ALGEBRAS OF GENERALIZED SINGULAR INTEGRAL OPERATORS WITH CAUCHY KERNEL

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Abstract. For bounded Lebesgue measurable functions f, g, ϕ and ψ on the unit circle, $P_+fP_+ + P_-gP_+ + P_+\phi P_- + P_-\psi P_-$ is called a generalized singular integral operator (GSIO) on $L^2(\mathbb{T})$, where P_+ is the Riesz projection, $P_- = I - P_+$. In this paper, we relate GSIOs to a number of operators, including Cauchy singular integral operator, (dual) truncated Toeplitz operator, Foguel-Hankel operator, multiplication operator, Toeplitz plus Hankel operator etc. We establish the short exact sequences associated of the C^*-algebras generated by GSIOs with bounded or quasi-continuous symbols. As a consequence we obtain the spectra of various classes of GSIOs, the spectral inclusion theorem and comput the Fredholm index of GSIOs. Moreover, we gave the necessary and sufficient conditions for invertibility(Fredholmness) of GSIOs via Winer-Hopf factorization.

1. Introduction

Let $\mathbb{D} = \{\xi \in \mathbb{C} : |\xi| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $\mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}$ be its boundary. Riemann-Hilbert boundary problem [25] on the unit circle can be reformulated as follows.

Given functions α, β, h on $\mathbb{T}$, find two analytic functions $f_+ \in \mathrm{Hol}(\mathbb{D})$ and $f_- \in \mathrm{Hol}(\mathbb{C} \setminus \overline{\mathbb{D}})(f_-(\infty) = 0)$ such that

$$\alpha f_+ + \beta f_- = h \tag{1.1}$$

on $\mathbb{T}$.

H^2 denotes the classical Hardy space of the open unit disk $\mathbb{D}$, we let $L^2 = L^2(\mathbb{T}), L^\infty = L^\infty(\mathbb{T})$ denote the usual Lebesgue spaces on the unit circle [12]. P_+ is the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2, P_- = I - P_+$. Suppose that $h \in L^2(\mathbb{T}), f_+ \in H^2$ and $f_- \in L^2(\mathbb{T}) \ominus H^2 = \bar{z}\overline{H^2}$. Put $f = f_+ + f_-$, the equation (1.1) becomes

$$S_{\alpha,\beta} f = h, \quad \text{where } S_{\alpha,\beta} = \alpha P_+ + \beta P_-.$$

$S_{\alpha,\beta}$ is called the singular integral operator with Cauchy kernel on $L^2(\mathbb{T})$, and

$$\left(S_{\alpha,\beta} f\right)(z) = \frac{\alpha(z) + \beta(z)}{2} f(z) + \frac{\alpha(z) - \beta(z)}{2} \frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(\xi)}{\xi - z} d\xi.$$

Riemann-Hilbert boundary problem is considered solved if one has found conditions for the operator $S_{\alpha,\beta}$ to be Fredholm or invertible. Most results about these operators can be found in [14, 15]. We are interested in the algebra of

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singular integral operator, but the adjoint of $R_{\alpha,\beta}$ is no longer a singular integral operator. Naturally, one can define the generalized singular integral operator.

Given a linear space X, we denote by X_N the linear space of all $N-$dimensional vectors with components from X and let $X_{N\times N}$ denote the linear space of $N\times N$ matrices with entries from X.

Definition 1.1. If $H=\left(\begin{smallmatrix} f & \phi \\ g & \psi \end{smallmatrix}\right)\in L^2_{2\times 2}(\mathbb{T})$, the generalized singular integral operator (GSIO) with symbol H is the operator R_H is defined by

$$R_H x = P_+ f P_+ x + P_- g P_+ x + P_+ \phi P_- x + P_- \psi P_- x.$$

for each $x \in L^2(\mathbb{T})$.

The significance of GSIOs comes from the following special cases.

