On the Hausdorff Measure of $\mathbb{R}^n$ with the Euclidean Topology

Marco Bagnara[*], Luca Gennaioli[†], Giacomo Maria Leccese [‡], Eliseo Luongo [§]

Abstract

In this paper we answer a question raised by David H. Fremlin about the Hausdorff measure of $\mathbb{R}^2$ with respect to a distance inducing the Euclidean topology. In particular we prove that the Hausdorff n-dimensional measure of $\mathbb{R}^n$ is never 0 when considering a distance inducing the Euclidean topology. Finally, we show via counterexamples that the previous result does not hold in general if we remove the assumption on the topology.

Keywords: Hausdorff measure; Euclidean topology
MSC: 28A75, 28A78

1 Introduction

The aim of this paper is to answer an open question stated by D. H. Fremlin in his famous book [3], which can be found moreover on
https://www1.essex.ac.uk/maths/people/fremlin/answer.pdf:

$$\begin{aligned}&\textit{let us consider a metric } \rho \textit{ on } \mathbb{R}^2 \textit{ inducing the Euclidean topology,}\\ &\textit{is it possible that } \mathcal{H}^2_\rho\left(\mathbb{R}^2\right) = 0?\end{aligned} \tag{Q}$$

By $\mathcal{H}^n_\rho$ we denote the n-dimensional Hausdorff measure according to Definition 1 below. We give an answer to this problem in full generality, since our proof is valid in $\mathbb{R}^n$, $\forall n \geq 1$, showing that such a behaviour cannot happen. On the other hand, we will show in Remark 9 that, when the metric does not induce the usual Euclidean topology, counterexamples can be found.

Before stating our main theorem in Section 2, we recall in this introductory section some classical tools for convenience of the reader (see [2], [4] for further details).

Definition 1 (Hausdorff measure) *Let (X, d) be a metric space. We define the n-dimensional Hausdorff outer measure of $A \in \mathcal{P}(X)$ as*

$$\mathcal{H}^n_d(A) := \sup_{\delta>0} \mathcal{H}^n_{\delta,d}(A), \qquad \textit{with} \tag{1}$$

$$\mathcal{H}^n_{\delta,d}(A) := \inf\Big\{\sum_{i\in I} \operatorname{diam}(A_i)^n \ : \ A \subseteq \cup_{i\in I} A_i,\ \operatorname{diam}(A_i) \leq \delta\Big\}, \tag{2}$$

where $diam(U) = \sup_{x,y\in U} d(x, y)$ and I is an at most countable collection of indices.

Remark 2 *The usual definition of Hausdorff measure is given scaling the result by a dimensional constant that, for instance, in the Euclidean case is equal to $2^{-n}\omega_n$, where ω_n is the volume of the unit n-ball. We opted to overlook the constant in order to simplify the notation. Clearly Theorem 7 is not affected by this choice.*

To prove our result we will exploit the following well-known theorem.

Theorem 3 (Dini) *Let (K, d) be a compact metric space. Let $f_n : K \to \mathbb{R}$ be continuous functions such that*

$$f_n \leq f_{n+1} \qquad \forall n \in \mathbb{N} \tag{3}$$

and assume that

$$f(x) = \lim_{n\to+\infty} f_n(x) \qquad \forall x \in K, \tag{4}$$

exists and the function $f : K \to \mathbb{R}$ is also continuous. Then $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on K.

[*]**Marco Bagnara**: Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy, e-mail: marco.bagnara@sns.it
[†]**Corresponding Author: Luca Gennaioli**: SISSA, Via Bonomea, 265, 34136 Trieste, Italy, e-mail: luca.gennaioli@sissa.it
[‡]**Giacomo Maria Leccese**: SISSA, Via Bonomea, 265, 34136 Trieste, Italy, e-mail: giacomomaria.leccese@sissa.it
[§]**Eliseo Luongo**: Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy, e-mail: eliseo.luongo@sns.it

Moreover, we briefly recall the definition and some properties of the Brouwer Degree. See for instance [1] for a complete treatment of this topic.

