

SELF-SIMILAR SOLUTION FOR HARDY OPERATOR

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ABSTRACT. We describe the large-time asymptotics of solutions to the heat equation for the fractional Laplacian with added subcritical or even critical Hardy-type potential. The asymptotics is governed by a self-similar solution of the equation, obtained as a normalized limit at the origin of the kernel of the corresponding Feynman-Kac semigroup.

1. INTRODUCTION

1.1. Main results and structure of the paper. Let $d \in \mathbb{N} := \{1, 2, \dots\}$, $\alpha \in (0, 2)$ and $\alpha < d$. We consider the semigroup \tilde{P}_t , $t > 0$, of the following Hardy operator on \mathbb{R}^d ,

$$(1.1) \quad \Delta^{\alpha/2} + \kappa|x|^{-\alpha}.$$

We call κ , and $\Delta^{\alpha/2} + \kappa|x|^{-\alpha}$, subcritical if $\kappa < \kappa^*$, critical if $\kappa = \kappa^*$ and supercritical if $\kappa > \kappa^*$. Here $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ is the fractional Laplacian,

$$\kappa^* := \frac{2^\alpha \Gamma((d + \alpha)/4)^2}{\Gamma((d - \alpha)/4)^2},$$

and $\Gamma(t) = \int_0^\infty y^{t-1} e^{-y} dy$ is the Gamma function. It is well known that κ^* is the best constant in the Hardy inequality for the quadratic form of $\Delta^{\alpha/2}$, see Herbst [20, Theorem 2.5], Beckner [2, Theorem 2] or Yafaev [44, (1.1)]; see also Frank and Seiringer [15, Theorem 1.1] and Bogdan, Dyda and Kim [5, Proposition 5]. Following [5, Section 4], for $\beta \in [0, d - \alpha]$ we let

$$(1.2) \quad \kappa_\beta = \frac{2^\alpha \Gamma((\beta + \alpha)/2) \Gamma((d - \beta)/2)}{\Gamma(\beta/2) \Gamma((d - \beta - \alpha)/2)},$$

where $\kappa_0 = \kappa_{d-\alpha} = 0$, according to the convention $1/\Gamma(0) = 0$. The function $\beta \mapsto \kappa_\beta$ is increasing on $[0, (d - \alpha)/2]$, decreasing on $[(d - \alpha)/2, d - \alpha]$, and $\kappa_\beta = \kappa_{d-\alpha-\beta}$, see [5, Proof of Proposition 5]. The maximal or *critical* value of κ_β is, therefore, $\kappa_{(d-\alpha)/2} = \kappa^*$, and for each $\kappa \in [0, \kappa^*]$ there is a unique number δ such that

$$(1.3) \quad 0 \leq \delta \leq (d - \alpha)/2 \quad \text{and} \quad \kappa = \kappa_\delta = \frac{2^\alpha \Gamma((\delta + \alpha)/2) \Gamma((d - \delta)/2)}{\Gamma(\delta/2) \Gamma((d - \delta - \alpha)/2)}.$$

In what follows, δ and κ shall satisfy (1.3). We let $h(x) = h_\delta(x) := |x|^{-\delta}$, $x \in \mathbb{R}^d$. By Bogdan, Grzywny, Jakubowski and Pilarczyk [6], the Schrödinger operator (1.1) has heat kernel \tilde{p} with singularity at the origin in \mathbb{R}^d and sharp explicit estimates (given by (2.12) below). The first main result of the present paper is a description of the limiting behavior of \tilde{p} , as follows.

Theorem 1.1. *The limit $\Psi_t(x) := \lim_{y \rightarrow 0} \frac{\tilde{p}(t, x, y)}{h(y)}$ exists whenever $0 \leq \delta \leq \frac{d-\alpha}{2}$, $t > 0$, $x \in \mathbb{R}^d$.*

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The proof of Theorem 1.1 is given in Section 3. The function $\Psi_t(x)$ is a self-similar semigroup solution of the heat equation for the Hardy operator, as we assert in (5.2) and (5.3) below. It has an important application to large-time asymptotics of the semigroup \tilde{P}_t , which we now present. To this end we consider Doob-conditioned and weighted L^q spaces. Let

$$H = \max\{1, h\}.$$

As usual, $L^1 = L^1(\mathbb{R}^d, dx)$, $L^1(H) = L^1(\mathbb{R}^d, H(x) dx)$, etc. We have $L^1(H) = \{f/H : f \in L^1\} = L^1(h) \cap L^1$. We then define, for $1 \leq q < \infty$,

$$(1.4) \quad \|f\|_{q,h} := \|f/h\|_{L^q(h^2)} = \left(\int_{\mathbb{R}^d} |f(x)|^q h^{2-q}(x) dx \right)^{\frac{1}{q}} = \|f\|_{L^q(h^{2-q})},$$

and, for $q = \infty$,

$$\|f\|_{\infty,h} := \operatorname{ess\,sup}_{x \in \mathbb{R}^d} |f(x)|/h(x).$$

Of course, $\|f\|_{2,h} = \|f\|_2$ and $\|f\|_{1,h} = \|f\|_{L^1(h)}$. For $f \in L^1(H)$ we let

$$(1.5) \quad \tilde{P}_t f(x) := \int_{\mathbb{R}^d} \tilde{p}(t, x, y) f(y) dy, \quad t > 0, x \in \mathbb{R}^d \setminus \{0\}.$$

Our second main result is the following large-time asymptotics for \tilde{P}_t .

Theorem 1.2. *If $f \in L^1(H)$, $A = \int_{\mathbb{R}^d} f(x)h(x) dx$, $u(t, x) = \tilde{P}_t f(x)$, and $q \in [1, \infty)$, then*

$$(1.6) \quad \lim_{t \rightarrow \infty} t^{\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \|u(t, \cdot) - A\Psi_t\|_{q,h} = 0.$$

The structure of the paper is as follows. The proof of Theorem 1.2 is given at the end of Section 5, where we also show that the result is optimal. In Section 3 we state and prove Theorem 3.1, of which Theorem 1.1 is a direct consequence. In Section 4 we discuss the Feynman-Kac semigroup \tilde{P}_t from the point of view of functional analysis, in particular we prove hypercontractivity of the semigroup in Theorem 4.6, which is then used in Section 5. The last main result of the paper, Theorem 6.4 in Section 6, gives an explicit formula for the potential $\int_0^\infty \Psi_t(x) dt$ of the self-similar solution. Notably, Theorem 6.4 and Corollary 6.3 further the integral analysis which is the foundation of [6]. They were inspired by one of our earlier attempts to prove Theorem 1.1 and are particularly interesting for $\kappa = \kappa^*$, see (6.3).

1.2. Motivation and methods. The classical result of Baras and Goldstein [1] asserts the existence of nontrivial nonnegative solutions of the heat equation $\partial_t = \Delta + \kappa|x|^{-2}$ in \mathbb{R}^d for (subcritical) $\kappa \in [0, (d-2)^2/4]$, and non-existence of such solutions for (supercritical) $\kappa > (d-2)^2/4$. Later on, the upper and lower bounds for the heat kernel of the subcritical Hardy operator $\Delta + \kappa|x|^{-2}$ were obtained by Liskevich and Sobol [31], Milman and Semenov [32, 33], Moschini and Tesi [34], Filippas, Moschini and Tertikas [13].

The classical Hardy operator $\Delta + \kappa|x|^{-2}$ plays a distinctive role in limiting and self-similar phenomena in probability [36] and partial differential equations [37]. This is related to the scaling of the corresponding heat kernel, which is the same as for the Gauss-Weierstrass kernel, and to the asymptotics at the origin in \mathbb{R}^d , which is very different. Such applications motivate our work on the Hardy perturbation of the fractional Laplacian. In fact the paper [6] was a preparation for the present work, which now comes to fruition.

The strategy of the proof of Theorem 1.1 is to prove and use the existence of a stationary density of a corresponding Ornstein-Uhlenbeck semigroup. Then the large-time asymptotics of the Ornstein-Uhlenbeck semigroup yields the asymptotics of \tilde{p} at the origin. To the best of our knowledge the approach is new and should apply to other heat kernels with scaling.

Let us also comment on Theorem 1.2. We start by recalling the initial value problem for the classical heat equation,

$$(1.7) \quad \begin{cases} \partial_t u(x, t) = \Delta u(x, t), & x \in \mathbb{R}^d, \quad t > 0, \\ u(x, 0) = f(x). \end{cases}$$

For $f \in L^1(\mathbb{R}^d)$ the following is asymptotics is well-known:

$$(1.8) \quad \lim_{t \rightarrow \infty} t^{\frac{d}{2}(1-\frac{1}{p})} \|u(t, \cdot) - M g_t\|_{L^p(\mathbb{R}^d)} = 0,$$

see, e.g., Giga, Giga and Saal [16, Theorem in Sect. 1.1.4] or Duoandikoetxea and Zuazua [12]). Here $p \in [1, \infty]$, $M := \int_{\mathbb{R}^d} f(x) dx$, $u(t, x) = g_t * f(x)$ is the semigroup solution of (1.7), and $g_t(x) = (4\pi t)^{-d/2} \exp(-|x|^2/(4t))$ is the Gauss-Weierstrass kernel. Of course, we have *scaling*: $g_t(x) = t^{-d/2} g_1(t^{-1/2}x)$, that is, the function is *self-similar*. The function also satisfies the first equation in (1.7). We can consider (1.8) as a statement about the universality of the self-similar solution $g_t(x)$ for the large-time behavior of all solutions to (1.7).

Theorem 1.2 gives an analogous result for $u(t, x) = \tilde{P}_t f(x)$ and the initial value problem

$$(1.9) \quad \begin{cases} \partial_t u(x, t) = (\Delta^{\alpha/2} + \kappa|x|^{-\alpha}) u(x, t), & x \in \mathbb{R}^d, \quad t > 0, \\ u(x, 0) = f(x). \end{cases}$$

for κ satisfying (1.3) and $f \in L^1(H)$. The use of h is novel in this setting – this and connections to the classical literature will be discussed in more detail in Section 5.

1.3. General conventions. We tend to use “:=” to indicate definitions, e.g., $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$, $a_+ := a \vee 0$ and $a_- := (-a) \vee 0$. Throughout, we only consider Borel measurable functions and Borel measures. As usual, integrals are considered well-defined when the integrands are nonnegative or absolutely integrable with respect to a given measure. In the case of integral kernels, the corresponding integrals should at least be well-defined pointwise almost everywhere (*a.e.*). For $x \in \mathbb{R}^d$ and $r > 0$ we define $B(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\}$, the ball with center at x and radius r . We write $f \approx g$, which we call *approximation* or *comparison*, and say f and g are *comparable*, if f, g are nonnegative functions, $c^{-1}g \leq f \leq cg$ with some constant c , that is a number in $(0, \infty)$. The values of constants may change without notice from line to line in a chain of estimates. Of course, we shall also use constants in inequalities (one-sided comparisons of functions), e.g., $f \leq cg$. We occasionally write $c = c(a, \dots, z)$ to assert that the constant c may be so selected as to depend only on a, \dots, z . As usual, for $1 \leq p \leq \infty$, $L^p := L^p(\mathbb{R}^d, dx)$, with norm $\|\cdot\|_p$, and $L^p(g) := L^p(\mathbb{R}^d, g dx)$ with norm $\|f\|_{L^p(g)}$ and nonnegative (weight) function g .

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2. PRELIMINARIES

2.1. Fractional Laplacian. Let

$$\nu(y) = \frac{\alpha 2^{\alpha-1} \Gamma((d+\alpha)/2)}{\pi^{d/2} \Gamma(1-\alpha/2)} |y|^{-d-\alpha}, \quad y \in \mathbb{R}^d.$$

The coefficient is so chosen that

$$(2.1) \quad \int_{\mathbb{R}^d} [1 - \cos(\xi \cdot y)] \nu(y) dy = |\xi|^\alpha, \quad \xi \in \mathbb{R}^d,$$

see, e.g., Bogdan, Byczkowski, Kulczycki, Ryznar, Song, and Vondraček [4, (1.28)]. The fractional Laplacian for (smooth compactly supported) *test functions* $\varphi \in C_c^\infty(\mathbb{R}^d)$ is

$$\Delta^{\alpha/2}\varphi(x) = \lim_{\varepsilon \downarrow 0} \int_{|y|>\varepsilon} [\varphi(x+y) - \varphi(x)] \nu(y) dy, \quad x \in \mathbb{R}^d.$$

Many authors use the notation $-(-\Delta)^{\alpha/2}$ for the operator. In terms of the Fourier transform, $\widehat{\Delta^{\alpha/2}\varphi}(\xi) = -|\xi|^\alpha \widehat{\varphi}(\xi)$, see, e.g., [4, Section 1.1.2] or [26].

2.2. The semigroup of $\Delta^{\alpha/2}$. We consider the convolution semigroup of functions

$$(2.2) \quad p_t(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t|\xi|^\alpha} e^{-ix \cdot \xi} d\xi, \quad t > 0, x \in \mathbb{R}^d.$$

According to (2.1) and the Lévy-Khinchine formula, each p_t is a radial probability density function and $\nu(y) dy$ is the Lévy measure of the semigroup, see, e.g., [4]. From (2.2) we have

$$(2.3) \quad p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha}x).$$

It is well-known that $p_1(x) \approx 1 \wedge |x|^{-d-\alpha}$ (see [4], Bogdan, Grzywny and Ryznar [7, remarks after Theorem 21] or [26]), so

$$(2.4) \quad p_t(x) \approx t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}, \quad t > 0, x \in \mathbb{R}^d.$$

Since $\alpha < d$, we get (the Riesz kernel)

$$(2.5) \quad \int_0^\infty p_t(x) dt = \mathcal{A}_{d,\alpha} |x|^{\alpha-d}, \quad x \in \mathbb{R}^d,$$

where

$$(2.6) \quad \mathcal{A}_{d,\alpha} = \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2}) 2^\alpha \pi^{d/2}},$$

see, e.g., [4, Section 1.1.2]. We denote

$$p(t, x, y) = p_t(y - x), \quad t > 0, x, y \in \mathbb{R}^d.$$

Clearly, p is symmetric:

$$p(t, x, y) = p(t, y, x), \quad t > 0, x, y \in \mathbb{R}^d,$$

and satisfies the Chapman-Kolmogorov equations:

$$(2.7) \quad \int_{\mathbb{R}^d} p(s, x, y) p(t, y, z) dy = p(t+s, x, z), \quad x, z \in \mathbb{R}^d, s, t > 0.$$

We denote, as usual, $P_t g(x) = \int_{\mathbb{R}^d} p(t, x, y) g(y) dy$. The fractional Laplacian extends to the generator of the semigroup $\{P_t\}_{t>0}$ on many Banach spaces, see, e.g., [26].

