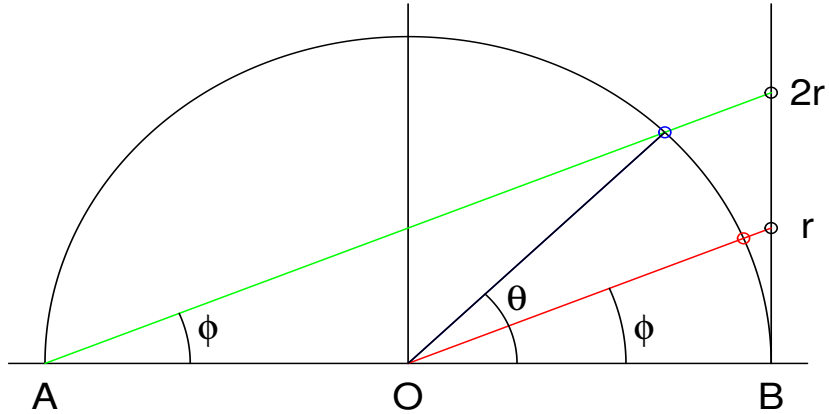


Figure 1: Two projections, gnomonic and stereographic, from the circle to the vertical line tangent to the circle at point B. If  $\varphi = \theta/2$ , then  $r = \tan \varphi = \tan \theta/2$  is both the gnomonic projection of  $\varphi$  and the stereographic projection of  $\theta$ .

Since  $d\mathbf{v} = d\mathbf{w}$  both represent Lebesgue measure in  $\mathbb{R}^{q-1}$ , (28) and (29) can be combined to describe effect of angle doubling on the sphere,

$$[d\mathbf{y}] = 2^{q-1} \cos^{q-2} \varphi [d\mathbf{x}].$$

The reason for the cosine factor is straightforward to understand intuitively. For example, consider the case  $q = 3$  corresponding to the usual sphere. For a constant value of a colatitude, the longitude can range between 0 and  $2\pi$ , and the corresponding points on the sphere lie on a small circle. If  $\varphi$  is near  $\pi/2$ , the corresponding small circle for  $\mathbf{x}$  is near the equator, a circle with circumference  $2\pi$ . However, the corresponding value of  $\theta = 2\varphi$  is near  $\pi$  and the corresponding small circle for  $\mathbf{y}$  lies near the south pole with circumference close to 0.

Figure 1 illustrates the two projections on the circle, where  $\theta = 2\varphi$ . The gnomonic projection of  $\varphi$  is obtained by following the ray from the origin O through  $(\cos \varphi, \sin \varphi)^T$  to the vertical line tangent to the circle at B. Stereographic projection of  $\theta$  is obtained by following the ray from A through  $(\cos \theta, \sin \theta)^T$  to the same vertical line and dividing the result by 2. Note the stereographic projection of  $\theta$  is the same as the gnomonic projection of  $\varphi$ .

## 6 The ACG distribution on the sphere

This section takes a closer look at the ACG distribution on the sphere  $S_{q-1}$ ,  $q \geq 2$  and in particular derives its behavior under gnomonic projection. First it is useful to recall some results about quadratic forms.



## 6.1 Review of quadratic forms in the multivariate normal distribution

Let  $\mathbf{x} = (\mathbf{x}_1^T, \mathbf{x}_2^T)^T$  be a  $q$ -dimensional vector partitioned into two parts of dimensions  $q_1$  and  $q_2$ . Similarly partition a  $q \times q$  positive definite matrix as

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}.$$

If  $\mathbf{x}$  follows a multivariate normal distribution,  $\mathbf{x} \sim N_q(\mathbf{0}, \boldsymbol{\Sigma})$ , then  $\mathbf{x}_1 \sim N_{q_1}(\mathbf{0}, \boldsymbol{\Sigma}_{11})$  and  $\mathbf{x}_2|\mathbf{x}_1 \sim N_{q_2}(\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{x}_1, \boldsymbol{\Sigma}_{22.1})$  (e.g. Mardia et al., 1979, p. 63), where  $\boldsymbol{\Sigma}_{22.1} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}$ . Writing the joint density of  $\mathbf{x}$  as a product of a marginal and a conditional density,  $f(\mathbf{x}) = f_1(\mathbf{x}_1)f(\mathbf{x}_2|\mathbf{x}_1)$  yields an identity for quadratic forms,

$$Q = Q_1 + Q_{2.1} \quad (30)$$

where

$$\begin{aligned} Q &= \mathbf{x}^T \boldsymbol{\Sigma}^{-1} \mathbf{x} \\ Q_1 &= \mathbf{x}_1^T \boldsymbol{\Sigma}_{11}^{-1} \mathbf{x}_1, \\ Q_{2.1} &= (\mathbf{x}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{x}_1)^T \boldsymbol{\Sigma}_{22.1}^{-1} (\mathbf{x}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{x}_1). \end{aligned} \quad (31)$$

