

EXISTENCE AND UNIQUENESS OF LIMITS AT INFINITY FOR HOMOGENEOUS SOBOLEV FUNCTIONS

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ABSTRACT. We establish the existence and uniqueness of limits at infinity along infinite curves outside a zero modulus family for functions in a homogeneous Sobolev space under the assumption that the underlying space is equipped with a doubling measure which supports a Poincaré inequality. We also characterize the settings where this conclusion is nontrivial. Secondly, we introduce notions of weak polar coordinate systems and radial curves on metric measure spaces. Then sufficient and necessary conditions for existence of radial limits are given. As a consequence, we characterize the existence of radial limits in certain concrete settings.

1. INTRODUCTION

Let (X, d, μ) be a metric measure space with metric d and Borel regular measure μ . A locally rectifiable curve $\gamma : [0, \infty) \rightarrow X$ is an *infinite curve* if $\gamma \setminus B \neq \emptyset$ for all balls B . Then $\int_\gamma ds = \infty$ if γ is an infinite curve. We write Γ^∞ for the collection of all infinite curves and denote by $\dot{N}^{1,p}(X)$, $1 \leq p < \infty$, the collection of all locally integrable functions that have a p -integrable upper gradient on (X, d, μ) . Here the notion of upper gradients is given in Section 2.2.

The aim of this paper is to study the existence and uniqueness of the limit

$$(1.1) \quad \lim_{t \rightarrow \infty} u(\gamma(t))$$

for $\gamma \in \Gamma^\infty$ and for $u \in \dot{N}^{1,p}(X)$. We say that *the existence and uniqueness* of (1.1) hold for p -a.e $\gamma \in \Gamma^\infty$ if, for every $u \in \dot{N}^{1,p}(X)$, there exists $c \in \mathbb{R}$ such that

$$\lim_{t \rightarrow \infty} u(\gamma(t)) \text{ exists and } \lim_{t \rightarrow \infty} u(\gamma(t)) = c \text{ for } p\text{-a.e } \gamma \in \Gamma^\infty.$$

Here the notion of p -a.e curve is given in Section 2.2.

Towards uniqueness, we employ an annular chain property at a given based point O , see Definition 2.3. This chain property holds, for example, if there exists a constant $C \geq 1$ for which any pair of points in $B(O, r) \setminus B(O, r/2)$ can be joined by a curve in $B(O, Cr) \setminus B(O, r/C)$, see Section 2.5 for more details. This holds especially when X is annularly quasiconvex as defined in [HKST15, Section 8.3].

Our first result deals with the limit (1.1) for p -a.e $\gamma \in \Gamma^\infty$.

Theorem 1.1. *Let $1 \leq p < \infty$. Suppose that (X, d, μ) is a doubling metric measure space that supports a p -Poincaré inequality. Assume that X has the annular chain property. Then the existence and uniqueness of (1.1) hold for p -a.e $\gamma \in \Gamma^\infty$.*

In Theorem 1.1, the uniqueness does not hold without some additional assumption besides doubling and Poincaré whose definitions are given in Section 2.3. For example, on a space with at least two ends one easily constructs a Lipschitz function in $\dot{N}^{1,p}(X)$ so that the tail value of two of the ends is a different constant.

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A tree in [BBGS17, NgWa20] (or a weighted real line in [BBK06, BBS20]) is doubling, supports a Poincaré inequality and has more than one end.

The conclusion of Theorem 1.1 is nontrivial only when the p -modulus of Γ^∞ is strictly positive. Let us suppose that μ is a Q -Ahlfors regular measure as in Section 2.3 where $1 < Q \leq p < \infty$. Then the p -modulus of Γ^∞ vanishes, and hence there exists $u \in \dot{N}^{1,p}(X)$ such that $\lim_{t \rightarrow \infty} u(\gamma(t)) = \infty$ for every $\gamma \in \Gamma^\infty$, see [HKST15, Page 135]. One can actually characterize the settings where the p -modulus of the family Γ^∞ is strictly positive. This is the content of Theorem 1.2 below.

Let O be a fixed point in X and let $A_{2^j}(O)$ be the annuli $B(O, 2^{j+1}) \setminus B(O, 2^j)$ for $j \in \mathbb{N}$. We define

$$\mathcal{R}_p(O) := \sum_{j \in \mathbb{N}} (2^j)^{\frac{p}{p-1}} \mu^{\frac{1}{1-p}}(A_{2^j}(O)) \quad \text{if } p > 1 \quad \text{and} \quad \mathcal{R}_1(O) := \sup_{j \in \mathbb{N}} 2^j \mu^{-1}(A_{2^j}(O)).$$

The finiteness of $\mathcal{R}_p(O)$ is then a volume growth condition. A reader familiar with classification theory should recognize this as a condition towards p -hyperbolicity [HoKo01, Hol99, Gri99].

Theorem 1.2. *Let $1 \leq p < \infty$. Suppose that (X, d, μ) is a complete doubling metric measure space that supports a p -Poincaré inequality. Assume that X has the annular chain property at O . Then the following statements are equivalent:*

- I. $\mathcal{R}_p(O) < \infty$.
- II. $\text{Mod}_p(\Gamma^\infty) > 0$.
- III. *For every $u \in \dot{N}^{1,p}(X)$, there exists an infinite curve $\gamma \in \Gamma^\infty$ such that $\lim_{t \rightarrow \infty} u(\gamma(t))$ exists.*

The space $X = [0, \infty) \subset \mathbb{R}$ equipped with the Lebesgue measure and the usual distance is doubling and supports a p -Poincaré inequality for all $p \geq 1$. It also has the annular chain property and there is (modulo reparametrizations and translations) only one injective infinite curve. Moreover, $\mathcal{R}_p(O) < \infty$ precisely when $p = 1$, in which case every function in $\dot{N}^{1,1}(X)$ has a (unique) limit along this curve. Hence the existence of a single “good” curve is the best one can obtain in this generality. However, for \mathbb{R}^n , $n \geq 2$, equipped with the Lebesgue measure and Euclidean distance, $\mathcal{R}_p(O) < \infty$ precisely when $p < n$ and one then obtains a unique limit along the radial half-line in the direction $\xi \in S^{n-1}$, for almost every ξ with respect to the surface measure on the boundary S^{n-1} of the unit ball. This is a consequence of the existence of spherical (or polar) coordinates. Towards establishing limits along a recognizable family of infinite curves, let us introduce an abstract version of polar coordinates.

Let \mathbb{S} be a nonempty set (a set of indices), $O \in X$ and consider collections $\Gamma^O(\mathbb{S})$ consisting of $\gamma_\xi \in \Gamma^\infty$, $\xi \in \mathbb{S}$, with $\gamma_\xi(0) = O$. We say that X has a *weak polar coordinate system* at the *coordinate point* O if there exist a set \mathbb{S} of indices with a Radon probability measure σ on \mathbb{S} , a choice of $\Gamma^O(\mathbb{S})$, a function $h : X \rightarrow [0, \infty)$, and a constant $\mathcal{C} > 0$ such that

$$(1.2) \quad \int_{\mathbb{S}} \int_{\gamma_\xi^O} |f|h \, ds d\sigma \leq \mathcal{C} \int_X |f| d\mu \quad \text{for every integrable function } f.$$

We then call h a *coordinate weight* and each $\gamma_\xi^O \in \Gamma^O(\mathbb{S})$ a *radial curve* in the direction $\xi \in \mathbb{S}$ from O . Towards the existence of (1.1) along radial curves, consider a polar coordinate (1.2) at O . We set

$$R_p(h, O) := \int_{X \setminus B(O, 1)} h^{\frac{p}{p-1}} d\mu \quad \text{if } p > 1 \quad \text{and} \quad R_1(h, O) := \|h^{-1}\|_{L^\infty_\mu(X \setminus B(O, 1))}.$$

Given $\xi \in F \subset \mathbb{S}$ with $\liminf_{t \rightarrow \infty} d(O, \gamma_\xi^O(t)) > 1$, there is $t_\xi > 0$ such that $d(O, \gamma_\xi^O(t_\xi)) = 1$ and $d(O, \gamma_\xi^O(t)) > 1$ for all $t > t_\xi$. Let $\hat{\gamma}_\xi^O : [0, \infty) \rightarrow X$ be the infinite curve starting from $\gamma_\xi^O(t_\xi)$ defined by $\hat{\gamma}_\xi^O(t) := \gamma_\xi^O(t + t_\xi)$ for $t \geq 0$. The collection of all these infinite curves $\hat{\gamma}_\xi^O$ with respect to $\xi \in F$ satisfying $\liminf_{t \rightarrow \infty} d(O, \gamma_\xi^O(t)) > 1$ is denoted by $\hat{\Gamma}^O(F)$.