2.3. Schrödinger perturbation by Hardy potential. We recall elements of the integral analysis of [5] and [6], which was used to handle the heat kernel \tilde{p} of $\Delta^{\alpha/2} + \kappa|x|^{-\alpha}$. Thus,

$$f_\beta(t) := c_\beta t_+^{(d-\alpha-\beta)/\alpha}, \quad t \in \mathbb{R},$$

for $\beta \in (0, d)$. The constant c_β is so chosen that

$$(2.8) \quad h_\beta(x) := \int_0^\infty f_\beta(t) p_t(x) dt = |x|^{-\beta}, \quad x \in \mathbb{R}^d.$$

The existence of such $c_\beta \in (0, \infty)$ follows from (2.3) and the estimate $p_1(x) \approx 1 \wedge |x|^{-d-\alpha}$. Of course, $f'_\beta(t) = c_\beta \frac{d-\alpha-\beta}{\alpha} t^{(d-2\alpha-\beta)/\alpha}$. Accordingly, for $\beta \in (0, d-\alpha)$ we may define

$$q_\beta(x) = \frac{1}{h_\beta(x)} \int_0^\infty f'_\beta(t) p_t(x) dt, \quad x \in \mathbb{R}_0^d,$$

where $\mathbb{R}_0^d := \mathbb{R}^d \setminus \{0\}$. By [5, (26)]¹,

$$q_\beta(x) = \kappa_\beta |x|^{-\alpha},$$

where κ_β is defined by (1.2). In what follows we keep our notation from (1.3), that is, we let

$$(2.9) \quad \delta \in [0, (d - \alpha)/2], \quad \kappa = \kappa_\delta, \quad h(x) = h_\delta(x) = |x|^{-\delta}, \quad q(x) = q_\delta(x) = \kappa |x|^{-\alpha}.$$

To wit, the case of $\delta \in (0, (d - \alpha)/2]$ is covered by the discussion of β above, and $\delta = 0$ yields the trivial $\kappa = 0$, $q = 0$ and $h = 1$. We then define the Schrödinger perturbation of p by q :

$$(2.10) \quad \tilde{p} = \tilde{p}_\delta = \sum_{n=0}^{\infty} p_n.$$

Here for $t > 0$ and $x, y \in \mathbb{R}^d$ we let $p_0(t, x, y) = p(t, x, y)$ and then proceed by induction:

$$(2.11) \quad \begin{aligned} p_n(t, x, y) &= \int_0^t \int_{\mathbb{R}^d} p(s, x, z) q(z) p_{n-1}(t-s, z, y) dz ds \\ &= \int_0^t \int_{\mathbb{R}^d} p_{n-1}(s, x, z) q(z) p(t-s, z, y) dz ds, \quad n \geq 1. \end{aligned}$$

Of course, $\tilde{p}_0 = p$. By (2.4), for $t > 0$ and $y \in \mathbb{R}^d$ we have

$$\begin{aligned} p_1(t, 0, y) &= \int_0^t \int_{\mathbb{R}^d} p(s, 0, z) q(z) p(t-s, z, y) dz ds \\ &\geq c_{t,y} \int_0^{t/2} \int_{|z| < s^{1/\alpha}} s^{-d/\alpha} |z|^{-\alpha} dz ds = \infty. \end{aligned}$$

By symmetry, $p_1(t, x, 0) = \infty$, too, for all $x \in \mathbb{R}^d$ and $t > 0$, therefore $\tilde{p}(t, x, y) = \infty$ if $x = 0$ or $y = 0$. By [6, Theorem 1.1], the above discussion of $x = 0$ and $y = 0$ and the usual notational conventions we have for all $x, y \in \mathbb{R}^d$, $t > 0$,

$$(2.12) \quad \tilde{p}(t, x, y) \approx \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) \left(1 + t^{\delta/\alpha} |x|^{-\delta} \right) \left(1 + t^{\delta/\alpha} |y|^{-\delta} \right).$$

Clearly, if $0 < t_1 < t_2 < \infty$, then

$$(2.13) \quad \tilde{p}(s, x, y) \approx \tilde{p}(t, x, y), \quad x, y \in \mathbb{R}^d, \quad t_1 \leq s, t \leq t_2$$

(the comparability constant does depend on t_1 , t_2 and d , α , δ). Recall that

$$(2.14) \quad H(x) = 1 \vee h(x) = 1 \vee |x|^{-\delta} \approx 1 + |x|^{-\delta}, \quad x \in \mathbb{R}^d.$$

Thus, we can reformulate (2.12) as follows,

$$(2.15) \quad \tilde{p}(t, x, y) \approx p(t, x, y) H(t^{-1/\alpha} x) H(t^{-1/\alpha} y), \quad t > 0, \quad x, y \in \mathbb{R}^d.$$

By [6] and the above conventions, \tilde{p} is a symmetric time-homogeneous transition density on \mathbb{R}^d , in particular the Chapman-Kolmogorov equations hold:

$$(2.16) \quad \int_{\mathbb{R}^d} \tilde{p}(s, x, z) \tilde{p}(t, z, y) dz = \tilde{p}(t+s, x, y), \quad x, y \in \mathbb{R}^d, \quad s, t > 0.$$

The following Duhamel formulae hold for p and \tilde{p} ,

$$(2.17) \quad \tilde{p}(t, x, y) = p(t, x, y) + \int_0^t \int_{\mathbb{R}^d} p(s, x, z) q(z) \tilde{p}(t-s, z, y) dz ds$$

$$(2.18) \quad = p(t, x, y) + \int_0^t \int_{\mathbb{R}^d} \tilde{p}(s, x, z) q(z) p(t-s, z, y) dz ds, \quad t > 0, \quad x, y \in \mathbb{R}^d.$$

In passing we refer to Bogdan, Hansen and Jakubowski [8] and Bogdan, Jakubowski and Sydor [9] for a general setting of Schrödinger perturbations of transition semigroups and other families of integral kernels.

¹The exponent $(d - \alpha - \beta)/\alpha$ in the definition of f is denoted β in [5, Corollary 6].

The function \tilde{p} is self-similar, i.e., has the following scaling [6, Lemma 2.2]:

$$(2.19) \quad \tilde{p}(t, x, y) = t^{-d/\alpha} \tilde{p}(1, t^{-1/\alpha} x, t^{-1/\alpha} y), \quad t > 0, \quad x, y \in \mathbb{R}^d.$$

This is the same scaling as for p . Furthermore, if T is a (linear) isometry of \mathbb{R}^d , then for all $t > 0$, $x, y \in \mathbb{R}^d$ we have $p(t, Tx, Ty) = p_t(T(y-x)) = p_t(y-x) = p(t, x, y)$, because p_t is radial. Of course, q is radial, too. By the change of variables $z = Tv$ and induction,

$$\begin{aligned} p_n(t, Tx, Ty) &= \int_0^t \int_{\mathbb{R}^d} p(s, Tx, z) q(z) p_{n-1}(t-s, z, Ty) dz ds \\ &= \int_0^t \int_{\mathbb{R}^d} p(s, Tx, Tv) q(Tv) p_{n-1}(t-s, Tv, Ty) dv ds \\ &= \int_0^t \int_{\mathbb{R}^d} p(s, x, v) q(v) p_{n-1}(t-s, v, y) dv ds = p_n(t, x, y), \quad n \geq 1. \end{aligned}$$

Therefore,

$$(2.20) \quad \tilde{p}(t, Tx, Ty) = \tilde{p}(t, x, y), \quad t > 0, \quad x, y \in \mathbb{R}^d.$$

2.4. Doob's conditioning. Recall that $\delta \in [0, (d-\alpha)/2]$ and $\kappa = \kappa_\delta$, $h(x) = h_\delta(x) = |x|^{-\delta}$, \tilde{p} depends on δ . By [6, Theorem 3.1] and the preceding discussions, the function h is *invariant* in the following sense:

$$(2.21) \quad \int_{\mathbb{R}^d} \tilde{p}(t, x, y) h(y) dy = h(x), \quad t > 0, \quad x \in \mathbb{R}^d.$$

We define the following Doob-conditioned (renormalized) kernel

$$(2.22) \quad \rho_t(x, y) = \frac{\tilde{p}(t, x, y)}{h(x)h(y)}, \quad t > 0, \quad x, y \in \mathbb{R}_0^d$$

(later on we shall extend ρ to $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$). We consider the integral weight $h^2(x)$, $x \in \mathbb{R}^d$. Doob-type conditioning is also called Davies' method, see Murugan and Saloff-Coste [35], and h is sometimes called desingularizing weight, see Milman and Norsemen [32]. By (2.21),

$$(2.23) \quad \int_{\mathbb{R}^d} \rho_t(x, y) h^2(y) dy = 1, \quad x \in \mathbb{R}_0^d, \quad t > 0.$$

By (2.7),

$$\begin{aligned} \int_{\mathbb{R}^d} \rho_s(x, y) \rho_t(y, z) h^2(y) dy &= \int_{\mathbb{R}^d} \frac{\tilde{p}(s, x, y)}{h(x)h(y)} \frac{\tilde{p}(t, y, z)}{h(y)h(z)} h^2(y) dy \\ (2.24) \quad &= \frac{1}{h(x)h(z)} \tilde{p}(t+s, x, z) = \rho_{t+s}(x, z), \quad x, z \in \mathbb{R}_0^d, \quad s, t > 0. \end{aligned}$$

We see that ρ is a symmetric time-homogeneous transition probability density on \mathbb{R}_0^d with the reference measure $h^2(y) dy$. For nonnegative $f \in L^1(h^2)$, by Fubini-Tonelli and (2.23),

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y) \rho_t(x, y) h^2(y) dy h^2(x) dx = \int_{\mathbb{R}^d} f(y) h^2(y) dy,$$

so the operators

$$\mathcal{R}_t f(x) := \int_{\mathbb{R}^d} f(y) \rho_t(x, y) h^2(y) dy, \quad t > 0,$$

are contractions on $L^1(h^2)$. By (2.19), ρ is self-similar: for $t > 0$ and $x, y \in \mathbb{R}^d$ we have

$$(2.25) \quad \rho_t(x, y) = \frac{t^{-d/\alpha} \tilde{p}(1, t^{-1/\alpha} x, t^{-1/\alpha} y)}{t^{-\delta/\alpha} h(t^{-1/\alpha} x) t^{-\delta/\alpha} h(t^{-1/\alpha} y)} = t^{\frac{2\delta-d}{\alpha}} \rho_1(t^{-1/\alpha} x, t^{-1/\alpha} y),$$

hence

$$(2.26) \quad \rho_{st}(t^{1/\alpha} x, t^{1/\alpha} y) = t^{\frac{2\delta-d}{\alpha}} \rho_s(x, y), \quad s > 0.$$

For each (linear) isometry T of \mathbb{R}^d , $h \circ T = h$, thus by (2.20) we get

$$(2.27) \quad \rho_t(Tx, Ty) = \rho_t(t, x, y), \quad t > 0, \quad x, y \in \mathbb{R}_0^d.$$

By (2.12),

$$(2.28) \quad \rho_1(x, y) \approx \left(1 \wedge |x - y|^{-d-\alpha}\right) \left(1 + |x|^\delta\right) \left(1 + |y|^\delta\right), \quad x, y \in \mathbb{R}_0^d.$$

Notably, ρ_1 is not bounded (consider large $x = y$). By (2.25),

$$(2.29) \quad \rho_t(x, y) \approx \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right) \left(t^{\delta/\alpha} + |x|^\delta\right) \left(t^{\delta/\alpha} + |y|^\delta\right), \quad t > 0, \quad x, y \in \mathbb{R}_0^d.$$

We also note that if $0 < t_1 < t_2 < \infty$, then

$$(2.30) \quad \rho_s(x, y) \approx \rho_t(x, y), \quad x, y \in \mathbb{R}_0^d, \quad t_1 \leq s, t \leq t_2$$

(the comparability constant depends on t_1, t_2 and d, α, δ). By (2.28), for every $M \in (0, \infty)$,

$$(2.31) \quad \rho_1(x, y) \approx (1 + |y|)^{-d-\alpha+\delta}, \quad 0 < |x| \leq M, \quad y \in \mathbb{R}_0^d.$$

Clearly, the right-hand side is integrable with respect to $h^2(y) dy$ and bounded. Our aim is to prove the continuity of ρ_1 , notably at $x = 0$.

3. LIMITING BEHAVIOUR AT THE ORIGIN

Here is a full-fledged variant of Theorem 1.1.

