

ON q -COMMUTING CO-EXTENSIONS AND q -COMMUTANT LIFTING

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ABSTRACT. Consider a nonzero contraction T and a bounded operator X satisfying $TX = qXT$ for a complex number q . There are some interesting results in the literature on q -commuting dilation and q -commutant lifting of such pair (T, X) when $|q| = 1$. Here we improve a few of them to the class of scalars q satisfying $|q| \leq \frac{1}{\|T\|}$.

1. INTRODUCTION

Throughout the paper we consider only bounded operators acting on complex Hilbert spaces. A contraction is an operator with norm not greater than 1. The aim of this paper is to contribute to the study of dilation and lifting of q -commuting and q -intertwining operators.

Definition 1.1. For a complex number q , a pair of operators (T_1, T_2) acting on a Hilbert space \mathcal{H} is said to be q -commuting if $T_1 T_2 = q T_2 T_1$. Also, for $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$, the pair (T_1, T_2) is said to be q -intertwining by an operator $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ if $AT_1 = qT_2A$.

A nontrivial step in operator theory is the isometric (or unitary) dilation of a contraction due to Sz.-Nagy, [10] which states the following: for any contraction T acting on a Hilbert space \mathcal{H} , there is a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and an isometry (or a unitary) V on \mathcal{K} such that $T^n = P_{\mathcal{H}} V^n|_{\mathcal{H}}$ for all $n \geq 0$ and that the dilation is minimal in the sense that

$$\mathcal{K} = \bigvee_{n=0}^{\infty} V^n \mathcal{H} = \overline{\text{span}} \{V^n h : h \in \mathcal{H}, n \geq 0\}.$$

Moreover, such a minimal dilation is unique upto isomorphism and V^* is the minimal co-isometric extension of T^* , that is, \mathcal{H} is invariant under V^* and $V^*|_{\mathcal{H}} = T^*$. Thus, (V, \mathcal{K}) is the minimal isometric lift of (T, \mathcal{H}) . Ando, [1] extended Sz.-Nagy's result to a pair of commuting contractions T_1, T_2 acting on \mathcal{H} . Indeed, he showed that there are commuting isometries V_1, V_2 on $\mathcal{K} \supseteq \mathcal{H}$ such that $T_1^{n_1} T_2^{n_2} = P_{\mathcal{H}} V_1^{n_1} V_2^{n_2}|_{\mathcal{H}}$ for all $n_1, n_2 \geq 0$. However, no further generalization was possible as was proved by S. Parrott in [12] via a counter example showing that a triple of commuting contractions (T_1, T_2, T_3) may not dilate to a commuting triple of isometries (V_1, V_2, V_3) . Sarason, [13] showed that if the minimal co-isometric extension V of T is such that V^* is a unilateral shift of multiplicity one, then any commutant X of T has a norm preserving extension Y such that $YV = VY$. Sz.-Nagy and Foias generalized the seminal work of Sarason for an arbitrary contraction T which is known as the classical commutant lifting theorem.

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Theorem 1.2. *Let T be a contraction on a Hilbert space \mathcal{H} and let (V, \mathcal{K}) be the minimal isometric (or minimal unitary) dilation of T . If R commutes with T , then there exists an operator S commuting with V such that $\|R\| = \|S\|$ and $RT^n = P_{\mathcal{H}}SV^n|_{\mathcal{H}}$ for all $n \geq 0$.*

Sz.-Nagy and Foias, [6, 7] proved a variant of this theorem replacing the minimal isometric dilation by the minimal co-isometric extension of T , which further was established independently by Douglas, Muhly and Pearcy in [4]. After a few decades, Sebestyén generalized the classical commutant lifting theorem to the q -commuting setup for $q = \pm 1$ in the following way.

Theorem 1.3 (Theorems 2 & 3, [15]). *Let $q = 1$ or -1 , $T \in \mathcal{B}(\mathcal{H})$ be any contraction and X be an operator on \mathcal{H} satisfying $TX = qXT$. If V acting on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ is the minimal unitary (or isometric) dilation of T , then there exists an operator Y_q on \mathcal{K} such that Y_q dilates X , $\|Y_q\| = \|X\|$ and $VY_q = qY_qV$.*

Recently Keshari and Mallick, [8] improved Sebestyén's result for q -commuting and q -intertwining operators with $|q| = 1$. Also, Mallick and Sumesh have established a more general operator theoretic version of the same results in [9].

The main results of this paper, Theorems 3.3, 3.5, 4.3, 4.6 show that these existing results can be further generalized for any complex number q satisfying $|q| \leq 1/\|T\|$. Needless to mention that it includes all scalars q from the closed unit disk $\overline{\mathbb{D}}$. We also provide independent proofs to some existing results in this direction.

2. EXAMPLES AND PREPARATORY RESULTS

We begin with a pair of examples of q -commuting of operators. In a similar fashion one can construct examples of q -intertwining pairs of operators.

Example 2.1. Let q be any non zero complex number. Now let $T_1 = \begin{pmatrix} 1 & 0 \\ 1/2 & q \end{pmatrix}$ and $T_2 = \begin{pmatrix} 0 & 0 \\ 1/4 & 0 \end{pmatrix}$ in $M_2(\mathbb{C})$. Then

$$T_1T_2 = \begin{pmatrix} 0 & 0 \\ q/4 & 0 \end{pmatrix}$$

and

$$T_2T_1 = \begin{pmatrix} 0 & 0 \\ 1/4 & 0 \end{pmatrix}.$$

Hence clearly $T_1T_2 = qT_2T_1$. In fact for any complex numbers a, b and d taking $T_1 = \begin{pmatrix} a & 0 \\ b & qa \end{pmatrix}$ and $T_2 = \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix}$ in $M_2(\mathbb{C})$ gives an example of a q -commuting pair of operators (T_1, T_2) .

The following example was given in [8].

Example 2.2 ([8]). Let $H^2(\mathcal{E}) = H^2(\mathbb{D}) \otimes \mathcal{E}$ be the \mathcal{E} -valued Hardy Hilbert space (similarly, $L_a^2(\mathbb{D}, \mathcal{E}) = L_a^2 \otimes \mathcal{E}$ denote the \mathcal{E} -valued Bergman space). Let q be any complex number of modulus 1. Let $T_2 = M_z \otimes I_{\mathcal{E}}$ and $T_1 = C_q \otimes I_{\mathcal{E}}$ on $H^2(\mathcal{E})$ (similarly on $L_a^2(\mathbb{D})$) be such that M_z is a multiplication operator on Hardy space $H^2(\mathbb{D})$ and C_q on $H^2(\mathbb{D})$ is defined as

$$C_q(f)(z) = f(qz) \text{ for } f \in H^2(\mathbb{D}) \text{ (similarly } f \in L_a^2(\mathbb{D})), z \in \mathbb{D}.$$

Then it can be easily verified that $T_1T_2 = qT_2T_1$.

To see more examples of q -commuting operators, an interested reader can refer [8]. The following dual version of Parrot's theorem ([11], Theorem 1) on quotient norms plays an important role in this paper.

Theorem 2.3 ([15], Theorem 1). *Let K and K' be Hilbert spaces, $H \subseteq K$ and $H' \subseteq K'$ be subspaces, and $X : H \rightarrow K'$ and $X' : H' \rightarrow K$ be given bounded linear transformations. Then there exists operator $Y : K \rightarrow K'$ extending X so that Y^* extends X' if and only if the following identity holds true:*

$$\langle Xh, h' \rangle = \langle h, X'h' \rangle \quad \text{for all } h \in H \text{ and } h' \in H'.$$

Moreover Y can be of norm $\max\{\|X\|, \|X'\|\}$ possible at most.

The following result from [3] will be useful.