In order to state our next result, we introduce the following properties:

1. For every $u \in \dot{N}^{1,p}(X)$, there is a radial curve γ such that $\liminf_{t \rightarrow \infty} u(\gamma(t))$ is finite.
2. The limit (1.1) exists for σ -a.e $\xi \in \mathbb{S}$, i.e. the limit

$$\lim_{t \rightarrow \infty} u(\gamma_\xi^O(t)) \text{ exists for } \sigma\text{-a.e } \xi \in \mathbb{S}, \text{ for every } u \in \dot{N}^{1,p}(X).$$

3. The existence and uniqueness of (1.1) are obtained for σ -a.e $\xi \in \mathbb{S}$, i.e. for every $u \in \dot{N}^{1,p}(X)$, there exists $c \in \mathbb{R}$ such that

$$\lim_{t \rightarrow \infty} u(\gamma_\xi^O(t)) \text{ exists and } \lim_{t \rightarrow \infty} u(\gamma_\xi^O(t)) = c \text{ for } \sigma\text{-a.e } \xi \in \mathbb{S}.$$

4. $\text{Mod}_p(\hat{\Gamma}^O(\mathbb{S})) > 0$.
5. If $F \subseteq \mathbb{S}$ satisfies $\sigma(F) > 0$, then $\text{Mod}_p(\hat{\Gamma}^O(F)) > 0$.

We show that each of $\{1., 2., 3., 4., 5.\}$ is “between” $\mathcal{R}_p(O) < \infty$ and $R_p(h, O) < \infty$.

Theorem 1.3. *Let $1 \leq p < \infty$. Suppose that (X, d, μ) is a doubling metric measure space that supports a p -Poincaré inequality. Assume that X has the annular chain property at O . Suppose that X has a weak polar coordinate system $(\mathbb{S}, \sigma, \Gamma^O, h)$ at O as in (1.2). Then*

- I. *The condition $R_p(h, O) < \infty$ is sufficient for all of $\{1., 2., 3., 4., 5.\}$.*
- II. *The condition $\mathcal{R}_p(O) < \infty$ is necessary for each of $\{1., 2., 3., 4., 5.\}$.*

Theorem 1.3 leads us to compare the finiteness of $R_p(h, O)$ and $\mathcal{R}_p(O)$. First of all, it immediately follows from Theorem 1.3 that the finiteness of $R_p(h, O)$ guarantees the finiteness of $\mathcal{R}_p(O)$. We do not know a simple direct proof for this. On the other hand, the finiteness of $\mathcal{R}_p(O)$ does not in general yield the finiteness of $R_p(h, O)$. To see this, simply consider \mathbb{R}^n , $n \geq 2$, with the Euclidean distance and Lebesgue measure, usual spherical coordinates (with normalized measure on S^{n-1}) but with the coordinate weight $h = \chi_{\mathbb{R}^n \setminus B(0,1)}$. On the other hand, for example, if

$$h(x) \gtrsim \sum_{j \in \mathbb{N}} \frac{\mu(A_{2^j}(O))}{(2^j)^p} \chi_{A_{2^j}(O)}(x)$$

for all $x \in X$, then $R_p(h, O) \lesssim \mathcal{R}_p(O)$.

Let us next consider the Muckenhoupt \mathcal{A}_p -weighted space (\mathbb{R}^n, d_E, w) , where $n \geq 2$ and O is the origin. Here d_E is the Euclidean distance, and w is a Muckenhoupt \mathcal{A}_p -weight with the associate measure $d\mu = w dx$, i.e. there is a constant $C \geq 1$ such that for every ball $B \subset \mathbb{R}^n$,

$$(1.3) \quad \left(\int_B w dx \right)^{\frac{1}{p}} \left(\int_B w^{\frac{1}{1-p}} dx \right)^{\frac{p}{p-1}} \leq C \text{ if } p > 1,$$

and

$$(1.4) \quad \left(\int_B w dx \right) \|w^{-1}\|_{L^\infty(B)} \leq C \text{ if } p = 1.$$

Then μ is doubling and supports a p -Poincaré inequality, see for instance [HKM06]. The annular chain property at O is satisfied and the usual (normalized) spherical coordinate system satisfies (1.2) at O . We show that $R_p(h, O) \approx \mathcal{R}_p(O)$ in Example 4.3. In particular, $R_p(h, O) < \infty$ if and only if $1 \leq p < n$ for the unweighted space \mathbb{R}^n . It follows that Theorem 1.3 recovers some of the conclusions in [EKN22, Usp61].

Next, we consider a Q -Ahlfors regular space X that supports a p -Poincaré inequality, where $1 \leq p < \infty$ and $1 \leq Q < \infty$. Suppose that X has the annular chain property at O and satisfies the strong form

$$\int_{X \setminus B(O,1)} |f| d\mu = \int_{\mathbb{S}} \int_1^\infty |f(\gamma_\xi(t))| t^{Q-1} dt d\sigma(\xi) \text{ (for every integrable function } f)$$

of our weak polar coordinate system. Then each of $\{1., 2., 3., 4., 5.\}$ is equivalent to $1 \leq p < Q$ if $1 < Q < \infty$, and equivalent to $p = 1$ if $Q = 1$ since

$$(1.5) \quad \mathcal{R}_p(O) < \infty \implies \begin{cases} 1 \leq p < Q & \text{if } 1 < Q < \infty, \\ p = 1 & \text{if } Q = 1. \end{cases} \implies R_p(h, O) < \infty$$

where

$$R_p(h, O) \approx \begin{cases} \int_1^\infty r^{\frac{Q-1}{1-p}} dr & \text{if } p > 1, \\ \|r^{1-Q}\|_{L^\infty([1, \infty))} & \text{if } p = 1, \end{cases} \quad \text{and} \quad \mathcal{R}_p(O) \approx \begin{cases} \sum_{j \in \mathbb{N}} (2^j)^{\frac{p-Q}{p-1}} & \text{if } p > 1, \\ \sup_{j \in \mathbb{N}} (2^j)^{1-Q} & \text{if } p = 1. \end{cases}$$

In particular, by [BaTy02, Ty20] every polarizable Carnot group G (especially every group of Heisenberg type) admits a polar coordinate system (2.3) that satisfies (1.2). Since the Haar measure on G is Q -Ahlfors regular for the homogeneous dimension $Q > 1$ of G and supports a p -Poincaré inequality for each p , each of $\{1., 2., 3., 4., 5.\}$ is equivalent to $1 \leq p < Q < \infty$.

We then obtain the following characterization.

Corollary 1.4. *Let $1 \leq p < \infty$ and let $1 < Q < \infty$. Then*

- I. *Each of $\{1., 2., 3., 4., 5.\}$ is equivalent to $R_p(h, O) < \infty$ (or $\mathcal{R}_p(O) < \infty$) on the Muckenhoupt \mathcal{A}_p -weighted Euclidean spaces.*
- II. *Each of $\{1., 2., 3., 4., 5.\}$ is equivalent to $1 \leq p < Q < \infty$ on polarizable Carnot groups.*

The organization of this paper is as follows. In Section 2, we introduce polar coordinates and recall the basic notions on metric measure spaces. In Sections 3-4, proofs of Theorem 1.1-1.2-1.3 are given.

Throughout this paper, we use the following conventions. We denote by O the base point in the annular chain property and also refer by O to the coordinate point in the polar coordinate (1.2). The notation $A \lesssim B$ ($A \gtrsim B$) means that there is a constant $C > 0$ only depending on the data such that $A \leq C \cdot B$ ($A \geq C \cdot B$), and $A \approx B$ means that $A \lesssim B$ and $A \gtrsim B$. For each locally integrable function f and for every measurable subset $A \subset X$ of positive measure, we let $f_A := \int_A f d\mu = \frac{1}{\mu(A)} \int_A f d\mu$.