Theorem 4 (Brouwer Degree) *There exists only one function, called Brouwer Degree and denoted by* deg*, from the set of couples (D, f), where $D \subset \mathbb{R}^n$ is open and bounded and $f : \bar{D} \to \mathbb{R}^n$ is continuous with $0 \notin f(\partial D)$, into the set $\mathbb{Z}$, which satisfies the following three properties:*

- *(Normalization)* $\deg[\mathrm{id}, D] = 1$ *if* $0 \in D$.
- *(Additivity)* $\deg[f, D] = \deg[f, D_1] + \deg[f, D_2]$ *if D_1 and D_2 are disjoint open subsets of D such that* $0 \notin f(\bar{D} \setminus (D_1 \cup D_2))$.
- *(Homotopy invariance) If $F \in C([0, 1] \times \bar{D}, \mathbb{R}^n)$ and $0 \notin F([0, 1] \times \partial D)$, then $\deg[F(t, \cdot), D]$ is independent of $t \in [0, 1]$.*

Definition 5 *If $D \subset \mathbb{R}^n$ is open and bounded, $f \in C(\bar{D}, \mathbb{R}^n)$ and $z \notin f(\partial D)$, the Brouwer degree $\deg[f, D, z]$ is defined by $\deg[f, D, z] = \deg[f(\cdot) - z, D]$.*

Proposition 6 *If $z \notin f(\bar{D})$, then $\deg[f, D, z] = 0$.*
Equivalently, if $\deg[f, D, z] \neq 0$, there exists at least one $x \in D$ such that $f(x) = z$.

2 Main results

We are now in the position to state our main theorem.

Theorem 7 *Let $(\mathbb{R}^n, \rho)$ be a metric space with ρ inducing the Euclidean topology, then $\mathcal{H}^n_\rho(\mathbb{R}^n) > 0$.*

Proof. Assume by contradiction that there exists a distance ρ in $\mathbb{R}^n$ such that $\mathcal{H}^n_\rho(\mathbb{R}^n) = 0$. We denote by $\mathbb{B}(0, 1)$ the closed unit ball with respect to Euclidean metric and we consider the identity map

$$\mathrm{id} : (\mathbb{B}(0, 1), \rho) \longrightarrow (\mathbb{B}(0, 1), \mathrm{d}_{\mathrm{eucl}}). \tag{5}$$

Such a map is an homeomorphism by assumption, but it carries no metric information a priori. Let us write

$$\mathrm{id}(x) = \big(\pi_1(x), \dots, \pi_n(x)\big) \tag{6}$$

and define

$$\pi_i^\varepsilon(x) := \min_{z \in \mathbb{B}(0,1)} \left[\pi_i(z) + \frac{1}{\varepsilon}\rho(x, z)\right] \qquad \forall i = 1, ..., n \quad \forall x \in \mathbb{B}(0, 1), \tag{7}$$

where we are using that $\mathbb{B}(0, 1)$ is compact also for the metric ρ. The latter functions are Lipschitz, since they are the infimum of a family of equi-Lipschitz functions, more precisely

$$|\pi_i^\varepsilon(x) - \pi_i^\varepsilon(y)| \leq \frac{1}{\varepsilon}\rho(x, y) \quad \forall x, y \in \mathbb{B}(0, 1). \tag{8}$$

Such functions converge pointwise to the components of the identity in the compact ball $\mathbb{B}(0, 1)$ as $\varepsilon \to 0$. Indeed, consider a sequence $(z_\varepsilon)_{\varepsilon>0} \subseteq \mathbb{B}(0, 1)$ such that

$$\pi_i^\varepsilon(x) = \pi_i(z_\varepsilon) + \frac{1}{\varepsilon}\rho(x, z_\varepsilon). \tag{9}$$

This sequence is bounded and by compactness it admits a convergent subsequence. Due to equation (9) and the bound

$$1 \geq \pi_i \geq \pi_i^\varepsilon \geq -1, \tag{10}$$

it follows that $\lim_{\varepsilon\to 0} \rho(z_\varepsilon, x) = 0$, which means that the whole sequence converges to x, leading to the desired pointwise convergence. Now, since we have $\pi_i^\varepsilon(x) \geq \pi_i^{\varepsilon+\gamma}(x)$ for every $\gamma, \varepsilon > 0$ and $\forall x \in \mathbb{B}(0, 1)$, by Dini's theorem π_i^ε converges uniformly to π_i on $\mathbb{B}(0, 1)$ for every $i = 1, ..., n$. Summing up we have obtained a sequence