Lemma 2.4 ([3], Theorem 1). *Suppose that $\mathcal{S}, \mathcal{H}, \mathcal{K}$ are Hilbert spaces and $A \in \mathcal{B}(\mathcal{S}, \mathcal{K})$ and $B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then there exists a contraction $Z \in \mathcal{B}(\mathcal{S}, \mathcal{H})$ satisfying $A = BZ$ if and only if $AA^* \leq BB^*$.*

Now we recall a generalization of the above lemma.

Theorem 2.5 ([4], Theorem 1). *Let $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$ and \mathcal{K} be Hilbert spaces, and for $0 \leq i \leq 2$ let A_i be an operator mapping \mathcal{H}_i into \mathcal{K} . Then there exist operators Z_1 and Z_2 that map \mathcal{H}_0 into \mathcal{H}_1 and \mathcal{H}_2 , respectively, and that satisfy the two conditions*

$$(1) \quad A_1 Z_1 + A_2 Z_2 = A_0,$$

$$(2) \quad Z_1^* Z_1 + Z_2^* Z_2 \leq I_{\mathcal{H}_0}$$

if and only if $A_1 A_1^* + A_2 A_2^* \geq A_0 A_0^*$.

We will make numerous applications of the following famous theorem due to Sz.-Nagy and Foias.

Theorem 2.6 (Sz.-Nagy & Foias, [6]). *Suppose that for $i = 1, 2$, T_i is a contraction acting on a Hilbert space \mathcal{H}_i and V_i is the unique minimal co-isometric extension of T_i acting on the Hilbert space $\mathcal{K}_i \supseteq \mathcal{H}_i$. Let X be an operator that maps \mathcal{H}_1 into \mathcal{H}_2 and satisfies the equation $XT_1 = T_2X$. Then there exists an operator Y mapping \mathcal{H}_1 into \mathcal{H}_2 such that*

$$YV_1 = V_2Y, \quad Y\mathcal{H}_1 \subseteq \mathcal{H}_2, \quad Y|_{\mathcal{H}_1} = X, \quad \|Y\| = \|X\|.$$

In [4], Douglas, Muhly and Pearcy gave an alternate proof to the above theorem by an application of the commutant lifting theorem.

Let S_0, T_0 be contractions acting on Hilbert spaces \mathcal{H}_0 and \mathcal{K}_0 respectively. Let U and V be the unique minimal co-isometric extensions of S_0 and T_0 acting on the Hilbert spaces \mathcal{H} and \mathcal{K} respectively. Let P_n and Q_n be the orthogonal projections of the spaces \mathcal{H} and \mathcal{K} to the corresponding subspaces $\bigvee_{k=0}^n U^{*k}(\mathcal{H}_0)$ and $\bigvee_{k=0}^n V^{*k}(\mathcal{K}_0)$ respectively, for $n = 0, 1, 2, \dots$. The following theorem is a generalization of the commutant lifting theorem due to Sz. Nagy and Foias.

Theorem 2.7 ([16], Theorem 3). *Let S_0 on Hilbert space \mathcal{H}_0 and T_0 on Hilbert space \mathcal{K}_0 be any contractions $R : \mathcal{H} \rightarrow \mathcal{H}$ be any contraction operator that commutes with all of the projections $\{P_n\}_{n=0}$, let $X_0 : \mathcal{H} \rightarrow \mathcal{K}$ be an operator that satisfies*

$$T_0 X_0 = X_0 S_0 R P_0.$$

Then there exists an operator $X : \mathcal{H} \rightarrow \mathcal{K}$ which extends X_0 has the same norm as X_0 and intertwines V and UR :

$$VX = XUR$$

3. THE q -COMMUTING AND q -INTERTWINING CO-ISOMETRIC EXTENSIONS

In this Section, we assume the existence of a co-isometric extension of a non-zero contraction T and find an extension of its q -commutant when $0 < q \leq \frac{1}{\|T\|}$. For a contraction T acting on a Hilbert space \mathcal{H} , if (V, \mathcal{K}) is the minimal co-isometric extension of T then $\mathcal{K} = \bigvee_{n=0}^{\infty} (V)^{*n} \mathcal{H}$. Suppose $K_n = \bigvee_{m=0}^n V^{*m} \mathcal{H}$. We begin with the simpler case when q belongs to the deleted closed disk $\overline{\mathbb{D}} \setminus \{0\}$.

Theorem 3.1. *Let T_1 be a contraction and T_2 be an operator on a Hilbert space \mathcal{H} . Suppose $T_1 T_2 = q T_2 T_1$, where $0 < |q| \leq 1$. Let V on \mathcal{K} be the minimal co-isometric extension of T_1 . Then there exists a bounded linear operator $S : \mathcal{K} \rightarrow \mathcal{K}$ such that*

- (1) $VS = qSV$
- (2) $\|S\| = \|T_2\|$ and
- (3) \mathcal{H} is invariant under S and $S|_{\mathcal{H}} = T_2$.

Proof. This result follows as a consequence of Theorem 2.7. Considering $qI_{\mathcal{K}}$ in place of R , T_1 in place of T_0 as well as S_0 and T_2 in place of X_0 in Theorem 2.7, there exists a bounded linear operator $S : \mathcal{K} \rightarrow \mathcal{K}$ such that the conditions (1) and (2) hold and $S|_{\mathcal{H}} = T_2$. As a consequence, for any $h \in \mathcal{H}$,

$$V^n S^m h = V^n T_2^m h = T_1^n T_2^m h.$$

Hence (3) holds and the proof is complete. \square

Removing the minimality constraint from Theorem 3.1, we obtain the following generalized version.

Theorem 3.2. *Let T_1 be a contraction and T_2 be an operator on a Hilbert space \mathcal{H} . Suppose $T_1 T_2 = q T_2 T_1$, where $0 < |q| \leq 1$. Let V on \mathcal{K} be a co-isometric extension of T_1 . Then there exists a bounded linear operator $S : \mathcal{K} \rightarrow \mathcal{K}$ such that*

- (1) $VS = qSV$
- (2) $\|S\| = \|T_2\|$ and
- (3) \mathcal{H} is invariant under S and $S|_{\mathcal{H}} = T_2$.

Proof. The idea of proof is same as that of Corollary 4.1 in [4]. Let $\mathcal{K}' \subseteq \mathcal{K}$ be the smallest reducing subspace for V that contains \mathcal{H} . If we put $V' = V|_{\mathcal{K}'}$, then (V', \mathcal{K}') is the minimal co-isometric extension of T . Thus by Theorem 3.1, there exists $S' \in \mathcal{B}(\mathcal{K}')$ such that $\|S'\| = \|T_2\|$, $V'S' = qS'V'$, \mathcal{H} is invariant under S' and $S'|_{\mathcal{H}} = T_2$. Now we define $S : \mathcal{K} \rightarrow \mathcal{K}$ as $S(k) = S'(k)$ for any $k \in \mathcal{K}'$ and $S(k) = 0$ on $\mathcal{K} \ominus \mathcal{K}'$. Hence clearly S satisfies properties (1), (2) and (3) as desired. This completes the proof. \square

In Theorem 3.1, in particular if T_2 is also a contraction, then one might expect S to be a co-isometric extension of T_2 . But for $|q| \neq 1$, this is not possible as $\|VS\| = 1$ and $\|qSV\| = |q|$. Although, we can expect the existence of co-isometric extensions V_1, V_2, qV_q of T_1, T_2, qT_1 respectively, such that $V_1 V_2 = qV_2 V_q$. In this Section, we obtain such V_1, V_2, qV_q when T_1, T_2 are q -commuting contractions with $\|T_2\| < 1$ and $0 < |q| \leq \frac{1}{\|T_1\|}$, see Theorem 3.5. Indeed, this is one of the main theorems of this paper, but before going to that we state and prove another result which is also one of our main results.