(1) Multiplication operator on $L^2(\mathbb{T})$: if $f=g=\phi=\psi$, then R_H is the multiplication operator on $L^2(\mathbb{T})$.
(2) Hilbert transform: if $f=g=-\phi=-\psi=1$, then $R_{\left(\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right)} - 1\otimes 1 = P_+ - P_- - 1\otimes 1$ is the Hilbert transform $\mathbb{H}$ [12, Ch III].
(3) Singular integral operator: if $f=g=\alpha, \phi=\psi=\beta$, then $R_{\left(\begin{smallmatrix} \alpha & \beta \\ \alpha & \beta \end{smallmatrix}\right)} = S_{\alpha,\beta}$ is the singular integral operator. T. Nakazi and T. Yamamoto [21, 19, 20, 22, 23, 24] have study the boundedness and normality of $S_{\alpha,\beta}$, and calculate its norm, C. Gu [16] have study the algebraic properties of $S_{\alpha,\beta}$.
(4) Toeplitz plus Hankel operators: if $H=\left(\begin{smallmatrix} f & 0 \\ g & 0 \end{smallmatrix}\right)$, then $(I\oplus J)R_H|_{H^2} = T_f + \Gamma_g$, where $Jx(z)=\bar{z}x(\bar{z})$ for $x\in L^2(\mathbb{T})$.
(5) Foguel-Hankel operators: if $\phi\in L^\infty$ and $H=\left(\begin{smallmatrix} \bar{z} & \phi \\ 0 & \bar{z} \end{smallmatrix}\right)$, then R_H and Foguel-Hankel operator $\left(\begin{smallmatrix} T_z^* & X \\ 0 & T_z \end{smallmatrix}\right)$ are unitarily equivalent(see Section 2). Foguel-Hankel operators closely related to Halmos' problem[17](whether or not any polynomially bounded operator on a Hilbert space H is similar to a contraction). J. Bourgain [4] has shown that R_H is similar to a contraction if $\phi^{'}\in BMOA$, A. Aleksandrov and V. Peller [1] have shown that if R_H is polynomially bounded then $\phi^{'}\in BMOA$. G. Pisier [27] and K. Davidson and V. Paulsen[7] give a negative answer to Halmos' problem via vector-Foguel-Hankel operators.
(6) (Dual) Truncated Toeplitz operators: let$u$ is an inner function, suppose $f\in L^\infty(\mathbb{T})$ and $H=\left(\begin{smallmatrix} f & u\bar{f} \\ uf & f \end{smallmatrix}\right)$, then R_H is unitary equivalent to the dual truncated Toeplitz operator D_f[8, 28, 29], furthermore, R_H is equivalent after extension to truncated Toeplitz operator for invertible symbol [6, Theorem 6.1].

Given a closed unital subalgebra $A\subset L^\infty(\mathbb{T})$, the C^*-algebra $\mathfrak{R}_A$ is defined by

$$\mathfrak{R}_A = \operatorname{clos}\left\{ \sum_{i=1}^n \prod_{j=1}^m R_{H_{ij}} \middle| H_{ij}\in A_{2\times 2}\right\}.$$

In fact, $\mathfrak{R}_A$ equals the C^*-algebra generated by $\{R_{\alpha,\beta}\big|\alpha,\beta\in A\}$ and $\{R^*_{\phi,\psi}\big|\phi,\psi\in A\}$. In this paper, we explore the structure of the C^*-algebra $\mathfrak{R}_{L^\infty(\mathbb{T})}$.

The earliest result on the C^*-algebra $\mathfrak{R}_{PC(\mathbb{T})}$ due to Gokhberg and Krupnik[13], where $PC(\mathbb{T})$ denote the algebra of all piecewise continuous and left continuous functions on $\mathbb{T}$. They proved that the sequence

$$0 \longrightarrow \mathfrak{C}(L^2(\mathbb{T})) \longrightarrow \mathfrak{R}_{PC(\mathbb{T})} \longrightarrow \mathscr{S} \longrightarrow 0.$$

is exact. The algebra $\mathscr{S}$ consist of matrix-valued functions of second order $\mathrm{M(t,\mu)} = (\alpha_{\mathrm{jk}}(\mathrm{t},\mu))^2_{j,\mathrm{k}}$ with the following properties:

- $\alpha_{11}(t,\mu), \alpha_{22}(t, 1-\mu), \alpha_{12}(t,\mu), \alpha_{21}(t,\mu) \in C(\mathbb{T}\times[0,1])$,
- $\alpha_{12}(t,0) = \alpha_{21}(t,0) = \alpha_{12}(t,1) = \alpha_{21}(t,1) = 0 \quad \forall t \in \mathbb{T}$.

