OPTIMAL RANGE OF HAAR MARTINGALE TRANSFORMS AND ITS APPLICATIONS

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Abstract. Let $(\mathcal{F}_n)_{n\geq 0}$ be the standard dyadic filtration on $[0,1]$. Let $\mathbb{E}_{\mathcal{F}_n}$ be the conditional expectation from $L_1 = L_1[0,1]$ onto $\mathcal{F}_n$, $n \geq 0$, and let $\mathbb{E}_{\mathcal{F}_{-1}} = 0$. We present the sharp estimate for the distribution function of the martingale transform T defined by

$$Tf = \sum_{m=0}^{\infty} \left(\mathbb{E}_{\mathcal{F}_{2m}} f - \mathbb{E}_{\mathcal{F}_{2m-1}} f \right), \ f \in L_1,$$

in terms of the classical Calderón operator. As an application, for a given symmetric function space E on $[0,1]$, we identify the symmetric space $\mathcal{S}_E$, the optimal Banach symmetric range of martingale transforms/Haar basis projections acting on E.

1. Introduction

Recall that the Haar system is formed by the functions $h_{0,0}(t) = h_1(t) = 1$,

$$h_{n,k}(t) = h_{2^n+k}(t) = \begin{cases} 1, t \in \Delta_{n+1}^{2k-1} \\ -1, t \in \Delta_{n+1}^{2k} \\ 0, \text{ for all other } t \in [0,1], \end{cases}$$

where $n = 0,1,\ldots$, $k = 1,\ldots,2^n$ and $\Delta_m^j = ((j-1)2^{-m}, j2^{-m})$, $m = 1,2,\ldots$, $j = 1,\ldots,2^m$. It is well known (see e.g., [17, Ch. 3] or [20, Proposition II.2.c.1]) that this system is a basis in $L_p = L_p[0,1]$ for all $1 \leq p < \infty$ and even in every separable symmetric function space [20, Proposition II.2.c.1]. Moreover, according to a remarkable result due to Paley [22] (see also [20, Theorem II.2.c.5.] or [17, § 3.3]):

> The Haar system $\{h_n\}_{n=1}^{\infty}$ is an unconditional basis in L_p for every $1 < p < \infty$.

This result turned to be extremely rich in its connections with many important problems of interest in analysis and probability theory. In particular, it served as the starting point for in-depth research undertaken by Burkholder, who has obtained sharp inequalities of Paley type for general classes of martingale transforms (see [6, 7, 8]).

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Nowadays, martingale transforms provides insights not only into probability and statistics but also into harmonic analysis, geometry of various classes of Banach spaces, operator algebras and mathematical physics (see e.g. [3, 9, 28, 13] and references therein). It is worth to note that the properties of the transformed martingales differ markedly from those of initial martingales (see e.g. the remark at the very beginning of [8], "... there do exist small martingales with large transforms"). The main result of this paper, the sharp estimate for the distribution function of the Haar martingale transform in terms of the classical Calderón operator, indicates that Burkholder's remark remains relevant also in this setting.

We detail now our setting. Let $(\mathcal{F}_n)_{n\geq 0}$ be the standard dyadic filtration on $[0,1]$. Given an arbitrary sequence $\epsilon = \{\epsilon_n\}_{n\geq 0}$, with $\epsilon_n \in \{-1,0,1\}$, $n \geq 0$, we consider a class of special martingale transforms T_ϵ defined by

$$T_\epsilon x = \sum_{n\geq 0} \epsilon_n \cdot (\mathbb{E}_n x - \mathbb{E}_{n-1} x),\ x \in L_1, \tag{1.1}$$

where $\mathbb{E}_n$ is the conditional expectation from L_1 onto $\mathcal{F}_n$, $n \geq 0$, and the series are understood in the sense of convergence in measure.

The classical Calderón operator S is defined by

$$(Sx)(t) := \frac{1}{t}\int_0^t x(s)\,ds + \int_t^1 \frac{x(s)}{s}\,ds,\ x \in L_1. \tag{1.2}$$

Our main interest in this paper lies in the comparison of the distribution functions of elements $|T_\epsilon x|$ and $|S(x)|$, or equivalently, of their decreasing right-continuous rearrangements $\mu(T_\epsilon x)$ and $\mu(Sx)$, respectively. Any martingale transform T_ϵ of the form (1.1) is a contraction in L_2 and is of weak type $(1,1)$ with constant 2 (this can be derived from [17, Theorem 3.3.7] or [21, Theorem 5.1]). Moreover, it is self-adjoint in the sense that

$$\int_0^1 T_\epsilon x(s) y(s)\,ds = \int_0^1 x(s) T_\epsilon y(s)\,ds, \quad x, y \in L_2.$$

Therefore, T_ϵ has an upper pointwise estimate given by the operator S: there exists a constant C_{abs} such that

$$\mu(T_\epsilon x) \leq C_{\mathrm{abs}} S\mu(x),\ \forall \epsilon = \{\epsilon_m\}_{m\geq 0} \text{ and } \forall x \in L_1 \tag{1.3}$$

(see e.g. [10, Appendix], [14, Proposition 5.2.2, p. 50], [29], [2, Example 4.15] and [15], [31] for a more general setting). The estimates of the type (1.3) are well known not only for transforms T_ϵ but also for other classical operators such as the Hilbert transform and the conjugate-function operator (see, for instance, [30, § 2] and [4, Theorem 3.6.10]), which, in fact, admit a converse. However, the case of the converse estimate for martingale transforms remains open, and we state it here as follows

Is estimate (1.3) *optimal?*

A similar question was also raised in [12] in the non-commutative setting. The main result of the present paper not only answers this question in the affirmative, but it also shows that the required optimality is achieved in fact by just *one* operator $T = T_\epsilon$ with $\epsilon = \{1, 0, 1, 0, \cdots\}_{m\geq 0}$, i.e.

$$Tx := \sum_{m\geq 0} (\mathbb{E}_{2m} x - \mathbb{E}_{2m-1} x),\ x \in L_1. \tag{1.4}$$

In order to state the main result, Theorem 1 below, we recall that the dilation operator σ_s, $s > 0$ (on the linear space of all measurable functions on $[0,1]$) is defined by $\sigma_s x(t) = x(t/s)\chi_{[0,1]}(t/s)$, $t \in [0,1]$.

Theorem 1. *For every function $x \in L_1$ there exists $f \in L_1$ such that*

$$|f| \le 3\sigma_4\mu(x) \ \ \text{and} \ \ \sigma_{\frac{1}{8}} S\mu(x) \le 12\mu(Tf).$$

Recall that Paley's result for L_p-spaces was later extended (see e.g. [20, II.2.c] or [18, Theorem II.9.6]) to the setting of separable symmetric function spaces having non-trivial Boyd indices (equivalently, separable interpolation spaces between L_p and L_q, for some $1 < p \le q < \infty$ [20]). The same condition is equivalent to the boundedness of the Calderón operator S on a symmetric function space (see e.g. [4, Chapter 3, Theorem 6.10 and Corollary 6.11]). Therefore, an immediate consequence of Theorem 1 is the fact that the unconditionality of the Haar basis in a symmetric function space E can be equivalently restated in the terms of the boundedness of the operator T in E (cf. [19]). Moreover, this result allows us to identify the optimal Banach symmetric range of martingale transforms on E, for any given symmetric function space E on $[0,1]$.