Theorem 3.1. *The function ρ has a continuous extension to $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and*

$$(3.1) \quad \rho_t(0, y) := \lim_{x \rightarrow 0} \rho_t(x, y), \quad t > 0, \quad y \in \mathbb{R}_0^d,$$

satisfies:

$$(3.2) \quad \rho_t(0, y) = t^{\frac{2\delta-d}{\alpha}} \rho_1(0, t^{-1/\alpha}y), \quad t > 0, \quad y \in \mathbb{R}_0^d,$$

$$(3.3) \quad \int_{\mathbb{R}^d} \rho_t(0, y) \rho_s(y, z) h^2(y) dy = \rho_{t+s}(0, z), \quad s, t > 0, \quad z \in \mathbb{R}_0^d.$$

The proof of Theorem 3.1 is given below in this section. Let us explain the line of attack. Given (3.1) and taking the limit in (2.24) as $x \rightarrow 0$, we should get (3.3). By (2.25) we should then obtain (3.2) and

$$(3.4) \quad \int_{\mathbb{R}^d} t^{\frac{2\delta-d}{\alpha}} \rho_1(0, t^{-1/\alpha}y) \rho_s(y, z) h^2(y) dy = (t+s)^{\frac{2\delta-d}{\alpha}} \rho_1(0, (t+s)^{-1/\alpha}z), \quad s, t > 0, \quad z \in \mathbb{R}_0^d.$$

Changing variables $u = (t+s)^{-1/\alpha}z$ and $x = t^{-1/\alpha}y$, we see that (3.4) is equivalent to

$$(3.5) \quad \int_{\mathbb{R}^d} \rho_1(0, x) (t+s)^{\frac{d-2\delta}{\alpha}} \rho_s(t^{1/\alpha}x, (t+s)^{1/\alpha}u) h^2(x) dx = \rho_1(0, u).$$

By (2.26), this is the same as

$$(3.6) \quad \int_{\mathbb{R}^d} \rho_{\frac{s}{s+t}} \left((t/(s+t))^{1/\alpha} x, u \right) \rho_1(0, x) h^2(x) dx = \rho_1(0, u).$$

In what follows we shall *define* $\rho_1(0, \cdot)$ as a solution to the integral equation (3.6) and then essentially reverse the above reasoning. Additionally, to simplify the notation and arguments we introduce an auxiliary Ornstein-Uhlenbeck-type semigroup.

3.1. Ornstein-Uhlenbeck semigroup. For $f \geq 0$ we let

$$(3.7) \quad L_t f(y) = \int_{\mathbb{R}^d} l_t(x, y) f(x) h^2(x) dx,$$

where

$$(3.8) \quad l_t(x, y) = \rho_{1-e^{-t}}(e^{-t/\alpha}x, y), \quad t > 0, \quad x, y \in \mathbb{R}_0^d.$$

By (2.24) and (2.26),

$$\begin{aligned} \int_{\mathbb{R}^d} l_s(x, y) l_t(y, z) h^2(y) dy &= \int_{\mathbb{R}^d} \rho_{1-e^{-s}}(e^{-s/\alpha}x, y) \rho_{1-e^{-t}}(e^{-t/\alpha}y, z) h^2(y) dy \\ &= \int_{\mathbb{R}^d} \rho_{1-e^{-s}}(e^{-s/\alpha}x, y) (e^{-t})^{\frac{2\delta-d}{\alpha}} \rho_{e^{t-1}}(y, e^{t/\alpha}z) h^2(y) dy \\ &= (e^{-t})^{\frac{2\delta-d}{\alpha}} \rho_{e^{t-e^{-s}}}(e^{-s/\alpha}x, e^{t/\alpha}z) = \rho_{1-e^{-s-t}}(e^{-(s+t)/\alpha}x, z) = l_{t+s}(x, z). \end{aligned}$$

Thus $l_t(x, y)$ is a transition density on \mathbb{R}_0^d with respect to the measure $h^2(y) dy$. By Fubini's theorem, $\{L_t\}_{t>0}$ is a semigroup of operators on $L^1(h^2)$, an Ornstein-Uhlenbeck-type semigroup, see [39, solution to E 18.17 on p. 462–463]. It shall be a major technical tool in our development.

3.2. Stationary density. If $\varphi \geq 0$ and $\int \varphi(x) h^2(x) dx = 1$, then we say that φ is a *density*. By Fubini-Tonelli Theorem and (2.23), for $t > 0$ and $f \geq 0$, we have

$$(3.9) \quad \int_{\mathbb{R}^d} L_t f(y) h^2(y) dy = \int_{\mathbb{R}^d} f(x) h^2(x) dx.$$

Thus, the operators L_t preserve densities. So they are Markov, see Komorowski [22], Lasota and Mackey [28], Lasota and York [29] for this setting.

We say that a density φ is *stationary* for L_t if $L_t \varphi = \varphi$.

Theorem 3.2. *There is a unique stationary density φ for the operators L_t , $t > 0$.*

Proof. Fix $t > 0$ and let $P = L_t$ so that $P^k = L_{kt}$, $k = 1, 2, \dots$. By (3.8) and (2.25), for all $f \geq 0$ and $k \in \mathbb{N}$ we have

$$(3.10) \quad \begin{aligned} P^k f(u) &= \int_{\mathbb{R}^d} f(y) \rho_{1-e^{-kt}}(e^{-kt/\alpha}y, u) h^2(y) dy \\ &= \int_{\mathbb{R}^d} f(y) (e^{-kt})^{\frac{2\delta-d}{\alpha}} \rho_{e^{kt-1}}(y, e^{kt/\alpha}u) h^2(y) dy, \quad u \in \mathbb{R}_0^d. \end{aligned}$$

Let $B = \{x \in \mathbb{R}^d : 0 < |x| \leq 1\}$. We write $f \in F$ if

$$(3.11) \quad f(y) = \int_B \rho_1(x, y) \mu(dx)$$

for some subprobability measure μ concentrated on B . Then, by (2.24) and (2.25),

$$(3.12) \quad \begin{aligned} P^k f(u) &= \int_B (e^{-kt})^{\frac{2\delta-d}{\alpha}} \int_{\mathbb{R}^d} \rho_1(x, y) \rho_{e^{kt-1}}(y, e^{kt/\alpha}u) h^2(y) dy \mu(dx) \\ &= \int_B (e^{-kt})^{\frac{2\delta-d}{\alpha}} \rho_{e^{kt}}(x, e^{kt/\alpha}u) \mu(dx) \\ &= \int_B \rho_1(e^{-kt/\alpha}x, u) \mu(dx) \\ &= \int_B \rho_1(x, u) \tilde{\mu}(dx) \approx (1 + |u|)^{-d-\alpha+\delta}, \quad u \in \mathbb{R}_0^d, \end{aligned}$$

where $\tilde{\mu}$ is a subprobability measure concentrated on $e^{-kt/\alpha}B \subset B$. Thus, $P^k F \subset F$. We note that the comparison (3.12) is independent of k , see (2.31). We next argue that the hypotheses

of [22, Theorem 3.1] hold true for P and F : (C1) – the lower bound, and (C2) – the uniform absolute continuity. Indeed, by (3.12),

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \int_B P^k f(u) h^2(u) \, du \approx \int_B (1 + |u|)^{-d-\alpha+\delta} h^2(u) \, du > 0,$$

which yields (C1) in [22]. Also, (C2) therein is satisfied because

$$\int_A P^k f(y) h^2(y) \, dy \leq C \int_A (1 + |y|)^{-d-\alpha+\delta} h^2(y) \, dy \rightarrow 0,$$

uniformly in k as $\int_A h^2(y) \, dy \rightarrow 0$, due to the integrability of $(1 + |y|)^{-d-\alpha+2\delta}$. By [22, Theorem 3.1], a density φ exists satisfying $L_t \varphi = P\varphi = \varphi$. Moreover, the density φ is unique. Indeed, if ψ is a probability density with respect to $h^2(x) \, dx$ and $P\psi = \psi$, then $r := \varphi - \psi$ satisfies $Pr = r$, too. If $r = 0$ *a.e.*, then we are done. Otherwise, $\int_{\mathbb{R}^d} r_+(x) h^2(x) \, dx = \int_{\mathbb{R}^d} r_-(x) h^2(x) \, dx > 0$. Then, on the one hand $|Pr| = |r|$, on the other hand for *a.e.* $x \in \mathbb{R}^d$,

$$\begin{aligned} Pr(x) &= Pr_+(x) - Pr_-(x) \\ &= \int_{\mathbb{R}^d} r_+(y) l_t(y, x) h^2(y) \, dy - \int_{\mathbb{R}^d} r_-(y) l_t(y, x) h^2(y) \, dy \end{aligned}$$

and both terms are nonzero, because l_t is positive. Therefore ,

$$|Pr(x)| < Pr_+(x) \vee Pr_-(x) \leq P|r|(x).$$

By this and (3.9),

$$\int_{\mathbb{R}^d} |r(x)| h^2(x) \, dx = \int_{\mathbb{R}^d} |Pr(x)| h^2(x) \, dx < \int_{\mathbb{R}^d} P|r|(x) h^2(x) \, dx = \int_{\mathbb{R}^d} |r(x)| h^2(x) \, dx.$$

We obtain a contradiction, so $r = 0$, and $\psi = \varphi$ in $L^1(h^2)$. In passing we note that the above argument is a part of the proof of Doob's theorem [10, Theorem 4.2.1].

Since the operators L_t , $t > 0$, commute, they have the same stationary density, that is φ . Indeed, if $s > 0$ and $Q = L_s$, then $P(Q\varphi) = QP\varphi = (Q\varphi)$, and $Q\varphi$ is a density, so by the uniqueness, $Q\varphi = \varphi$. \square

In passing we like to refer the interested reader to additional literature on the existence and uniqueness of stationary densities and measures. The book of Foguel [14] gives a concise introduction to ergodic theory of Markov processes, in particular to the L^1 setting. Stationary measures and densities are discussed in Da Prato and Zabczyk [10, Remark 3.1.3], [29, Theorem 3.1], Lasota [27, Theorem 6.1], and Komorowski, Peszat, Szarek [23]. A nice presentation of asymptotic stability (and periodicity) of Markov operators is given by Komornik [21], see also Lasota and Mackey [28, p. 373, 11.9.4], Komorowski and Tyrcha [24], and Stettner [41].

We note that the existence of a stationary density in our setting would also follow from the – perhaps more constructive – approach using the weak compactness of the set of functions given by (3.11) and Schauder-Tychonoff fixed point theorem, see, e.g., Rudin [38, 5.28 Theorem], see also [38, Section 3.8], the Dunford-Pettis theorem in Voigt [43, Chapter 15] and (3.12) above. We could also use the Krylov-Bogolioubov theorem, see, e.g., [28] or the minicourse of Hairer [19]; and it is always worthy to check with Doob [11], especially that his example (0.9) touches upon the classical Ornstein-Uhlenbeck semigroup.

In what follows, φ denotes the stationary density for the operators L_t , $t > 0$ (in Lemma 3.5 below we verify that φ can be defined pointwise so as to be continuous).

3.3. Asymptotic stability. The state space \mathbb{R}_0^d is a Polish space and l_t is positive. By Theorem 3.2 and the results of Kulik and Scheutzov [25, Theorem 1 and Remark 2], for every $x \in \mathbb{R}_0^d$ we get

$$(3.13) \quad \int_{\mathbb{R}^d} |l_t(x, y) - \varphi(y)| h^2(y) \, dy \rightarrow 0 \text{ as } t \rightarrow \infty.$$

This is the (large-time) asymptotic stability of the semigroup aforementioned in the title of this subsection. Let A be a bounded subset of \mathbb{R}_0^d and $x, x_0 \in A$. By Theorem 3.2 and (2.28),

$$(3.14) \quad \begin{aligned} \int_{\mathbb{R}^d} |l_{1+t}(x, y) - \varphi(y)| h^2(y) dy &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} l_1(x, z) (l_t(z, y) - \varphi(y)) h^2(z) dz \right| h^2(y) dy \\ &\leq c \int_{\mathbb{R}^d} l_1(x_0, z) \int_{\mathbb{R}^d} |l_t(z, y) - \varphi(y)| h^2(y) dy h^2(z) dz. \end{aligned}$$

We have $I_t(z) := \int_{\mathbb{R}^d} |l_t(z, y) - \varphi(y)| h^2(y) dy \rightarrow 0$ as $t \rightarrow \infty$. Furthermore, for every $z \in \mathbb{R}_0^d$, $I_t(z) \leq \int_{\mathbb{R}^d} (l_t(z, y) + \varphi(y)) h^2(y) dy = 2$. Of course, $\int_{\mathbb{R}^d} 2l_1(x_0, z) h^2(z) dz = 2 < \infty$. By the dominated convergence theorem, the iterated integral in (3.14) converges to 0, therefore the convergence in (3.13) is uniform for $x \in A$. In terms of ρ , (3.13) reads as follows: uniformly in $x \in A$,

$$(3.15) \quad \int_{\mathbb{R}^d} \left| \rho_{1-e^{-t}}(e^{-t/\alpha}x, y) - \varphi(y) \right| h^2(y) dy \rightarrow 0 \text{ as } t \rightarrow \infty.$$

As a consequence we obtain the following spatial convergence in $L^1(h^2)$.

Lemma 3.3. *We have $\int_{\mathbb{R}^d} |\rho_1(x, y) - \varphi(y)| h^2(y) dy \rightarrow 0$ as $x \rightarrow 0$.*

Proof. It suffices to make cosmetic changes to (3.15). Of course, $e^{-t/\alpha}x \rightarrow 0$ and $1 - e^{-t} \rightarrow 1$ as $t \rightarrow \infty$. By scaling,

$$\rho_{1-e^{-t}}(e^{-t/\alpha}x, y) = (1 - e^{-t})^{(2\delta-d)/\alpha} \rho_1((e^t - 1)^{-1/\alpha}x, (1 - e^{-t})^{-1/\alpha}y),$$

so

$$(3.16) \quad \int_{\mathbb{R}^d} \left| \rho_1((e^t - 1)^{-1/\alpha}x, (1 - e^{-t})^{-1/\alpha}y) - \varphi(y) \right| h^2(y) dy \rightarrow 0 \text{ as } t \rightarrow \infty.$$

By the continuity of dilations on $L^1(dx)$,

$$\int_{\mathbb{R}^d} \left| \varphi((1 - e^{-t})^{-1/\alpha}y) h^2(y) - \varphi(y) h^2(y) \right| dy \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Using the triangle inequality and changing variables in (3.16) we get

$$\int_{\mathbb{R}^d} \left| \rho_1((e^t - 1)^{-1/\alpha}z, y) - \varphi(y) \right| h^2(y) dy \rightarrow 0 \text{ as } t \rightarrow \infty$$

uniformly in $z \in A$ for bounded $A \subset \mathbb{R}_0^d$. We take A as the unit sphere, for $x \in \mathbb{R}_0^d$ we write $x = (e^t - 1)^{-1/\alpha}z$, where $t = \ln(1 + |x|^{-\alpha})$ and $z = x/|x| \in A$, and we obtain the result. \square

By Lemma 3.3 and (2.31) we obtain the following estimate.