This paper is organized as follows. In section 2, we presents some preliminaries and basic properties of GISO. In section 3 and section 4, we establish the short exact sequences associated of the C^*-algebras generated by GISO with bounded symbols or quasicontinuous symbols, and obtain the essential spectrum of GISO and index forumla. In section 5, we establish vector we obtain the necessary and sufficient conditions for invertibility and Fredholmness of GSIO via equivalence after extension and Winer-Hopf factorization. In the last section, corresponding results apply for the spectrum of singular integral operators, Foguel-Hankel operators and dual truncated Toeplitz operators.

2. Preliminaries

The generalized singular integral operator $R_{\begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix}}$ can be expressed as an operator matrix with respect to the decomposition $L^2(\mathbb{T}) = H^2 \oplus \bar{z}\overline{H^2}$, the result is of the form

$$\begin{pmatrix} T_f & H^*_{\bar{\phi}} \\ H_g & \tilde{T}_\psi \end{pmatrix}, \tag{2.1}$$

where T_f denote the Toeplitz operator on H^2 such that

$$T_f x = P_+(fx), \quad x \in H^2;$$

H_g denote the Hankel operator on H^2 such that

$$H_g x = P_-(gx), \quad x \in H^2;$$

$H^*_{\bar{\phi}}$ denote the adjoint of Hankel operator such that

$$H^*_{\bar{\beta}} y = P_+(\phi y), \quad y \in \bar{z}\overline{H^2};$$

$\tilde{T}_\psi$ denote the dual Toeplitz operator on $\bar{z}\overline{H^2}$ such that

$$\tilde{T}_\psi y = P_-(\psi y), \quad y \in \bar{z}\overline{H^2}.$$

Converse, if an operator T on $L^2(\mathbb{T})$ has form (2.1), then T is a GSIO. Moreover, the generalized singular integral operator $R_{\begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix}}$ is unitarily equivalent to an operator matrix on H^2_2. To illustrate this, we need to introduce two useful operators and their properties. For $x \in L^2$, define

$$Vx(z) = \bar{z}\overline{x(z)};$$
$$Jx(z) = \bar{z}x(\bar{z}).$$

Note that V is an anti-unitary operator and U is an unitary operator, and they have the following properties:

(1) $\langle Vx, Vy\rangle = \langle y, x\rangle, \quad \langle Ux, Uy\rangle = \langle x, y\rangle$;
(2) $VM_fV = M_{\bar{f}}, \quad UM_fU = M_{\tilde{f}}, \quad$ where $\tilde{f}(z) = f(\bar{z})$;
(3) $VP_- = P_+V, \quad UP_- = P_+U$;
(4) $VH^2 = \bar{z}\overline{H^2}, \quad UH^2 = \bar{z}\overline{H^2}$;
(5) $Uz^n = Vz^n = \bar{z}^{n+1}$.

Using the operator U, for $g \in L^2$, we can define the Hankel operator on H^2 by

$$\Gamma_g = UH_g.$$

The operator $\begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix} : L^2 = H^2 \oplus \bar{z}\overline{H^2} \to H^2 \oplus H^2$ is unitary. A simple computation gives

$$\begin{aligned} &\begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} T_f & H^*_{\bar{\phi}} \\ H_g & S_\psi \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix} \\ =& \begin{pmatrix} T_f & H^*_{\bar{\phi}}U \\ UH_g & US_\psi U \end{pmatrix} \\ =& \begin{pmatrix} T_f & \Gamma^*_{\bar{\phi}} \\ \Gamma_g & UP_-M_\psi P_-U \end{pmatrix} \\ =& \begin{pmatrix} T_f & \Gamma^*_{\bar{\phi}} \\ \Gamma_g & P_+UM_\psi UP_+ \end{pmatrix} \\ =& \begin{pmatrix} T_f & \Gamma^*_{\bar{\phi}} \\ \Gamma_g & P_+M_{\tilde{\psi}}P_+ \end{pmatrix} \\ =& \begin{pmatrix} T_f & \Gamma^*_{\bar{\phi}} \\ \Gamma_g & T_{\tilde{\psi}} \end{pmatrix} \\ =& \begin{pmatrix} T_f & \Gamma_{\tilde{\phi}} \\ \Gamma_g & T_{\tilde{\psi}} \end{pmatrix}. \end{aligned}$$