2. POLAR COORDINATES AND PRELIMINARIES

In Section 2.1, we introduce polar coordinates. We recall the basic notions of modulus, doubling, Poincaré inequalities, and Lebesgue points in Sections 2.2-2.3-2.4.

2.1. Polar coordinates. Let (X, d) be a metric space. A *curve* is a nonconstant continuous mapping from an interval $I \subseteq \mathbb{R}$ into X . The *length* of a curve γ is denoted by $l(\gamma)$. A curve γ is said to be a *rectifiable curve* if its length is finite. Similarly, γ is a *locally rectifiable curve* if its restriction to each compact subinterval of I is rectifiable. Each rectifiable curve γ will be parameterized by arc length and hence the *line integral* over γ of a Borel function f on X is

$$\int_\gamma f ds = \int_0^{l(\gamma)} f(\gamma(t)) dt.$$

If γ is locally rectifiable, then we set

$$\int_\gamma f ds = \sup \int_{\gamma'} f ds$$

where the supremum is taken over all rectifiable subcurves γ' of γ . Let $\gamma : [0, \infty) \rightarrow X$ be a locally rectifiable curve, parameterized by arc length. Then

$$\int_\gamma f ds = \int_0^\infty f(\gamma(t)) dt.$$

A locally rectifiable curve γ is an *infinite curve* if $\gamma \setminus B \neq \emptyset$ for all balls B . Then $\int_\gamma ds = \infty$. We denote by Γ^∞ the collection of all infinite curves.

Let \mathbb{S} be a nonempty set (a set of indices). Given a point O , we consider collections $\Gamma^O(\mathbb{S})$ of infinite curves with parameter space \mathbb{S} starting from O , namely

$$\Gamma^O(\mathbb{S}) = \{\gamma_\xi^O \in \Gamma^\infty : \gamma_\xi^O(0) = O, \xi \in \mathbb{S}\}.$$

We say that (X, d, μ) has a *weak polar coordinate system* at the *coordinate point* O if there is a choice of a pair $(\mathbb{S}, \Gamma^O(\mathbb{S}))$ with a Radon probability measure σ on \mathbb{S} , a *coordinate weight* $h : X \rightarrow [0, \infty)$, and a constant $C > 0$ such that

$$(2.1) \quad \int_{\mathbb{S}} \int_{\gamma_\xi^O} |f| h ds d\sigma \leq C \int_X |f| d\mu \quad \text{for every integrable function } f.$$

Each $\gamma_\xi^O \in \Gamma^O(\mathbb{S})$ is called a *radial curve* with respect to $\xi \in \mathbb{S}$ (starting from O). Notice that we have not assumed h to be strictly positive. In applications it will be important to require h to be strictly positive at least almost everywhere in the union of the images of γ_ξ^O .

Let us give examples of weak polar coordinates. First of all, in the n -dimensional Euclidean space \mathbb{R}^n , where $n \geq 2$, one has the usual spherical coordinate system at a given point O : there is a constant $C(n) > 0$ depending on n such that

$$(2.2) \quad \int_{\mathbb{R}^n} f(x) dx = C(n) \int_{S^{n-1}} \int_0^\infty f(O + r \cdot \xi) r^{n-1} dr d\mathcal{H}^{n-1} \quad \text{for every integrable function } f.$$

Here \mathcal{H}^{n-1} is the (normalized) $(n-1)$ -Hausdorff measure and S^{n-1} is the unit sphere centered at O . For these coordinates, (2.1) holds as an identity.

In [BaTy02, Ty20], Balogh and Tyson produced a polar coordinate system at the origin on polarizable Carnot groups G : there is a family of horizontal curves $\gamma_\xi : [0, \infty) \rightarrow G$ where ξ ranges over a certain compact unit sphere $S \subset G$, and a positive Radon measure σ on S so that

$$(2.3) \quad \int_G f(g) dg = \int_S \int_0^\infty f(\gamma_\xi(t)) t^{Q-1} dt d\sigma(\xi)$$

is valid for every integrable function f . Here the integral on the left is taken with respect to the Haar measure in the group and Q denotes the homogeneous dimension of G . The unit sphere S is the level set $\{g \in G : N(g) = 1\}$ for a certain homogeneous norm N in G .

In [KNZ22, Lemma 3.1], a polar coordinate at the root O on a K -regular tree is given. Let X be a K -regular tree with its boundary ∂X . We equip X with a radially weighted distance λ and a radially weighted measure μ . Then there is a uniform measure ν on ∂X such that

$$\int_X f d\mu \approx \int_{\partial X} \int_{[O, \xi)} f(x) \frac{K^{|x|} \mu(x)}{\lambda(x)} ds(x) d\nu(\xi)$$

for every integrable function f . Here $[O, \xi) := \gamma_\xi$ is the unique geodesic ray from the root O to $\xi \in \partial X$.

In the above examples, one actually has two-sided estimates for the terms in (2.1). This is not necessarily the case for our weak polar coordinate systems. For example, let $X := \{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_2 \leq 1\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : |x_2| \leq x_1\}$ and equip it with the Lebesgue measure and the Euclidean distance. Let $S_1 = [-1, 1]$ and set $\gamma_\xi^1(t) = \begin{cases} (-t, t \cdot \xi), & 0 \leq t \leq 1 \\ (-t, \xi), & t > 1 \end{cases}$, when $\xi \in [-1, 1]$. Define $h_1(x_1, x_2) = \begin{cases} 1, & x_1 < -1 \\ 0, & \text{otherwise.} \end{cases}$

Then

$$\int_{S_1} \int_0^\infty |f(\gamma_\xi^1(t))| h_1(\gamma_\xi^1(t)) dt d\xi \leq \int_X |f(x_1, x_2)| dx_1 dx_2 \quad \text{for every integrable function } f.$$

This is a weak polar coordinate system. A second weak polar coordinate system on X is obtained by taking $S_2 = [-\pi/4, \pi/4]$, $\gamma_\xi^2(t) = e^{i \cdot \xi}$ and $h_2(x_1, x_2) = \|(x_1, x_2)\| \chi_{\{(x_1, x_2) \in X : x_2 \geq 0\}}(x_1, x_2)$. Then

$$\int_{S_2} \int_0^\infty |f(\gamma_\xi^2(t))| h_2(\gamma_\xi^2(t)) dt d\xi \leq \int_X |f(x_1, x_2)| dx_1 dx_2 \quad \text{for every integrable function } f.$$

For a third one, we take $S_3 = S_1 \cup S_2$, $h_3 = h_1 + h_2$ and $\gamma_\xi^3 = \gamma_\xi^1 \cup \gamma_\xi^2$. Then

$$\int_{S_3} \int_0^\infty |f(\gamma_\xi^3(t))| h_3(\gamma_\xi^3(t)) dt d\xi \leq \int_X |f(x_1, x_2)| dx_1 dx_2 \quad \text{for every integrable function } f.$$

In [KM98, Section 3], there is a weak polar coordinate system on the Cantor ∞ -diamond. Let us describe it. Let E_i be the usual Cantor set in the unit interval $[i, i+1]$ obtained by first taking out the middle interval of length $1/3$ and leaving two intervals of length $1/3$ and then continuing inductively. The Cantor ∞ -diamond, denoted by X , is obtained by replacing each of the complementary intervals of E_i by a square having that interval as one of its diagonals. Thus we have a line of diamonds along the unit interval, and they are joined up by E_i . We consider the map $F : [0, \infty) \times [-1, 1] \rightarrow X$ defined by

$$F(x, y) = \left(x, \delta(x) \tan\left(\pi \frac{y}{4}\right) \right).$$

Here $\delta(x)$ is the distance from x to $E := \bigcup_{i=0}^\infty E_i$. Then the map F is simply the vertical projection on $E \times [-1, 1]$ and it is one-to-one, locally bi-Lipschitz on $([0, \infty) \setminus E) \times [-1, 1]$. The Jacobian of F at (x, y) , denoted by $J_F(x, y)$, is $\frac{\pi \delta(x)}{4 \cos^2(\pi y/4)}$. Let Γ be the family of curves γ_y , $-1 < y < 1$, defined by

$$\gamma_y(x) = F(x, y) \quad \text{for all } y \in [-1, 1].$$

Then γ_y is 4-Lipschitz for each $y \in [-1, 1]$. We denote by σ the 1-Lebesgue measure on $[-1, 1]$. By the change of variables formula, we have

$$\begin{aligned} \int_\Gamma \int_{\gamma} f \cdot (J_F \circ F^{-1}) ds d\left(\frac{\sigma}{2}\right)(\gamma) &= \frac{1}{2} \int_{[-1, 1]} \left(\int_{[0, \infty)} (f \circ \gamma_y)(x) \cdot (J_F \circ F^{-1} \circ \gamma_y)(x) \cdot |\dot{\gamma}_y(x)| dx \right) dy \\ &\leq 2 \int_{[-1, 1]} \int_{[0, \infty)} (f \circ F)(x, y) \cdot J_F(x, y) dx dy \\ &= 2 \int_X f d\mathcal{L}^2 \end{aligned}$$

for each positive integrable function f on X with respect to the 2-Lebesgue measure \mathcal{L}^2 . Here the inequality is obtained since γ_y is 4-Lipschitz for each $y \in [-1, 1]$ and $(J_F \circ F^{-1} \circ \gamma_y)(x) = (J_F \circ F^{-1})(F(x, y)) = J_F(x, y)$.