Theorem 3.3. *Let T be a non-zero contraction on Hilbert space \mathcal{H} and $X \in \mathcal{B}(\mathcal{H})$ be such that $TX = qXT$ for $0 < |q| \leq \frac{1}{\|T\|}$. Suppose (V, \mathcal{K}') and (qV_q, \mathcal{K}) are the minimal co-isometric extensions of T and qT respectively. Then there exists an operator $Y \in \mathcal{B}(\mathcal{K}, \mathcal{K}')$ such that*

$$VY = qYV_q, Y(\mathcal{H}) \subseteq \mathcal{H}, Y|_{\mathcal{H}} = X \text{ and } \|X\| = \|Y\|.$$

Proof. We follow the idea of the proof of Theorem 2 in [15]. Since (V, \mathcal{K}') and (qV_q, \mathcal{K}) are the minimal co-isometric extensions of T and qT respectively, we have

$$\mathcal{K}' = \bigvee_{n=0}^{\infty} V^{*n} \mathcal{H} \text{ and } \mathcal{K} = \bigvee_{n=0}^{\infty} (qV_q)^{*n} \mathcal{H}.$$

Let $K_n = \bigvee_{\ell=0}^n (qV_q)^{* \ell} \mathcal{H}$ and $K'_n = \bigvee_{m=0}^n V^{*m} \mathcal{H}$. Then $\bigcup_{n=0}^{\infty} K_n$ is dense in \mathcal{K} and $\bigcup_{n=0}^{\infty} K'_n$ is dense in \mathcal{K}' . Consider the orthogonal projections $P_n : \mathcal{K} \rightarrow K_n$ and $P'_n : \mathcal{K}' \rightarrow K'_n$ for all $n \geq 0$. Clearly

$$P_{n+1}(qV_q)^* = (qV_q)^* P_n \text{ and } P'_{n+1} V^* = V^* P'_n. \quad (3.1)$$

We construct the required operator $Y : \mathcal{K} \rightarrow \mathcal{K}'$ inductively by finding operators $Y_n : K_n \rightarrow K'_n$ for all $n \in \mathbb{N}$.

Claim. For any $h_1, h_2 \in \mathcal{H}$, $\langle (qV_q)^* X^* V(V^* h_1), h_2 \rangle = \langle (V^* h_1), X h_2 \rangle$.

Proof of Claim. Since V and qV_q are the minimal co-isometric extensions of T and qT respectively, then for $h_1, h_2 \in \mathcal{H}$ we have

$$\begin{aligned} \langle (qV_q)^* X^* V(V^* h_1), h_2 \rangle &= \langle (qV_q)^* X^* h_1, h_2 \rangle = \langle X^* h_1, (qV_q) h_2 \rangle = \langle X^* h_1, qT h_2 \rangle \\ &= \langle h_1, qXT h_2 \rangle \\ &= \langle h_1, TX h_2 \rangle \text{ (since } TX = qXT) \\ &= \langle h_1, VX h_2 \rangle \\ &= \langle V^* h_1, X h_2 \rangle. \end{aligned}$$

This completes the proof of the claim.

Therefore, by Theorem 2.3, there exists an operator $Y_1^* \in \mathcal{B}(K'_1, K_1)$ such that $Y_1|_{\mathcal{H}} = X$, $Y_1^*|_{V^* \mathcal{H}} = (qV_q)^* X^* V|_{V^* \mathcal{H}}$ and $\|Y_1\| \leq \max\{\|X\|, \|(qV_q)^* X^* V|_{V^* \mathcal{H}}\|\} = \|X\|$. Since Y_1 is an extension of X , we have $\|Y_1\| = \|X\|$. Also, we have

$$\begin{aligned} Y_1^* P'_1 V^* &= Y_1^* V^* P'_0 && \text{(from (3.1))} \\ &= (qV_q)^* X^* V V^* P'_0 && \text{(since } Y_1^*|_{V^* \mathcal{H}} = (qV_q)^* X^* V|_{V^* \mathcal{H}}) \\ &= (qV_q)^* X^* P'_0. \end{aligned} \quad (3.2)$$

Suppose there is Y_{n-1} such that $Y_{n-1}^* : K'_{n-1} \rightarrow K_{n-1}$ satisfies

$$\begin{aligned} (a) \quad &Y_{n-1}|_{K_{n-2}} = Y_{n-2}, \\ (b) \quad &Y_{n-1}^*|_{V^* K'_{n-2}} = (qV_q)^* Y_{n-2}^* V|_{V^* K'_{n-2}}, \\ (c) \quad &Y_{n-1}^* P'_{n-1} V^* = (qV_q)^* Y_{n-2}^* P'_{n-2}, \\ (d) \quad &\|Y_{n-1}\| = \|Y_{n-2}\|. \end{aligned} \quad (3.3)$$

Claim. For any $h_1 \in K'_{n-1}$ and $h_2 \in K_{n-1}$, we have $\langle (qV_q)^* Y_{n-1}^* V(V^* h_1), h_2 \rangle = \langle V^* h_1, Y_{n-1} h_2 \rangle$.

Proof of Claim. Suppose $h_1 \in K'_{n-1}$ and $h_2 \in K_{n-1}$. Then

$$\begin{aligned}
\langle (qV_q)^* Y_{n-1}^* V(V^* h_1), h_2 \rangle &= \langle (qV_q)^* Y_{n-1}^* h_1, h_2 \rangle \\
&= \langle P_{n-1} (qV_q)^* Y_{n-1}^* h_1, h_2 \rangle \text{ (since } (qV_q)^* K_{n-1} \subseteq K_n) \\
&= \langle (qV_q)^* P_{n-2} Y_{n-1}^* h_1, h_2 \rangle \text{ (using (3.1))} \\
&= \langle h_1, Y_{n-1} P_{n-2} (qV_q) h_2 \rangle \\
&= \langle P'_{n-2} h_1, Y_{n-1} P_{n-2} (qV_q) h_2 \rangle \text{ (since } Y_{n-1} K_{n-2} \subseteq K'_{n-2}) \\
&= \langle P'_{n-2} h_1, Y_{n-2} P_{n-2} (qV_q) h_2 \rangle \text{ (using (a))} \\
&= \langle Y_{n-2}^* P'_{n-2} h_1, P_{n-2} (qV_q) h_2 \rangle \\
&= \langle Y_{n-2}^* P'_{n-2} h_1, (qV_q) h_2 \rangle \text{ (since } Y_{n-2}^* K'_{n-2} \subseteq K_{n-2}) \\
&= \langle (qV_q)^* Y_{n-2}^* P'_{n-2} h_1, h_2 \rangle \\
&= \langle Y_{n-1}^* P'_{n-1} V^* h_1, h_2 \rangle \text{ (using (c))} \\
&= \langle V^* h_1, Y_{n-1} h_2 \rangle.
\end{aligned}$$

The last inequality follows from the fact that $Y_{n-1} h_2 \in K'_{n-1}$, for any $h_2 \in K_{n-1}$. This completes the proof of claim.

Hence, Theorem 2.3 guarantees the existence of $Y_n : K_n \rightarrow K'_n$ satisfying

- (i) $Y_n|_{K_{n-1}} = Y_{n-1}$,
- (ii) $Y_n^*|_{V^* K'_{n-1}} = (qV_q)^* Y_{n-1}^* V|_{V^* K'_{n-1}}$,
- (iii) $Y_n^* P'_n V^* = (qV_q)^* Y_{n-1}^* P'_{n-1}$,
- (iv) $\|Y_n\| = \|Y_{n-1}\| = \|X\|$.

Note that condition-(iii) above is obtained in the following way:

$$Y_n^* P'_n V^* = Y_n^* V^* P'_{n-1} = (qV_q)^* Y_{n-1}^* V V^* P'_{n-1} = (qV_q)^* Y_{n-1}^* P'_{n-1},$$

where the second last equality follows from condition-(c) of (3.3). Thus, by induction the result holds for any natural number n . Now define $Y_0 : \bigcup_{n=0}^{\infty} \mathcal{K}_n \rightarrow \bigcup_{n=0}^{\infty} \mathcal{K}'_n$ such that $Y_0|_{K_n} = Y_n$. This

is well defined as $Y_n|_{K_{n-1}} = Y_{n-1}$. Since $\bigcup_{n=0}^{\infty} K'_n$ is dense in \mathcal{K}' , by continuity Y_0 extends to an operator $Y : \mathcal{K} \rightarrow \mathcal{K}'$. From (iii) we have $V Y_n P_n = Y_{n-1} P_{n-1} qV_q$ and one can easily check that $Y_n P_n$ converges to Y in the strong operator topology. Therefore, $VY = qYV_q$. Clearly $\|Y\| = \|X\|$. This completes the proof of the theorem. \square

If we drop the minimality conditions on co-isometric extension in Theorem 3.3, then we have the following result.