This shows that the operator $R_{\left(\begin{smallmatrix} f & \phi \\ g & \psi \end{smallmatrix}\right)} : L^2 \to L^2$ is unitary equivalent to

$$\begin{pmatrix} T_f & \Gamma_{\tilde{\phi}} \\ \Gamma_g & T_{\tilde{\psi}} \end{pmatrix} : H^2 \oplus H^2 \to H^2 \oplus H^2.$$

Therefore, $R(\bar{z}, 0, \phi, \bar{z})$ is unitary equivalent to the Foguel-Hankel operator[4]

$$\begin{pmatrix} T^*_z & \Gamma_{\tilde{\phi}} \\ 0 & T_z \end{pmatrix}.$$

Example 2.1. For $\alpha, \beta \in L^\infty$, the truncated singular integral operator

$$S^u_{\alpha,\beta}x = \alpha P_u x + \beta Q_u x, \quad x \in L^2.$$

It can be write as an operator matrix with respect to the decomposition $L^2(\mathbb{T}) = H^2 \oplus \bar{z}\overline{H^2}$,

$$\begin{aligned}&\begin{pmatrix} T_\alpha + T_{(\beta-\alpha)u}T_{\bar{u}} & H^*_{\bar{\beta}} \\ H_\alpha + H_{(\beta-\alpha)u}T_{\bar{u}} & S_\beta \end{pmatrix}\\ =&\begin{pmatrix} T_\alpha & H^*_{\bar{\beta}} \\ H_\alpha & S_\beta \end{pmatrix} + \begin{pmatrix} T_{(\beta-\alpha)u}T_{\bar{u}} & 0 \\ H_{(\beta-\alpha)u}T_{\bar{u}} & 0 \end{pmatrix}\\ =&\begin{pmatrix} T_\alpha & H^*_{\bar{\beta}} \\ H_\alpha & S_\beta \end{pmatrix} + \begin{pmatrix} T_{(\beta-\alpha)u} & 0 \\ H_{(\beta-\alpha)u} & 0 \end{pmatrix}\begin{pmatrix} T_{\bar{u}} & 0 \\ 0 & I \end{pmatrix}\end{aligned}$$

Example 2.2. Asymmetric dual truncated Toeplitz operator $D^{\theta,\alpha}_\phi : (K^2_\theta)^\perp \to (K^2_\alpha)^\perp$ is unitarily equivalent to some general singular integral operator. Let $h, g \in H^2$, we have

$$\begin{aligned} D^{\theta,\alpha}_\phi(\theta h + \bar{z}\bar{g}) &= (P_- + \alpha P_+ \bar{\alpha})\phi(\theta h + \bar{z}\bar{g}) \\ &= \alpha P_+ \bar{\alpha}\phi\theta P_+ h + P_-\phi\theta P_+ h + \alpha P_+\bar{\alpha}\phi\bar{z}\bar{g} + P_-\phi\bar{z}\bar{g} \end{aligned}$$

or

$$D^{\theta,\alpha}_\phi \begin{pmatrix} M_\theta & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} h \\ \bar{z}\bar{g} \end{pmatrix} = \begin{pmatrix} M_\alpha & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} T_{\bar{\alpha}\theta\phi} & H^*_{\bar{\phi}\alpha} \\ H_{\phi\theta} & S_\phi \end{pmatrix}\begin{pmatrix} h \\ \bar{z}\bar{g} \end{pmatrix}$$

. Hence

$$D^{\theta,\alpha}_\phi = \begin{pmatrix} M_\alpha & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} T_{\bar{\alpha}\theta\phi} & H^*_{\bar{\phi}\alpha} \\ H_{\phi\theta} & S_\phi \end{pmatrix}\begin{pmatrix} M_{\bar{\theta}} & 0 \\ 0 & I \end{pmatrix},$$

where $\begin{pmatrix} M_{\bar{\theta}} & 0 \\ 0 & I \end{pmatrix} : (K^2_\theta)^\perp \to L^2$ and $\begin{pmatrix} M_{\bar{\theta}} & 0 \\ 0 & I \end{pmatrix} : L^2 \to (K^2_\alpha)^\perp$ are unitary.

We begin our study of GSIO by considering some elementary properties.

Proposition 2.3. *Let* $H = \begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} \in L^2_{2\times 2}(\mathbb{T})$.