Finally, we give a definition of Semmes-type families which are related to our weak polar coordinate system.

Definition 2.1 (A Semmes type family of infinite curves). *Let (X, d, μ) be a metric measure space with metric d and measure μ . Given $O \in X$, a family Γ of infinite curves starting from O is called Semmes-type if there exist a constant $C > 0$ and a Radon probability measure σ on Γ such that*

$$(2.4) \quad \int_\Gamma \int_{\gamma} f ds d\sigma(\gamma) \leq C \int_X f(x) \frac{d(x, O)}{\mu(B(O, d(x, O)))} d\mu(x)$$

for every positive measurable function f on X for which the right-hand side of (2.4) is finite.

For instance, a family of all radial curves is a Semmes type family on \mathbb{R}^n by the usual spherical coordinate system (2.2) or on a polarizable Carnot group by the coordinate system (2.3). This definition is naturally

modified from the existence of families of rectifiable curves joining pairs of points on metric measure spaces by Semmes in [Sem96].

2.2. Modulus. Let Γ be a family of curves in a metric measure space (X, d, μ) . Given $1 \leq p < \infty$, the p -modulus of Γ , denoted $\text{Mod}_p(\Gamma)$, is defined by

$$\text{Mod}_p(\Gamma) := \inf \int_X \rho^p d\mu$$

where the infimum is taken over all Borel functions $\rho : X \rightarrow [0, \infty]$ satisfying $\int_\gamma \rho ds \geq 1$ for every locally rectifiable curve $\gamma \in \Gamma$. A family of curves is called p -exceptional if it has p -modulus zero. We say that a property holds for p -a.e curve if the collection of curves for which the property fails is p -exceptional.

Let u be a locally integrable function. A Borel function $\rho : X \rightarrow [0, \infty]$ is said to be an *upper gradient* of u if

$$(2.5) \quad |u(x) - u(y)| \leq \int_\gamma \rho ds$$

for every rectifiable curve γ connecting x and y . Then we have that (2.5) holds for all compact subcurves of $\gamma \in \Gamma^\infty$. We say that ρ is a p -weak upper gradient of u if (2.5) holds for p -a.e rectifiable curve. In what follows, we denote by g_u the *minimal upper gradient* of u , which is unique up to measure zero and which is minimal in the sense that $g_u \leq \rho$ a.e. for every p -integrable p -weak upper gradient ρ of u . In [Haj03], the existence and uniqueness of such a minimal upper gradient are given.

The notion of upper gradients is due to Heinonen and Koskela [HeKo98], we refer interested readers to [Björn, Haj03, HeKo98, Sha00] for a more detailed discussion on upper gradients.

2.3. Doubling and Poincaré inequalities. Let (X, d) be a metric space. A Borel regular measure μ is called *doubling* if every ball in X has finite positive measure and if there exists a constant $C \geq 1$ such that for all balls $B(x, r)$ with radius $r > 0$ and center at $x \in X$,

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Let $1 \leq Q < \infty$. A Borel regular measure μ is said to be Q -Ahlfors regular if there exists a constant $C \geq 1$ such that for all balls $B(x, r)$ with radius $r > 0$ and center at $x \in X$,

$$\frac{r^Q}{C} \leq \mu(B(x, r)) \leq Cr^Q.$$

Hence if μ is Q -Ahlfors regular for some $1 \leq Q < \infty$, then μ is a doubling measure.

Let $1 \leq p < \infty$. We say that a measure μ supports a p -Poincaré inequality if every ball in X has finite positive measure and if there exist constants $C > 0$ and $\lambda \geq 1$ such that

$$\int_{B(x, r)} |u - u_{B(x, r)}| d\mu \leq Cr \left(\int_{\lambda \cdot B(x, r)} \rho^p d\mu \right)^{\frac{1}{p}}$$

for all balls $B(x, r)$ with radius $r > 0$ and center at $x \in X$, and for all pairs (u, ρ) satisfying (2.5) such that u is integrable on balls. Here $\lambda \cdot B(x, r) := B(x, \lambda \cdot r)$ and λ is called the scaling constant of p -Poincaré inequality or the scaling factor of p -Poincaré inequality. Since $u \in \dot{N}^{1,p}(X)$ is integrable on balls, the p -Poincaré inequality makes sense for any pair (u, ρ_u) where ρ_u is an upper gradient of u .

2.4. Lebesgue points. A point $x \in X$ is called a Lebesgue's point of u if $\lim_{r \rightarrow 0} \int_{B(x,r)} |u(y) - u(x)| d\mu(y) = 0$ where $B(x, r)$ is the ball with radius r and center at x . Let N_u be a set of all points $x \in X$ such that x is not a Lebesgue point of u .

Theorem 2.2 (Lebesgue differentiation theorem, see for instance [HKST15, Page 77]). *We have $\mu(N_u) = 0$ for every locally integrable function u on X .*

2.5. Chain conditions. In this paper, we employ the following annular chain property.

Definition 2.3. *Let $\lambda \geq 1$. We say that X satisfies an annular λ -chain condition at O if the following holds. There are constants $c_1 \geq 1, c_2 \geq 1, \delta > 0$ and a finite number $M < \infty$ so that given $r > 0$ and points $x, y \in B(O, r) \setminus B(O, r/2)$, one can find balls B_1, B_2, \dots, B_k with the following properties:*

1. $k \leq M$.
2. $B_1 = B(x, r/(\lambda c_1)), B_k = B(y, r/(\lambda c_1))$ and the radius of each B_i is $r/(\lambda c_1)$ for $1 \leq i \leq k$.
3. $B_i \subset B(O, c_2 r) \setminus B(O, r/c_2)$ for $1 \leq i \leq k$.
4. For each $1 \leq i \leq k - 1$, there is a ball $D_i \subset B_i \cap B_{i+1}$ with radius δr .

If X satisfies an annular λ -chain condition at O for every $\lambda \geq 1$, we say that X has the annular chain property.

If X satisfies an annular λ -chain condition at O for some $\lambda \geq 1$ then it also satisfies an annular λ' -chain condition at O for all λ' with $1 \leq \lambda' \leq \lambda$ by taking $c_1(\lambda') := c_1(\lambda) \frac{\lambda}{\lambda'} \geq 1$. Here $c_1(\lambda)$ is the constant c_1 with respect to λ as in Definition 2.3. It follows that X has the annular chain property if and only if there is a sequence λ_k with $\lim_{k \rightarrow +\infty} \lambda_k = +\infty$ such that X satisfies an annular λ_k -chain condition at O for each $k \in \mathbb{N}$.

Lemma 2.4. *Let μ be doubling on (X, d) . Suppose that there is a constant $c_0 \geq 1$ so that for every $r > 0$, each pair x, y of points in $B(O, r) \setminus B(O, r/2)$ can be joined by a curve in $B(O, c_0 r) \setminus B(O, r/c_0)$. Then X has the annular chain property.*

Proof. We follow ideas in the proof of [HaKo00, Theorem 7.2]. Let $\lambda \geq 1$. Let $\gamma_{x,y}$ be a curve in $B(O, c_0 r) \setminus B(O, r/c_0)$ joining $x, y \in B(O, r) \setminus B(O, r/2)$. We consider the collection of all balls $B(w, r/(100\lambda))$ with $w \in B(O, c_0 r) \setminus B(O, r/c_0)$. As μ is doubling, by the $5B$ -covering lemma, we find a cover of $\gamma_{x,y}$ consisting of k of these balls, say D_1, D_2, \dots, D_k , with k depending only on c_0, λ and the doubling constant, so that the following properties hold:

- (1) $\gamma_{x,y} \subset \bigcup_{i=1}^k 5D_i$.
- (2) $\{D_i\}_{i=1}^k$ are pairwise disjoint.
- (3) $5D_i \cap 5D_{i+1} \neq \emptyset$ for $1 \leq i \leq k - 1$.