Corollary 3.4. $\varphi(y) \approx (1 + |y|)^{-d-\alpha+\delta}$ for almost all $y \in \mathbb{R}^d$.

By the next result we may actually consider φ as defined pointwise.

Lemma 3.5. *After modification on a set of Lebesgue measure zero, φ is a continuous radial function on \mathbb{R}^d and $\varphi(y) \approx (1 + |y|)^{-d-\alpha+\delta}$ for all $y \in \mathbb{R}^d$.*

Proof. By Theorem 3.2, $\varphi = L_1\varphi$ (a.e.), so it suffices to prove that $L_1\varphi$ is a.e. equal to a continuous function on \mathbb{R}^d . By (2.30) and (2.31), $c(1 + |y|)^{-d-\alpha+\delta}$ is an integrable majorant of $l_1(x, y)$ for bounded x . Furthermore, φ is essentially bounded (a.e.), by Corollary 3.4. The function $l_1(x, y)$ is continuous in $y \in \mathbb{R}_0^d$ so, by the dominated convergence theorem, $L_1\varphi(y)$ is continuous for $y \in \mathbb{R}_0^d$. It remains to prove the convergence of $L_1\varphi(y)$ to a finite limit as

$y \rightarrow 0$. We let $t > 0$ and $y \rightarrow 0$. By scaling, changing variables, the symmetry of ρ_1 and Lemma 3.3,

$$\begin{aligned}
 L_t \varphi(y) &= \int_{\mathbb{R}_0^d} \rho_{1-e^{-t}}(e^{-t/\alpha}x, y) \varphi(x) h^2(x) dx \\
 &= \int_{\mathbb{R}_0^d} e^{t(d-2\delta)/\alpha} \rho_1(z, (1-e^{-t/\alpha})^{1/\alpha}y) \varphi((e^t-1)^{1/\alpha}z) h^2(z) dz \\
 &\rightarrow \int_{\mathbb{R}_0^d} e^{t(d-2\delta)/\alpha} \varphi(z) \varphi((e^t-1)^{1/\alpha}z) h^2(z) dz \\
 (3.17) \quad &= \int_{\mathbb{R}_0^d} \varphi((e^{-t})^{1/\alpha}x) \varphi((1-e^{-t})^{1/\alpha}x) h^2(x) dx < \infty.
 \end{aligned}$$

Since $-2d - 2\alpha + 2\delta < -d - 3\alpha < -d$, the finiteness in (3.17) follows from Corollary 3.4. This proves the continuity of the extension of $L_1\varphi$ on the whole of \mathbb{R}^d . The rest of the lemma follows from the continuity, Lemma 3.3 and (2.27), and from Corollary 3.4. \square

Needless to say, the continuous modification of φ is unique and pointwise defined for every $x \in \mathbb{R}^d$. From now on the extension shall be denoted by φ . By the equality in (3.17),

$$(3.18) \quad \varphi(0) = \int_{\mathbb{R}_0^d} \varphi(\lambda^{1/\alpha}x) \varphi((1-\lambda)^{1/\alpha}x) h^2(x) dx$$

for every $\lambda \in [0, 1]$, including the endpoint cases, since φ is a density. In particular,

$$(3.19) \quad \varphi(0) = \int_{\mathbb{R}^d} \varphi(2^{-1/\alpha}x)^2 h^2(x) dx.$$

3.4. Regularization of ρ . We are now in a position to prove convergence of $\rho_1(x, y)$ as $x \rightarrow 0$ to a finite limit. By scaling, Chapman-Kolmogorov, Lemma 3.3 and the boundedness of φ , for $y \in \mathbb{R}_0^d$ and $\mathbb{R}_0^d \ni x \rightarrow 0$ we get

$$\begin{aligned}
 \rho_1(x, y) &= 2^{\frac{d-2\delta}{\alpha}} \rho_2(2^{1/\alpha}x, 2^{1/\alpha}y) \\
 &= 2^{\frac{d-2\delta}{\alpha}} \int \rho_1(2^{1/\alpha}x, z) \rho_1(z, 2^{1/\alpha}y) h^2(z) dz \\
 &\rightarrow 2^{\frac{d-2\delta}{\alpha}} \int \varphi(z) \rho_1(z, 2^{1/\alpha}y) h^2(z) dz \\
 &= \int \varphi(z) \rho_{1/2}(2^{-1/\alpha}z, y) h^2(z) dz = L_{\ln 2} \varphi(y) = \varphi(y).
 \end{aligned}$$

Thus, for every $y \neq 0$,

$$(3.20) \quad \rho_1(0, y) := \lim_{x \rightarrow 0} \rho_1(x, y) = \varphi(y).$$

By scaling, for all $t > 0$ we get

$$(3.21) \quad \rho_t(0, y) := \lim_{x \rightarrow 0} \rho_t(x, y) = t^{\frac{2\delta-d}{\alpha}} \rho_1(0, t^{-1/\alpha}y) = t^{\frac{2\delta-d}{\alpha}} \varphi(t^{-1/\alpha}y).$$

By (2.23), for $x \neq 0$ we have

$$\int_{\mathbb{R}^d} \rho_1(x, y) |y|^{-2\delta} dy = 1.$$

By (2.31) and (3.20),

$$(3.22) \quad \rho_1(x, y) |y|^{-2\delta} \approx (1 + |y|)^{-d-\alpha+\delta} |y|^{-2\delta}, \quad |x| < 1, \quad y \neq 0.$$

Thus, applying (3.20) and the dominated convergence theorem, we get

$$(3.23) \quad \int_{\mathbb{R}^d} \rho_t(0, y) h^2(y) dy = 1, \quad t > 0.$$

By the symmetry of ρ_t and (3.21), for all $t > 0$, $x \in \mathbb{R}_0^d$ we also have

$$(3.24) \quad \rho_t(x, 0) := \lim_{y \rightarrow 0} \rho_t(x, y) = \rho_t(0, x) = t^{\frac{2\delta-d}{\alpha}} \rho_1 \left(t^{-1/\alpha} x, 0 \right) = t^{\frac{2\delta-d}{\alpha}} \varphi \left(t^{-1/\alpha} x \right).$$

Lemma 3.6. *The function ρ has a unique continuous positive extension to $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.*

Proof. By Chapman-Kolmogorov and the symmetry of ρ_1 , for $x, y \neq 0$, we have

$$(3.25) \quad \rho_1(x, y) = \int_{\mathbb{R}^d} \rho_{1/2}(z, x) \rho_{1/2}(z, y) dz.$$

Recall that ρ_1 is continuous on $\mathbb{R}_0^d \times \mathbb{R}_0^d$. This and (3.21) yield for all $z \in \mathbb{R}_0^d$ that $\rho_{1/2}(z, x) \rightarrow \rho_{1/2}(z, x_0)$ and $\rho_{1/2}(z, y) \rightarrow \rho_{1/2}(z, y_0)$ if $x \rightarrow x_0 \in \mathbb{R}^d$ and $y \rightarrow y_0 \in \mathbb{R}^d$, hence

$$(3.26) \quad \rho_1(x, y) \rightarrow \int_{\mathbb{R}^d} \rho_{1/2}(z, x_0) \rho_{1/2}(z, y_0) dz,$$

by the dominated convergence theorem, since for bounded $x, y \neq 0$, by and (2.30) and (2.31),

$$(3.27) \quad \rho_{1/2}(z, x) \rho_{1/2}(z, y) \approx (1 + |z|)^{-2d-2\alpha+2\delta}.$$

The latter function is integrable because $-2d - 2\alpha + 2\delta < -d - 3\alpha < -d$. Furthermore, (3.27) implies that the limit in (3.26) is positive and, by (3.25), it is an extension of ρ_1 to $\mathbb{R}^d \times \mathbb{R}^d$, which we shall denote by ρ_1 again. In view of (2.25),

$$(3.28) \quad t^{\frac{2\delta-d}{\alpha}} \rho_1(t^{-1/\alpha} x, t^{-1/\alpha} y), \quad x, y \in \mathbb{R}^d, \quad t > 0,$$

is finite and defines a continuous extension of ρ_t for each $t > 0$. The extension is clearly positive and jointly continuous in t, x, y . It is also unique, since $(0, \infty) \times \mathbb{R}_0^d \times \mathbb{R}_0^d$ is dense in $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. \square

We recall that the boundedness of ρ_1 near the origin readily follows from (2.28). On the other hand the continuity turned out difficult for us to capture directly or indirectly. For instance we considered this connection in [3]. The proof presented above seems to be underpinned by self-regularization of functions satisfying (implicit) integral equations.

In what follows, ρ will denote the continuous extension of ρ to $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

Corollary 3.7. $\rho_1(0, 0) = \lim_{x, y \rightarrow 0} \rho_1(x, y) \in (0, \infty)$.

The following Chapman-Kolmogorov equations hold, so ρ is a transition density on \mathbb{R}^d .

Corollary 3.8. *For all $s, t > 0$, $x, y \in \mathbb{R}^d$, we have $\int_{\mathbb{R}^d} \rho_s(x, z) \rho_t(z, y) h^2(z) dz = \rho_{s+t}(x, y)$.*

Proof. Since $(1 + |z|)^{-2d-2\alpha+2\delta} |z|^{-2\delta}$ is integrable, we can use (3.20) and (2.24). \square

Proof of Theorem 3.1. The first statement of the theorem is proved in Lemma 3.6, (3.2) is proved more generally as (3.28), and (3.3) is proved more generally in Corollary 3.8. See also (3.20) and (3.21) for a more explicit expression of (3.1). \square

Proof of Theorem 1.1. The result is immediate from (2.22) and (3.1). \square

We can now extend (2.29) as follows.

Corollary 3.9.

$$\rho_t(x, y) \approx \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right) \left(t^{\delta/\alpha} + |x|^\delta \right) \left(t^{\delta/\alpha} + |y|^\delta \right), \quad t > 0, \quad x, y \in \mathbb{R}^d.$$

Example 3.10. For $\kappa = 0$ we have $H = h = 1$, $\rho = \tilde{p} = p$, $l_t(x, y) = p_{1-e^{-t}}(e^{-t/\alpha} x, y)$, $\varphi(x) = p_1(0, x)$, and $\Psi_t(x) = p_t(0, x)$.

4. FUNCTIONAL ANALYSIS OF \tilde{P}_t

We recall that $\tilde{P}_t\varphi(x) := \int_{\mathbb{R}^d} \tilde{p}(t, x, y)\varphi(y) dy$. By (2.21), $\tilde{P}_t h = h$ for all $t > 0$.

Lemma 4.1. $\{\tilde{P}_t\}_{t>0}$ is a contraction semigroup on $L^1(h)$ and for every $f \in L^1(h)$, we have

$$(4.1) \quad \int_{\mathbb{R}^d} \tilde{P}_t f(x)h(x) dx = \int_{\mathbb{R}^d} f(x)h(x) dx, \quad t > 0.$$

Proof. The semigroup property of \tilde{P}_t follows from (2.16). Let $f \geq 0$ be a measurable function. By Fubini-Tonelli, the symmetry of \tilde{p} and (2.21),

$$\int_{\mathbb{R}^d} \tilde{P}_t f(x)h(x) dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{p}(t, x, y)f(y)h(x) dy dx = \int_{\mathbb{R}^d} h(y)f(y) dy.$$

For arbitrary $f \in L^1(h)$ we write $f = f_+ - f_-$ and use the nonnegative case. \square

Lemma 4.2. $\{P_t\}_{t>0}$ is a strongly continuous contraction semigroup on $L^1(h)$.

Proof. Since $p \leq \tilde{p}$, for nonnegative function f and $t > 0$ we get $\|P_t f\|_{L^1(h)} \leq \|\tilde{P}_t f\|_{L^1(h)} \leq \|f\|_{L^1(h)}$, so the contractivity follows as in the proof of Lemma 4.1.

By the contractivity and the semigroup property of P_t , it suffices to prove strong continuity as $t \rightarrow 0$. Let $f \in L^1(h)$. Then $g := fh \in L^1$. There are functions $g_n \in C_c^\infty(\mathbb{R}_0^d)$ such that $\|g - g_n\|_{L^1} \rightarrow 0$ as $n \rightarrow \infty$. Let $f_n = g_n/h$. Of course, $f_n \in C_c^\infty(\mathbb{R}_0^d)$ and $\|f - f_n\|_{L^1(h)} = \|g - g_n\|_{L^1}$ for every n , and, by the contractivity, we have

$$\begin{aligned} \|P_t f - f\|_{L^1(h)} &\leq \|P_t f - P_t f_n\|_{L^1(h)} + \|P_t f_n - f_n\|_{L^1(h)} + \|f_n - f\|_{L^1(h)} \\ &\leq 2\|f - f_n\|_{L^1(h)} + \|P_t f_n - f_n\|_{L^1(h)}. \end{aligned}$$

Therefore, it suffices to prove that $\|P_t f - f\|_{L^1(h)} \rightarrow 0$ as $t \rightarrow 0$ for every $f \in C_c^\infty(\mathbb{R}_0^d)$. To this end take $K \in (0, \infty)$ such that $\text{supp } f \subset B(0, K/2)$. Then,

$$\|P_t f - f\|_{L^1(h)} \leq \int_{|x| \leq K} |P_t f(x) - f(x)|h(x) dx + \int_{|x| > K} P_t |f|(x)h(x) dx.$$

When $t \rightarrow 0$, the above converges to 0 because $P_t f \rightarrow f$ uniformly, h is locally integrable and $P_t |f|(x) \leq ct(1 + |x|)^{-d-\alpha}$ on $B(0, K)^c$, see (2.4). The proof is complete. \square

Lemma 4.3. $\{\tilde{P}_t\}_{t>0}$ is a strongly continuous contraction semigroup on $L^1(h)$.