(1) R_H is bounded on $L^2(\mathbb{T})$ if and only if $f, \psi \in L^\infty$ and $g_-, (\bar{\phi})_- \in \mathrm{BMO}(\mathbb{T})$. Where $\mathrm{BMO}(\mathbb{T}) = L^\infty(\mathbb{T}) + \mathbb{H}L^\infty(\mathbb{T})$.
(2) If R_H is bounded, then R_H is zero if and only if $f = \psi = 0$ and $g, \bar{\phi} \in H^2$.
(3) If R_H is bounded, then R_H is compact if and only if $f = \psi = 0$ and $g, \bar{\phi} \in H^\infty + C(\mathbb{T})$.
(4) If R_H is bounded, then $R(f, g, \phi, \psi)$ is self-adjoint if and only if f and ψ are real valued, and $g - \bar{\phi} \in H^2$.
(5) If R_H is bounded and positive, then f and ψ are positive and $g - \bar{\phi} \in H^2$.
(6) If R_H is bounded, then R_H is complex symmetric operator for V if and only if $f = \psi$, where $Vf(z) = \bar{z}\bar{f}(z)$.

Proof. (1)-(3) Clearly R_H is bounded (resp. zero, compact) if and only if $T_f, H^*_{\bar{\phi}}, H_g$ and S_ψ are bounded (resp. zero, compact). Toeplitz operator T_a is bounded [9, 7.8] (resp., zero, compact[5, p.94]) if and only if its symbol a is bounded(resp., zero, zero), Hankel operator H_a is bounded[26, Theorem 1.3](resp., zero, compact [26, Theorem 5.5]) if and only if $a_- \in BMO$ (resp.,$a \in H^2, a \in H^\infty + C(\mathbb{T})$), the conclusion follows.

(4) By the matrix represention (2.1), we have R_H is self-adjoint if and only if

$$\begin{pmatrix} T_f & H^*_{\bar{\phi}} \\ H_g & S_\psi \end{pmatrix} = \begin{pmatrix} T_{\bar{f}} & H^*_g \\ H_{\bar{\phi}} & S_{\bar{\psi}} \end{pmatrix}$$

if and only if $T_f = T_{\bar{f}}, H_g = H_{\bar{\phi}}$ and $S_\psi = S_{\bar{\psi}}$. $T_f = T_{\bar{f}}$ is equivalent to f is real, $H_g = H_{\bar{\phi}}$ is equivalent to $g - \bar{\phi} \in H^2$, $S_\psi = S_{\bar{\psi}}$ is equivalent to ψ is real.

(4) If R_H is positive, then

$$\begin{aligned} 0 &\leq \langle R_H k_z, k_z \rangle \\ &= \langle (P_+ f P_+ + P_- g P_+ + P_+ \phi P_- + P_- \psi P_-) k_z, k_z \rangle \\ &= \langle (P_+ f P_+ + P_- g P_+ + P_+ \phi P_- + P_- \psi P_-) P_+ k_z, P_+ k_z \rangle \\ &= \langle P_+ (P_+ f P_+ + P_- g P_+ + P_+ \phi P_- + P_- \psi P_-) P_+ k_z, k_z \rangle \\ &= \langle P_+ f P_+ k_z, k_z \rangle \\ &= \langle f k_z, k_z \rangle \\ &= \int_0^{2\pi} f(e^{i\theta}) |k_z(e^{i\theta})|^2 \frac{d\theta}{2\pi}, \end{aligned} \tag{2.2}$$

where $k_z(\omega) = \frac{\sqrt{1-|z|^2}}{1-\bar{z}\omega}$ is the normalized reproducing kernel of H^2. The last equality is the Poisson integral of f, so f is positive almost everywhere on $\mathbb{T}$. Similarly,

$$\begin{aligned} 0 &\leq \langle R(f, g, \phi, \psi) \bar{z} \bar{k}_z, \bar{z} \bar{k}_z \rangle \\ &= \langle (P_+ f P_+ + P_- g P_+ + P_+ \phi P_- + P_- \psi P_-) P_- \bar{z} \bar{k}_z, P_- \bar{z} \bar{k}_z \rangle \\ &= \langle P_- \psi P_- \bar{z} \bar{k}_z, P_- \bar{z} \bar{k}_z \rangle \\ &= \langle \psi k_z, k_z \rangle \\ &= \int_0^{2\pi} \psi(e^{i\theta}) |k_z(e^{i\theta})|^2 \frac{d\theta}{2\pi}, \end{aligned}$$

so ψ is positive almost everywhere on $\mathbb{T}$. Since positive opertor is self-adjoint and (4),$g - \bar{\phi} \in H^2$.