Corollary 3.4. *Let T be a contraction on Hilbert space \mathcal{H} and $X \in \mathcal{B}(\mathcal{H})$ be such that $TX = qXT$ for $0 < |q| \leq \frac{1}{\|T\|}$. Suppose (V, \mathcal{K}') and (qV_q, \mathcal{K}) are co-isometric extensions of T and qT respectively. Then there exists an operator $Y \in \mathcal{B}(\mathcal{K}, \mathcal{K}')$ such that*

$$VY = qYV_q, Y(\mathcal{H}) \subseteq \mathcal{K}', Y|_{\mathcal{H}} = X \text{ and } \|X\| = \|Y\|.$$

Proof. Let $\mathcal{K}_1 \subseteq \mathcal{K}$ and $\mathcal{K}'_1 \subseteq \mathcal{K}'$ be the smallest reducing subspaces containing \mathcal{H} , for qV_q and V respectively. Let $V_1 = V|_{\mathcal{K}_1}$ and $qV_2 = (qV_q)|_{\mathcal{K}'_1}$. Then V_1 and qV_2 are minimal co-isometric extensions of T and qT respectively. By Theorem 3.3, there exists $Y_1 \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}'_1)$ such that $V_1 Y_1 = qY_1 V_2$, $Y_1 \mathcal{H} \subseteq \mathcal{H}$, $Y_1|_{\mathcal{H}} = X$ and $\|Y_1\| = \|X\|$. Define $Y : \mathcal{K} \rightarrow \mathcal{K}'$ as $Y(k) = Y_1(k)$ for $k \in \mathcal{K}_1$ and $Y(k) = 0$ for $k \in \mathcal{K} \ominus \mathcal{K}_1$. Clearly $Y(\mathcal{H}) \subseteq \mathcal{H}'$, $Y|_{\mathcal{H}} = X$ and $\|X\| = \|Y\|$.

As qV_q reduces \mathcal{K}_1 , for $k \in \mathcal{K} \ominus \mathcal{K}_1$, $qV_q(k) \in \mathcal{K} \ominus \mathcal{K}_1$. Hence $VY(k) = 0$, $qV_q(k) = 0$ and the proof is complete. \square

Now we are in a position to present one of our main results.

Theorem 3.5. *Let T_1 be any contraction and T_2 be a strict contraction on Hilbert space \mathcal{H} , such that $T_1T_2 = qT_2T_1$ where $0 < |q| \leq \frac{1}{\|T_1\|}$. Then there exist a Hilbert space \mathcal{K} containing \mathcal{H} and co-isometric extensions X_1, X_2, qX_q on \mathcal{K} of T_1, T_2, qT_1 such that $X_1X_2 = qX_2X_q$. Moreover, if T_1 is also a strict contraction then X_1, X_2, qX_q are pure co-isometries.*

Proof. Let $(V, \widetilde{\mathcal{H}})$ and $(qV_q, \widetilde{\mathcal{H}})$ be co-isometric extensions of T_1 and qT_1 respectively (indeed, we can take $\widetilde{\mathcal{H}} = \ell^2(\mathcal{H})$). By Corollary 3.4, there exists an operator X on $\widetilde{\mathcal{H}}$ such that

- (1) $X|_{\mathcal{H}} = T_2$;
- (2) $\|X\| = \|T_2\|$;
- (3) $VX = qXV_q$.

Since X is a strict contraction so D_{X^*} is invertible on $\widetilde{\mathcal{H}}$. Consider the Hilbert space $\mathcal{K} = \bigoplus_0^\infty \widetilde{\mathcal{H}}$. We embed \mathcal{H} in \mathcal{K} via a map $h \mapsto (h, 0, 0, \dots)$. Let X_2, X_1 and qX_q be operators on \mathcal{K} defined as follows

$$X_2 = \begin{pmatrix} X & D_{X^*} & 0 & 0 & \cdots \\ 0 & 0 & I_{\widetilde{\mathcal{H}}} & 0 & \cdots \\ 0 & 0 & 0 & I_{\widetilde{\mathcal{H}}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$X_1 = \begin{pmatrix} V & 0 & 0 & \cdots \\ 0 & D_{X^*}^{-1}VD_{X^*} & 0 & \cdots \\ 0 & 0 & D_{X^*}^{-1}VD_{X^*} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$qX_q = \begin{pmatrix} qV_q & 0 & 0 & \cdots \\ 0 & D_{X^*}^{-1}VD_{X^*} & 0 & \cdots \\ 0 & 0 & D_{X^*}^{-1}VD_{X^*} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Clearly, X_2, X_1 and qX_q are bounded linear operators on \mathcal{K} and by construction X_2 is a pure co-isometry. Now we prove that X_1 and qX_q are co-isometries. Since V and qV_q are co-isometries, it suffices to show that

$$(D_{X^*}^{-1}VD_{X^*})(D_{X^*}^{-1}VD_{X^*})^* = I_{\widetilde{\mathcal{H}}}. \quad (3.4)$$

Using $qXV_q = VX$, $VV^* = I_{\widetilde{\mathcal{H}}}$ and $(qV_q)(qV_q)^* = I_{\widetilde{\mathcal{H}}}$ we have

$$\begin{aligned} (D_{X^*}^{-1}VD_{X^*})(D_{X^*}^{-1}VD_{X^*})^* &= D_{X^*}^{-1}VD_{X^*}^2V^*D_{X^*}^{-1} \\ &= D_{X^*}^{-1}V(I_{\widetilde{\mathcal{H}}} - XX^*)V^*D_{X^*}^{-1} \\ &= D_{X^*}^{-1}(I_{\widetilde{\mathcal{H}}} - VX(VX)^*)D_{X^*}^{-1} \\ &= D_{X^*}^{-1}(I_{\widetilde{\mathcal{H}}} - qXV_q(qXV_q)^*)D_{X^*}^{-1} \\ &= D_{X^*}^{-1}(I_{\widetilde{\mathcal{H}}} - XX^*)D_{X^*}^{-1} \\ &= I_{\widetilde{\mathcal{H}}}. \end{aligned}$$

By direct computation and using $VX = qXV_q$, we get $X_1X_2 = qX_2X_q$. We have that V, qV_q and X are extensions of T_1, qT_1 and T_2 respectively. Thus X_1, qX_q and X_2 are extensions of T_1, qT_1 and T_2 respectively.

Again if T_1 is a strict contraction then we can assume V to be a pure co-isometric extension of T_1 and we have

$$(D_{X^*}^{-1}VD_{X^*})^n = D_{X^*}^{-1}V^nD_{X^*}. \quad (3.5)$$

Since V and qV_q are pure, we obtain from equations (3.4) and (3.5) that X_1 and qX_q are pure co-isometric extensions of V and qV_q respectively. The proof is now complete. \square

By an application of Theorem 3.5, we obtain the following two lemmas.