If  $q_1 = 1$ ,  $q_2 = q - 1$ , then  $\mathbf{x}_1 = x_1$  is a scalar,  $\boldsymbol{\Sigma}_{11} = \sigma_{11}$  is a scalar and  $\boldsymbol{\Sigma}_{21} = \boldsymbol{\sigma}_{21}$  is a vector. This case will be useful in the next section when studying gnomonic projection.

## 6.2 Basic properties of the ACG distribution

This section reviews some basic facts about the ACG distribution. Let  $\boldsymbol{\Sigma}$  be a symmetric  $q \times q$  positive definite matrix with inverse  $\boldsymbol{\Omega} = \boldsymbol{\Sigma}^{-1}$ . The angular central Gaussian (ACG) distribution on  $S_{q-1}$  is defined by the density (with respect to the uniform measure on  $S_{q-1}$ ) by

$$f_{\text{ACG}}(\mathbf{x}) = f_{\text{ACG}}(\mathbf{x}; \boldsymbol{\Omega}) = \pi_q^{-1} |\boldsymbol{\Omega}|^{1/2} / (\mathbf{x}^T \boldsymbol{\Omega} \mathbf{x})^{q/2}. \quad (32)$$

The parameter  $\boldsymbol{\Omega}$  is defined up to a multiplicative scalar. If  $\boldsymbol{\Omega}$  has spectral decomposition  $\boldsymbol{\Omega} = \boldsymbol{\Gamma} \boldsymbol{\Delta} \boldsymbol{\Gamma}^T$  where  $\boldsymbol{\Gamma}$  is an orthogonal containing the eigenvectors and  $\boldsymbol{\Delta}$  is a diagonal matrix containing the eigenvalues, then it is possible to separate out the orientation and the concentration parts of the model. The ACG distribution is antipodally symmetric,  $f_{\text{ACG}}(\mathbf{x}) = f_{\text{ACG}}(-\mathbf{x})$ .

If  $q = 2$  and  $\boldsymbol{\Omega} = \text{diag}(b^2, 1)$  is a diagonal matrix with  $0 < b < 1$ , then the density in polar coordinates reduces to (2). A similar expansion can be carried out in higher dimensions  $q > 2$ . Suppose  $\boldsymbol{\Omega}$  is partitioned as

$$\boldsymbol{\Omega} = \begin{bmatrix} \omega_{11} & \boldsymbol{\omega}_{21}^T \\ \boldsymbol{\omega}_{21} & \boldsymbol{\Omega}_{22} \end{bmatrix}$$

and partition a unit vector  $\mathbf{x} \in S_{q-1}$  as in (20). The quadratic form becomes

$$\mathbf{x}^T \boldsymbol{\Omega} \mathbf{x} = \omega_{11} \cos^2 \varphi + 2 \sin \varphi \cos \varphi (\boldsymbol{\omega}_{21}^T \mathbf{u}) + \sin^2 \varphi \mathbf{u}^T \boldsymbol{\Omega}_{22} \mathbf{u}. \quad (33)$$

If, in addition,  $\boldsymbol{\omega}_{21} = \mathbf{0}$ , then  $\omega_{11}$  is an eigenvalue. If  $\omega_{11}$  is the smallest eigenvalue, then the density has its modes at  $\varphi = 0, \pi$ .