In Section 4.1, we introduce the least receptacle $\mathcal{S}_E$ of the Calderón operator S on a quasi-Banach symmetric function space E such that $E \subset L_1$ and, as an application of Theorem 1, we show that the optimal symmetric quasi-Banach range of T on such a space E coincides with $\mathcal{S}_E$ (see Theorem 16 and Corollary 17). Moreover, in Section 4.3, we prove the following:

Corollary 2. *Assume that $E \subset L_1$ and F are quasi-Banach symmetric function spaces on $(0,1)$. The following statements are equivalent:*

(1) The martingale transform T is bounded from E into F.
(2) The Hilbert transform[1] H is bounded from E into F.
(3) The Calderón operator S is bounded from E into F.

If, in addition, E is separable, then each of the statements (1) — (3) is equivalent to the following:

(4) The projections $P_A : L_2 \to L_2$, $A \subset \mathbb{N}$, defined by setting

$$P_A h_i = \begin{cases} h_i, & i \in A \\ 0, & i \notin A \end{cases}.$$

extend to bounded linear mappings from E into F. Moreover,

$$\sup_{A \subset \mathbb{N}} \|P_A\|_{E\to F} < \infty.$$

In the case when E has non-trivial Boyd indices, the space $\mathcal{S}_E$ coincides with E and from this angle, the result of Corollary 2 extends and complements classical results of Paley and others cited above.

[1] As usual, the Hilbert transform H on $[0,1]$ is defined by the principal-value integral:

$$Hx(t) := \lim_{\delta\to 0} \int_{|t-s|\ge\delta} \frac{x(s)}{t-s}\, ds, \ \ t \in [0,1]$$

(equivalently, in this context we can consider the conjugate-function operator $x \mapsto \tilde{x}$, see e.g. [4, p. 160]).

In the special case, when the space E is a Lorentz space $E = \Lambda_\phi(0,1)$ with ϕ satisfying some natural conditions., we provide a precise identification of the space $\mathcal{S}_E$ as another Lorentz space (see Section 4.2 and Corollary 20).

In conclusion, we apply our results to the theory of narrow operators (see [26, 25]). In Section 4.4, we show that the identity operator on every separable quasi-Banach symmetric function space E is a sum of two narrow operators (given by basis projections with respect to the Haar basis) bounded from E into $\mathcal{S}_E$. This application extends the known result (see [26, 24, 25]) that the identity operator on a separable symmetric space E with non-trivial Boyd indices is a sum of two narrow operators bounded in E, which plays an important role in the theory of narrow operators.

2. Preliminaries

2.1. **Decreasing Rearrangement.** Let (I, m) denote the measure space $I = (0,1)$ equipped with the Lebesgue measure m. Denote by $S(0,1)$ the space of all measurable real-valued functions on (I, m) (more precisely, classes of functions which coincide almost everywhere).

For $x \in S(0,1)$, we denote by $\mu(x) = \mu(t; x)$ the decreasing right-continuous rearrangement of the function $|x|$ (see e.g. [20, II, p. 117] or [4, p. 29]), that is,

$$\mu(t;x) := \inf\{s \geq 0 :\ m(\{u \in [0,1] :\ |x(u)| > s\}) \leq t\}, \quad t \in I.$$

2.2. **Symmetric (Quasi-)Banach Function Spaces.** For the general theory of symmetric Banach function spaces (resp. quasi-Banach spaces), we refer the reader to [4, 18, 20] (resp. to [16]).

Definition 3. *We say that a (quasi-)normed space* $(E, \|\cdot\|_E)$ *is a symmetric (quasi-)normed function space on* $[0,1]$ *if the following hold:*
(a) E *is a subset of* $S(0,1)$;
(b) *If* $x \in E$ *and if* $y \in S(0,1)$ *are such that* $|y| \leq |x|$, *then* $y \in E$ *and* $\|y\|_E \leq \|x\|_E$;
(c) *If* $x \in E$ *and if* $y \in S(0,1)$ *are such that* $\mu(y) = \mu(x)$, *then* $y \in E$ *and* $\|y\|_E = \|x\|_E$.
If, in addition, $(E, \|\cdot\|_E)$ *is a (quasi-)Banach space, then* $(E, \|\cdot\|_E)$ *is called a symmetric (quasi-)Banach function space.*

For each $s > 0$, the dilation operator σ_s given by $\sigma_s x(t) = x(t/s)\chi_{[0,1]}(t/s)$, $t \in [0,1]$, is well defined and bounded on every (quasi-)Banach symmetric function space E.

The Boyd indices [20, 18] of a Banach symmetric function space E are defined by

$$\alpha(E) = \lim_{s\to 0} \frac{\ln \|\sigma_s\|_{E\to E}}{\ln s},\ \ \beta(E) = \lim_{s\to\infty} \frac{\ln \|\sigma_s\|_{E\to E}}{\ln s}.$$

In general, $0 \leq \alpha(E) \leq \beta(E) \leq 1$.

2.3. **Calderón operator.** The classical Hardy (or Cesaro) operator C and its (formal) dual C^*[2] are defined by setting

$$(Cx)(s) := \frac{1}{s}\int_0^s x(u)\,du$$

[2]For any $x, y \in L_2$, we have $\int_0^1 (Cx)(s)y(s)\,ds = \int_0^1 x(s)(C^*y)(s)\,ds$.

and

$$(C^*x)(s) := \int_s^1 \frac{x(u)}{u} du,$$

respectively [11]. It is well known that $C : L_1 \to L_{1,\infty}$ and $C^* : L_1 \to L_1$, where the quasi-Banach symmetric space $L_{1,\infty} := L_{1,\infty}(0,1)$ consists of all functions $x \in S(0,1)$ such that the quasi-norm

$$\|x\|_{L_{1,\infty}} := \sup_{0<t\le 1} t\mu(t;x)$$

is finite.

One can easily see that the Calderón operator S (see (1.2)) satisfies the following equality

$$(Sx)(t) = (Cx)(t) + (C^*x)(t), \; x \in L_1.$$

3. Proof of Theorem 1

Let $\{\mathcal{F}_n\}_{n\ge 0}$ be the standard dyadic filtration on $[0,1]$. Let $\mathbb{E}_n$ be the conditional expectation from L_1 onto $\mathcal{F}_n$, $n \ge 0$, and assume for convenience that $\mathbb{E}_{-1} = 0$.

Also, we denote $I_n := (2^{-n-1}, 2^{-n})$, $J_n := (0, 2^{-n})$, $n \ge 0$, and set

$$E = \bigcup_{n\ge 0} I_{2n}. \tag{3.1}$$

Recall (see (1.4)) that the martingale transform T is defined by the formula

$$Tf = \sum_{\substack{m\ge 0 \\ m \text{ is even}}} (\mathbb{E}_m f - \mathbb{E}_{m-1} f).$$

3.1. **Pointwise upper estimate: the case of the operator C^*.** In this subsection, we were inspired by the proof of Theorem 1 in [19]; see also [1, Chapter 13.2].

For a measurable function $x \in L_1$, we define a function f_1 by setting

$$f_1 = \sum_{n=0}^{\infty} (-1)^{n+1} \mu\left(2^{-n-1}; x\right) h_{n,1}, \tag{3.2}$$

where $\{h_{n,1}\}_{n\ge 0}$ is a subsequence of the Haar system $\{h_{n,k}\}$. Note that

$$h_{n,1} := \chi_{J_{n+1}} - \chi_{I_n} = \chi_{(0,2^{-n-1})} - \chi_{(2^{-n-1},2^{-n})}, \quad n \ge 0. \tag{3.3}$$

The following proposition delivers a pointwise upper estimate for an element $C^*(\mu(x))$ in terms of the operator T and the function f_1 introduced above.