Lemma 3.6. *Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ be contractions such that $T_1 \neq 0$, $\|T_2\| < 1$ and $T_1T_2 = qT_2T_1$ for some $0 < |q| \leq \frac{1}{\|T_1\|}$. Suppose \tilde{X}_2 is a co-isometric extension of T_2 on $\tilde{\mathcal{H}}$. Then there exist co-isometric extensions V_1, V_2 and qV_q of T_1, \tilde{X}_2 and qT_1 on a Hilbert space $\mathcal{H} \supseteq \tilde{\mathcal{H}}$ respectively such that $V_1V_2 = qV_2V_q$.*

Proof. Let $\tilde{X}_2 = X_0 \oplus Y_2$ on $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 \oplus \mathcal{R}_2$, where X_0 is the minimal co-isometric extension of T_2 . By Theorem 3.5, there exist co-isometric extensions Z_1, Z_2, qZ_q of T_1, T_2, qT_1 respectively on some Hilbert space \mathcal{L} such that $Z_1Z_2 = qZ_2Z_q$. Let $\mathcal{H} = \mathcal{L} \oplus \mathcal{R}_2$, $X_2 = Z_2 \oplus Y_2, X_1 = Z_1 \oplus I$ and $qX_q = qZ_q \oplus I$. Clearly, $X_1X_2 = qX_2X_q$, X_1, X_2, qX_q are co-isometries and $X_1|_{\mathcal{H}} = T_1, qX_q|_{\mathcal{H}} = qT_1$. Since minimal co-isometric extension of a contraction is unique up to isomorphism and every co-isometric extension of a contraction contains a minimal co-isometric extension, Z_2 is a co-isometric extension of X_0 . Therefore, X_2 is a co-isometric extension of \tilde{X}_2 . This completes the proof. \square

Lemma 3.7. *Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ be contractions such that $T_1 \neq 0$, $\|T_2\| < 1$ and $T_1T_2 = qT_2T_1$ for some $0 < |q| \leq \frac{1}{\|T_1\|}$. Suppose \tilde{X}_1 on $\tilde{\mathcal{H}}_1$ and \tilde{qX}_q on $\tilde{\mathcal{H}}_2$ are co-isometric extensions of T_1 and qT_1 respectively. Then there exist co-isometric extensions X_1, X_2 and qX_q of \tilde{X}_1, T_2 and \tilde{qX}_q on a Hilbert space $\mathcal{H} \supseteq \tilde{\mathcal{H}}$ respectively such that $X_1X_2 = qX_2X_q$.*

Proof. Suppose $\tilde{X}_1 = X_{10} \oplus Y_1$ on $\tilde{\mathcal{H}}_1 = K_{10} \oplus \mathcal{R}_1$ and $\tilde{qX}_q = X_{q0} \oplus Y_2$ on $\tilde{\mathcal{H}}_2 = K_{20} \oplus \mathcal{R}_2$, where X_{10} and X_{q0} are the minimal co-isometric extensions of T_1 and qT_1 respectively. By Theorem 3.5, there exist co-isometric extensions Z_1, Z_2, qZ_q of T_1, T_2, qT_1 respectively on \mathcal{L} such that $Z_1Z_2 = qZ_2Z_q$. Let $\mathcal{H} = \mathcal{L} \oplus \mathcal{R}_1 \oplus \mathcal{R}_2$, $X_2 = Z_2 \oplus I \oplus I, X_1 = Z_1 \oplus Y_1 \oplus Y_2$ and $qX_q = qZ_q \oplus Y_1 \oplus Y_2$. Clearly, $X_1X_2 = qX_2X_q$, X_1, X_2, qX_q are co-isometries and $X_2|_{\mathcal{H}} = T_2$. Further X_1 and qX_q are co-isometric extensions of \tilde{X}_1 and \tilde{qX}_q respectively. This is because, X_{10}, X_{q0} are the minimal co-isometric extensions of T_1, qT_1 respectively and also $\tilde{X}_1, \tilde{qX}_q$ are co-isometric extensions of T_1, qT_1 respectively. \square

We conclude this Section with a similar co-extension result for q -intertwining operators when q is a complex number of modulus one.

Theorem 3.8. *Suppose $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is a strict contraction and $T_i \in \mathcal{B}(\mathcal{H}_i)$ is a contraction for $i = 1, 2$ such that $AT_1 = qT_2A$, where $|q| = 1$. Then there exist co-isometric extensions $Y \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ of A and $X_i \in \mathcal{B}(\mathcal{K}_i)$ of T_i for $i = 1, 2$ such that $YX_1 = qX_2Y$. Moreover, Y is a pure co-isometry and if T_i is a strict contraction then X_i is pure co-isometry.*

Proof. Consider

$$\tilde{T} = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \text{ on } \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Suppose $\tilde{V} = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$, where V_i on $\ell^2(\mathcal{H}_i)(= \mathfrak{L}_i)$ is a co-isometric extension of T_i for $i = 1, 2$.

Clearly $\tilde{A}\tilde{T} = q\tilde{T}\tilde{A}$ and \tilde{V} is a co-isometric extension of \tilde{T} on $K = \mathfrak{L}_1 \oplus \mathfrak{L}_2$. By Corollary 3.2, there exists an operator \tilde{Y} in $\mathcal{B}(K)$ such that

$$\tilde{Y}\tilde{V} = q\tilde{V}\tilde{Y}, \quad \tilde{Y}|_{\mathcal{H}_1 \oplus \mathcal{H}_2} = \tilde{A}, \quad \|\tilde{Y}\| = \|\tilde{A}\| = \|A\|.$$

The block matrix of \tilde{Y} with respect to $\mathfrak{L}_1 \oplus \mathfrak{L}_2$ is $\begin{pmatrix} Z_1 & Z_2 \\ B & Z_3 \end{pmatrix}$. Hence we have $BV_1 = qV_2B$. Now clearly $\|B\| \leq \|\tilde{Y}\| = \|A\|$. Further $\tilde{Y}|_{\mathcal{H}_1 \oplus \mathcal{H}_2} = \tilde{A}$ implies that $B|_{\mathcal{H}_1} = A$. Therefore, $\|A\| \leq \|B\|$ and hence $\|B\| = \|A\| < 1$. Since $\|B\|$ is less than 1, D_{B^*} is invertible. Consider the following operators

$$Y = \begin{pmatrix} B & \frac{1}{q}D_{B^*} & 0 & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & 0 & \cdots \\ 0 & 0 & 0 & I & 0 & \cdots \\ 0 & 0 & 0 & 0 & I & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} : \mathfrak{L}_1 \oplus \ell^2(\mathfrak{L}_2) \rightarrow \mathfrak{L}_2 \oplus \ell^2(\mathfrak{L}_2),$$

$$X_1 = \begin{pmatrix} V_1 & 0 & 0 & 0 & \cdots \\ 0 & qD_{B^*}^{-1}V_2D_{B^*} & 0 & 0 & \cdots \\ 0 & 0 & D_{B^*}^{-1}V_2D_{B^*} & 0 & \cdots \\ 0 & 0 & 0 & D_{B^*}^{-1}V_2D_{B^*} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ on } \mathfrak{L}_1 \oplus \ell^2(\mathfrak{L}_2),$$

$$X_2 = \begin{pmatrix} V_2 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{q}D_{B^*}^{-1}V_2D_{B^*} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{q}D_{B^*}^{-1}V_2D_{B^*} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{q}D_{B^*}^{-1}V_2D_{B^*} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ on } \mathfrak{L}_2 \oplus \ell^2(\mathfrak{L}_2).$$

It can be easily verified that X_1, X_2 and Y satisfy the required conditions. The proof is complete. \square

4. q -COMMUTANT LIFTING

In the previous Section, we assumed the existence of co-isometric extension of a contraction and established a few q -commutant lifting results. Here we study the existence of a q -commutant of the minimal isometric lift of a contraction. We begin with a few preparatory results.