Let $B_i := 20D_i$. Then the four properties as in Definition 2.3 can be checked to hold for a subcollection of the balls B_i when $\delta = 1/(100\lambda), c_1 = 1/5, c_2 = 10c_0$. \square

Corollary 2.5. *If X is annularly quasiconvex as defined in [HKST15, Section 8.3], then X has the annular chain property.*

It especially follows from Corollary 2.5 together with [HKST15, Theorem 9.4.1] that every complete metric measure space that is Q -Ahlfors regular and supports a p -Poincaré inequality for some $1 \leq p < Q$ has the annular chain property.

3. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1. Throughout this section, we always assume that (X, d, μ) is doubling and supports a p -Poincaré inequality with scaling factor $\lambda \geq 1$, and that $u \in \dot{N}^{1,p}(X)$ with ρ_u a p -integrable

upper gradient of u . We assume that X satisfies an annular λ -condition at O . Let $B(x, r)$ be the ball with radius $r > 0$ and center at $x \in X$. We set $\tau B(x, r) := B(x, \tau r)$, $A_{\lambda 2^{j+1}} := B(O, \lambda 2^{j+2}) \setminus B(O, \lambda 2^{j+1})$, $c_2 \cdot A_{\lambda 2^{j+1}} := B(O, c_2 \lambda 2^{j+3}) \setminus B(O, 2^j / (c_2 \lambda))$ for $\tau > 0$, $j \in \mathbb{N}$ where c_2 is as in Definition 2.3. Let us begin with useful lemmas, established using ideas from [HeKo98, HaKo95, HaKo00].

Lemma 3.1. *Let $1 \leq p < \infty$ and $j \in \mathbb{N}$. Suppose that E, F are two subsets of $A_{\lambda 2^{j+1}}$ such that $|u(x) - u(y)| \geq 1$ for all $x \in E, y \in F$, and that $|u(x) - u_{B(x, 2^j/c_1)}| \leq 1/5$, $|u(y) - u_{B(y, 2^j/c_1)}| \leq 1/5$ for some $x \in E, y \in F$, where $c_1 \geq 1$ is as in Definition 2.3. Then*

$$(3.1) \quad \mu(c_2 \cdot A_{\lambda 2^{j+1}}) \lesssim (2^j)^p \int_{c_2 \cdot A_{\lambda 2^{j+1}}} \rho_u^p d\mu.$$

Proof. We have that $1 \leq |u(x) - u(y)| \leq 1/5 + |u_{B(x, 2^j/c_1)} - u_{B(y, 2^j/c_1)}| + 1/5$. Hence $1 \lesssim |u_{B(x, 2^j/c_1)} - u_{B(y, 2^j/c_1)}|$. Let $\{B_i\}_{i=1}^k$ satisfy the four properties in Definition 2.3. Since μ is doubling and supports a p -Poincaré inequality, it follows that

$$1 \lesssim |u_{B(x, 2^j/c_1)} - u_{B(y, 2^j/c_1)}| \lesssim \sum_{i=1}^k \int_{B_i} |u - u_{B_i}| d\mu \lesssim \sum_{i=1}^k 2^j \left(\int_{\lambda B_i} \rho_u^p d\mu \right)^{\frac{1}{p}}.$$

Hence there is an index i such that $1 \lesssim 2^j \left(\int_{\lambda B_i} \rho_u^p d\mu \right)^{\frac{1}{p}}$. By the doubling property, we obtain that

$$\mu(c_2 \cdot A_{\lambda 2^{j+1}}) \lesssim \mu(\lambda B_i) \lesssim (2^j)^p \int_{\lambda B_i} \rho_u^p d\mu \leq (2^j)^p \int_{c_2 \cdot A_{\lambda 2^{j+1}}} \rho_u^p d\mu$$

which is (3.1). \square

Lemma 3.2. *Let $1 \leq p < \infty$ and $j \in \mathbb{N}$. Suppose that E is a subset of $A_{\lambda 2^{j+1}}$ such that $|u(x) - u_{B(x, 2^j/c_1)}| > 1/5$ holds for every $x \in E$, where $c_1 \geq 1$ is as in Definition 2.3. Then*

$$(3.2) \quad \mu(E) \lesssim (2^j)^p \int_{c_2 \cdot A_{\lambda 2^{j+1}}} \rho_u^p d\mu.$$

Proof. Let $x \in E$. Set $B_i = B(x, 2^{j-i}/c_1)$, $i \in \mathbb{N}$. By Theorem 2.2, we may assume that every $x \in E$ is a Lebesgue point of u . Since μ is doubling and supports a p -Poincaré inequality, we have that

$$(3.3) \quad \sum_{i=0}^{\infty} 2^{-i} \approx \frac{1}{5} < |u(x) - u_{B(x, 2^j/c_1)}| \leq \sum_{i=0}^{\infty} |u_{B_i} - u_{B_{i+1}}| \lesssim \sum_{i=0}^{\infty} 2^{j-i} \left(\int_{\lambda B_i} \rho_u^p d\mu \right)^{\frac{1}{p}}.$$

Thus there is an index i_x such that $2^{-i_x} \lesssim 2^{j-i_x} \left(\int_{\lambda B_{i_x}} \rho_u^p d\mu \right)^{\frac{1}{p}}$ and so

$$(3.4) \quad \mu(\lambda B_{i_x}) \lesssim (2^j)^p \int_{\lambda B_{i_x}} \rho_u^p d\mu.$$

Hence E has a cover $\{\lambda B_{i_x} : (3.4) \text{ holds}\}_{x \in E}$. Using the $5B$ -covering lemma, there is a pairwise disjoint collection $\{\lambda B_{i_{x_k}}\}_{k=1}^{\infty}$ such that $E \subset \bigcup_{k=1}^{\infty} 5\lambda B_{i_{x_k}}$. By (3.4), since μ is doubling, we obtain that

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(5\lambda B_{i_{x_k}}) \lesssim \sum_{k=1}^{\infty} \mu(\lambda B_{i_{x_k}}) \lesssim \sum_{k=1}^{\infty} (2^j)^p \int_{\lambda B_{i_{x_k}}} \rho_u^p d\mu \leq (2^j)^p \int_{c_2 \cdot A_{\lambda 2^{j+1}}} \rho_u^p d\mu$$

which is (3.2). Here the last inequality of above is given since $\{\lambda B_{i_{x_k}}\}_{k=1}^{\infty}$ are pairwise disjoint in $c_2 \cdot A_{\lambda 2^{j+1}}$. The claim follows. \square

Proof of Theorem 1.1. Let $A := \left\{ \gamma \in \Gamma^\infty : \int_\gamma \rho_u ds = \infty \right\}$. We then obtain from [HKST15, Lemma 5.2.8] that $\text{Mod}_p(A) = 0$ since $u \in \dot{N}^{1,p}(X)$, and hence $\lim_{t \rightarrow \infty} u(\gamma(t))$ exists for all $\gamma \in \Gamma^\infty \setminus A$. It remains to show the uniqueness of c in (1.1) for p -a.e $\gamma \in \Gamma^\infty$. We argue by contradiction. By adding a suitable constant to u and finally by multiplying u by another suitable constant, we may assume that there exist two subfamilies of Γ^∞ , denoted Γ_E, Γ_F , and $\delta > 0$ such that

$$(3.5) \quad \text{Mod}_p(\Gamma_E) \geq \delta > 0, \text{Mod}_p(\Gamma_F) \geq \delta > 0, \lim_{t \rightarrow \infty} u(\gamma(t)) \geq 2, \lim_{t \rightarrow \infty} u(\gamma'(t)) \leq 0 \text{ for all } \gamma \in \Gamma_E, \gamma' \in \Gamma_F.$$