Proposition 4. *Let $x \in L_1$. If E and f_1 are as in (3.1) and (3.2), respectively, then*

$$(Tf_1)\chi_E \ge \frac{1}{2\log(2)} \chi_E \cdot \sigma_{\frac{1}{2}} C^* \mu(x).$$

We split the proof of Proposition 4 into several steps.

The first lemma is just a simple observation. We provide a short proof for the reader's convenience.

Lemma 5. *Let $a_n \in \mathbb{R}$, $n \ge 0$. We have*

$$\sum_{n=0}^{\infty} a_n h_{n,1} = -a_0 \chi_{I_0} + \sum_{m=1}^{\infty} \left(\left(\sum_{n=0}^{m-1} a_n \right) - a_m \right) \chi_{I_m}.$$

Proof. By the definitions of $h_{n,1}$, I_n and J_n, we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_n h_{n,1} &\overset{(3.3)}{=} \sum_{n=0}^{\infty} a_n \left(\chi_{J_{n+1}} - \chi_{I_n}\right) \\ &= \sum_{n=0}^{\infty} a_n \chi_{J_{n+1}} - \sum_{n=0}^{\infty} a_n \chi_{I_n} \\ &= \sum_{n=0}^{\infty} a_n \sum_{m=n+1}^{\infty} \chi_{I_m} - \sum_{m=0}^{\infty} a_m \chi_{I_m} \\ &= \sum_{m=1}^{\infty} \chi_{I_m} \sum_{n=0}^{m-1} a_n - \sum_{m=0}^{\infty} a_m \chi_{I_m} \\ &= -a_0 \chi_{I_0} + \sum_{m=1}^{\infty} \left(\left(\sum_{n=0}^{m-1} a_n \right) - a_m \right) \chi_{I_m}.\end{aligned}$$

This completes the proof. □

For the sake of convenience, we observe the following standard result.

Lemma 6. *Let $\{b_n\}_{n \geq 1} \subset \mathbb{R}$ be a sequence with alternating signs and with increasing absolute values. We have*

$$\left| \left(\sum_{n=1}^{m-1} b_n \right) - b_m \right| \leq 2|b_m|.$$

Lemma 7. *Let $x \in L_1$. If f_1 is as in (3.2), then we have*

$$|f_1| \leq 2\sigma_2 \mu(x).$$

Proof. Let

$$b_n := (-1)^{n+1} \mu \left(2^{-n-1}; x\right), \quad n \geq 1.$$

This is a sequence with alternating signs and with increasing absolute values. By the definition of f_1, we have

$$f_1 = \sum_{n=1}^{\infty} b_n h_{n,1}.$$

By Lemma 5, we have

$$f_1|_{I_m} = \Big(\sum_{n=1}^{m-1} b_n \Big) - b_m, \quad m \geq 1.$$

By Lemma 6, we have

$$|f_1| \Big|_{I_m} \leq 2|b_m| = 2\mu(2^{-m-1}, x) \leq 2\sigma_2 \mu(x) \Big|_{I_m}, \quad m \geq 1.$$

A combination of these inequalities yields $|f_1| \leq 2\sigma_2 \mu(x)$ on every I_m, $m \geq 1$, and, therefore, on $(0, \frac{1}{2})$. On the interval $(\frac{1}{2}, 1)$, we have

$$\left| f_1 \chi_{(\frac{1}{2},1)} \right| \overset{(3.2)}{=} \mu\left(0; x\right) \leq \mu(x) \chi_{(\frac{1}{2},1)}.$$

This completes the proof. □

Proof of Proposition 4. By definitions (1.4) and (3.3), we have $Th_{n,1} = h_{n,1}$ for every odd natural number n and $Th_{n,1} = 0$ for every even natural number n. Therefore, we have

$$Tf_1 = \sum_{n=0}^{\infty}(-1)^{n+1}\mu\left(2^{-n-1};x\right)Th_{n,1} = \sum_{\substack{n\geq 1\\ n \text{ is odd}}}\mu\left(2^{-n-1};x\right)h_{n,1} = \sum_{n\geq 1}c_n h_{n,1},$$

where

$$c_n = \begin{cases}\mu\left(2^{-n-1};x\right), & n \text{ is odd};\\ 0, & n \text{ is even}.\end{cases}$$

By Lemma 5, for even m,

$$Tf_1\big|_{I_m} = \left(\sum_{n=0}^{m-1}c_n\right) - c_m = \sum_{\substack{0\leq n\leq m-1\\ n \text{ is odd}}}\mu(2^{-n-1};x). \tag{3.4}$$

For an even natural number $m \geq 1$, it follows that

$$\begin{aligned}\sum_{\substack{0\leq n\leq m-1\\ n \text{ is odd}}}\mu(2^{-n-1};x) &\geq \frac{1}{2}\sum_{n=0}^{m-1}\mu(2^{-n-1};x)\\ &= \frac{1}{2\log(2)}\sum_{n=0}^{m-1}\int_{2^{-n-1}}^{2^{-n}}\mu(2^{-n-1};x)\frac{ds}{s}\\ &\geq \frac{1}{2\log(2)}\sum_{n=0}^{m-1}\int_{2^{-n-1}}^{2^{-n}}\mu(s;x)\frac{ds}{s}\\ &= \frac{1}{2\log(2)}\int_{2^{-m}}^{1}\mu(s;x)\frac{ds}{s}\\ &= \frac{1}{2\log(2)}(C^*\mu(x))(2^{-m}).\end{aligned} \tag{3.5}$$

By (3.4) and (3.5), we have

$$Tf_1\big|_{I_m} \geq \frac{1}{2\log(2)}\left(\sigma_{\frac{1}{2}}C^*\mu(x)\right)\Big|_{I_m}, \quad m\geq 1 \text{ is even}.$$

This completes the proof. □

3.2. **Pointwise upper estimate: the case of the operator C.** As above, we denote $I_n = (2^{-n-1}, 2^{-n})$, $J_n = (0, 2^{-n})$, $n \geq 0$. For any integer $n \geq 0$, we define the function g_n by setting

$$g_n := \sum_{k=0}^{\infty}2^{-(n-k)_+}\cdot\chi_{I_k}, \tag{3.6}$$

where $u_+ = \begin{cases}u, & \text{if } u\geq 0;\\ 0, & \text{if } u<0.\end{cases}$

For any $x \in L_1$, we define the function f_2 by

$$f_2 := \sum_{\substack{n\geq 0\\ n \text{ is even}}}\mu\left(2^{-n-2};x\right)\chi_{I_n}. \tag{3.7}$$

Now, we state the main result of this subsection.

Proposition 8. *Let $x \in L_1$. If E and f_2 are as in (3.1) and (3.7), respectively, then*

$$(Tf_2) \cdot \chi_E \geq \frac{1}{6}\chi_E \cdot C\mu(x).$$

We split the proof of Proposition 8 into several steps.