Proof. The contractivity is resolved in Lemma 4.1. Since P_t is strongly continuous as $t \rightarrow 0$, by (2.18) it suffices to consider nonnegative $f \in L^1(h)$ and verify that

$$I_t := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \tilde{p}(s, x, z)q(z)p(t-s, z, y)f(y) dz ds dy h(x) dx \rightarrow 0$$

as $t \rightarrow 0$. By (2.21) and [6, Lemma 3.3],

$$\begin{aligned} I_t &= \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} h(z)q(z)p(t-s, z, y)f(y) dz ds dy \\ &= \int_{\mathbb{R}^d} (h(y) - P_t h(y))f(y) dy = \|f\|_{L^1(h)} - \|P_t f\|_{L^1(h)} \rightarrow 0, \end{aligned}$$

indeed, because of Lemma 4.2. \square

We consider the resolvent operators $R_\lambda = \int_0^\infty e^{-\lambda t} P_t dt$ and $\tilde{R}_\lambda = \int_0^\infty e^{-\lambda t} \tilde{P}_t dt$ for $\lambda > 0$. The contractivities of P_t and \tilde{P}_t yield the following result.

Corollary 4.4. $\lambda \tilde{R}_\lambda$ and λR_λ are contractions on $L^1(h)$.

Let \mathcal{L} be the $L^1(h)$ -generator of the semigroup P_t , with domain \mathcal{D} , and let $\tilde{\mathcal{L}}$ be the $L^1(h)$ -generator of the semigroup \tilde{P}_t , with domain $\tilde{\mathcal{D}}$. The following result is consistent with (1.1).

Proposition 4.5. *We have $\mathcal{D} \subset \tilde{\mathcal{D}}$ and $\tilde{\mathcal{L}}f = \mathcal{L}f + qf$ for $f \in \mathcal{D}$.*

Proof. Let $\lambda > 0$. Recall that $f \in \mathcal{D}$ if and only if $f = R_\lambda g$ for some (unique) $g \in L^1(h)$ and then $\mathcal{L}f = \lambda R_\lambda g - g$. The same holds for \tilde{R}_λ . By integrating (2.18) with respect to $e^{-\lambda t} dt$ and using Fubini-Tonelli and the following self-explanatory notation we get

$$\begin{aligned} \tilde{R}_\lambda(x, y) &= R_\lambda(x, y) + \int_0^\infty e^{-\lambda t} \int_0^t \int_{\mathbb{R}^d} \tilde{p}(s, x, z) q(z) p(t-s, z, y) dz ds dt \\ &= R_\lambda(x, y) + \int_0^\infty \int_s^\infty \int_{\mathbb{R}^d} e^{-\lambda s} \tilde{p}(s, x, z) q(z) e^{-\lambda(t-s)} p(t-s, z, y) dz dt ds \\ &= R_\lambda(x, y) + \int_{\mathbb{R}^d} \tilde{R}_\lambda(x, z) q(z) R_\lambda(z, y) dz, \quad x, y \in \mathbb{R}^d. \end{aligned}$$

Accordingly, for $f \geq 0$, we get $\tilde{R}_\lambda f = R_\lambda f + \tilde{R}_\lambda q R_\lambda f$. In particular, $\tilde{R}_\lambda q R_\lambda f \in L^1(h)$ if $f \in L^1(h)$. Moreover, $q R_\lambda$ is a bounded operator on $L^1(h)$. Indeed, if $0 \leq f \in L^1(h)$, then using Fubini-Tonelli, the identity $e^{-\lambda t} = \int_t^\infty \lambda e^{-\lambda s} ds$ and [6, Lemma 3.3], we get

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^d} q(x) R_\lambda f(x) h(x) dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^\infty e^{-\lambda t} p(t, x, y) q(x) h(x) f(y) dt dy dx \\ &= \int_0^\infty \int_{\mathbb{R}^d} \lambda e^{-\lambda s} f(y) \left(\int_0^s \int_{\mathbb{R}^d} p(t, x, y) q(x) h(x) dx dt \right) dy ds \\ &\leq \int_0^\infty \lambda e^{-\lambda s} ds \int_{\mathbb{R}^d} h(y) f(y) dy = \int_{\mathbb{R}^d} h(y) f(y) dy. \end{aligned}$$

It follows that $R_\lambda = \tilde{R}_\lambda(I - q R_\lambda)$ on $L^1(h)$, and so

$$\mathcal{D} = R_\lambda(L^1(h)) \subset \tilde{R}_\lambda(L^1(h)) = \tilde{\mathcal{D}}.$$

To prove that $\tilde{\mathcal{L}}\varphi = \mathcal{L}\varphi + q\varphi$ for $\varphi \in \mathcal{D}$, we let $\psi = \mathcal{L}\varphi$, $\varphi, \psi \in L^1(h)$, that is $\varphi = \lambda R_\lambda f - R_\lambda \psi$. It is enough to prove that $\tilde{\mathcal{L}}f = \psi + qf$, that is

$$(4.2) \quad f = \lambda \tilde{R}_\lambda f - \tilde{R}_\lambda q f.$$

But the right-hand-side of (4.2) is

$$\lambda R_\lambda f + \tilde{R}_\lambda \lambda q R_\lambda f - R_\lambda \psi - R_\lambda q R_\lambda \psi - \tilde{R}_\lambda q f = f + \tilde{R}_\lambda q f - \tilde{R}_\lambda q f = f.$$

The proof is complete. \square

Recall the notation of (1.4). Our proof of the large-time asymptotics of \tilde{P}_t hinges on the following hypercontractivity result (for $q = 1$ see Lemma 4.1 and for $q = \infty$ see Lemma 4.7).

Theorem 4.6. *If $1 < q < \infty$ then for all $t > 0$ and nonnegative functions f on \mathbb{R}^d ,*

$$(4.3) \quad \|\tilde{P}_t f\|_{q, h} \leq C t^{-\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \|f\|_{L^1(h)} + C t^{-\frac{d-2\delta}{\alpha}(1-\frac{1}{q})-\frac{\delta}{\alpha}} \|f\|_{L^1(\mathbb{R}^d)}.$$

Proof. We first prove the inequality (4.3) for $t = 1$. Let $q' = \frac{q}{q-1}$. It is enough to verify that for every function $g \geq 0$ on \mathbb{R}^d ,

$$(4.4) \quad \int_{\mathbb{R}^d} \tilde{P}_1 f(x) g(x) h^{2-q}(x) dx \leq C \|f\|_{L^1(H)} \|g\|_{L^{q'}(h^{2-q})},$$

since then, by the duality of $L^p(h^{2-q})$, we get

$$(4.5) \quad \|\tilde{P}_1 f\|_{q, h} = \|\tilde{P}_1 f\|_{L^q(h^{2-q})} = \sup_{\|b\|_{L^{q'}(h^{2-q})} = 1} \left| \int_{\mathbb{R}^d} \tilde{P}_1 f(x) b(x) h^{2-q}(x) dx \right| \leq C \|f\|_{L^1(H)},$$

as needed. To prove (4.4), by (2.12) it is enough to estimate

$$(4.6) \quad \begin{aligned} I(f, g) &:= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} H(x) H(y) p(1, x, y) f(y) g(x) h^{2-q}(x) dy dx \\ &= I(f \mathbf{1}_B, g \mathbf{1}_B) + I(f \mathbf{1}_B, g \mathbf{1}_{B^c}) + I(f \mathbf{1}_{B^c}, g \mathbf{1}_B) + I(f \mathbf{1}_{B^c}, g \mathbf{1}_{B^c}), \end{aligned}$$

where $B = B(0, 1) \subset \mathbb{R}^d$. By Hölder's inequality,

$$\begin{aligned} I(f\mathbf{1}_B, g\mathbf{1}_B) &\leq \int_B \int_B h(x)h(y)p(1, x, y)f(y)g(x) dy h(x)^{2-q} dx \\ &\leq c \left(\int_B h(y)f(y) dy \right) \left(\int_B h(x)g(x)h^{2-q}(x) dx \right) \\ &\leq c \|f\|_{L^1(H)} \left(\int_B g^{q'}(x)h^{2-q}(x) dx \right)^{\frac{1}{q'}} \left(\int_B h^2(x) dx \right)^{\frac{1}{q}} \\ &\leq C \|f\|_{L^1(H)} \|g\|_{L^{q'}(h^{2-q})}, \end{aligned}$$

where $\int_B h^2(x) dx < \infty$, because $2\delta \leq d - \alpha < d$. We next deal with the second term in (4.6),

$$\begin{aligned} (4.7) \quad I(f\mathbf{1}_B, g\mathbf{1}_{B^c}) &\leq \int_B \int_{B^c} f(y)h(y)p(1, x, y)g(x)h^{2-q}(x) dx dy \\ &\leq c \int_B \int_{B^c} f(y)h(y)|x|^{-d-\alpha} g(x)h^{2-q}(x) dx dy \\ &\leq c \|f\|_{L^1(H)} \int_{B^c} |x|^{-d-\alpha} g(x)h^{2-q}(x) dx \\ &\leq c \|f\|_{L^1(H)} \left(\int_{B^c} g^{q'}(x)h^{2-q}(x) dx \right)^{\frac{1}{q'}} \left(\int_{B^c} |x|^{-(d+\alpha)q} h^{2-q}(x) dx \right)^{\frac{1}{q}} \\ &\leq C \|f\|_{L^1(H)} \|g\|_{L^{q'}(h^{2-q})}. \end{aligned}$$

The last integral in (4.7) is finite because $0 \leq \delta \leq (d - \alpha)/2$, so $|x|^{-(d+\alpha)q - \delta(2-q)} \leq |x|^{-d-\alpha} \in L^1(B^c, dx)$ for every $q \in (1, +\infty)$. We then estimate the third term,

$$\begin{aligned} I(f\mathbf{1}_{B^c}, g\mathbf{1}_B) &\leq \int_{B^c} \int_B f(y)h(y)p(1, x, y)g(x)h^{2-q}(x) dx dy \\ &\leq c \int_{B^c} \int_B f(y)h(y)|y|^{-d-\alpha} g(x)h^{2-q}(x) dx dy \\ &\leq c \|f\|_{L^1(H)} \int_B g(x)h^{2-q}(x) dx \\ &\leq c \|f\|_{L^1(H)} \left(\int_B g^{q'}(x)h^{2-q}(x) dx \right)^{\frac{1}{q'}} \left(\int_B h^{2-q}(x) dx \right)^{\frac{1}{q}}, \end{aligned}$$

and the last integral is finite, as before. For the fourth term in (4.6) we note that

$$\begin{aligned} \int_{\mathbb{R}^d} p(1, x, y)g(x)h^{2-q}(x) dx &\leq \left(\int_{\mathbb{R}^d} g^{q'}(x)h^{2-q}(x) dx \right)^{\frac{1}{q'}} \left(\int_{\mathbb{R}^d} p^q(1, x, y)h^{2-q}(x) dx \right)^{\frac{1}{q}} \\ &\leq \|g\|_{L^{q'}(h^{2-q})} \left(\int_{\mathbb{R}^d} p_1^q(x)h^{2-q}(x) dx \right)^{\frac{1}{q}}, \end{aligned}$$

by rearrangement inequalities [30, Sec.3]. The last integral is finite, because

$$\begin{aligned} p_1^q(x)h^{2-q}(x) &\leq C|x|^{-\delta}, \quad x \in B, \\ p_1^q(x)h^{2-q}(x) &\leq C|x|^{-(d+\alpha)q - \delta(2-q)} \leq |x|^{-d-\alpha}, \quad x \in B^c. \end{aligned}$$

Thus,

$$\begin{aligned} I(f\mathbf{1}_{B^c}, g\mathbf{1}_{B^c}) &\leq \int_{B^c} \int_{B^c} p(1, x, y)f(y)g(x)h^{2-q}(x) dx dy \\ &\leq C \|g\|_{L^{q'}(h^{2-q})} \int_{B^c} f(y) dy \leq C \|f\|_{L^1(H)} \|g\|_{L^{q'}(h^{2-q})}. \end{aligned}$$

Therefore (4.6) is bounded above by $C \|f\|_{L^1(H)} \|g\|_{L^{q'}(h^{2-q})}$, which yields (4.3) for $t = 1$.

By this and a change of variables, for all $t > 0$ we get

$$\begin{aligned}
 (4.8) \quad \|\tilde{P}_t f\|_{q,h} &= t^{\frac{d-\delta(2-q)}{\alpha q}} \|\tilde{P}_1 f(t^{1/\alpha} \cdot)\|_{q,h} \\
 &\leq C t^{\frac{d-\delta(2-q)}{\alpha q}} \left(\|f(t^{1/\alpha} \cdot)\|_{L^1(h)} + \|f(t^{1/\alpha} \cdot)\|_{L^1} \right) \\
 &= C t^{\frac{d-\delta(2-q)}{\alpha q}} \left(t^{-\frac{d-\delta}{\alpha}} \|f\|_{L^1(h)} + t^{-\frac{d}{\alpha}} \|f\|_{L^1} \right) \\
 &= C t^{-\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \|f\|_{L^1(h)} + C t^{-\frac{d-2\delta}{\alpha}(1-\frac{1}{q})-\frac{\delta}{\alpha}} \|f\|_{L^1}.
 \end{aligned}$$

The proof of (4.3) is complete. \square

We note in passing that (4.3) for $t = 1$ is equivalent to (4.5) but H is incompatible with the scaling (4.8), so the long form of (4.3) seems inevitable. Here is a more trivial bound.