(5) By the definition of complex symmetric operator [11], we have R_H is complex symmetric with the conjugation V if and only if $V R_H V = R^*_H$. Using the properties of V yields

$$\begin{aligned} & V R_H V \\ =& V (P_+ f P_+ + P_- g P_+ + P_+ \phi P_- + P_- \psi P_-) V \\ =& V P_+ f P_+ V + V P_- g P_+ V + V P_+ \phi P_- V + V P_- \psi P_- V \\ =& P_- V f V P_- + P_+ V g V P_- + P_- V \phi V P_+ + P_+ V \psi V P_+ \\ =& P_- \bar{f} P_- + P_+ \bar{g} P_- + P_- \bar{\phi} P_+ + P_+ \bar{\psi} P_+ \\ =& \begin{pmatrix} T_{\bar{\psi}} & H^*_g \\ H_{\bar{\phi}} & \tilde{T}_{\bar{f}} \end{pmatrix}. \end{aligned} \tag{2.3}$$

On the other hand, $R_H^* = \begin{pmatrix} T_{\bar{f}} & H_g^* \\ H_{\bar{\phi}} & S_{\bar{\psi}} \end{pmatrix}$. It follows that $VR_HV = R_H^*$ holds if and only if $T_{\bar{f}} = T_{\bar{\psi}}$ and $S_{\bar{f}} = S_{\bar{\psi}}$ hold if and only if $f = \psi$. □

3. C^*−ALGEBRAS $\mathfrak{R}_{L^\infty}$

Recall the C^*−algebra $\mathfrak{R}_{L^\infty}$ is defined by

$$\mathfrak{R}_{L^\infty} = \text{clos}\left\{ \sum_{i=1}^{n} \prod_{j=1}^{m} R_{H_{ij}} \middle| H_{ij} \in L^\infty_{2\times 2}(\mathbb{T}) \right\}.$$

Let $\mathfrak{S}\mathfrak{R}_{L^\infty}$ be the closed ideal of $\mathfrak{R}_{L^\infty}$ generated by operators of the form

$$R_{\begin{pmatrix} f_1 & \phi_1 \\ g_1 & \psi_1 \end{pmatrix}} R_{\begin{pmatrix} f_2 & \phi_2 \\ g_2 & \psi_2 \end{pmatrix}} - R_{\begin{pmatrix} f_1 f_2 & \phi \\ g & \psi_1\psi_2 \end{pmatrix}} \tag{3.1}$$

where$f_i, g_i, \phi_i, \psi_i, g, \phi$ are in $L^\infty(\mathbb{T})(i = 1, 2)$. Furthermore, the C^*−algebra $\mathfrak{R}_{L^\infty}$ equals the algebra generated by Riesz projection and all multiplication operators with $L^\infty(\mathbb{T})$ symbols, i.e.

$$\mathfrak{R}_{L^\infty} = \text{clos span}\left\{P, M_\phi | \phi \in L^\infty(\mathbb{T})\right\}.$$

Next,we will establish the symbol map of $\mathfrak{R}_{L^\infty}$ with the normalized reproducing kernel of H^2.

Lemma 3.1. *Let* $H_i = \left(\begin{smallmatrix} f_i & \phi_i \\ g_i & \psi_i \end{smallmatrix}\right) \in \bigcap_{p\geq 1} L^p_{2\times 2}(\mathbb{T}), i \in \mathbb{Z}_+$.

(1) The radial limit

$$\lim_{r\to 1^-} \left\langle R_{H_1} \cdots R_{H_m} k_{r\xi}, k_{r\xi} \right\rangle = f_1(\xi) \cdots f_m(\xi) \quad a.e. \text{ on } \mathbb{T}.$$

(2) The radial limit

$$\lim_{r\to 1^-} \left\langle R_{H_1} \cdots R_{H_m} \bar{z}\bar{k}_{r\xi}, \bar{z}\bar{k}_{r\xi} \right\rangle = \psi_1(\xi) \cdots \psi_m(\xi) \quad a.e. \text{ on } \mathbb{T}.$$

(3) If $g, \phi \in \cap_{p\geq 1} L^p(\mathbb{T})$, *then* $\prod_{i=1}^n R_{H_i} - R_{\begin{pmatrix} \Pi_{i=1}^n f_i & \phi \\ g & \Pi_{i=1}^n \psi_i \end{pmatrix}} \in \mathfrak{S}\mathfrak{R}_{L^\infty}$.