### 6.3 ACG distribution under gnomonic projection

Under gnomonic projection, (30) and (31) can be used to show that the ACG distribution on the sphere is transformed to a multivariate Cauchy distribution in  $\mathbb{R}^{q-1}$ . To verify this result, recall the identities in (4). Then the quadratic form  $Q = Q(\mathbf{x}) = \mathbf{x}^T \boldsymbol{\Sigma}^{-1} \mathbf{x}$ , after dividing by  $\cos^2 \varphi = 1/(1+r^2)$ , becomes

$$\begin{aligned} (1+r^2)Q &= \omega_{11} + 2\mathbf{v}^T \boldsymbol{\omega}_{21} + \mathbf{v}^T \boldsymbol{\Omega}_{22} \mathbf{v} \\ &= \omega_{11} - \boldsymbol{\omega}_{21}^T \boldsymbol{\Omega}_{22}^{-1} \boldsymbol{\omega}_{21} + (\mathbf{v} + \boldsymbol{\Omega}_{22}^{-1} \boldsymbol{\omega}_{21})^T \boldsymbol{\Omega}_{22} (\mathbf{v} + \boldsymbol{\Omega}_{22}^{-1} \boldsymbol{\omega}_{21}) \\ &= \sigma_{11}^{-1} + (\mathbf{v} - \boldsymbol{\sigma}_{21}/\sigma_{11})^T \boldsymbol{\Sigma}_{22.1}^{-1} (\mathbf{v} - \boldsymbol{\sigma}_{21}/\sigma_{11}), \end{aligned} \quad (34)$$

using the identities  $\boldsymbol{\sigma}_{21}/\sigma_{11} = -\boldsymbol{\Omega}_{22}^{-1} \boldsymbol{\omega}_{21}$ ,  $\sigma_{11}^{-1} = \omega_{11} - \boldsymbol{\omega}_{21}^T \boldsymbol{\Omega}_{22}^{-1} \boldsymbol{\omega}_{21}$  and  $\boldsymbol{\Sigma}_{22.1}^{-1} = \boldsymbol{\Omega}_{22}$  for the inverse of a partitioned matrix (e.g., Mardia et al., 1979, p. 459). Without loss of generality we can rescale  $\boldsymbol{\Sigma}$  so that  $\sigma_{11} = 1$ .

The  $(q-1)$ -dimensional multivariate  $t$ -distribution, with location parameter  $\boldsymbol{\mu}$ , scatter matrix  $\mathbf{B}$  and degrees of freedom  $\kappa > 0$ , written  $t_{q-1}(\boldsymbol{\mu}, \mathbf{B}, \kappa)$ , has density proportional to

$$f(\mathbf{v}) \propto \{1 + \kappa^{-1}(\mathbf{v} - \boldsymbol{\mu})^T \mathbf{B}^{-1}(\mathbf{v} - \boldsymbol{\mu})\}^{-(q-1+\kappa)/2} \quad (35)$$

(e.g. Mardia et al., 1979). If  $\kappa = 1$  the distribution is known as the multivariate Cauchy distribution.

Using (28), (32) and (34) to give the p.d.f. of the ACG( $\boldsymbol{\Sigma}$ ) distribution after gnomonic projection yields

$$f_{\text{ACG,gnomonic}}(\mathbf{v}) \propto Q^{-q/2} \cos^q \varphi = Q^{-q/2} (1+r^2)^{-q/2},$$

with respect to Lebesgue measure  $d\mathbf{v}$  in the tangent plane, which is the same as (35) with  $\kappa = 1$ . That is, the gnomonic projection follows a multivariate Cauchy distribution  $t_{q-1}(\boldsymbol{\sigma}_{21}, \boldsymbol{\Sigma}_{22.1}^{-1}, 1)$ .

## 7 The spherical Cauchy distribution

Kato & McCullagh (2020) have defined the *spherical Cauchy (SC) distribution* on  $S_{q-1}$  to have the p.d.f.

$$f_{\text{SC}}(\mathbf{y}; \lambda, \boldsymbol{\mu}_0) = \pi_q^{-1} \left\{ \frac{1 - \lambda^2}{P(\mathbf{y})} \right\}^{q-1}, \quad P(\mathbf{y}) = 1 + \lambda^2 - 2\lambda \mathbf{y}^T \boldsymbol{\mu}_0, \quad \mathbf{y} \in S_{q-1}. \quad (36)$$

Here  $0 \leq \lambda < 1$  is a measure of concentration and  $\boldsymbol{\mu}_0$  is a unit  $q$ -vector representing the modal direction. When  $q = 2$ , the SC distribution reduces to the WC distribution (1).