Lemma 4.7. $\|\tilde{P}_t f\|_{\infty,h} \leq \|f\|_{\infty,h}$.

Proof. For $x \in \mathbb{R}_0^d$ we have $|\tilde{P}_t f(x) dx| \leq \|f\|_{\infty,h} \int_{\mathbb{R}^d} \tilde{P}_t(x,y) h(y) dy \leq h(x) \|f\|_{\infty,h}$. \square

5. ASYMPTOTIC BEHAVIOR FOR LARGE TIME

This section is devoted to the proof of Theorem 1.2. For the sake of comparison let us discuss the following classical analogue of (1.9),

$$(5.1) \quad \begin{cases} \partial_t u(x,t) = (\Delta + \kappa|x|^{-2}) u(x,t), & x \in \mathbb{R}^d, t > 0, \\ u(x,0) = f(x), \end{cases}$$

where $d \geq 3$ and $\kappa \in \mathbb{R}$. As we already mentioned in Section 1.2, the Cauchy problem (5.1) was popularized by Baras and Goldstein [1], who discovered that (5.1) has no positive local-in-time solutions if $\kappa > (d-2)^2/4$, which is called the instantaneous blow up. See also Goldstein and Kombe [17] for a simple proof via Harnack inequality. Vázquez and Zuazua [42] then studied the large time behavior of solutions to (5.1) with $0 < \kappa \leq (d-2)^2/4$. Using a weighted version of the Hardy-Poincaré inequality, they proved in [42, Theorem 10.3] the stabilization of some solutions toward the following self-similar solution of (5.1)

$$V(x,t) = t^{\sigma-\frac{d}{2}} |x|^{-\sigma} e^{-\frac{|x|^2}{4t}},$$

where $\sigma = \frac{d-2}{2} - \sqrt{(d-2)^2/4 - \kappa}$. Then, Pilarczyk [37] proved that if $u = u(t,x)$ is the solution of (5.1) with $\kappa \in (-\infty, (d-2)^2/4)$, then for every $1 \leq q \leq \infty$,

$$\lim_{t \rightarrow \infty} t^{\frac{d}{2}(1-\frac{1}{q})-\frac{\sigma}{2}} \|u(\cdot,t) - AV(\cdot,t)\|_{q,\varphi_\sigma(t)} = 0.$$

Here $f \in L^1(\mathbb{R}^d) \cap L^1(|x|^{-\sigma} dx)$,

$$A = \frac{\int_{\mathbb{R}^d} |x|^{-\sigma} f(x) dx}{\int_{\mathbb{R}^d} |x|^{-2\sigma} e^{-\frac{|x|^2}{4}} dx},$$

$\varphi_\sigma(x,t) = 1 \vee (\sqrt{t}/|x|)^\sigma$, and

$$\|g\|_{q,\varphi_\sigma(t)} = \begin{cases} \left(\int_{\mathbb{R}^d} |g(x)|^q \varphi_\sigma^{2-q}(x,t) dx \right)^{1/q} & \text{for } 1 \leq q < \infty, \\ \text{ess sup}_{x \in \mathbb{R}^d} |g(x)| / \varphi_\sigma(x,t) & \text{for } q = \infty. \end{cases}$$

In Theorem 1.2 we extend the results of Vázquez and Zuazua [42] and Pilarczyk [37] to the Cauchy problem (1.9). In fact we also propose a novel description of the asymptotics, using simpler, time-independent norms $\|f\|_{q,h}$ defined in (1.4).

We now return to the setting of (1.9) and (2.9). The solutions to (1.9) will be defined as $u(t, x) = \tilde{P}_t f(x)$, $t > 0$, $x \in \mathbb{R}^d$ for $f \in L^1(H)$. By Theorem 1.1,

$$\Psi_t(x) := \lim_{y \rightarrow 0} \frac{\tilde{p}(t, x, y)}{h(y)} = \rho_t(0, x)h(x) = t^{\frac{2\delta-d}{\alpha}} \varphi(t^{-1/\alpha}x)h(x), \quad t > 0, \quad x \in \mathbb{R}^d.$$

In particular, $\Psi_t(0) = \infty$. Since, for $s > 0$,

$$\rho_{t+s}(0, x) = \int_{\mathbb{R}^d} \rho_t(0, y)\rho_s(y, x)h^2(y) dy = \int_{\mathbb{R}^d} \rho_t(0, y)h(y)\tilde{p}(s, y, x)/h(x) dy,$$

we get the following evolution property

$$(5.2) \quad \int_{\mathbb{R}^d} \tilde{p}(s, y, x)\Psi_t(y) dy = \Psi_{t+s}(x).$$

By (3.28) we also have

$$(5.3) \quad \Psi_t(x) = t^{\frac{\delta-d}{\alpha}} \Psi_1(t^{-1/\alpha}x), \quad t > 0, \quad x \in \mathbb{R}^d.$$

We summarize (5.2) and (5.3) by saying that $\Psi_t(x)$ is a self-similar semigroup solution of (1.9). By (3.23),

$$(5.4) \quad \int_{\mathbb{R}^d} \Psi_t(x)h(x) dx = \int_{\mathbb{R}^d} \Psi_1(x)h(x) dx = 1.$$

Furthermore, $\Psi_t(x)$ is a mild solution of (1.9) since it satisfies the following Duhamel formula.

Lemma 5.1.

$$(5.5) \quad \Psi_t(x) = \int_0^t \int_{\mathbb{R}^d} \Psi_r(z)q(z)p(t-r, x, z) dz dr, \quad t > 0, \quad x \in \mathbb{R}^d.$$

Proof. By (2.12), we have

$$\frac{\tilde{p}(t, x, y)}{h(y)} \approx (|y|^\delta + t^{\delta/\alpha})(1 + t^{\delta/\alpha}|x|^{-\delta})p(t, x, y),$$

hence

$$(5.6) \quad \Psi_t(x) \approx t^{\delta/\alpha}(1 + t^{\delta/\alpha}|x|^{-\delta})(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}), \quad t > 0, \quad x \in \mathbb{R}^d.$$

Now, by (5.2) and (2.17) for $0 < s < t$ we obtain

$$\begin{aligned} \Psi_t(x) &= \int_{\mathbb{R}^d} \tilde{p}(t-s, x, y)\Psi_s(y) dy \\ &\quad + \int_0^{t-s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p(r, x, y)q(y)\tilde{p}(t-s-r, y, z)\Psi_s(z) dz dy dr \\ &= \int_{\mathbb{R}^d} p(t-s, x, y)\Psi_s(y) dy + \int_0^{t-s} \int_{\mathbb{R}^d} p(r, x, y)q(y)\Psi_{t-r}(\cdot, y) dy dr \\ (5.7) \quad &= \int_{\mathbb{R}^d} p(t-s, x, y)\Psi_s(y) dy + \int_s^t \int_{\mathbb{R}^d} p(t-r, x, y)q(y)\Psi_r(y) dy dr. \end{aligned}$$

Using (5.6) and (5.3), we get

$$\begin{aligned} 0 &\leq \limsup_{s \rightarrow 0} \int_{\mathbb{R}^d} p(t-s, x, z)\Psi_s(z) dz \leq c \limsup_{s \rightarrow 0} (t-s)^{-d/\alpha} \int_{\mathbb{R}^d} \Psi_s(z) dz \\ &= c \limsup_{s \rightarrow 0} (t-s)^{-d/\alpha} \int_{\mathbb{R}^d} s^{(\delta-d)/\alpha} \Psi_1(s^{-1/\alpha}z) dz \\ &= ct^{-d/\alpha} \int_{\mathbb{R}^d} \Psi_1(z) dz \lim_{s \rightarrow 0} s^{\delta/\alpha} = 0. \end{aligned}$$

Therefore, taking $s \rightarrow 0$ in (5.7), we obtain (5.5). \square

Remark 5.2. By the self-similarity (5.3), the definition of $\|\cdot\|_{q,h}$, the change of variables $y = t^{-1/\alpha}x$, and (5.6), we have for all $t > 0$

$$(5.8) \quad 0 < \|\Psi_t\|_{q,h} = t^{-\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \|\Psi_1\|_{q,h} < \infty,$$

see (4.8). It follows that

$$(5.9) \quad \|\Psi_{st}\|_{q,h} = t^{-\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \|\Psi_s\|_{q,h}$$

and

$$(5.10) \quad \|\Psi_t\|_{1,h} = \|\Psi_1\|_{1,h} = 1,$$

which is the same as (5.4). Furthermore,

$$t^{-\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \|u(t, \cdot) - A\Psi_t\|_{q,h} = \|t^{\frac{d-\delta}{\alpha}} u\left(t, t^{\frac{1}{\alpha}} \cdot\right) - A\Psi_1\|_{q,h},$$

in analogy with the results of Pilarczyk [37] and Vázquez and Zuazua [42].

Remark 5.3. Similar arguments yield the following result, to be proved in a forthcoming paper,

$$(5.11) \quad \lim_{t \rightarrow \infty} t^{\frac{d}{\alpha}(1-\frac{1}{q})-\frac{\delta}{\alpha}} \|u(t, \cdot) - A\Psi_t(\cdot)\|_{q,H_t} = 0,$$

where $1 \leq q \leq \infty$, $H_t(z) = H(t^{-1/\alpha}z)$ and the norms are defined by

$$\|f\|_{q,H_t} = \begin{cases} \left(\int_{\mathbb{R}^d} |f(x)|^q H_t^{2-q}(x) dx \right)^{\frac{1}{q}} & \text{for } 1 \leq q < \infty, \\ \sup_{x \in \mathbb{R}^d} |f(x)|/H_t(x) & \text{for } q = \infty. \end{cases}$$

The asymptotics (5.11) is an exact analogue of the result in [37, Theorem 2.1].

Remark 5.4. By (4.1) and (5.4), A in Theorem 1.2 satisfies $\int_{\mathbb{R}^d} (\tilde{P}_t f(x) - A\Psi_s(x))h(x) dx = 0$ for all $t \geq 0, s > 0$.

Remark 5.5. By Lemma 4.1 we have $\int_{\mathbb{R}^d} u(t, x)h(x) dx = \int_{\mathbb{R}^d} f(x)h(x) dx$ for all $t \geq 0$. Here is a rather heuristic alternative argument for the equality: If we multiply (1.9) by the function h and integrate, then indeed we get

$$\frac{d}{dt} \int_{\mathbb{R}^d} |x|^{-\delta} u(t, x) dx = - \int_{\mathbb{R}^d} (-\Delta)^{\alpha/2} u(t, x) |x|^{-\delta} dx + \kappa_\delta \int_{\mathbb{R}^d} |x|^{-\delta-\alpha} u(t, x) dx.$$

Using the Fourier symbol of $\Delta^{\alpha/2}$, Parseval's relation and (2.9) we get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} |x|^{-\delta} u(t, x) dx &= - \frac{2^\alpha \pi^{\alpha+\delta-d/2} \Gamma\left(\frac{d-\delta}{2}\right)}{\Gamma\left(\frac{\delta}{2}\right)} \int_{\mathbb{R}^d} |\xi|^{\alpha+\delta-d} \hat{u}(t, \xi) d\xi + \kappa_\delta \int_{\mathbb{R}^d} |x|^{-\delta-\alpha} u(t, x) dx \\ &= \left(\kappa_\delta - \kappa_\delta\right) \int_{\mathbb{R}^d} |x|^{-\delta-\alpha} u(t, x) dx = 0, \end{aligned}$$

see [18, Theorem 2.2.14 p.102] and [40, Lemma 2 p.117].

Recall that \tilde{P}_t is defined in (1.5). The next lemma is crucial for the proof of Theorem 1.2.

Lemma 5.6. *If $f \in L^1(H)$, $\int_{\mathbb{R}^d} f(x)h(x) dx = 0$ and $1 \leq q < \infty$, then*

$$(5.12) \quad \lim_{t \rightarrow \infty} t^{\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \|\tilde{P}_t f\|_{q,h} = 0.$$

If, additionally, f has compact support, then (5.12) is true for $q = \infty$, too.

Proof. First, we consider a compactly supported function ψ such that $\psi \in L^1(H)$ and

$$(5.13) \quad \int_{\mathbb{R}^d} \psi(x)h(x) dx = 0,$$

and we intend to prove (5.12) with f replaced by ψ .

Step 1. Case $q = \infty$.

By the definition of the norm $\|\cdot\|_{\infty,h}$,

$$I(t) := t^{\frac{d-2\delta}{\alpha}} \|\tilde{P}_t \psi\|_{\infty,h} = t^{\frac{d-2\delta}{\alpha}} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} \rho_t(x,y)h(y)\psi(y) dy \right|.$$

Using the condition (5.13), we obtain

$$I(t) = t^{\frac{d-2\delta}{\alpha}} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} (\rho_t(x,y) - \rho_t(x,0)) h(y)\psi(y) dy \right|.$$

We fix $\omega > 0$. Since ψ has a compact support, for sufficiently large $t > 0$ we get

$$\begin{aligned} I(t) &= t^{\frac{d-2\delta}{\alpha}} \sup_{x \in \mathbb{R}^d} \left| \int_{|y| \leq t^{\frac{1}{\alpha}} \omega} (\rho_t(x,y) - \rho_t(x,0)) h(y)\psi(y) dy \right| \\ &\leq t^{\frac{d-2\delta}{\alpha}} \sup_{\substack{x \in \mathbb{R}^d \\ |y| \leq \omega t^{\frac{1}{\alpha}}}} |\rho_t(x,y) - \rho_t(x,0)| \int_{|y| \leq \omega t^{\frac{1}{\alpha}}} h(y)|\psi(y)| dy, \end{aligned}$$

so by scaling of ρ ,

$$\begin{aligned} I(t) &\leq \sup_{\substack{x \in \mathbb{R}^d \\ |y| \leq \omega t^{\frac{1}{\alpha}}}} \left| \rho_1\left(t^{-\frac{1}{\alpha}}x, t^{-\frac{1}{\alpha}}y\right) - \rho_1\left(t^{-\frac{1}{\alpha}}x, 0\right) \right| \int_{|y| \leq t^{\frac{1}{\alpha}} \omega} h(y)|\psi(y)| dy \\ &\leq \sup_{\substack{x \in \mathbb{R}^d \\ |y| \leq \omega}} |\rho_1(x,y) - \rho_1(x,0)| \int_{\mathbb{R}^d} h(y)|\psi(y)| dy. \end{aligned}$$

By Lemma 3.6 and (2.31) we choose ω small enough to have

$$\sup_{\substack{x \in \mathbb{R}^d \\ |y| \leq \omega}} |\rho_1(x,y) - \rho_1(x,0)| < \varepsilon.$$

This proves (5.12) in the considered setting.