Proposition 4.1. *Let $T_1 \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}'_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}'_2)$ be contractions. Suppose X is an operator from \mathcal{H}_1 to \mathcal{H}'_2 . Then*

$$Y = \begin{pmatrix} T_1 & 0 \\ X & T_2 \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}'_1 \oplus \mathcal{H}'_2$$

is a contraction if and only if $X = D_{T_2^}CD_{T_1}$ for some contraction $C : \mathcal{H}_1 \rightarrow \mathcal{H}'_2$.*

Proof. This is a variant of Proposition 2.2 in [4] and we implement a similar idea to prove it. The operator $Y = \begin{pmatrix} T_1 & 0 \\ X & T_2 \end{pmatrix}$ is a contraction if and only if $Y^*Y \leq \begin{pmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & I_{\mathcal{H}_2} \end{pmatrix}$, that is, if and only if

$$\begin{aligned} & \left(\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ X & T_2 \end{pmatrix} \right)^* \left(\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ X & T_2 \end{pmatrix} \right) \leq \begin{pmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & I_{\mathcal{H}_2} \end{pmatrix} \\ \iff & \begin{pmatrix} T_1^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ X & T_2 \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ X & T_2 \end{pmatrix} \leq \begin{pmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & I_{\mathcal{H}_2} \end{pmatrix} \\ \iff & \begin{pmatrix} T_1^*T_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ X & T_2 \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ X & T_2 \end{pmatrix} \leq \begin{pmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & I_{\mathcal{H}_2} \end{pmatrix} \\ \iff & \begin{pmatrix} 0 & 0 \\ X & T_2 \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ X & T_2 \end{pmatrix} \leq \begin{pmatrix} I_{\mathcal{H}_1} - T_1^*T_1 & 0 \\ 0 & I_{\mathcal{H}_2} \end{pmatrix}. \end{aligned} \quad (4.1)$$

By Lemma 2.4, Equation (4.1) holds if and only if there exists a contraction

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}'_1 \oplus \mathcal{H}'_2$$

such that

$$\begin{pmatrix} 0 & 0 \\ X & T_2 \end{pmatrix}^* = \begin{pmatrix} D_{T_1} & 0 \\ 0 & I_{\mathcal{H}_2} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}^* = \begin{pmatrix} D_{T_1}Z_{11}^* & D_{T_1}Z_{21}^* \\ Z_{12}^* & Z_{22}^* \end{pmatrix}. \quad (4.2)$$

Equation (4.2) holds if and only if there exists a contraction Z such that $Z_{12} = 0$, $Z_{22} = T_2$, $X = Z_{21}D_{T_1}$ and $D_{T_1}Z_{11}^* = 0$. This is equivalent to the existence of a contraction $Z' = \begin{pmatrix} 0 & 0 \\ Z_{21} & T_2 \end{pmatrix}$ satisfying $X = Z_{21}D_{T_1}$. Now Z' is a contraction if and only if $Z'Z'^* \leq I_{\mathcal{H}'_1 \oplus \mathcal{H}'_2}$ if and only if $Z_{21}Z_{21}^* + T_2T_2^* \leq I_{\mathcal{H}'_2}$, that is, $Z_{21}Z_{21}^* \leq I_{\mathcal{H}'_2} - T_2T_2^*$. Thus, we have that $Y = \begin{pmatrix} T_1 & 0 \\ X & T_2 \end{pmatrix}$ is a contraction if and only if there exists

$$Z' = \begin{pmatrix} 0 & 0 \\ Z_{21} & T_2 \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}'_1 \oplus \mathcal{H}'_2$$

satisfying $Z_{21}Z_{21}^* \leq I_{\mathcal{H}'_2} - T_2T_2^*$ and $X = Z_{21}D_{T_1}$. By Lemma 2.4, $Z_{21}Z_{21}^* \leq I_{\mathcal{H}'_2} - T_2T_2^*$ holds if and only if there exists a contraction $C : \mathcal{H}_1 \rightarrow \mathcal{H}'_2$ such that $Z_{21} = D_{T_2}C$. Therefore, Y is a contraction if and only if $X = D_{T_2}CD_{T_1}$ for some contraction C and the proof is complete. \square

Following the same technique as in the proof of Theorem 3 in [4], we obtain a generalized version of the above result for intertwining operators.

Theorem 4.2. *Let $T \in \mathcal{B}(\mathcal{H}_1)$, $T' \in \mathcal{B}(\mathcal{H}'_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}'_1)$ be contractions such that $T_2T = T'T_2$. Suppose $V = \begin{pmatrix} T & 0 \\ S & 0 \end{pmatrix}$ on $\mathcal{H}_1 \oplus \mathcal{H}_2$, $V' = \begin{pmatrix} T' & 0 \\ S' & 0 \end{pmatrix}$ on $\mathcal{H}'_1 \oplus \mathcal{H}'_2$ such that $T^*T + S^*S = I_{\mathcal{H}_1}$ and $T'^*T' + S'^*S' = I_{\mathcal{H}'_1}$. Then there exists $Y = \begin{pmatrix} T_2 & 0 \\ A & B \end{pmatrix}$ from $\mathcal{H}_1 \oplus \mathcal{H}_2$ to $\mathcal{H}'_1 \oplus \mathcal{H}'_2$ such that $YV = V'Y$ and $\|Y\| = \|T_2\|$.*

Proof. Without loss of generality assume $\|T_2\| = 1$. We want to find a matrix $Y = \begin{pmatrix} T_2 & 0 \\ A & B \end{pmatrix}$ from $\mathcal{H}_1 \oplus \mathcal{H}_2$ to $\mathcal{H}'_1 \oplus \mathcal{H}'_2$ such that $YV = V'Y$, that is, to find A and B such that

$$AT + BS = S'T_2. \quad (4.3)$$

For such a Y to be contraction, due to Proposition 4.1 it suffices to show that A, B are contractions and

$$A = D_{B^*}CD_{T_2} \quad (4.4)$$

for some contraction $C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}'_2)$. We first construct K from \mathcal{H}_1 to \mathcal{H}'_2 such that $A = KD_{T_2}$. Thus (4.3) gives us

$$KD_{T_2}T + BS = S'T_2.$$

This implies

$$T^*D_{T_2}K^* + S^*B^* = T_2^*S'^* \quad (4.5)$$

In order to find K^* and B^* satisfying (4.5), due to Theorem 2.5, it suffices to show that

$$(T^*D_{T_2})(T^*D_{T_2})^* + S^*S \geq T_2^*S'^*S'T_2.$$

Since $T_2^*T_2 \leq I_{\mathcal{H}_1}$, $T^*T + S^*S = I_{\mathcal{H}_1}$ and $T'^*T' + S'^*S' = I_{\mathcal{H}'_1}$, we have

$$\begin{aligned} (T^*D_{T_2})(T^*D_{T_2})^* + S^*S &= T^*(I - T_2^*T_2)T + S^*S \\ &\geq T^*(I - T_2^*T_2)T + S^*S - (I_{\mathcal{H}_1} - T_2^*T_2) \\ &= T_2^*T_2 - T^*T_2^*T_2T \\ &= T_2^*T_2 - T_2^*T'^*T'T_2 \\ &= T_2^*(I_{\mathcal{H}'_1} - T'^*T')T_2 \\ &= T_2^*S'^*S'T_2. \end{aligned}$$

Therefore, by Theorem 2.5, there exist operators $K : \mathcal{H}_1 \rightarrow \mathcal{H}'_2$ and $B : \mathcal{H}_2 \rightarrow \mathcal{H}'_2$ satisfying (4.5) and $KK^* + BB^* \leq I_{\mathcal{H}'_2}$. This implies that B is a contraction and $KK^* \leq I_{\mathcal{H}'_2} - BB^*$. Again by applying Lemma 2.4, there exists a contraction C from \mathcal{H}_1 to \mathcal{H}'_2 such that $K = D_{B^*}C$. Clearly $\|Y\| \geq \|T_2\| = 1$. Since T_2, B are contractions and $A = D_{B^*}CD_{T_2}$ for some contraction C , it follows from Proposition 4.1 that $\|Y\| \leq 1$. This completes the proof. \square

Suppose T is a contraction on \mathcal{H} and (V, \mathcal{K}) is an isometric lift of T . Let $\mathcal{K}_1 = \bigvee_{n=0}^{\infty} V^n \mathcal{H}$ and $K_n = \bigvee_{l=0}^n V^l \mathcal{H}$. Then $(V|_{\mathcal{K}_1}, \mathcal{K}_1)$ is the minimal isometric lift of T . Again, any two minimal isometric dilations of T are unitarily equivalent. Thus, without loss of generality we may consider the Schaeffer's minimal isometric dilation (V, \mathcal{K}) of T , where $\mathcal{K} = \mathcal{H} \oplus \mathcal{D}_T \oplus \mathcal{D}_T \oplus \dots$ and

$$V = \begin{pmatrix} T & 0 & 0 & 0 & \dots \\ D_T & 0 & 0 & 0 & \dots \\ 0 & I_{\mathcal{D}_T} & 0 & 0 & \dots \\ 0 & 0 & I_{\mathcal{D}_T} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The following result is an analogue of Theorem 3.3 while we consider the minimal isometric lift of a contraction instead of a co-isometric extension. This is another main result of this paper.