Lemma 9. *Let $x \in L_1$. We have*

$$|f_2| \leq \sigma_4 \mu(x).$$

Proof. Observe that

$$f_2 \overset{(3.7)}{=} \sum_{\substack{n\geq 0\\ n \text{ is even}}} \mu(2^{-n-2};x)\chi_{I_n} \leq \sum_{\substack{n\geq 0\\ n \text{ is even}}} \mu(2^{-n-2};x)(\chi_{I_n} + \chi_{I_{n+1}}) \leq \sigma_4\mu(x).$$

□

Lemma 10. *Let $n \geq 0$ be an even number. If E and g_n are as in (3.1) and (3.6), respectively, then*

$$(T\chi_{I_n})\chi_E \geq \frac{1}{3} g_n \chi_E,$$

where $I_n = (2^{-n-1}, 2^{-n})$.

Proof. Let $\epsilon = \{(-1)^n\}_{n\geq 0}$ and consider the operator T_ϵ as in (1.1), i.e.

$$T_\epsilon f = \sum_{m\geq 0} (-1)^m (\mathbb{E}_m f - \mathbb{E}_{m-1} f), \quad f \in L_1.$$

Clearly,

$$\mathbb{E}_m \chi_{I_n} = \begin{cases} \chi_{I_n}, & m \geq n+1; \\ 2^{m-1-n}\chi_{J_m}, & m \leq n, \end{cases} \tag{3.8}$$

and hence

$$\mathbb{E}_m \chi_{I_n} = \mathbb{E}_{m-1}\chi_{I_n}, \quad m \geq n+2.$$

Since $\mathbb{E}_{-1} = 0$, it follows that

$$\begin{aligned} T_\epsilon \chi_{I_n} &= \sum_{m=0}^{n+1} (-1)^m (\mathbb{E}_m \chi_{I_n} - \mathbb{E}_{m-1}\chi_{I_n}) \\ &= \sum_{m=0}^{n+1} (-1)^m \mathbb{E}_m \chi_{I_n} - \sum_{m=0}^{n} (-1)^{m-1} \mathbb{E}_m \chi_{I_n} \\ &\overset{(3.8)}{=} (-1)^{n+1}\chi_{I_n} + 2\sum_{m=0}^{n} (-1)^m 2^{m-1-n}\chi_{J_m}. \end{aligned} \tag{3.9}$$

By the definition of J_m and I_m, we have

$$\chi_{J_m} = \sum_{k=m}^{\infty} \chi_{I_k}.$$

Therefore,

$$
\begin{aligned}
\sum_{m=0}^{n}(-1)^m 2^{m-1-n}\chi_{J_m} &= \sum_{m=0}^{n}(-1)^m 2^{m-1-n}\sum_{k=m}^{\infty}\chi_{I_k} \\
&= \sum_{k=0}^{\infty}\chi_{I_k}\sum_{m=0}^{\min\{k,n\}}(-1)^m 2^{m-1-n} \\
&= 2^{-n-1}\sum_{k=0}^{\infty}\chi_{I_k}\cdot\frac{(-2)^{\min\{k,n\}+1}-1}{-3}.
\end{aligned}
\tag{3.10}
$$

Now, we arrive at

$$
\begin{aligned}
T_\epsilon\chi_{I_n} &\overset{(3.9)}{=} (-1)^{n+1}\chi_{I_n} + 2\sum_{m=0}^{n}(-1)^m 2^{m-1-n}\chi_{J_m} \\
&\overset{(3.10)}{=} (-1)^{n+1}\chi_{I_n} + 2^{-n}\sum_{k=0}^{\infty}\chi_{I_k}\cdot\frac{(-2)^{\min\{k,n\}+1}-1}{-3} \\
&= (-1)^{n+1}\chi_{I_n} + 2^{-n}\sum_{k=0}^{n-1}\chi_{I_k}\cdot\frac{(-2)^{k+1}-1}{-3} + 2^{-n}\sum_{k=n}^{\infty}\chi_{I_k}\cdot\frac{(-2)^{n+1}-1}{-3}.
\end{aligned}
$$

If n is even, then

$$
T_\epsilon\chi_{I_n} + \chi_{I_n} = 2^{-n}\sum_{k=0}^{n-1}\chi_{I_k}\cdot\frac{(-1)^k 2^{k+1}+1}{3} + 2^{-n}\sum_{k=n}^{\infty}\chi_{I_k}\cdot\frac{2^{n+1}+1}{3}.
\tag{3.11}
$$

Thus, for any even number $n \geq 0$, we have

$$
\begin{aligned}
(T_\epsilon\chi_{I_n} + \chi_{I_n})\cdot\chi_E &\overset{(3.11)}{=} \left(2^{-n}\sum_{k=0}^{n-1}\chi_{I_k}\cdot\frac{2^{k+1}+1}{3} + \sum_{k=n}^{\infty}\chi_{I_k}\cdot\frac{2+2^{-n}}{3}\right)\cdot\chi_E \\
&\geq \left(2^{-n}\sum_{k=0}^{n-1}\chi_{I_k}\cdot\frac{2^{k+1}}{3} + \sum_{k=n}^{\infty}\chi_{I_k}\cdot\frac{2}{3}\right)\cdot\chi_E \\
&= \frac{2}{3}\chi_E\cdot\left(\sum_{k=0}^{\infty}2^{-(n-k)_+}\cdot\chi_{I_k}\right).
\end{aligned}
$$

Since $T_\epsilon + \mathrm{id} = 2T$ (see (1.4)), the assertion follows. $\square$

Lemma 11. *For every $n \geq 0$, we have*

$$
g_n \geq \frac{1}{2}C\chi_{J_n},
$$

where g_n is defined by formula (3.6) and $J_n = (0, 2^{-n})$.

Proof. If $t \in I_k$, $k \geq n$, then we have $g_n(t) = 1$ and $(C\chi_{J_n})(t) = 1$. If $t \in I_k = (2^{-k-1}, 2^{-k})$, $k < n$, then $t \notin J_n$, and therefore,

$$
(C\chi_{J_n})(t) = \frac{m(J_n)}{t} = \frac{1}{2^n t} \leq 2^{k+1-n} = 2\cdot 2^{k-n} = 2g_n(t),
$$

and the desired inequality follows. $\square$

Proof of Proposition 8. Applying successively the definition of f_2 (see (3.7)), Lemma 10 and Lemma 11, we obtain

$$
\begin{aligned}
(Tf_2)\chi_E &= \sum_{\substack{n\geq 0\\ n \text{ is even}}} \mu(2^{-n-2};x)(T\chi_{I_n})\chi_E \\
&\geq \frac{1}{3}\sum_{\substack{n\geq 0\\ n \text{ is even}}} \mu(2^{-n-2};x)g_n\chi_E \\
&\geq \frac{1}{6}\chi_E\cdot C\left(\sum_{\substack{n\geq 0\\ n \text{ is even}}} \mu(2^{-n-2};x)\chi_{J_n}\right). \qquad (3.12)
\end{aligned}
$$

Recall that $I_n=(2^{-n-1},2^{-n})$ and $J_n=(0,2^{-n})$. Observe that

$$
\sum_{\substack{n\geq 0\\ n \text{ is even}}} \mu(2^{-n-2};x)\chi_{J_n} \geq \sum_{\substack{n\geq 0\\ n \text{ is even}}} \mu(2^{-n-2};x)(\chi_{I_n}+\chi_{I_{n+1}}) \geq \mu(x),
$$

which together with (3.12) yields the assertion. □

3.3. **Proof of Theorem 1.** Here, we complete the proof of Theorem 1, which is a simple consequence of the estimates obtained in the previous subsections.