Step 2. Case $q = 1$.

By the definition of the norm $\|\cdot\|_{1,h}$,

$$J(t) := \|\tilde{P}_t \psi\|_{1,h} = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \tilde{p}(t,x,y)h(x)\psi(y) dy \right| dx = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \rho_t(x,y)h^2(x)\psi(y)h(y) dy \right| dx.$$

Applying (5.13), we get

$$J(t) \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\rho_t(x,y) - \rho_t(x,0)| h^2(x)|\psi(y)|h(y) dy dx.$$

We fix $\omega > 0$ and notice that

$$J(t) \leq \int_{\mathbb{R}^d} \int_{|y| \leq \omega t^{\frac{1}{\alpha}}} |\rho_t(x,y) - \rho_t(x,0)| h^2(x)|\psi(y)|h(y) dy dx,$$

for sufficiently large t , since the function ψ has a compact support. Then

$$J(t) \leq \sup_{|y| \leq \omega t^{\frac{1}{\alpha}}} \|\rho_t(\cdot,y) - \rho_t(\cdot,0)\|_{L^1(h^2)} \int_{\mathbb{R}^d} h(y)|\psi(y)| dy.$$

By scaling of ρ , substituting $x = t^{\frac{1}{\alpha}}z$ we obtain

$$\begin{aligned} & \sup_{|y| \leq \omega t^{\frac{1}{\alpha}}} \|\rho_t(\cdot, y) - \rho_t(\cdot, 0)\|_{L^1(h^2)} \\ &= t^{\frac{2\delta-d}{\alpha}} \sup_{|y| \leq \omega t^{\frac{1}{\alpha}}} \int \left| \rho_1\left(t^{-1/\alpha}x, t^{-1/\alpha}y\right) - \rho_1\left(t^{-1/\alpha}x, 0\right) \right| h^2(x) dx \\ &= \sup_{|y| < \omega} \|\rho_1(\cdot, y) - \rho_1(\cdot, 0)\|_{L^1(h^2)}. \end{aligned}$$

Applying Lemma 3.3, for $\omega > 0$ sufficiently small we get

$$J(t) \leq \varepsilon \|\psi\|_{L^1(h)}.$$

Step 3. Case $q \in (1, \infty)$.

By the definition of the norm $\|\cdot\|_{q,h}$ and Hölder inequality,

$$\begin{aligned} t^{\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \|\tilde{P}_t \psi\|_{q,h} &= t^{\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \left(\int_{\mathbb{R}^d} |\tilde{P}_t \psi(x)/h(x)|^{q-1} |\tilde{P}_t \psi(x)h(x)| dx \right)^{\frac{1}{q}} \\ &\leq \left(t^{\frac{d-2\delta}{\alpha}} \|\tilde{P}_t \psi\|_{\infty,h} \right)^{1-\frac{1}{q}} \left(\|\tilde{P}_t \psi\|_{1,h} \right)^{\frac{1}{q}}. \end{aligned}$$

Both factors converge to zero as $t \rightarrow \infty$ by *Step 1.* and *Step 2.*

Step 4. General initial datum.

Let $R > 0$, $c_R = \int_{|x| \leq R} f(x)h(x) dx / \int_{|x| \leq R} h(x) dx$ and $\psi_R(x) = (f(x) - c_R)\mathbf{1}_{|x| \leq R}$. Of course, ψ_R is compactly supported and

$$\int_{\mathbb{R}^d} h(x)\psi_R(x) dx = 0.$$

Furthermore,

$$\begin{aligned} \|f - \psi_R\|_{L^1(h)} &= |c_R| \int_{|x| \leq R} h(x) dx + \int_{|x| > R} h(x)|f(x)| dx \\ &= \left| \int_{|x| \leq R} h(x)f(x) dx \right| + \int_{|x| > R} h(x)|f(x)| dx \rightarrow 0 \end{aligned}$$

as $R \rightarrow \infty$, due to the assumption (5.13) and the condition $f \in L^1(h)$. Let $\varepsilon > 0$ and choose $R > 0$ so large that

$$\|f - \psi_R\|_{1,h} < \varepsilon.$$

For $q = 1$, by using the triangle inequality and Lemma 4.1, we have

$$\begin{aligned} \|\tilde{P}_t f\|_{1,h} &\leq \|\tilde{P}_t \psi_R\|_{1,h} + \|\tilde{P}_t(f - \psi_R)\|_{1,h} \\ &\leq \|\tilde{P}_t \psi_R\|_{1,h} + \|f - \psi_R\|_{1,h}. \end{aligned}$$

Hence, by *Step 2.* of this proof we get

$$(5.14) \quad \limsup_{t \rightarrow \infty} \|\tilde{P}_t f\|_{1,h} \leq \varepsilon,$$

which completes the verification of (5.12) in this case.

If $1 < q < \infty$, then using the triangle inequality and Theorem 4.6, we obtain

$$\begin{aligned} t^{\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \|\tilde{P}_t f\|_{q,h} &\leq t^{\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \|\tilde{P}_t \psi_R\|_{q,h} + t^{\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \|\tilde{P}_t(f - \psi_R)\|_{q,h} \\ &\leq t^{\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \|\tilde{P}_t \psi_R\|_{q,h} + C\|f - \psi_R\|_{1,h} + Ct^{-\frac{\delta}{\alpha}} \|f - \psi_R\|_{L^1(\mathbb{R}^d)}. \end{aligned}$$

By *Step 3.* of this proof,

$$(5.15) \quad \limsup_{t \rightarrow \infty} t^{\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \|\tilde{P}_t f\|_{q,h} \leq 2C\varepsilon,$$

which is valid both for $\delta > 0$ and $\delta = 0$. This completes the proof of (5.12) for $q \in (1, \infty)$. \square

We prove Theorem 1.2 immediately after the following discussion of *solutions* to (1.9).

Remark 5.7. By (5.7) we conclude that $\Psi_t(x)$ is a mild solution of (1.9). In passing we note that, by (5.6), $\Psi_t(x) \rightarrow 0$ when $t \rightarrow 0^+$ and $x \neq 0$. Further, if $f \in L^1(H)$, $t > 0$, $x \in \mathbb{R}_0^d$ and $u(t, x) = \tilde{P}_t f(x)$, then

$$(5.16) \quad u(t, x) = \int_{\mathbb{R}^d} p(t, x, y) f(y) dy + \int_0^t \int_{\mathbb{R}^d} p(t-r, x, y) q(y) u(r, y) dy dr,$$

as follows from (2.17). This justifies calling u in Theorem 1.2 solution to (1.9). By definition, u can also be called the semigroup solution and (5.2) yields an analogue for $\Psi_t(x)$.

Proof of Theorem 1.2. By (2.16), (5.2), Remark 5.4 and Lemma 5.6,

$$\begin{aligned} \lim_{t \rightarrow \infty} t^{\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \|u(t, \cdot) - A\Psi_t\|_{q,h} &= \lim_{t \rightarrow \infty} t^{\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \|u(t+1, \cdot) - A\Psi_{t+1}\|_{q,h} \\ &= \lim_{t \rightarrow \infty} t^{\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \|\tilde{P}_t (\tilde{P}_1 f - A\Psi_1)\|_{q,h} = 0. \end{aligned}$$

□

Theorem 1.2 is optimal, as asserted by the following two observations.

Proposition 5.8. *Let $q \in [1, \infty)$ and $\tau : [0, \infty) \rightarrow [0, \infty)$ be increasing with $\lim_{t \rightarrow \infty} \tau(t) = \infty$. Then there is $f \in L^1(H)$ such that $\int_{\mathbb{R}^d} f(x) h(x) dx = 1$ and $u := \tilde{P}_t f$ satisfies*

$$(5.17) \quad \lim_{t \rightarrow \infty} \tau(t) t^{\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \|u(t, \cdot) - \Psi_t\|_{q,h} = \infty.$$

Proof. The proof builds on the fact that $\|\Psi_1 - \Psi_s\|_{q,h} > 0$ for $s > 1$, which, as we shall see below, is a consequence of the scaling of $\Psi_t(x)$.

Without loss of generality we may assume that τ is continuous, strictly increasing and $\tau(0) = 0$, by replacing $\tau(t)$ by $t/(t+1) \left[t/(t+1) + \frac{1}{t} \int_0^t \tau(s) ds \right] \leq \tau(t) + 1$, see also (1.6). In particular, τ^{-1} is well defined : $[0, \infty) \rightarrow [0, \infty)$. For $n \in \mathbb{N}$ we let $t_n = \tau^{-1}(2^{2n})$. Let $f = \sum_{n=1}^{\infty} 2^{-n} \Psi_{t_n}$. According to (5.10), $\int_{\mathbb{R}^d} f(x) h(x) dx = \|f\|_{1,h} = 1$. By (5.2),

$$u(t, x) := \tilde{P}_t f(x) = \sum_{n=1}^{\infty} 2^{-n} \Psi_{t+t_n}(x) \geq 0, \quad t > 0, x \in \mathbb{R}^d.$$

We remark that if $\tau(t) \geq 1$, then $t_n \geq t$ is equivalent to $n \geq \lceil \frac{1}{2} \log_2 \tau(t) \rceil$. In this case, by (5.9), and triangle inequality, we get

$$\begin{aligned} (5.18) \quad t^{\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \|u(t, \cdot) - \Psi_t\|_{q,h} &= \left\| \sum_{n=1}^{\infty} 2^{-n} \Psi_{1+t_n/t} - \Psi_1 \right\|_{q,h} \\ &\geq \|\Psi_1\|_{q,h} - \sum_{n=1}^{\infty} 2^{-n} \|\Psi_{1+t_n/t}\|_{q,h} = \|\Psi_1\|_{q,h} \sum_{n=1}^{\infty} 2^{-n} \left(1 - (1+t_n/t)^{-\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \right) \\ &\geq \|\Psi_1\|_{q,h} \left(1 - 2^{-\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \right) \sum_{t_n \geq t} 2^{-n} = \|\Psi_1\|_{q,h} \left(1 - 2^{-\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \right) 2^{1 - \lceil \frac{1}{2} \log_2 \tau(t) \rceil} \\ &\geq \|\Psi_1\|_{q,h} \left(1 - 2^{-\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} \right) \tau(t)^{-1/2}. \end{aligned}$$

The case of $q \in (1, \infty)$ in (5.17) is resolved, because $1 - 2^{-\frac{d-2\delta}{\alpha}(1-\frac{1}{q})} > 0$ in this case.

We next consider $q = 1$. Then, starting from (5.18), we get

$$\begin{aligned}
 \|u(t, \cdot) - \Psi_t\|_{1,h} &= \left\| \sum_{n=1}^{\infty} 2^{-n} \Psi_{1+t_n/t} - \Psi_1 \right\|_{1,h} \\
 &\geq \int_{B(0,1)} \left| \Psi_1(x) - \sum_{n=1}^{\infty} 2^{-n} \Psi_{1+t_n/t}(x) \right| h(x) \, dx \\
 &\geq \int_{B(0,1)} \left(\Psi_1(x) - \sum_{n=1}^{\infty} 2^{-n} \Psi_{1+t_n/t}(x) \right) h(x) \, dx \\
 (5.19) \quad &= \sum_{n=1}^{\infty} 2^{-n} \left(\int_{B(0,1)} \Psi_1(x) h(x) \, dx - \int_{B(0,1)} \Psi_{1+t_n/t}(x) h(x) \, dx \right).
 \end{aligned}$$

By the same change of variables as in (5.8), for all $t > 0$ and $U \subset \mathbb{R}^d$ we get

$$(5.20) \quad \int_U \Psi_t(x) h(x) \, dx = \int_{t^{-1/\alpha} U} \Psi_1(x) h(x) \, dx.$$

Therefore by (5.19),

$$\begin{aligned}
 \|u(t, \cdot) - \Psi_t\|_{1,h} &\geq \sum_{n=1}^{\infty} 2^{-n} \int_{B(0,1) \setminus B(0,(1+t_n/t)^{-1/\alpha}} \Psi_1(x) h(x) \, dx \\
 &\geq \int_{B(0,1) \setminus B(0,2^{-1/\alpha})} \Psi_1(x) h(x) \, dx \sum_{t_n \geq t} 2^{-n}.
 \end{aligned}$$

By (5.6), $\int_{B(0,1) \setminus B(0,2^{-1/\alpha})} \Psi_1(x) h(x) \, dx > 0$, and we conclude as before. \square

Remark 5.9. We note that (1.6) does not hold for $q = \infty$. Indeed, let $f_n(x) = \mathbf{1}_{B(x_n,1)}(x)$, where $\{x_n\}$ is a sequence in \mathbb{R}^d such that $|x_n| = 2^n$. Then, by (2.4), for $t > 1$ we have

$$\tilde{P}_t f_n(x_n) \geq \int_{B(x_n,1)} p(t, x_n, y) \, dy \geq ct^{-d/\alpha}.$$

Let $f(x) = \sum_{n=2}^{\infty} f_n(x)$. Clearly $f \in L^1(h)$ and $\tilde{P}_t f(x_n) \geq ct^{-d/\alpha}$. Moreover, by (5.6) and (2.4), $\Psi_t(x_n) \leq ct^{\delta/\alpha} (1 + t^{-\delta/\alpha} |x_n|^{-\delta}) t |x_n|^{-d-\alpha}$. Hence for every $t > 1$ we actually have,

$$\|u(t, \cdot) - A\Psi_t\|_{\infty,h} = \sup_{x \in \mathbb{R}^d} |\tilde{P}_t f(x) - A\Psi_t(x)|/h(x) \geq c_t \sup_{n \in \mathbb{N}} (|x_n|^\delta - |x_n|^{\delta-d-\alpha}) = \infty.$$

6. THE POTENTIAL OF THE SELF-SIMILAR SOLUTION

In this section we assume that $0 \leq \kappa \leq \kappa^*$, except that on several occasions we explicitly exclude the critical case $\kappa = \kappa^*$. As usual, κ and δ are related by (2.9).