Write  $\boldsymbol{\mu}_0 = (\mu_1, \boldsymbol{\mu}_2^T)^T$  where  $\mu_1$  is a scalar and  $\boldsymbol{\mu}_2$  is a  $(q-1)$ -vector and  $\mu_1^2 + \boldsymbol{\mu}_2^T \boldsymbol{\mu}_2 = 1$ . Then, similarly to the expansion in (24), the quantity  $P(\mathbf{y})$  in (23) can

be written in stereographic coordinates  $\mathbf{v}$  as

$$\begin{aligned}
P &= P(\mathbf{y}) = 1 + \lambda^2 - 2\lambda\mathbf{y}^T\boldsymbol{\mu}_0 \\
&= (1 + \lambda^2) - 2\lambda(\mu_1 \cos \theta + \mathbf{u}^T\boldsymbol{\mu}_2 \sin \theta) \\
&= \frac{1}{1+r^2}\{(1 + \lambda^2)(1 + r^2) - 2\lambda[(1 - r^2)\mu_1 + 2\mathbf{v}^T\boldsymbol{\mu}_2]\} \\
&= \frac{1}{1+r^2}\{\gamma + \delta r^2 - 4\lambda\mathbf{v}^T\boldsymbol{\mu}_2\} \\
&= \frac{1}{1+r^2}\{\gamma - (4\lambda^2/\delta)\boldsymbol{\mu}_2^T\boldsymbol{\mu}_2 + \delta(\mathbf{v} - (2\lambda/\delta)\boldsymbol{\mu}_2)^T(\mathbf{v} - (2\lambda/\delta)\boldsymbol{\mu}_2)\} \\
&= \frac{\gamma^*}{1+r^2}\{1 + (\mathbf{v} - \mathbf{m})^T(\mathbf{v} - \mathbf{m})/\sigma^2\}, \tag{37}
\end{aligned}$$

where in the fourth line

$$\gamma = 1 + \lambda^2 - 2\lambda\mu_1, \quad \delta = 1 + \lambda^2 + 2\lambda\mu_1,$$

and in the final line

$$\gamma^* = \gamma - (4\lambda^2/\delta)\boldsymbol{\mu}_2^T\boldsymbol{\mu}_2 = (1 - \lambda^2)/\delta, \quad \mathbf{m} = (2\lambda/\delta)\boldsymbol{\mu}_2, \quad \sigma = (1 - \lambda^2)/\delta.$$

In addition the identities  $\mathbf{v}^T\mathbf{v} = r^2\mathbf{u}^T\mathbf{u} = r^2$ ,  $\cos^2\varphi = 1/(1 - r^2)$ ,  $\cos\theta = (1 - r^2)/(1 + r^2)$ , and  $\sin\theta = 2\sin\varphi\cos\varphi = (2\tan\varphi)/(1 + r^2)$  have been used.

Using the change of variables formula (29), the distribution of the stereographic projection of  $\mathbf{y}$  has density

$$f_{\gamma,\text{stereo}}(\mathbf{v}) \propto P^{-(q-1)}\cos^{2(q-1)}(\theta/2) = \{(1 + r^2)P\}^{-(q-1)},$$

which as a function of  $\mathbf{v}$  can be identified with the density of the multivariate  $t$ -distribution  $t_{q-1}(\mathbf{m}, (q-1)^{-1}\sigma^2\mathbf{I}_{q-1}, q-1)$  distribution with  $\kappa = q-1$  degrees of freedom. Note the identification is valid even  $\boldsymbol{\mu}_0 \neq \mathbf{e}_1$ , i.e. even if the the mode of the SC distribution does not lie in the direction of the first coordinate axis. This result was proved in Kato & McCullagh (2020); see also McCullagh (1996) for a deeper study of the circular case.

Note the factor  $(1 + r^2)^{-(q-1)}$  in the density has the right power to combine with  $P^{-(q-1)}$  in the density. This property explains why the SC distribution was defined by raising  $P$  to the power  $-(q-1)$ , and not some other power, in (36).

When  $q \neq 2$ , the SC distribution can never be identified with the ACG distribution under angle doubling. In particular, the gnomonic projections of an ACG distribution follows a multivariate Cauchy distribution (i.e. a multivariate  $t$ -distribution with 1 degree of freedom). In contrast, the stereographic projection of an SC distribution follows a multivariate  $t$ -distribution with  $q-1$  degrees of freedom.