(4) If $T \in \mathfrak{S}\mathfrak{R}_{L^\infty}$, *then*

$$\lim_{r\to 1} \langle T k_{r\xi}, k_{r\xi} \rangle = 0,$$
$$\lim_{r\to 1} \langle T \bar{z}\bar{k}_{r\xi}, \bar{z}\bar{k}_{r\xi} \rangle = 0.$$

(5) The uniform limit of GSIO *is also a* GSIO.

Proof. (1) We will prove this lemma by induction on m. For $m = 1$, applying (2.2), we obtain

$$\langle R_{H_1} k_{r\xi}, k_{r\xi} \rangle = \int_0^{2\pi} f_1(e^{i\theta}) |k_{r\xi}(e^{i\theta})|^2 \frac{d\theta}{2\pi}$$

where $|k_{r\xi}|^2$ is the Poisson kernel for $r\xi \in \mathbb{D}$. By Fatou's theorem,

$$\lim_{r\to 1} \langle R_{H_1} k_{r\xi}, k_{r\xi} \rangle = f_1(\xi)$$

for almost all $\xi \in \mathbb{T}$.

Let $m \geq 2$, assume the result true up to $n-1$. A simple computation gives

$$\begin{aligned}
&\langle R_{H_1}R_{H_2}\cdots R_{H_n}k_{r\xi},k_{r\xi}\rangle = \langle R_{H_2}\cdots R_{H_n}k_{r\xi}, R^*_{H_1}k_{r\xi}\rangle\\
=&\langle R_{H_2}\cdots R_n k_{r\xi},(P_+\bar{f}_1P_+ + P_+\bar{g}_1P_- + P_-\bar{\phi}_1P_+ + P_-\bar{\psi}_1P_-)k_{r\xi}\rangle\\
=&\langle R_{H_2}\cdots R_{H_n}k_{r\xi},(P_+\bar{f}_1P_+ + P_-\bar{\phi}_1P_+)k_{r\xi}\rangle\\
=&\langle R_{H_2}\cdots R_{H_n}k_{r\xi},P_+\bar{f}_1P_+k_{r\xi}\rangle + \langle R_{H_2}\cdots R_{H_n}k_{r\xi},P_-\bar{\phi}_1P_+k_{r\xi}\rangle\\
=&\langle R_{H_2}\cdots R_{H_n}k_{r\xi},P_+\bar{f}_1k_{r\xi}\rangle + \langle R_{H_2}\cdots R_{H_n}k_{r\xi},P_-\bar{\phi}_1k_{r\xi}\rangle\\
=&\langle R_2\cdots R_{H_n}k_{r\xi},P_+(\bar{f}_{1+}+\bar{f}_{1-})k_{r\xi}\rangle + \langle R_{H_2}\cdots R_{H_n}k_{r\xi},P_-\bar{\phi}_1k_{r\xi}\rangle\\
=&\langle R_{H_2}\cdots R_{H_n}k_{r\xi},\bar{f}_{1+}(r\xi)k_{r\xi}+P_+\bar{f}_{1-}k_{r\xi}\rangle + \langle R_{H_2}\cdots R_{H_n}k_{r\xi},P_-\bar{\phi}_1k_{r\xi}\rangle\\
=&f_{1+}(r\xi)\langle R_{H_2}\cdots R_{H_n}k_{r\xi},k_{r\xi}\rangle + \langle R_{H_2}\cdots R_{H_n}k_{r\xi},P_+\bar{f}_{1-}k_{r\xi}\rangle\\
&+\langle R_{H_2}\cdots R_{H_n}k_{r\xi},P_-\bar{\phi}_1k_{r\xi}\rangle,
\end{aligned}$$