Lemma 12. *Let $x\in L_1$ and let $f:=f_1+f_2$, where f_1 and f_2 are defined in (3.2) and (3.7), respectively. We have*

$$
|f|\leq 3\sigma_4\mu(x) \text{ and } (Tf)\cdot\chi_E \geq \frac{1}{6}\chi_E\cdot\sigma_{\frac{1}{2}}S\mu(x).
$$

Proof. By Lemmas 7 and 9, we have

$$
|f|\leq |f_1|+|f_2|\leq 2\sigma_2\mu(x)+\sigma_4\mu(x)\leq 3\sigma_4\mu(x).
$$

On the other hand, Propositions 4 and 8 imply

$$
\begin{aligned}
(Tf)\cdot\chi_E &= (Tf_1)\cdot\chi_E+(Tf_2)\cdot\chi_E \\
&\geq \frac{1}{2\log(2)}\chi_E\cdot\sigma_{\frac{1}{2}}C^*\mu(x)+\frac{1}{6}\chi_E\cdot C\mu(x) \\
&\geq \frac{1}{6}\chi_E\cdot\sigma_{\frac{1}{2}}C^*\mu(x)+\frac{1}{6}\chi_E\cdot\sigma_{\frac{1}{2}}C\mu(x) \\
&= \frac{1}{6}\chi_E\cdot\sigma_{\frac{1}{2}}S\mu(x),
\end{aligned}
$$

and everything is done. □

Lemma 13. *If $y=\mu(y)\in S(0,1)$, then*

$$
\frac{1}{2}\sigma_{\frac{1}{4}}y\leq \mu(\chi_E\cdot y),
$$

where E is defined in (3.1).

Proof. Observe that from the definition of E it follows

$$
\chi_{E^c}\cdot y\leq \sigma_2(\chi_E\cdot y).
$$

Thus, by [4, Proposition 2.1.7], we have

$$\begin{aligned}\frac{1}{2}\sigma_{\frac{1}{4}}y = \frac{1}{2}\sigma_{\frac{1}{4}}\mu\big(\chi_{E^c}\cdot y + \chi_E\cdot y\big) &\le \frac{1}{2}\sigma_{\frac{1}{2}}\mu(\chi_{E^c}\cdot y) + \frac{1}{2}\sigma_{\frac{1}{2}}\mu(\chi_E\cdot y)\\ &\le \frac{1}{2}\mu(\chi_E\cdot y) + \frac{1}{2}\sigma_{\frac{1}{2}}\mu(\chi_E\cdot y) \le \mu(\chi_E\cdot y),\end{aligned}$$

and the proof is completed. □

Proof of Theorem 1. Let f be defined as in Lemma 12. Then, $|f| \le 3\sigma_4\mu(x)$. Moreover, by Lemmas 12 and 13, we have

$$\mu(Tf) \ge \mu((Tf)\cdot\chi_E) \ge \frac{1}{6}\mu(\chi_E\cdot\sigma_{\frac{1}{2}}S\mu(x)) \ge \frac{1}{12}\sigma_{\frac{1}{8}}S\mu(x).$$

□

4. Applications to the geometry of Banach spaces

4.1. **Optimal symmetric quasi-Banach range for the martingale transforms.** From Theorem 1 and estimate (1.3) it follows that the optimal symmetric Banach range of the martingale transform T on a quasi-Banach symmetric function space E coincides with that of the Calderón operator S on E. Thus, we arrive at the problem of a description of the least receptacle of the operator S acting on E. To solve the latter problem, we employ the description of the optimal symmetric range for the Calderón operator defined on a quasi-Banach symmetric space on $(0,\infty)$ given in [31].

For definitions related to quasi-Banach symmetric spaces on $(0,\infty)$ (which differ only slightly from those in the case $[0,1]$) we refer the reader to the books [4, 18, 20]. In particular, $S(0,\infty)$ is the set of all measurable functions x on $(0,\infty)$ such that $m(\{t : |x(t)| > s\})$ is finite for some $s > 0$.

Recall that

$$L_{1,\infty}(0,\infty) := \{f \in S(0,\infty) : \|f\|_{L_{1,\infty}(0,\infty)} := \sup_{t>0} t\mu(t;f) < \infty\}$$

and

$$\Lambda_{\log}(0,\infty) := \left\{x \in S(0,\infty) : \|x\|_{\Lambda_{\log}(0,\infty)} := \int_0^\infty \mu(s;x)\frac{ds}{s+1} < \infty\right\}.$$

The Calderón operator (on the semiaxis) is given by

$$(S_\infty x)(t) := \frac{1}{t}\int_0^t x(s)ds + \int_t^\infty x(s)\frac{ds}{s},\ x \in \Lambda_{\log}(0,\infty).$$

For convenience of the reader, we describe first shortly the main result in [31]. Further, we still denote a symmetric function space on $[0,1]$ by E, while the notation $E(0,\infty)$ will be reserved for symmetric function spaces on $(0,\infty)$.

Given quasi-Banach symmetric space $E(0,\infty)$ such that $E(0,\infty) \subset \Lambda_{\log}(0,\infty)$, we define the linear space $\mathcal{S}_E(0,\infty)$ by

$$\mathcal{S}_E(0,\infty) = \{x \in (L_{1,\infty}+L_\infty)(0,\infty) : \exists y \in E(0,\infty),\ \mu(x) \le S_\infty\mu(y)\}, \tag{4.1}$$

equipped with the functional

$$x \mapsto \|x\|_{\mathcal{S}_{E(0,\infty)}} := \inf\{\|y\|_E : \mu(x) \le S\mu(y)\}.$$

Theorem 14. [31, Theorem 26] *Let $E(0,\infty) \subset \Lambda_{\log}(0,\infty)$ be a quasi-Banach symmetric space on $(0,\infty)$. We have*

(i) $(\mathcal{S}_E(0,\infty), \|\cdot\|_{\mathcal{S}_E}(0,\infty))$ *is a quasi-Banach symmetric function space.*
(ii) $\mathcal{S}_E(0,\infty)$ *is the optimal symmetric quasi-Banach range for the operator* S *on* $E(0,\infty)$.

Below, we obtain a similar identification of the optimal symmetric range for the Calderón operator on a given quasi-Banach symmetric space on $(0,1)$.

Definition 15. *Let* E *be a quasi-Banach symmetric space on* $(0,1)$ *such that* $E \subset L_1$. *Define the linear space*

$$\mathcal{S}_E = \{x \in L_{1,\infty} = L_{1,\infty}(0,1) : \exists y \in E,\ \mu(x) \le S\mu(y)\}, \tag{4.2}$$

and equip it with the functional

$$x \mapsto \|x\|_{\mathcal{S}_E} := \inf\{\|y\|_E : \mu(x) \le S\mu(y)\}.$$

Theorem 16. *Let* $E \subset L_1$ *be a quasi-Banach symmetric space on* $(0,1)$. *We have*
(i) $(\mathcal{S}_E, \|\cdot\|_{\mathcal{S}_E})$ *is a quasi-Banach symmetric function space.*
(ii) $\mathcal{S}_E$ *is the optimal symmetric quasi-Banach range for the operator* S *on* E.