If $\kappa < \kappa^*$ then, by (5.3) and (5.6), there is $c \in (0, \infty)$, such that

$$c \int_0^\infty \Psi_s(x) \, ds = |x|^{\delta+\alpha-d}, \quad x \in \mathbb{R}^d.$$

Furthermore,

$$\begin{aligned}
 \int_{\mathbb{R}^d} \tilde{p}_t(x, y) |y|^{\delta+\alpha-d} \, dy &= c \int_0^\infty \int_{\mathbb{R}^d} \Psi_s(y) \tilde{p}_t(x, y) \, dy \, ds = c \int_0^\infty \Psi_{t+s}(x) \, ds \\
 &= c \int_0^\infty \Psi_s(x) \, ds - c \int_0^t \Psi_s(x) \, ds = |x|^{\delta+\alpha-d} - c \int_0^t \Psi_s(x) \, ds.
 \end{aligned}$$

Hence, for $t > 0$ and $y \in \mathbb{R}^d \setminus \{0\}$,

$$c \int_0^t \Psi_s(x) \, ds = |x|^{\delta+\alpha-d} - \int_{\mathbb{R}^d} \tilde{p}_t(x, y) |y|^{\delta+\alpha-d} \, dy.$$

The main goal of this section is to calculate the constant c , and derive a similar formula for $\kappa = \kappa^*$ (see Corollary 6.3).

Lemma 6.1. *For $t > 0$ and $x \in \mathbb{R}^d$ we have*

$$\Psi_t(x) = \lim_{\beta \rightarrow 0^+} \frac{\Gamma(d/2)}{2\pi^{d/2}} \int_{\mathbb{R}^d} \beta |z|^{\beta+\delta-d} \tilde{p}(t, x, z) dz.$$

Proof. Let $x \in \mathbb{R}_0^d$. For $z \in \mathbb{R}_0^d$ and $\beta > 0$, put $f_\beta(z) = \frac{\Gamma(d/2)}{2\pi^{d/2}} \beta |z|^{\beta-d}$. Let $\varepsilon > 0$ and $\beta \rightarrow 0$. We have

$$\beta \int_{B(0, \varepsilon)} |z|^{\beta-d} dz = d|B(0, 1)|\beta \int_0^\varepsilon r^{\beta-d} r^{d-1} dr = d|B(0, 1)|\varepsilon^\beta \rightarrow \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

Furthermore, by (2.12) and the dominated convergence theorem for every $x \in \mathbb{R}_0^d$ we get

$$\lim_{\beta \rightarrow 0^+} \beta \int_{|z| > \varepsilon} \frac{\tilde{p}(t, x, z)}{h(z)} |z|^{\beta-d} dz = 0.$$

Then, since $z \mapsto \tilde{p}(t, x, z)/h(z)$ has a continuous extension to \mathbb{R}^d with the value $\Psi_t(x)$ at the origin, we get

$$\Psi_t(x) = \lim_{\beta \rightarrow 0^+} \int_{\mathbb{R}^d} \frac{\tilde{p}(t, x, z)}{h(z)} f_\beta(z) dz.$$

The result follows (in the case of $x = 0$, the statement is trivial). \square

Lemma 6.2. *For $t > 0$ and $x \in \mathbb{R}_0^d$,*

$$\int_0^t \Psi_s(x) ds = \lim_{\beta \rightarrow 0^+} \frac{\Gamma(d/2)}{2\pi^{d/2}} \frac{\beta}{\kappa_{\delta+\beta} - \kappa_\delta} \left(|x|^{-(d-\delta-\alpha-\beta)} - \int_{\mathbb{R}^d} \tilde{p}(t, x, z) |z|^{-(d-\delta-\alpha-\beta)} dz \right).$$

Proof. Fix $x \in \mathbb{R}_0^d$ and $t > 0$. By (2.4), there is a constant $c = c(d, \alpha)$ such that

$$\begin{aligned} \int_{\mathbb{R}^d} p(s, x, z) |z|^{\gamma-d} dz &\leq \int_{|z| > |x|/2} p(s, x, z) (|x|/2)^{\gamma-d} dz + c \int_{|z| \leq |x|/2} \frac{s}{|x|^{d+\alpha}} |z|^{\gamma-d} dz \\ &\leq (|x|/2)^{\gamma-d} + \frac{c}{\gamma} t |x|^{\gamma-d-\alpha} \leq c \frac{t+1}{\gamma} (1 + |x|^{-d-\alpha}), \quad s \in (0, t), \gamma \in (0, d). \end{aligned}$$

Then, by (2.15), there is $c' = c'(d, \alpha, t)$ such that

$$\begin{aligned} \int_{\mathbb{R}^d} \beta \tilde{p}(s, x, z) |z|^{\beta+\delta-d} dz &\leq c' H(x) \beta \int_{\mathbb{R}^d} p(s, x, z) (|z|^{\beta+\delta-d} + |z|^{\beta-d}) dz \\ &\leq c' H(x) (1 + |x|^{-d-\alpha}), \quad \beta \in (0, d - \delta). \end{aligned}$$

Therefore, by Lemma 6.1, the dominated convergence theorem and [6, Theorem 3.1],

$$\begin{aligned} \int_0^t \Psi_s(x) ds &= \lim_{\beta \rightarrow 0^+} \frac{\Gamma(d/2)}{2\pi^{d/2}} \int_0^t \int_{\mathbb{R}^d} \beta \tilde{p}(s, x, z) |z|^{\beta+\delta-d} dz ds \\ (6.1) \quad &= \lim_{\beta \rightarrow 0^+} \frac{\Gamma(d/2)}{2\pi^{d/2}} \frac{\beta}{\kappa_{d-\delta-\alpha-\beta} - \kappa_\delta} \left(|x|^{-(d-\delta-\alpha-\beta)} - \int_{\mathbb{R}^d} \tilde{p}(t, x, z) |z|^{-(d-\delta-\alpha-\beta)} dz \right). \end{aligned}$$

The result follows from the symmetry of κ . \square

Corollary 6.3. *For $0 \leq \delta < \frac{d-\alpha}{2}$, we have*

$$(6.2) \quad \int_0^t \Psi_s(x) ds = \frac{\Gamma(d/2)}{2\pi^{d/2} \kappa'_\delta} \left(|x|^{-(d-\delta-\alpha)} - \int_{\mathbb{R}^d} \tilde{p}(t, x, z) |z|^{-(d-\delta-\alpha)} dz \right), \quad t > 0, x \in \mathbb{R}_0^d.$$

For $\delta = \frac{d-\alpha}{2}$,

$$(6.3) \quad \int_0^t \Psi_s(x) ds = \frac{\Gamma(d/2)}{\pi^{d/2} \kappa''_\delta} \left(|x|^{-\delta} \ln |x| - \int_{\mathbb{R}^d} \tilde{p}(t, z, x) |z|^{-\delta} \ln |z| dz \right), \quad t > 0, x \in \mathbb{R}_0^d.$$

Here, κ'_δ and κ''_δ are the first and the second derivatives of κ_δ , respectively.

Proof. For $0 \leq \delta < \frac{d-\alpha}{2}$ the statement follows directly from Lemma 6.2, (2.12) and the dominated convergence theorem. For $\delta = \frac{d-\alpha}{2}$, we have $\kappa'_\delta = 0$ and

$$(6.4) \quad \int_{\mathbb{R}^d} \tilde{p}(t, x, z) |z|^{\delta+\alpha-d} dz = \int_{\mathbb{R}^d} \tilde{p}(t, x, z) |z|^{-\delta} dz = |x|^{-\delta} = |x|^{\delta+\alpha-d},$$

see (2.21), so the difference in the parentheses on the right hand side of (6.1) tends to 0. So, as $\beta \rightarrow 0$, we use (6.4), (2.12) and the dominated convergence theorem, obtaining

$$\begin{aligned} \int_0^t \Psi_s(x) ds &= \lim_{\beta \rightarrow 0^+} \frac{\Gamma(d/2)}{2\pi^{d/2}} \frac{\beta^2}{\kappa_{\delta+\beta} - \kappa^*} \left(\frac{|x|^{\beta-\delta} - |x|^{-\delta}}{\beta} - \int_{\mathbb{R}^d} \tilde{p}(t, x, z) \frac{|z|^{\beta-\delta} - |z|^{-\delta}}{\beta} dz \right) \\ &= \frac{\Gamma(d/2)}{\pi^{d/2} \kappa''_\delta} \left(|x|^{-\delta} \ln |x| - \int_{\mathbb{R}^d} \tilde{p}(t, x, z) |z|^{-\delta} \ln |z| dz \right). \end{aligned}$$

□

Theorem 6.4. For $\delta \in [0, \frac{d-\alpha}{2})$ and $x \in \mathbb{R}^d$ we have

$$(6.5) \quad \int_0^\infty \Psi_s(x) ds = \frac{\Gamma(d/2)}{2\pi^{d/2} \kappa'_\delta} |x|^{\delta+\alpha-d},$$

and

$$(6.6) \quad \int_t^\infty \Psi_s(x) ds = \frac{\Gamma(d/2)}{2\pi^{d/2} \kappa'_\delta} \int_{\mathbb{R}^d} \tilde{p}(t, x, z) |z|^{\delta+\alpha-d} dz, \quad t > 0.$$

For $\delta = (d - \alpha)/2$,

$$(6.7) \quad \int_t^\infty \Psi_s(x) ds = \infty, \quad x \in \mathbb{R}^d, \quad t \geq 0.$$

Proof. We shall prove (6.5) by letting $t \rightarrow \infty$ in (6.2). To this end let $x \in \mathbb{R}_0^d$ and $T > 1$. By (6.2), $\int_{\mathbb{R}^d} \tilde{p}(t, x, z) |z|^{\delta+\alpha-d} dz$ is finite and decreases as t increases to ∞ . Hence, for every $\varepsilon > 0$ there is $R > 0$ such that for every $t > T$,

$$(6.8) \quad \int_{B(0,R)^c} \tilde{p}(t, x, z) |z|^{\delta+\alpha-d} dz \leq \int_{B(0,R)^c} \tilde{p}(T, x, z) |z|^{\delta+\alpha-d} dz < \varepsilon.$$

By (2.12), for $t > T$ we get

$$\begin{aligned} |z|^{\delta+\alpha-d} \tilde{p}(t, x, z) &\leq c |z|^{\delta+\alpha-d} (1 + t^{\delta/\alpha} |z|^{-\delta}) (1 + t^{\delta/\alpha} |x|^{-\delta}) t^{-d/\alpha} \\ &\leq c |z|^{\delta+\alpha-d} (1 + |z|^{-\delta}) (1 + |x|^{-\delta}) T^{-(d-2\delta)/\alpha}. \end{aligned}$$

By the dominated convergence theorem,

$$\lim_{t \rightarrow \infty} \int_{B(0,R)} \tilde{p}(t, x, z) |z|^{\delta+\alpha-d} dz = 0.$$

This and (6.8) yield (6.5). Then (6.6) follows by (6.2) and (6.5). If $x = 0$, then we trivially have infinity on both sides of (6.5) and (6.6). Finally, (6.7) follows from (5.6). □

Remark 6.5. We note that the function

$$(6.9) \quad \mu_t(x) := \int_0^t \Psi_s(x) ds$$

is self-similar, too. Namely, by (5.3) and changing variables $s = tu$ in (6.9), we get

$$\mu_t(x) = t^{(\alpha+\delta-d)/\alpha} \mu_1(t^{-1/\alpha} x), \quad t > 0, x \in \mathbb{R}^d.$$

Furthermore, μ satisfies the Duhamel formula,

$$\mu_t(x) = \int_0^t \int_{\mathbb{R}^d} \mu_s(z) q(z) p(t-s, x, z) dz ds.$$

Indeed, by (5.5) and Fubini-Tonelli,

$$\begin{aligned}
 \int_0^t \int_{\mathbb{R}^d} p(t-s, x, z) q(z) \mu_s(z) \, dz \, ds &= \int_0^t \int_{\mathbb{R}^d} p(t-s, x, z) q(z) \int_0^s \Psi_{s-r}(z) \, dr \, dz \, ds \\
 &= \int_0^t \int_r^t \int_{\mathbb{R}^d} p(t-s, x, z) q(z) \Psi_{s-r}(z) \, ds \, dz \, dr \\
 &= \int_0^t \int_0^{t-r} \int_{\mathbb{R}^d} p(t-r-s, x, z) q(z) \Psi_s(z) \, ds \, dz \, dr \\
 &= \int_0^t \Psi_{t-r}(z) \, dr = \mu_t(x).
 \end{aligned}$$

Of course, it is Ψ_t , not μ_t , that captures the large time asymptotics for the solutions of the equation (1.9) in Theorem 1.2. Interestingly, it seems feasible, if not easy, to construct μ_t directly as $\lim_{y \rightarrow 0} \int_0^t \tilde{p}(s, x, y)/h(y) \, ds$ and then attempt to define $\Psi_t(x) = \partial \mu_t(x)/\partial t$.

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