Theorem 4.3. *Let T_1 be a nonzero contraction and T_2 be an operator on a Hilbert space \mathcal{H} . Suppose $T_1T_2 = qT_2T_1$, where q is a complex number satisfying $0 < |q| \leq \frac{1}{\|T_1\|}$. Let V on \mathcal{K}' be the minimal isometric dilation of T_1 and V_q be an operator on \mathcal{K} such that qV_q is the minimal isometric dilation of qT_1 . Then there exists a bounded linear operator $W : \mathcal{K} \rightarrow \mathcal{K}'$ such that*

- (1) \mathcal{K} is invariant under W^* and $W^*|_{\mathcal{K}} = T_2^*$;
- (2) $\|W\| = \|T_2\|$ and
- (3) $VW = qWV_q$.

Proof. Since \mathcal{K}' and \mathcal{K} are the minimal isometric dilation spaces for T_1 and qT_1 respectively, without loss of generality we may assume that

$$\mathcal{K}' = \mathcal{H} \oplus \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_1} \oplus \dots \quad \text{and} \quad \mathcal{K} = \mathcal{H} \oplus \mathcal{D}_{qT_1} \oplus \mathcal{D}_{qT_1} \oplus \dots$$

Consider for all $n \in \mathbb{N} \cup \{0\}$,

$$\mathcal{K}'_n = \mathcal{H} \oplus \underbrace{\mathcal{D}_{T_1} \oplus \dots \oplus \mathcal{D}_{T_1}}_{n \text{ copies}} \oplus \{0\} \oplus \dots \quad \text{and} \quad \mathcal{K}_n = \mathcal{H} \oplus \underbrace{\mathcal{D}_{qT_1} \oplus \dots \oplus \mathcal{D}_{qT_1}}_{n \text{ copies}} \oplus \{0\} \oplus \dots$$

It can be easily verified that \mathcal{K}_n and \mathcal{K}'_n are invariant subspaces for $(qV_q)^*$ and V^* respectively. Hence define $V_n^* = V^*|_{\mathcal{K}'_n}$ and $(qV_q)_n^* = (qV_q)^*|_{\mathcal{K}_n}$. Since \mathcal{K}'_n is an invariant subspace for V_{n+1}^* , with respect to the decomposition $\mathcal{K}'_{n+1} = \mathcal{K}'_n \oplus \mathcal{D}_{T_1}$ the operator V_{n+1} has the block matrix form $V_{n+1} = \begin{pmatrix} V_n & 0 \\ S_n' & 0 \end{pmatrix}$. Similarly, \mathcal{K}_n is an invariant subspace for $(qV_q)_{n+1}^*$ hence, with respect to the decomposition $\mathcal{K}_{n+1} = \mathcal{K}_n \oplus \mathcal{D}_{qT_1}$ the operator $(qV_q)_{n+1}$ has the block matrix form $(qV_q)_{n+1} = \begin{pmatrix} (qV_q)_n & 0 \\ S_n & 0 \end{pmatrix}$. Note that $V_n^*V_n + S_n'^*S_n' = I_{\mathcal{K}'_n}$ and $(qV_q)_n^*(qV_q)_n + S_n^*S_n = I_{\mathcal{K}_n}$.

Now we find a sequence $\{W_n\}$ inductively using Theorem 4.2. We have $V_1 = \begin{pmatrix} T_1 & 0 \\ D_{T_1} & 0 \end{pmatrix}$ and $(qV_q)_1 = \begin{pmatrix} qT_1 & 0 \\ D_{qT_1} & 0 \end{pmatrix}$ satisfying $T_2(qT_1) = T_1T_2$, $T_1^*T_1 + D_{T_1}^2 = I_{\mathcal{H}}$ and $(qT_1)^*(qT_1) + D_{qT_1}^2 = I_{\mathcal{H}}$.

Therefore, by Theorem 4.2, there exists $W_1 = \begin{pmatrix} T_2 & 0 \\ A_1 & B_1 \end{pmatrix}$ from \mathcal{K}_1 to \mathcal{K}'_1 such that $W_1(qV_q)_1 = V_1W_1$, $W_1^*|_{\mathcal{K}'_0} = T_2^*$ and $\|W_1\| = \|T_2\|$.

Assume that for $1 \leq m \leq n-1$, there exists $W_m = \begin{pmatrix} W_{m-1} & 0 \\ A_m & B_m \end{pmatrix}$ from \mathcal{K}_m to \mathcal{K}'_m such that $W_m(qV_q)_m = V_mW_m$, $W_m^*|_{\mathcal{K}'_{m-1}} = W_{m-1}^*$ and $\|W_m\| = \|T_2\|$. Considering V_n in place of V' and $(qV_q)_n$ in place of V in Theorem 4.2, there exists $W_n = \begin{pmatrix} W_{n-1} & 0 \\ A_n & B_n \end{pmatrix}$ from \mathcal{K}_n to \mathcal{K}'_n such that $W_n(qV_q)_n = V_nW_n$, $W_n^*|_{\mathcal{K}'_{n-1}} = W_{n-1}^*$ and $\|W_n\| = \|T_2\|$. Hence by induction we have a sequence $\{W_n\}$ such that for each $n \in \mathbb{N}$, $W_n(qV_q)_n = V_nW_n$, $W_n^*|_{\mathcal{K}'_{n-1}} = W_{n-1}^*$ and $\|W_n\| = \|T_2\|$. We may consider W_n^* as an operator from \mathcal{K}' to \mathcal{K} by defining $W_n^*(k) = 0$ for $k \in \mathcal{K}' \ominus \mathcal{K}'_n$. Similarly V_n^* and $(qV_q)_n^*$ can be thought of as operators on \mathcal{K}' and \mathcal{K} by defining it to be 0 on \mathcal{K}'_n^\perp and \mathcal{K}_n^\perp respectively. Hence for all $n \in \mathbb{N}$, $(qV_q)_n^*W_n^* = W_n^*V_n^*$ from \mathcal{K}' to \mathcal{K} . For each $x \in \bigcup_{n=0}^{\infty} \mathcal{K}'_n$, which is dense in \mathcal{K}' , the sequence $\{W_n^*x\}$ is a Cauchy sequence and since $\|W_n^*\| = \|T_2^*\|$, by uniform boundedness principle the sequence $\{W_n^*\}$ converges strongly, say to the operator W^* from \mathcal{K}' to \mathcal{K} . Clearly $\|W\| = \|T_2\|$ and $W^*|_{\mathcal{H}} = T_2^*$. By construction $\{V_n^*\}$ and $\{(qV_q)_n^*\}$ converge strongly to V^* and $(qV_q)^*$ respectively. Hence $W(qV_q) = VW$. This completes the proof. \square

One of the assumptions of the previous theorem was $TX = qXT$ for $0 < |q| \leq \frac{1}{\|T\|}$. If we change the position of q , that is, if we consider $qTX = XT$ for $0 < |q| \leq \frac{1}{\|T\|}$, then we have the following analogous result.