Proof. (i). For simplicity of notations, we may assume that $\left\|\chi_{(0,1)}\right\|_E = 1$. Define a symmetric quasi-Banach function space $F(0,\infty)$ on $(0,\infty)$ by setting

$$F(0,\infty) := \Big\{x \in L_1(0,\infty) : \|x\|_{F(0,\infty)} := \left\|\mu(x)\chi_{(0,1)}\right\|_E + \|x\|_{L_1(0,\infty)} < \infty\Big\}.$$

We claim that for every x supported on $(0,1)$, we have

$$\frac{1}{4}\|x\|_{\mathcal{S}_{F(0,\infty)}} \le \|x\|_{\mathcal{S}_E} \le 2\|x\|_{\mathcal{S}_{F(0,\infty)}}. \tag{4.3}$$

Indeed, if $x \in \mathcal{S}_E$, then there exists $y \in E$ such that

$$\mu(x) \le S\mu(y) \text{ and } \|y\|_E \le 2\|x\|_{\mathcal{S}_E}.$$

Extending y to a function on $(0,\infty)$ by setting $y = 0$ on $(1,\infty)$, we still have $\mu(x) \le S_\infty\mu(y)$. Moreover, in view of the embedding $E \subset L_1$ with constant 1 (see e.g. [18, Theorem II.4.1]), it holds

$$\|y\|_{F(0,\infty)} = \|y\|_E + \|y\|_{L_1(0,\infty)} \le 2\|y\|_E \le 4\|x\|_{\mathcal{S}_E}.$$

Taking the infimum over all such y, we infer that $\|x\|_{\mathcal{S}_{F(0,\infty)}} \le 4\|x\|_{\mathcal{S}_E}$.

Next, let $x \in \mathcal{S}_{F(0,\infty)}$ with support in $(0,1)$ and let $y \in F(0,\infty)$ be such that $\mu(x) \le S_\infty\mu(y)$ and $\|y\|_{F(0,\infty)} \le 2\|x\|_{\mathcal{S}_{F(0,\infty)}}$ (see Theorem 14). Without loss of generality, we may assume that $y = \mu(y)$. Set

$$z(t) = \Big(y(t) + \int_1^\infty y(s)\frac{ds}{s}\Big)\chi_{(0,1)}(t), \quad t \in (0,1).$$

We have

$$\mu(t;x) \le (S_\infty y)(t) \le (Sz)(t), \quad t \in (0,1).$$

Also,

$$\begin{aligned}\|z\|_E &\le \left\|y\chi_{(0,1)}\right\|_E + \int_1^\infty y(s)\frac{ds}{s}\\ &\le \left\|y\chi_{(0,1)}\right\|_E + \|y\|_{L_1(0,\infty)}\\ &= \|y\|_{F(0,\infty)} \le 2\|x\|_{\mathcal{S}_{F(0,\infty)}}.\end{aligned}$$

Taking the infimum over all such z, we get $\|x\|_{\mathcal{S}_E} \le 2\|x\|_{\mathcal{S}_{F(0,\infty)}}$.

Clearly, $\|\cdot\|_{\mathcal{S}_E}$ is a homogeneous functional. Since $\|\cdot\|_{\mathcal{S}_F(0,\infty)}$ is a quasi-norm (see Theorem 14 above), it follows from (4.3) that $\|\cdot\|_{\mathcal{S}_E}$ is also a quasi-norm.

Let us now prove the completeness of $(\mathcal{S}_E, \|\cdot\|_{\mathcal{S}_E})$. Let $(x_n)_{n\geq 0}$ be a Cauchy sequence in $\mathcal{S}_E$. By (4.3), $(x_n)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{S}_F(0,\infty)$. By the completeness of $\mathcal{S}_F(0,\infty)$, we have that $x_n \to x$ in $\mathcal{S}_F(0,\infty)$. Clearly, x is also supported on $(0,1)$. Again using (4.3), we conclude that $x_n \to x$ in $\mathcal{S}_E$. On the other hand, by the definition of $\|\cdot\|_{\mathcal{S}_E}$, we obtain that the quasi-norm $\|\cdot\|_{\mathcal{S}_E}$ is symmetric.

(ii) From the definition of $\mathcal{S}_E$ it follows immediately that $\mathcal{S}_E$ is the minimal receptacle of the operator S in the category of quasi-Banach symmetric function spaces (see also [31, p.3549] for a full proof in the setting of $(0,\infty)$). □

The following result is a combination of Theorems 1 and 16 with estimate (1.3).

Corollary 17. *Assume that $E \subset L_1$ is quasi-Banach symmetric function space on $(0,1)$. Then, the space $\mathcal{S}_E$ generated by the Calderón operator S is the optimal symmetric quasi-Banach range for the martingale transform T on E.*

4.2. **Optimal symmetric Banach range for the martingale transforms in Lorentz spaces.** Here, we apply the results obtained in the preceding sections to present a description of the optimal symmetric Banach range of the martingale transforms on Lorentz function spaces on $[0,1]$.

Let $\phi : [0,1) \to [0,1)$ (respectively, $\phi : [0,\infty) \to [0,\infty)$) be an increasing concave function such that $\lim_{t\to 0+} \phi(t) = 0$ (or briefly $\phi(+0) = 0$). The Lorentz space Λ_ϕ (respectively, $\Lambda_\phi(0,\infty)$ is defined by setting

$$\Lambda_\phi := \left\{ x \in S(0,1) : \|x\|_{\Lambda_\phi} := \int_0^1 \mu(s;x)\, d\phi(s) < \infty \right\}$$

(respectively,

$$\Lambda_\phi(0,\infty) = \left\{ x \in S(0,\infty) : \|x\|_{\Lambda_\phi} := \int_0^\infty \mu(s;x)\, d\phi(s) < \infty \right\}).$$

In [32], the optimal Banach symmetric range of the Calderón operator S_∞ on Lorentz spaces $\Lambda_\phi(0,\infty)$ was determined. Let us state their main result.

Theorem 18. [32, Theorem 11] *Let $\phi : [0,\infty) \to [0,\infty)$ be an increasing concave function such that $\phi(0+) = 0$. Suppose the function*

$$\psi(u) := \inf_{w>1} \frac{\phi(uw)}{1+\log(w)}$$

satisfies $\lim_{t\to\infty} \frac{\psi(t)}{t} = 0$. Then, the conditions $\Lambda_\phi(0,\infty) \subset \Lambda_{\log}(0,\infty)$ and

$$\int_0^u \frac{\psi(t)}{t} dt + u \int_u^\infty \frac{\psi(t)}{t^2} dt \leq c_{\phi,\psi}\phi(u),\ \ u>0,$$

imply the following:

(i) *The Calderón operator $S_\infty : \Lambda_\phi(0,\infty) \to \Lambda_\psi(0,\infty)$ is bounded;*
(ii) *for every $x \in \Lambda_\psi(0,\infty)$, there exists $y \in \Lambda_\phi(0,\infty)$ such that $\mu(x) \leq S_\infty \mu(y)$ and $\|y\|_{\Lambda_\phi(0,\infty)} \leq 8\,\|x\|_{\Lambda_\psi(0,\infty)}$.*

We apply Theorem 18 to obtain a similar result for Lorentz function spaces on $[0,1]$, i.e., we determine the optimal range of the Calderón operator S on a Lorentz space Λ_ϕ as some Lorentz space Λ_ψ.