We assume that each curve γ in these two curve families is parameterized by arc length. Set $\Gamma_E^j = \{ \gamma \in \Gamma_E : u(\gamma(t)) \geq \frac{3}{2} \text{ for all } t \geq \lambda 2^j \}$. Then $\Gamma_E = \bigcup_j \Gamma_E^j$ and hence, by the subadditivity of the modulus,

$$\text{Mod}_p(\Gamma_E) \leq \sum_j \text{Mod}_p(\Gamma_E^j).$$

Hence (3.5) gives the existence of j_E and $\delta_E > 0$ so that $\text{Mod}_p(\Gamma_E^{j_E}) \geq \delta_E$. By arguing analogously for F , we find j_F so that $\text{Mod}_p(\Gamma_F^{j_F}) \geq \delta_F > 0$ and $u(\gamma(t)) \leq \frac{1}{2}$ when $\gamma \in \Gamma_F^{j_F}$ and $t \geq \lambda 2^{j_F}$. Let $j \geq \max\{j_E, j_F\}$. We define sets E_j, F_j by setting

$$E_j := \left\{ A_{\lambda 2^{j+1}} \cap \left(\bigcup_{\gamma \in \Gamma_E^{j_E}} \gamma \right) \right\}$$

and

$$F_j := \left\{ A_{\lambda 2^{j+1}} \cap \left(\bigcup_{\gamma \in \Gamma_F^{j_F}} \gamma \right) \right\}.$$

Then $u(x) \geq 3/2$ for all $x \in E_j$ and $u(x) \leq 1/2$ for all $x \in F_j$. Moreover, $(\lambda 2^{j+1})^p \text{Mod}_p(\Gamma_E^{j_E}) \leq \mu(E_j)$ and $(\lambda 2^{j+1})^p \text{Mod}_p(\Gamma_F^{j_F}) \leq \mu(F_j)$ since every curve in these families has a subcurve of length no less than $\lambda 2^{j+1}$ in $A_{\lambda 2^{j+1}}$ and hence $\chi_{E_j}/(\lambda 2^{j+1})$ and $\chi_{F_j}/(\lambda 2^{j+1})$ are admissible functions for computing $\text{Mod}_p(\Gamma_E^{j_E})$ and $\text{Mod}_p(\Gamma_F^{j_F})$, respectively.

Notice that $1 \leq |u(x) - u(y)| \leq |u(x) - u_{B(x, 2^j/c_1)}| + |u_{B(x, 2^j/c_1)} - u_{B(y, 2^j/c_1)}| + |u_{B(y, 2^j/c_1)} - u(y)|$ for all $x \in E_j, y \in F_j$ where c_1 is as in Definition 2.3. We will consider three cases corresponding to Lemma 3.1-3.2. Applying Lemma 3.1 for the pair (E_j, F_j) , the estimate (3.1) with $1 \leq |u(x) - u(y)|$ for all $x \in E_j, y \in F_j$ gives

$$\mu(c_2 \cdot A_{\lambda 2^{j+1}}) \lesssim (2^j)^p \int_{c_2 \cdot A_{\lambda 2^{j+1}}} \rho_u^p d\mu$$

if $|u(x) - u_{B(x, 2^j/c_1)}| \leq \frac{1}{5}$ and $|u(y) - u_{B(y, 2^j/c_1)}| \leq \frac{1}{5}$ hold for some $x \in E_j, y \in F_j$. Applying Lemma 3.2 for each E_j, F_j , the estimate (3.2) gives

$$\min\{\mu(E_j), \mu(F_j)\} \lesssim (2^j)^p \int_{c_2 \cdot A_{\lambda 2^{j+1}}} \rho_u^p d\mu$$

if either $|u(x) - u_{B(x, 2^j/c_1)}| > \frac{1}{5}$ holds for every $x \in E_j$ or $|u(y) - u_{B(y, 2^j/c_1)}| > \frac{1}{5}$ holds for every $y \in F_j$. Since both E_j and F_j are subsets of $c_2 \cdot A_{\lambda 2^{j+1}}$, the above estimates imply that

$$\min\{\mu(E_j), \mu(F_j)\} \lesssim (2^j)^p \int_{c_2 \cdot A_{\lambda 2^{j+1}}} \rho_u^p d\mu.$$

From our upper estimates on $\text{Mod}_p(\Gamma_E^{jE})$ and on $\text{Mod}_p(\Gamma_F^{jF})$ from above we conclude that

$$\min\{\text{Mod}_p(\Gamma_E^{jE}), \text{Mod}_p(\Gamma_F^{jF})\} \lesssim \int_{c_2 \cdot A_{\lambda^{2j+1}}} \rho_u^p d\mu.$$

By inserting the strictly positive lower bounds for these two modulus, we conclude that

$$0 < \min\{\delta_E, \delta_F\} \lesssim \int_{c_2 \cdot A_{\lambda^{2j+1}}} \rho_u^p d\mu.$$

Since $u \in \dot{N}^{1,p}(X)$, a contradiction follows by letting j tend to infinity. \square

4. PROOFS OF THEOREMS 1.2-1.3

We begin with a sequence of auxiliary lemmas whose proofs rely on arguments similar to those in [EKN22, KNZ22, Ngu22]. Fix $1 \leq p < \infty$.

Lemma 4.1. *Let (X, d, μ) be a doubling metric measure space that supports a p -Poincaré inequality. Assume that X has the annular chain property at O . Suppose that X has a weak polar coordinate system at O as in (1.2). If $R_p(h, O) < \infty$, then the existence and uniqueness of (1.1) hold for σ -a.e $\xi \in \mathbb{S}$.*

Proof. Let $u \in \dot{N}^{1,p}(X)$. By Theorem 1.1, there exists $c \in \mathbb{R}$ such that

$$(4.1) \quad \lim_{t \rightarrow \infty} u(\gamma(t)) \text{ exists and } \lim_{t \rightarrow \infty} u(\gamma(t)) = c \text{ for } p\text{-a.e } \gamma \in \Gamma^{+\infty}.$$

Let F be the collection of all $\xi \in \mathbb{S}$ such that $\lim_{t \rightarrow \infty} u(\gamma_\xi^O)$ does not exist or exists but is not equal to c . It suffices to prove that

$$(4.2) \quad \sigma(F) = 0.$$

We first set $F_{\text{bad}} := \{\xi \in F : \liminf_{t \rightarrow \infty} d(O, \gamma_\xi^O) \leq 1\}$. Let $\gamma_\xi^O \in F_{\text{bad}}$. By the definition of infinite curves, we have $\gamma_\xi^O \setminus B(O, r) \neq \emptyset$ for all $r > 0$ and hence there is a sequence $\{t_n\}_{n \in \mathbb{N}}$ such that $d(O, \gamma_\xi^O(t_n)) \geq n$. From $\liminf_{t \rightarrow \infty} d(O, \gamma_\xi^O) \leq 1$, it follows that there is also (after passing to a subsequence) a sequence $\{s_n\}_{n \in \mathbb{N}}$ with $\lim_{n \rightarrow \infty} s_n = \infty$ such that $d(O, \gamma_\xi^O(s_n)) \leq 1$ and $s_n \in (t_n, t_{n+1})$ for all $n \in \mathbb{N}$. Roughly speaking, γ_ξ^O oscillates infinitely often between the unit ball and far away parts of our space. Therefore, it is clear that

$$\int_{\gamma_\xi^O} \chi_{B(O,2) \setminus B(O,1)} ds = \infty$$

for each $\gamma_\xi^O \in F_{\text{bad}}$. Also, $h(x) > 0$ almost everywhere in the complement of $B(O, 1)$ because $R_p(h, O) < \infty$. We then obtain that for all $m \in \mathbb{N}$,

$$(4.3) \quad \begin{aligned} m \sigma(F_{\text{bad}}) &\leq \int_{F_{\text{bad}}} \int_{\gamma_\xi^O} \chi_{B(O,2) \setminus B(O,1)} ds d\sigma(\xi) \\ &\leq \int_{\mathbb{S}} \int_{\gamma_\xi^O} \chi_{B(O,2) \setminus B(O,1)} ds d\sigma(\xi) \\ &\leq C \int_X \frac{\chi_{B(O,2) \setminus B(O,1)}}{h} d\mu \\ &\leq C [\mu(B(O, 2) \setminus B(O, 1))]^{1/p} \left(\int_{B(O,2) \setminus B(O,1)} h^{\frac{p}{1-p}} d\mu \right)^{\frac{p-1}{p}} \\ &\leq C [\mu(B(O, 2) \setminus B(O, 1))]^{1/p} R_p^{\frac{p-1}{p}}(h, O) \end{aligned}$$

if $p > 1$, and similarly

$$(4.4) \quad m \sigma(F_{\text{bad}}) \leq \mathcal{C} \mu(B(O, 2) \setminus B(O, 1)) R_1(h, O)$$

by (1.2) and the Hölder inequality, and where $\mathcal{C} > 0$ is the constant of our weak polar coordinate system (1.2). Here the weak polar coordinate system (1.2) can be applied since $\frac{\chi_{B(O, 2) \setminus B(O, 1)}}{h}$ is integrable. Letting $m \rightarrow \infty$, the above estimates give

$$(4.5) \quad \sigma(F_{\text{bad}}) = 0$$

for $1 \leq p < \infty$ since the right-hand side of (4.3)-(4.4) is bounded.