$$F^\varepsilon = (\pi_1^\varepsilon, ..., \pi_n^\varepsilon) : (\mathbb{B}(0, 1), \rho) \longrightarrow (\mathbb{R}^n, \mathrm{d}_{\mathrm{eucl}}) \tag{11}$$

such that

$$\mathrm{d}_{\mathrm{eucl}}\big(F^\varepsilon(x), F^\varepsilon(y)\big) \leq C_\varepsilon \rho(x, y) \quad \forall x, y \in \mathbb{B}(0, 1) \tag{12}$$

with $C_\varepsilon > 0$ and such that it converges uniformly to the identity in $\mathbb{B}(0, 1)$. The following claim is of crucial importance.
Claim: there exists $\varepsilon > 0$ such that $F^\varepsilon(\mathbb{B}(0, 1))$ has non-empty interior.
Fix $\hat{\varepsilon} > 0$ such that

$$\sup_{x \in \mathbb{B}(0,1)} \mathrm{d}_{\mathrm{eucl}}(F^\varepsilon(x), x) \leq \frac{1}{2} \tag{13}$$

for every $\varepsilon \in [0, \hat{\varepsilon}]$ and consider the function

$$F : [0, \hat{\varepsilon}] \times \mathbb{B}(0,1) \to \mathbb{R}^n \tag{14}$$

defined by the relation $F(\varepsilon, \cdot) = F^\varepsilon$ for $\varepsilon > 0$ and $F(0, \cdot) = id$. We prove that the function F is a continuous function, or in other words that F is an homotopy between id and $F^{\hat{\varepsilon}}$. First we observe that for every $\varepsilon_n \nearrow \varepsilon$ in $(0, \hat{\varepsilon}]$, given z such that

$$F_i^\varepsilon(x) = \pi_i(z) + \frac{1}{\varepsilon}\rho(x, z), \tag{15}$$

then

$$\pi_i(z) + \frac{1}{\varepsilon_n}\rho(x, z) \geq F_i^{\varepsilon_n}(x) \geq F_i^\varepsilon(x) \tag{16}$$

and taking the limit for $n \to +\infty$, we obtain that $\lim_{n\to+\infty} F_i^{\varepsilon_n}(x) = F_i^\varepsilon(x)$. Also, for every $\varepsilon_n \searrow \varepsilon$ in $(0, \hat{\varepsilon}]$ and every $x \in \mathbb{B}(0,1)$, we have

$$1 \geq \pi_i(x) \geq F_i^\varepsilon(x) \geq F_i^{\varepsilon_n}(x) \geq -1, \tag{17}$$

therefore given z^n such that

$$F^{\varepsilon_n}(x) = \pi_i(z_n) + \frac{1}{\varepsilon_n}\rho(z_n, x), \tag{18}$$

up to a subsequence, we have that $z_n \to \hat{z}$ and, by equation (17), $\hat{z}$ realizes the minimum for $F_i^\varepsilon(x)$, thus we have $F_i^{\varepsilon_n}(x) \to F_i^\varepsilon(x)$. Therefore, for a generic sequence $\varepsilon_n \to \varepsilon$ in $(0, \hat{\varepsilon}]$ and every $x \in \mathbb{B}(0,1)$ fixed, up to subsequence we can assume $\varepsilon_n \searrow \varepsilon$ or $\varepsilon_n \nearrow \varepsilon$, hence we have that $F_i^{\varepsilon_n}(x) \to F_i^\varepsilon(x)$. In general, given $\varepsilon_n \to \varepsilon$ in $(0, \hat{\varepsilon}]$ and $x_n \to x$ in $\mathbb{B}(0,1)$, consider z_n satisfying equation (18) as before and observe that

$$\begin{aligned} |F_i^{\varepsilon_n}(x_n) - F_i^\varepsilon(x)| &\leq |F_i^{\varepsilon_n}(x_n) - F_i^{\varepsilon_n}(x)| + |F_i^{\varepsilon_n}(x) - F_i^\varepsilon(x)| \\ &\leq \frac{\rho(x_n, x)}{\varepsilon_n} + o(1) = o(1) \qquad \text{for } n \to +\infty. \end{aligned} \tag{19}$$