Let $\phi : [0,1) \to [0,1)$ be an increasing concave function such that $\phi(0+) = 0$. We set

$$\psi(u) := \inf_{1<w<\frac{1}{u}} \frac{\phi(uw)}{1+\log(w)}, \ u \in [0,1). \tag{4.4}$$

Theorem 19. *Let $\phi : [0,1) \to [0,1)$ be an increasing concave function such that $\phi(0+) = 0$ and let ψ be the function defined by the formula (4.4). If the inequality*

$$\int_0^u \frac{\psi(t)}{t}\,dt + u\int_u^1 \frac{\psi(t)}{t^2}\,dt \le c_{\phi,\psi}\phi(u), \quad u \in (0,1), \tag{4.5}$$

holds for some constant $c_{\phi,\psi}$, then:

(i) *The Calderón operator $S : \Lambda_\phi \to \Lambda_\psi$ is bounded;*
(ii) *for every $x \in \Lambda_\psi$, there exists $y \in \Lambda_\phi$ such that $\mu(x) \le S\mu(y)$ and $\|y\|_{\Lambda_\phi} \le 8\|x\|_{\Lambda_\psi}$.*

Proof. Without loss of generality, we may assume that $\phi(1) = 1$. We define the functions $\tilde{\phi}$ and $\tilde{\psi}$ on $(0,\infty)$ by setting

$$\tilde{\phi}(t) := \begin{cases} \phi(t), \ t \in (0,1) \\ 1+\log(t), \ t \ge 1 \end{cases}$$

and

$$\tilde{\psi}(u) := \inf_{w>1} \frac{\phi(uw)}{1+\log(w)}.$$

If $u \ge 1$, then we have

$$\tilde{\psi}(u) = \inf_{w>1} \frac{\phi(uw)}{1+\log(w)} = \inf_{w>1} \frac{1+\log(u)+\log(w)}{1+\log(w)} = 1. \tag{4.6}$$

Moreover, in the case when $0 < u < 1$

$$\begin{aligned}\tilde{\psi}(u) = \inf_{w>1} \frac{\phi(uw)}{1+\log(w)} &= \min\left\{\inf_{1<w<\frac{1}{u}} \frac{\phi(uw)}{1+\log(w)}, \ \inf_{w\ge\frac{1}{u}} \frac{\phi(uw)}{1+\log(w)}\right\} \\ &= \min\left\{\inf_{1<w<\frac{1}{u}} \frac{\phi(uw)}{1+\log(w)}, \ \inf_{w\ge\frac{1}{u}} \frac{1+\log(uw)}{1+\log(w)}\right\} \\ &= \min\left\{\inf_{1<w<\frac{1}{u}} \frac{\phi(uw)}{1+\log(w)}, \ 1\right\} \\ &= \min\{\psi(u), \ 1\}.\end{aligned}$$

One can easily verify that ψ is increasing on $(0,1)$ (see also the proof of Lemma 5 in [32]). Hence, $\psi(u) \le \psi(1) = 1$ whenever $0 < u < 1$. Thus,

$$\tilde{\psi}(u) = \psi(u) \ \text{ for all } \ 0 < u < 1.$$

Next, if $0 < u \le 1$, we have

$$\begin{aligned}\int_0^u \frac{\tilde\psi(t)}{t}dt + u\int_u^\infty \frac{\tilde\psi(t)}{t^2}dt &= \int_0^u \frac{\tilde\psi(t)}{t}dt + u\int_u^1 \frac{\tilde\psi(t)}{t^2}dt + u\int_1^\infty \frac{\tilde\psi(t)}{t^2}dt \\ &\overset{(4.5)}{\le} c_{\phi,\psi}\phi(u) + u\int_1^\infty \frac{\tilde\psi(t)}{t^2}dt \\ &\overset{(4.6)}{\le} c_{\phi,\psi}\phi(u) + u\int_1^\infty \frac{1}{t^2}dt \\ &= c_{\phi,\psi}\phi(u) + u \\ &\le (c_{\phi,\psi}+1)\tilde\phi(u),\end{aligned}$$

and in the case $u > 1$

$$\begin{aligned}\int_0^u \frac{\tilde\psi(t)}{t}dt + u\int_u^\infty \frac{\tilde\psi(t)}{t^2}dt &= \int_0^1 \frac{\tilde\psi(t)}{t}dt + \int_1^u \frac{\tilde\psi(t)}{t}dt + u\int_u^\infty \frac{\tilde\psi(t)}{t^2}dt \\ &\overset{(4.5)}{\le} c_{\phi,\psi}\phi(1) + \int_1^u \frac{\tilde\psi(t)}{t}dt + u\int_u^\infty \frac{\tilde\psi(t)}{t^2}dt \\ &\overset{(4.6)}{\le} c_{\phi,\psi}\phi(1) + \int_1^u \frac{1}{t}dt + u\int_u^\infty \frac{1}{t^2}dt \\ &\le c_{\phi,\psi}\phi(1) + \log(u) + 1 \\ &\le (c_{\phi,\psi}+1)\tilde\phi(u).\end{aligned}$$

Summarizing all, we obtain

$$\int_0^u \frac{\tilde\psi(t)}{t}dt + u\int_u^\infty \frac{\tilde\psi(t)}{t^2}dt \le (c_{\phi,\psi}+1)\tilde\phi(u),\ \ u > 0.$$

Thus, all the assumptions of Theorem 18 hold for the functions $\tilde\phi$ and $\tilde\psi$. Hence, in particular, the Calderón operator $S_\infty : \Lambda_{\tilde\phi}(0,\infty) \to \Lambda_{\tilde\psi}(0,\infty)$ is bounded. Therefore, for any $z \in \Lambda_\phi \subset \Lambda_{\tilde\phi}(0,\infty)$ (we extend z to a function on $(0,\infty)$ by setting $z = 0$ on $(1,\infty)$), we have

$$\|Sz\|_{\Lambda_\psi} \le \|S_\infty z\|_{\Lambda_{\tilde\psi}(0,\infty)} \le C\|z\|_{\Lambda_{\tilde\phi}(0,\infty)} = C\|z\|_{\Lambda_\phi},$$

which implies that S is bounded from Λ_ϕ in Λ_ψ.

To prove (ii), we take $x \in \Lambda_\psi \subset \Lambda_{\tilde\psi}(0,\infty)$. By Theorem 18, there is $y \in \Lambda_{\tilde\phi}(0,\infty)$ such that $\mu(x) \le S_\infty\mu(y)$ and

$$\|y\|_{\Lambda_{\tilde\phi}(0,\infty)} \le 8\,\|x\|_{\Lambda_{\tilde\psi}(0,\infty)}\,. \tag{4.7}$$

Without loss of generality, we may assume that $y = \mu(y)$. Then, if

$$z(t) := \left(y(t) + \int_1^\infty y(s)\,\frac{ds}{s}\right)\chi_{(0,1)}(t), \quad t \in (0,1),$$

we have

$$\mu(t;x) \le (S_\infty y)(t) \le (Sz)(t), \quad t \in (0,1).$$

On the other hand,

$$\begin{aligned}\|z\|_{\Lambda_\phi} &\le \left\|y\chi_{(0,1)}\right\|_{\Lambda_\phi} + \int_1^\infty y(s)\frac{ds}{s} \\ &= \|y\|_{\Lambda_{\tilde\phi}(0,\infty)} \\ &\overset{(4.7)}{\le} 8\,\|x\|_{\Lambda_{\tilde\psi}(0,\infty)} \\ &= 8\,\|x\|_{\Lambda_\psi}.\end{aligned}$$

This completes the proof of the theorem. □

Recall (see Corollary 17) that the space $(\mathcal{S}_{\Lambda_\phi}, \|\cdot\|_{\mathcal{S}_{\Lambda_\phi}})$ is the optimal symmetric quasi-Banach range for the martingale transform T on the space Λ_ϕ. The following result shows that, under the assumptions of Theorem 19, it can be identified as the Lorentz space Λ_ψ from this theorem.