Let $\gamma_\xi^O \in F \setminus F_{\text{bad}}$. Then $\liminf_{t \rightarrow \infty} d(O, \gamma_\xi^O(t)) > 1$, and so there is $t_\xi > 0$ such that $d(O, \gamma_\xi^O(t_\xi)) = 1$ and $\gamma_\xi^O(t) \cap B(O, 1) = \emptyset$ for all $t > t_\xi$. Let us define $\hat{\gamma}_\xi^O$, a subcurve of γ_ξ^O , by setting $\hat{\gamma}_\xi^O(t) = \gamma_\xi^O(t + t_\xi)$ for all $t \geq 0$. Let $\hat{\Gamma}^O(F)$ be the collection of all these infinite subcurves $\hat{\gamma}_\xi^O$ of γ_ξ^O with respect to $\xi \in F$ satisfying $\liminf_{t \rightarrow \infty} d(O, \gamma_\xi^O(t)) > 1$. Then, for all $\hat{\gamma}_\xi^O \in \hat{\Gamma}^O(F)$,

$$(4.6) \quad \hat{\gamma}_\xi^O \cap B(O, 1) = \emptyset$$

and

$$\lim_{t \rightarrow \infty} u(\gamma_\xi^O(t)) \text{ does not exist or is not equal to } c$$

since $\xi \in F$. By (4.1), we also have

$$(4.7) \quad \text{Mod}_p(\hat{\Gamma}^O(F)) = 0.$$

Let g be admissible for computing $\text{Mod}_p(\hat{\Gamma}^O(F))$. Then we may assume that $g = 0$ on $B(O, 1)$ since $\hat{\gamma}_\xi^O \cap B(O, 1) = \emptyset$ by (4.6). Suppose first that $p > 1$. By the same arguments as for (4.3)-(4.4), we obtain that

$$\sigma(F \setminus F_{\text{bad}}) \leq \mathcal{C} \left(\int_X g^p d\mu \right)^{1/p} R_p^{\frac{p-1}{p}}(h, O).$$

Since g is arbitrary, it follows that

$$\sigma(F \setminus F_{\text{bad}}) \leq \mathcal{C} \left(\text{Mod}_p(\hat{\Gamma}^O(F)) \right)^{1/p} R_p^{\frac{p-1}{p}}(h, O).$$

Combining with (4.5), we conclude that

$$(4.8) \quad \sigma(F) \leq \mathcal{C} \left(\text{Mod}_p(\hat{\Gamma}^O(F)) \right)^{1/p} R_p^{\frac{p-1}{p}}(h, O).$$

As $R_p(h, O) < \infty$ and $\text{Mod}_p(\hat{\Gamma}^O(F)) = 0$ by (4.7), we conclude that (4.2) holds. The case $p = 1$ follows via an analogous argument. \square

Lemma 4.2. *If $\mathcal{R}_p(O) = \infty$, then there is $u \in \dot{N}^{1,p}(X)$ such that $\liminf_{t \rightarrow \infty} u(\gamma(t)) = \infty$ for every $\gamma \in \Gamma^\infty$.*

Proof. By $\mathcal{R}_p(O) = \infty$, there exists a sequence $\{n_k\}_{k=1}^\infty$ (or $\{i_k\}_{k=1}^\infty$) with

$$(4.9) \quad n_1 < n_2 < \dots \text{ (or } i_1 < i_2 < \dots \text{) such that } \lim_{k \rightarrow \infty} n_k = \infty \text{ (or } \lim_{k \rightarrow \infty} i_k = \infty \text{),}$$

$$(4.10) \quad \sum_{i=n_k}^{n_{k+1}} (2^i)^{\frac{p}{p-1}} \mu^{\frac{1}{1-p}}(A_{2^i}) > 2^k \quad \text{if } p > 1 \text{ (or } 2^{i_k} \mu^{-1}(A_{2^{i_k}}) > 2^k \text{ if } p = 1 \text{)}.$$

Here $A_{2^i} := B(O, 2^{i+1}) \setminus B(O, 2^i)$. Let

$$g_p(x) = \sum_{k=1}^{\infty} \left(\sum_{i=n_k}^{n_{k+1}} \frac{(2^i)^{\frac{1}{p-1}} \mu^{\frac{1}{1-p}}(A_{2^i})}{\sum_{i=n_k}^{n_{k+1}} (2^i)^{\frac{p}{p-1}} \mu^{\frac{1}{1-p}}(A_{2^i})} \chi_{A_{2^i}}(x) \right) \text{ if } p > 1 \text{ (or } g_1(x) = \sum_{k=1}^{\infty} 2^{-i_k} \chi_{A_{2^{i_k}}}(x)).$$

We define $u(x) := \inf \int_{\gamma_{O,x}} g_p ds$ for $x \in X$ where the infimum is taken over all rectifiable curves $\gamma_{O,x}$ connecting O and x . Then g_p is an upper gradient of u , see for instance [HKST15, Page 188-189]. We have the fact (see for instance [HKST15, Proposition 5.1.11]) that $\int_{\gamma_{O,x} \cap A_{2^i}} ds \geq \text{diam}(\gamma_{O,x} \cap A_{2^i}) \gtrsim 2^i$ for every $\gamma_{O,x}$ with $d(O, x) \geq 2^{i+1}$. Here $\text{diam}(\gamma_{O,x} \cap A_{2^i})$ is the diameter of $\gamma_{O,x} \cap A_{2^i}$. We let $\gamma \in \Gamma^\infty$ and let $N > 1$. For all $x \in \gamma$ with $d(O, x) = N$, we have that

$$\begin{aligned} u(x) &= \inf_{\gamma_{O,x}} \int_{\gamma_{O,x}} g_p ds = \inf_{\gamma_{O,x}} \sum_{2^{n_{k+1}} \leq N} \sum_{i=n_k}^{n_{k+1}} \frac{(2^i)^{\frac{1}{p-1}} \mu^{\frac{1}{1-p}}(A_{2^i})}{\sum_{i=n_k}^{n_{k+1}} (2^i)^{\frac{p}{p-1}} \mu^{\frac{1}{1-p}}(A_{2^i})} \int_{\gamma_{O,x} \cap A_{2^i}} ds \\ &\gtrsim \sum_{2^{n_{k+1}} \leq N} \sum_{i=n_k}^{n_{k+1}} \frac{(2^i)^{\frac{1}{p-1}} \mu^{\frac{1}{1-p}}(A_{2^i})}{\sum_{i=n_k}^{n_{k+1}} (2^i)^{\frac{p}{p-1}} \mu^{\frac{1}{1-p}}(A_{2^i})} 2^i = \sum_{2^{n_{k+1}} \leq N} 1 \rightarrow \infty \text{ as } N \rightarrow \infty \text{ if } p > 1, \end{aligned}$$