Finally, consider the last case when $\varepsilon_n \to 0$ and $x_n \to x$ in $\mathbb{B}(0,1)$, then

$$|F_i^{\varepsilon_n}(x_n) - \pi_i(x)| \leq |F_i^{\varepsilon_n}(x_n) - \pi_i(x_n)| + |\pi_i(x_n) - \pi_i(x)| \leq o(1), \tag{20}$$

because of uniform convergence.
Now we consider the topological degree of the function $F^{\hat{\varepsilon}}$ with respect to the set $\mathbb{B}(0,1)$ and any point of $B(0, \frac{1}{2})$, the open ball of radius $\frac{1}{2}$. We recall that the map $F^{\hat{\varepsilon}}$ is homotopy equivalent to the map id, and observe that equation (13) implies that for any $y \in B(0, \frac{1}{2})$, we have $y \notin F^\varepsilon(\mathbb{B}(0,1) \setminus B(0,1))$. Therefore, we can apply homotopy invariance obtaining that

$$1 = \deg(\mathrm{id}, \mathbb{B}(0,1), y) = \deg(F^{\hat{\varepsilon}}, \mathbb{B}(0,1), y) \tag{21}$$

for every $y \in B(0, \frac{1}{2})$, hence, by Proposition 6, it follows $B(0, \frac{1}{2}) \subseteq F^{\hat{\varepsilon}}(\mathbb{B}(0,1))$ and this proves our claim.
Since $F^{\hat{\varepsilon}}(\mathbb{B}(0,1))$ contains a non-empty open set and $F^{\hat{\varepsilon}}$ is Lipschitz, we get

$$\mathcal{H}^n_{\mathrm{d_{eucl}}}\left(F^{\hat{\varepsilon}}(\mathbb{B}(0,1))\right) \leq C_\varepsilon^n \mathcal{H}^n_\rho(\mathbb{B}(0,1)) = 0, \tag{22}$$

which is a contradiction since the n-dimensional Hausdorff measure on $\mathbb{R}^n$ with the Euclidean distance gives positive measure to not empty open sets. ■

Remark 8 *The same proof of Theorem 7 can be adapted to prove that any nonempty open set A is such that $\mathcal{H}^n_\rho(A) > 0$.*

Remark 9 *Removing the assumption that ρ induces the Euclidean topology, counterexamples show that $\mathcal{H}^n_\rho(\mathbb{R}^n)$ might vanish. Consider, for instance, the metric space $(\mathcal{C}, d)$, where $\mathcal{C} \subset \mathbb{R}$ is the Cantor set and d denotes the usual one-dimensional Euclidean distance. Having $\mathcal{C}$ the cardinality of the continuum, there exist bijections $g_n : \mathcal{C} \to \mathbb{R}^n$. Then, define on $\mathbb{R}^n$ the metric $\rho(x, y) = d(g_n^{-1}(x), g_n^{-1}(y))$.*
Given any collection $(A_i)_{i\in\mathbb{N}}$ that covers $\mathcal{C}$, follows that $(g_n(A_i))_{i\in\mathbb{N}}$ covers $\mathbb{R}^n$ and $\mathrm{diam}(A_i) = \mathrm{diam}(g_n(A_i))\, \forall i \in \mathbb{N}$. Clearly, also the opposite direction applies. Therefore, we have

$$\mathcal{H}^n_\rho(\mathbb{R}^n) = \mathcal{H}^n_d(\mathcal{C}) = 0 \tag{23}$$

that shows a counterexample.

Remark 10 *Note that, under previous assumptions on ρ, it is not true in general that $\dim_H^\rho(\mathbb{R}^n) = n$. In fact, choosing $\rho(x, y) = d_{eucl}(x, y)^{1/2}$, the distance ρ induces the Euclidean topology, but in this case*

$$\mathcal{H}^s_{d_{eucl}}(A) = \mathcal{H}^{2s}_\rho(A)$$

for all $A \subseteq \mathbb{R}^n$, $s \geq 0$, see for example [2]. For this reason we get that $\dim_H^\rho(\mathbb{R}^n) = 2n$.

Acknowledgements

We would like to thank professor David H. Fremlin for proposing this problem and professor Stefano Bianchini for the inspiring and useful discussions we had about it.

Conflict of interest

Authors state no conflict of interest.

References

[1] G. Dinca and J. Mawhin. *Brouwer Degree*. Springer, 2021.

[2] H. Federer. *Geometric measure theory*. Springer, 2014.

[3] D. H. Fremlin. *Measure theory*, volume 4. Torres Fremlin, 2000.

[4] J. Jost. *Postmodern analysis*. Springer Science & Business Media, 2005.