Corollary 20. *If the assumptions of Theorem 19 hold, then we have $\mathcal{S}_{\Lambda_\phi} = \Lambda_\psi$. Thus, Λ_ψ is the optimal symmetric (quasi-)Banach range for the martingale transform T defined on the Lorentz space Λ_ϕ.*

Proof. First, by Theorem 19 (i), $S : \Lambda_\phi \to \Lambda_\psi$ is a bounded operator. Hence, it follows from Theorem 16 that $\mathcal{S}_{\Lambda_\phi} \subset \Lambda_\psi$.

To prove the converse inclusion, we assume that $x \in \Lambda_\psi$. Then, by Theorem 19 (ii) there exists $y \in \Lambda_\phi$ such that $\mu(x) \le S\mu(y)$. Hence, from the definition of the space $\mathcal{S}_{\Lambda_\phi}$ it follows $x \in \mathcal{S}_{\Lambda_\phi}$, and we conclude that $\Lambda_\psi \subset \mathcal{S}_{\Lambda_\phi}$. □

4.3. **Proof of Corollary 2.**

Proof of Corollary 2. Let $E \subset L_1$ and F be quasi-Banach symmetric function spaces on $(0,1)$. From the estimates obtained in Theorem 1 it follows that assertions (1) and (3) are equivalent. Moreover, it is well known that the Hilbert transform H is bounded from E in F if and only if so is $S : E \to F$ (see e.g. [4, Theorems 3.6.8 and 3.6.10] and the classical result [5, Theorem 2.1]). Therefore, we obtain that (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3). It remains to prove implications (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) whenever E is separable.

(3) $\Longrightarrow$ (4). Let $A \subset \mathbb{N}$. The operator P_A is a martingale transform with respect to the Haar filtration and so, by [6] (see also [17, Theorem 3.3.7], [20, II p.156] or [23]), P_A can be extended to a bounded linear operator from L_1 into $L_{1,\infty}$ with norm which does not depend on the set A. Therefore (see (1.3) and subsequent references), we have

$$\mu(P_A x) \le c_{\rm abs} S\mu(x), \quad x \in L_1, \quad A \subset \mathbb{N}.$$

Hence,

$$\|P_A x\|_F \le c_{\rm abs}\,\|S\mu(x)\|_F \le c_{\rm abs}\,\|S\|_{E\to F}\,\|x\|_E\,, \quad x \in E, \quad A \subset \mathbb{N}.$$

Finally, observe that implication (4) $\Longrightarrow$ (1) follows from the fact that

$$T = P_A, \quad A = \{1\} \bigcup \Big(\bigcup_{n\ge 1} \{2^{2n-1}+1, \cdots, 2^{2n}\} \Big).$$

□

Recall that any Lorentz space Λ_ϕ on $[0,1]$, with $\phi(+0)=0$, is separable (see e.g. [18, Lemma II.5.1]). Thus, the next result follows immediately from Corollary 2. It complements results in [32] (see also the motivation provided in [5, section 4]).

Corollary 21. *Let the assumptions of Theorem 19 hold. The following statements are equivalent:*

(1) The martingale transform T is bounded from Λ_ϕ into Λ_ψ.
(2) The Hilbert transform H is bounded from Λ_ϕ into Λ_ψ.
(3) The Calderón operator S is bounded from Λ_ϕ into Λ_ψ.
(4) Every Haar basis projection is bounded from Λ_ϕ into Λ_ψ.

4.4. **Narrow operators.** Let E be a quasi-Banach symmetric function space on $[0,1]$ and let X be an F-space [16, 26]. A bounded linear operator $T: E\to X$ is called *narrow* if for each set $A\subset(0,1)$ and arbitrary $\varepsilon>0$ there exists a sign x on A (i.e., x is a function supported on A and taking values in the set $\{-1,1\}$ on A) such that $\|Tx\|_X<\varepsilon$ [26, Proposition 1.9(ii)].

It is well known [26, 24, 25] that the identity operator on a separable symmetric space E is a sum of two narrow operators bounded on E whenever E has an unconditional basis (equivalently, E is an interpolation space between L_p and L_q for some $1<p<q<\infty$ [20, II. p.161]). The main result in this subsection is linked with the following open problem stated in [27]:

Assume that the identity operator id on a separable symmetric space E on $(0,1)$ may be represented as a sum of two narrow operators bounded on E. Does this imply that E has an unconditional basis?

In Theorem 22 below, we show that the identity operator on any separable quasi-Banach symmetric function space E such that $E\subset L_1$ is a sum of two narrow operators (basis projections), which are bounded from E into the optimal range $\mathcal{S}_E$ of the Calderón operator S on E (see Section 4.1). This extends the above-mentioned result for symmetric function spaces having non-trivial Boyd indices.

Theorem 22. *If E be a separable quasi-Banach symmetric function space on $(0,1)$ with $E\subset L_1$, then the identity operator* $\mathrm{id}: E\to E$ *is a sum of two narrow operators bounded from E into $\mathcal{S}_E$.*

Let T be the operator defined in (1.4). We write

$$\mathrm{id}=T+(\mathrm{id}-T).$$

To prove Theorem 22, it suffices to prove the following lemma.

Lemma 23. *Let E be a separable quasi-Banach symmetric function space on $(0,1)$ with $E\subset L_1$. Then, the operators $T,\mathrm{id}-T:\,E\to\mathcal{S}_E$ are narrow.*

Proof. As above, $h_{n,k}$'s are Haar functions. Recall that

$$Th_{n,k}=\begin{cases}h_{n,k}, & n\text{ is odd}\\ 0, & n\text{ is even}\end{cases},\quad (\mathrm{id}-T)h_{n,k}=\begin{cases}h_{n,k}, & n\text{ is even}\\ 0, & n\text{ is odd}\end{cases}.$$

We only prove the assertion for the operator T as the argument for $\mathrm{id}-T$ follows *mutatis mutandi*.

Since E is separable, it follows from [26, Lemma 1.12] that it suffices to prove that for any dyadic interval $\Delta_m^l=[\frac{l-1}{2^m},\frac{l}{2^m})$ for any $m=0,1,\cdots$ and $l=1,\cdots,2^m$, there exists $x\in E$ with $x^2=\chi_{\Delta_m^l}$ and $Tx=0$.

Observe that

$$\Delta_m^l = \Delta_{m+1}^{2l-1} + \Delta_{m+1}^{2l} = \Delta_{m+2}^{4l-3} + \Delta_{m+2}^{4l-2} + \Delta_{m+2}^{4l-1} + \Delta_{m+2}^{4l}.$$

Letting

$$x = \begin{cases} h_{m,l}, & \text{if } m \text{ is even;} \\ h_{m+1,2l-1} + h_{m+1,2l}, & \text{if } m \text{ is odd,} \end{cases}$$

we have $x^2 = \chi_{\Delta_m^l}$ and $Tx = 0$. This completes the proof. □

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