(or that $u(x) = \inf_{\gamma_{O,x}} \int_{\gamma_{O,x}} g_1 ds = \inf_{\gamma_{O,x}} \sum_{2^{i_k} \leq N} 2^{-i_k} \int_{\gamma_{O,x} \cap A_{2^{i_k}}} ds \gtrsim \sum_{2^{i_k} \leq N} 2^{-i_k} 2^{i_k} = \sum_{2^{i_k} \leq N} 1 \rightarrow \infty$ as $N \rightarrow \infty$). Hence $\liminf_{t \rightarrow \infty} u(\gamma(t)) = \infty$ for every $\gamma \in \Gamma^\infty$. It remains to show that g_p is p -integrable. Using (4.9)-(4.10), we have that

$$\int_X g_p^p d\mu = \sum_{k=1}^{\infty} \sum_{i=n_k}^{n_{k+1}} \int_{A_{2^i}} \left(\frac{(2^i)^{\frac{1}{p-1}} \mu^{\frac{1}{1-p}}(A_{2^i})}{\sum_{i=n_k}^{n_{k+1}} (2^i)^{\frac{p}{p-1}} \mu^{\frac{1}{1-p}}(A_{2^i})} \right)^p d\mu = \sum_{k=1}^{\infty} \frac{1}{\left(\sum_{i=n_k}^{n_{k+1}} (2^i)^{\frac{p}{p-1}} \mu^{\frac{1}{1-p}}(A_{2^i}) \right)^{p-1}} \leq \sum_{k=1}^{\infty} \frac{1}{2^{k(p-1)}}$$

if $p > 1$ (or that $\int_X g_1 d\mu = \sum_{k=1}^{\infty} \int_{A_{2^{i_k}}} 2^{-i_k} d\mu = \sum_{k=1}^{\infty} 2^{-i_k} \mu(A_{2^{i_k}}) \leq \sum_{k=1}^{\infty} \frac{1}{2^k}$). The claim follows. \square

Example 4.3. If $h(x) = |x - O|^{n-1} w(x) \chi_{\mathbb{R}^n \setminus B(O,1)}(x)$, where w is a classical Muckenhoupt \mathcal{A}_p -weight on \mathbb{R}^n and where $n \geq 2$, then $R_p(h, O) \approx \mathcal{R}_p(O)$.

Proof. Let $A_{2^i} = B(O, 2^{i+1}) \setminus B(O, 2^i)$. Notice that

$$R_p(h, O) = \sum_{i=0}^{\infty} \int_{A_{2^i}} |x - O|^{\frac{p(n-1)}{1-p}} w^{\frac{1}{1-p}}(x) dx \text{ if } p > 1 \text{ (or } R_1(h, O) = \sup_{i \in \mathbb{N}} \| |x - O|^{1-n} w^{-1}(x) \|_{L^\infty(A_{2^i})}).$$

Since $|x - O| \approx 2^i$ for any $x \in A_{2^i}$ and $|A_{2^i}| \approx (2^i)^n$, it follows that

$$R_p(h, O) \approx \sum_{i=0}^{\infty} (2^i)^{\frac{p(n-1)}{1-p} + n} \int_{A_{2^i}} w^{\frac{1}{1-p}}(x) dx \text{ if } p > 1 \text{ (or } R_1(h, O) \approx \sup_{i \in \mathbb{N}} (2^i)^{1-n} \|w^{-1}(x)\|_{L^\infty(A_{2^i})} \text{ if } p = 1).$$

Because w is a Muckenhoupt \mathcal{A}_p -weight, we have from (1.3)-(1.4) that

$$\left(\int_{A_{2^i}} w^{\frac{1}{1-p}}(x) dx \right) \approx \left(\int_{A_{2^i}} w(x) dx \right)^{\frac{1}{1-p}} \approx (2^i)^{\frac{-n}{1-p}} \mu^{\frac{1}{1-p}}(A_{2^i}) \text{ if } p > 1,$$

(or that $\|w^{-1}(x)\|_{L^\infty(A_{2^i})} \approx \left(\int_{A_{2^i}} w(x) dx \right)^{-1} \approx (2^i)^n \mu^{-1}(A_{2^i})$). Inserting these into the above formula of $R_p(h, O)$, we obtain the claim. \square

Lemma 4.4. Let (X, d, μ) be a doubling metric measure space that supports a p -Poincaré inequality. Assume that X is complete. If $\mathcal{R}_p(O) < \infty$ then $\text{Mod}_p(\Gamma^\infty) > 0$.

Proof. Since X is complete and doubling, and supports a p -Poincaré inequality, there exists a geodesic metric \hat{d} that is biLipschitz equivalent to d , see for instance in [HKST15, Corollary 8.3.16]. Let $\mathcal{R}_p(\hat{d}, O)$ be the version of $\mathcal{R}_p(O)$ in (X, \hat{d}, μ) and $B_{\hat{d}}(x, r)$ be the ball with radius r and center at x on (X, \hat{d}) . It follows that $\mathcal{R}_p(\hat{d}, O) < \infty$ since $\mathcal{R}_p(O) < \infty$. Since \hat{d} is geodesic we are allowed to employ Theorem 2.10 in [HoKo01]. This result gives us a constant $C > 0$ such that for all $i \in \mathbb{N}$

$$C \leq \text{Cap}_p(B_{\hat{d}}(O, 1), B_{\hat{d}}(O, 2^i)) \left(\mathcal{R}_p(\hat{d}, O) \right)^{p-1}$$

when $p > 1$ (or otherwise

$$C \leq \text{Cap}_1(B_{\hat{d}}(O, 1), B_{\hat{d}}(O, 2^i)) \mathcal{R}_1(\hat{d}, O).$$

Here $\text{Cap}_p(B_{\hat{d}}(O, 1), B_{\hat{d}}(O, 2^i))$ is the quantity

$$\text{Cap}_p(B_{\hat{d}}(O, 1), B_{\hat{d}}(O, 2^i)) := \inf \int_X g_u^p d\mu$$

where the infimum is taken over all functions $u : X \rightarrow \mathbb{R}$ with the minimal upper gradient g_u such that $u|_{B_{\hat{d}}(O, 1)} \equiv 1$ and $u|_{X \setminus B_{\hat{d}}(O, 2^i)} \equiv 0$. It follows from $\mathcal{R}_p(\hat{d}, O) < \infty$ that $\text{Cap}_p(B_{\hat{d}}(O, 1), B_{\hat{d}}(O, 2^i)) \geq \delta > 0$ for all $i \in \mathbb{N}$, and for some δ only depending on C and $\mathcal{R}_p(\hat{d}, O)$. By [HeKo98, Page 12], we obtain that

$$\text{Cap}_p(B_{\hat{d}}(O, 1), B_{\hat{d}}(O, 2^i)) \leq \text{Mod}_p(B_{\hat{d}}(O, 1), B_{\hat{d}}(O, 2^i)) \quad \text{for all } i \in \mathbb{N}.$$

Here $\text{Mod}_p(B_{\hat{d}}(O, 1), B_{\hat{d}}(O, 2^i))$ is the p -modulus of the family of all rectifiable curves connecting $B_{\hat{d}}(O, 1)$ and $X \setminus B_{\hat{d}}(O, 2^i)$. Then $\text{Mod}_p(B_{\hat{d}}(O, 1), B_{\hat{d}}(O, 2^i)) \geq \delta > 0$ for all $i \in \mathbb{N}$. It follows that X is p -hyperbolic in the sense of [Sha21, Definition 2.4]. Hence [Sha21, Theorem 4.2] yields that $\text{Mod}_p(\Gamma^\infty) > 0$. The claim follows. \square

Proof of Theorem 1.2. I. \Rightarrow II. is given by Lemma 4.4.

II. \Rightarrow III. is given by Theorem 1.1.

III. \Rightarrow I. is given by Lemma 4.2. \square

Proof of Theorem 1.3. $R_p(h, O) < \infty \Rightarrow 3.$ is given by Lemma 4.1.

3. \Rightarrow 2. \Rightarrow 1. is trivial.

1. $\Rightarrow \mathcal{R}_p(O) < \infty$ is given by Lemma 4.2.

$R_p(h, O) < \infty \Rightarrow 4.$ is given by the estimate (4.8).

$R_p(h, O) < \infty \Rightarrow 5.$ is given by the estimate (4.8).

4. $\Rightarrow \mathcal{R}_p(O) < \infty$ is given by Lemma 4.2 together with Theorem 1.1.

5. $\Rightarrow \mathcal{R}_p(O) < \infty$ is given by Lemma 4.2 together with Theorem 1.1. Indeed, if $\mathcal{R}_p(O) = \infty$ then $\text{Mod}_p(\hat{\Gamma}^O(F)) = 0$ for all subsets $F \subseteq \mathbb{S}$ with $\sigma(F) > 0$. \square

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