Theorem 4.4. *Let T be a contraction on Hilbert space \mathcal{H} and $X \in \mathcal{B}(\mathcal{H})$ be such that $qTX = XT$ for $0 < |q| \leq \frac{1}{\|T\|}$. Suppose (V, \mathcal{K}') and (qV_q, \mathcal{K}) are the minimal isometric dilations of*

T and qT respectively. Then there exists an operator $Y \in \mathcal{B}(\mathcal{H}', \mathcal{H})$ such that

$$YV = qV_qY, Y^*(\mathcal{H}) \subseteq \mathcal{H}, Y^*|_{\mathcal{H}} = X^* \text{ and } \|X\| = \|Y\|.$$

Proof. Note that $qTX = XT$ implies $\bar{q}X^*T^* = T^*X^*$. Since V, qV_q are the minimal isometric dilations of T, qT respectively, $V^*, (qV_q)^*$ are the minimal co-isometric extensions of $T^*, (qT)^*$ respectively. Then from Theorem 3.3, there exists Y^* from \mathcal{H} to \mathcal{H}' such that $V^*Y^* = Y^*\bar{q}V_q^*$, $Y^*|_{\mathcal{H}} = X^*$ and $\|X\| = \|Y\|$. That is, $YV = qV_qY$. \square

Here we have another consequence of the previous results.

Lemma 4.5. *Let $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$ be contractions and $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be such that $AT_1 = T_2A$. Assume that $V_1 \in \mathcal{B}(\mathcal{H}_1)$ and $V_2 \in \mathcal{B}(\mathcal{H}_2)$ are minimal isometric dilations of T_1 and T_2 respectively. Then there exists a bounded linear operator $B : \mathcal{H}'_1 \rightarrow \mathcal{H}_2$ such that*

- (1) B^* maps \mathcal{H}_2 to \mathcal{H}'_1 and $B^*|_{\mathcal{H}_2} = A^*$;
- (2) $\|B\| = \|A\|$;
- (3) $V_2B = BV_1$.

Proof. Clearly $AT_1 = T_2A$ implies that $A^*T_2^* = T_1^*A^*$. Further, V_i is the minimal isometric dilation of T_i implies that V_i^* is the minimal co-isometric extension of T_i^* . Hence by Theorem 2.6, there exists $Y \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}'_1)$ such that $YV_2^* = V_1^*Y$, $Y|_{\mathcal{H}_2} = A^*$ and $\|Y\| = \|A\|$. Now by considering $B = Y^*$ we get the desired result. \square

The next result provides an analogue of Theorem 4.3 in the unitary dilation setting. We consider here the minimal unitary dilation of a contraction T and find a q -commutant lift of a q -commutant of T . This is a refinement of Proposition 2.6 of [8] and is another main result of this paper.

Theorem 4.6. *Let T_1 be a nonzero contraction and T_2 be an operator on a Hilbert space \mathcal{H} . Suppose $T_1T_2 = qT_2T_1$ with $0 < |q| \leq \frac{1}{\|T_1\|}$. Let U on \mathcal{H}' be the minimal unitary dilation of T_1 and U_q be an operator on \mathcal{H} such that qU_q is the minimal unitary dilation of qT_1 . Then there exists a bounded linear operator $S : \mathcal{H} \rightarrow \mathcal{H}'$ such that*

- (1) $\|S\| = \|T_2\|$;
- (2) $US = qSU_q$ and
- (3) $T_1^n T_2 = P_{\mathcal{H}} U_q^n S|_{\mathcal{H}}$ and $T_2 T_1^n = P_{\mathcal{H}} S U_q^n|_{\mathcal{H}}$ for all $n \geq 0$.

Proof. Let (U_+, \mathcal{H}'_+) and $((qU_q)_+, \mathcal{H}_+)$ be the minimal isometric dilations of (T_1, \mathcal{H}) and (qT_1, \mathcal{H}) respectively. Let $(U_q)_+ = \frac{1}{q}(qU_q)_+$. Then by Theorem 4.3, there exists $S_+ : \mathcal{H}_+ \rightarrow \mathcal{H}'_+$ such that

- (i) \mathcal{H} is invariant under S_+^* and $S_+^*|_{\mathcal{H}} = T_2^*$;
- (ii) $\|S_+\| = \|T_2\|$ and
- (iii) $U_+ S_+ = qS_+(U_q)_+ = S_+(qU_q)_+$.

From (iii) we have that $S_+^* U_+^* = (qU_q)_+^* S_+^*$. Since (U_+, \mathcal{H}'_+) and $((qU_q)_+, \mathcal{H}_+)$ are the minimal isometric dilations of (U_+, \mathcal{H}'_+) and $((qU_q)_+, \mathcal{H}_+)$ respectively, by Lemma 4.5 there exists $S^* : \mathcal{H}' \rightarrow \mathcal{H}$ such that

- (a) S maps \mathcal{H}_+ to \mathcal{H}'_+ and $S|_{\mathcal{H}_+} = S_+$,
- (b) $\|S\| = \|S_+\| = \|T_2\|$,
- (c) $S^* U^* = (qU_q)^* S^*$.

From (c) we have that $US = qSU_q$. By (i) the block matrices of S_+^* with respect to the decompositions $\mathcal{K}'_+ = \mathcal{H} \oplus (\mathcal{K}'_+ \ominus \mathcal{H})$ and $\mathcal{K}_+ = \mathcal{H} \oplus (\mathcal{K}_+ \ominus \mathcal{H})$ are of the form $\begin{pmatrix} T_2^* & * \\ 0 & * \end{pmatrix}$, i.e.

$$S_+ = \begin{pmatrix} T_2 & 0 \\ * & * \end{pmatrix} : \mathcal{H} \oplus (\mathcal{K}_+ \ominus \mathcal{H}) \rightarrow \mathcal{H} \oplus (\mathcal{K}'_+ \ominus \mathcal{H}).$$

Again by (a) the block matrices of S with respect to the decompositions $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_+^\perp$ and $\mathcal{K}' = \mathcal{K}'_+ \oplus \mathcal{K}'_+^\perp$ are of the form $\begin{pmatrix} S_+ & * \\ 0 & * \end{pmatrix}$. Therefore, the block matrices of S with respect to the decompositions $\mathcal{K} = \mathcal{H} \oplus (\mathcal{K}_+ \ominus \mathcal{H}) \oplus \mathcal{K}_+^\perp$ and $\mathcal{K}' = \mathcal{H} \oplus (\mathcal{K}'_+ \ominus \mathcal{H}) \oplus \mathcal{K}'_+^\perp$ take the form $\begin{pmatrix} T_2 & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$. Again, since (U, \mathcal{K}') is the minimal unitary dilation of (T_1, \mathcal{H}) , the block matrices of U with respect to the decompositions $\mathcal{K} = \mathcal{H} \oplus (\mathcal{K}_+ \ominus \mathcal{H}) \oplus \mathcal{K}_+^\perp$ and $\mathcal{K}' = \mathcal{H} \oplus (\mathcal{K}'_+ \ominus \mathcal{H}) \oplus \mathcal{K}'_+^\perp$ have the form $\begin{pmatrix} T_1 & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$. Similarly, the block matrix of qU_q with respect to the decompositions $\mathcal{K} = \mathcal{H} \oplus (\mathcal{K}_+ \ominus \mathcal{H}) \oplus \mathcal{K}_+^\perp$ and $\mathcal{K}' = \mathcal{H} \oplus (\mathcal{K}'_+ \ominus \mathcal{H}) \oplus \mathcal{K}'_+^\perp$ would be $\begin{pmatrix} qT_1 & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$. Therefore, we conclude that

$$(qU_q)^n S = \begin{pmatrix} (qT_1)^n T_2 & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \quad \text{for all } n \geq 0$$

and consequently $T_1^n T_2 = P_{\mathcal{H}} U_q^n S|_{\mathcal{H}}$ for every non-negative integer n . Similarly, $T_2 T_1^n = P_{\mathcal{H}} S U^n|_{\mathcal{H}}$ for all $n \geq 0$ and the proof is complete. \square

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