## 8 Parameterizations and motivations for the wrapped Cauchy distribution on $S_1$

The  $\text{WC}(\lambda)$  distribution on the circle arises in a variety of settings in statistics. Here we give a brief review. The standard one-dimensional Cauchy distribution with scale parameter  $b^2$  and written  $t_1(0, b^2, 1)$  in (35), plays a key role in two of the settings.

Table 1: Various parameterizations of the wrapped Cauchy distribution

Number	Parameter	$A$	$B$	$C$	Setting
1	$0 \leq \lambda < 1$	$1 - \lambda^2$	$1 + \lambda^2$	$2\lambda$	wrapped Cauchy, AR(1)
2	$0 < b \leq 1$	$2b$	$1 + b^2$	$1 - b^2$	doubled ACG, stereographic projection
3	$0 < \mu \leq \pi/2$	$\sin \mu$	1	$\cos \mu$	angular rep
4	$0 \leq \alpha < 1/2$	$\sqrt{1 - 4\alpha^2}$	1	$2\alpha$	CAR(1)

- (a) *Angle doubling.* This topic has been the main theme of the paper. In particular, the  $WC(\lambda)$  distribution can be obtained from the  $ACG(b)$  distribution by angle doubling, where  $b$  and  $\lambda$  are related by (3).
- (b) *Stereographic projection.* As noted in Sections 6-7, the  $WC(\lambda)$  distribution can be obtained from the Cauchy distribution by inverse stereographic projection when  $b$  is related to  $\lambda$  by (3).
- (c) *Wrapping.* If  $Z \sim t_1(0, b^2, 1)$ , set  $\Theta = Z \bmod 2\pi$ . Recall the Cauchy distribution has Fourier transform  $\hat{f}(t) = \exp(-b|t|)$ ,  $t \in \mathbb{R}$ , and its wrapped version has Fourier coefficients  $\hat{f}(m)$ ,  $m \in \mathbb{Z}$ . Since the  $WC(\lambda)$  distribution has Fourier coefficients,  $\lambda^{|m|}$ ,  $m \in \mathbb{Z}$ , it follows that  $\Theta \sim WC(\lambda)$  distribution with  $\lambda = \exp(-b)$ . Note this value of  $\lambda$  is different from (b).
- (d) *AR(1) process.* Consider the first-order autoregression AR(1) model in time series,

$$X_{t+1} = \lambda X_t + \epsilon_t, \quad t \in \mathbb{Z},$$

where the innovation sequence  $\{\epsilon_t\}$  consists of independent identically distributed  $N(0, \sigma_\epsilon^2)$  random variables with  $\epsilon_t$  independent of  $X_s$ ,  $s < t$ . For  $|\lambda| < 1$ , the model describes a stationary Gaussian process with spectral density (after standardizing it to be a probability density) given by the  $WC(\lambda)$  density.

- (e) *CAR(1) process.* Consider the first-order conditional autoregression CAR(1) model, defined by the conditional distributions

$$X_t | \{X_s, s \neq t\} \sim N(\alpha(X_{t-1} + X_{t+1}), \sigma_\eta^2),$$

indexed by  $t \in \mathbb{Z}$ . For  $|\alpha| < 1/2$ , this model defines a stationary process which is the same as the stationary AR(1) process. The parameters are related by  $\alpha = \lambda/(1 + \lambda^2)$ .

Several of these settings involve different ways to parameterize the WC distribution. Note that the  $WC(\lambda)$  density for  $0 \leq \lambda < 1$  can be written in the form

$$f_{WC}(\theta; \lambda) = \frac{1}{2\pi} \frac{A}{B - C \cos \theta}, \quad \theta \in S_1, \quad (38)$$

where  $A, B > 0$  and  $C \geq 0$ . Provided  $B^2 = A^2 + C^2$ , the density integrates to 1. Further, the density is unchanged if the parameters are multiplied by the same

scalar constant. Hence, there is only one free parameter. Table 1 lists some common choices for  $A, B, C$ . Further, by interchanging  $A$  and  $C$ , as has already been done for Parameterizations 1 and 2, the number of parameterizations can be doubled.

Parameterization 1 is the standard representation. As noted in (a), Parameterization 2 is motivated by doubling the angle in the ACG distribution with its standard parameterization. As noted in (b), it is also motivated by the standard parameterization of the Cauchy distribution after inverse stereographic projection. Parameterization 3 is the simplest algebraically. Parameterization 4 is motivated by the CAR(1) model in (e).

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