where $f_{1+} = P_+f_1$, $f_{1-} = P_-f_1$. Note that

$$\begin{aligned}
&\langle R_{H_2}\cdots R_{H_n}k_{r\xi},P_+\bar{f}_{1-}k_{r\xi}\rangle = \langle f_{1-}P_+R_{H_2}\cdots R_{H_n}k_{r\xi},k_{r\xi}\rangle\\
=&\langle f_{1-}P_+R_{H_2}\cdots R_{H_n}k_{r\xi},P_+k_{r\xi}\rangle = \langle P_+f_{1-}P_+R_{H_2}\cdots R_{H_n}k_{r\xi},k_{r\xi}\rangle\\
=&\langle P_+f_{1-}P_+(P_+f_2P_+ + P_-g_2P_+ + P_+\phi_2P_- + P_-\psi_2P_-)R_3\cdots R_nk_{r\xi},k_{r\xi}\rangle\\
=&\langle (P_+f_{1-}P_+f_2P_+ + P_+f_{1-}P_+\phi_2P_-)R_{H_3}\cdots R_{H_n}k_{r\xi},k_{r\xi}\rangle\\
=&\langle (P_+f_{1-}f_2P_+ + P_+f_{1-}\phi_2P_-)R_{H_3}\cdots R_{H_n}k_{r\xi},k_{r\xi}\rangle\\
=&\langle R_{\left(\begin{smallmatrix} f_2f_{1-} & \phi_2f_{1-}\\ 0 & 0\end{smallmatrix}\right)}R_{H_3}\cdots R_{H_n}k_{r\xi},k_{r\xi}\rangle,
\end{aligned}$$

$$|\langle R_{H_2}\cdots R_{H_n}k_{r\xi},P_-\bar{\phi}_1k_{r\xi}\rangle| \leq \|R_{H_2}\cdots R_{H_n}\|\|k_{r\xi}\|\|P_-\bar{\phi}_1k_{r\xi}\|$$

and

$$\begin{aligned}
&\|P_-\bar{\phi}_1k_{r\xi}\|\\
=&\|P_-(\bar{\phi}_{1+}+\bar{\phi}_{1-})k_{r\xi}\|\\
=&\|P_-\bar{\phi}_{1+}k_{r\xi}\|\\
=&\|(I-P_+)(\bar{\phi}_{1+}k_{r\xi})\|\\
=&\left(\int_0^{2\pi}|\bar{\phi}_{1+}(e^{i\theta})-\bar{\phi}_{1+}(r\xi)|^2|k_{r\xi}(e^{i\theta})|^2\frac{d\theta}{2\pi}\right)^{\frac{1}{2}} \to 0, a.e.(r\to 1^-).
\end{aligned}$$

By induction hypothesis, the result holds.

(2)Using the properties of V, we have

$$\begin{aligned}
\langle R_{H_1}\cdots R_{H_m}\bar{z}\bar{k}_{r\xi},\bar{z}\bar{k}_{r\xi}\rangle =&\langle R_{H_1}\cdots R_{H_m}Vk_{r\xi},Vk_{r\xi}\rangle\\
=&\langle VVk_{r\xi},VR_{H_1}\cdots R_{H_m}Vk_{r\xi}\rangle\\
=&\langle k_{r\xi},VR_{H_1}\cdots R_{H_m}Vk_{r\xi}\rangle.\\
=&\overline{\langle (VR_{H_1}V)\cdots(VR_{H_m}V)k_{r\xi},k_{r\xi}\rangle.}
\end{aligned}$$

By (2.3), we have

$$VR_{H_i}V = R_{\left(\begin{smallmatrix}\bar{\psi}_i & \bar{g}_i\\ \bar{\phi}_i & \bar{f}_i\end{smallmatrix}\right)} \quad 1\leq i\leq m.$$

Hence Lemma 3.1 (1) implies the result.

(3) For $k=2$, by the definition 4.1,we have

$$R_{\begin{pmatrix} f_1 & \phi_1 \\ g_1 & \psi_1 \end{pmatrix}} R_{\begin{pmatrix} f_2 & \phi_2 \\ g_2 & \psi_2 \end{pmatrix}} - R_{\begin{pmatrix} f_1 f_2 & \phi \\ g & \psi_1\psi_2 \end{pmatrix}} \in \mathfrak{S}\mathfrak{R}_{L^\infty}.$$

Assume the result true up to $n-1$. Observe that

$$\begin{aligned}
&\prod_{i=1}^{n} R_{\begin{pmatrix} f_i & \phi_i \\ g_i & \psi_i \end{pmatrix}} - R_{\begin{pmatrix} \Pi_{i=1}^n f_i & \phi \\ g & \Pi_{i=1}^n \psi_i \end{pmatrix}} \\
=&\prod_{i=1}^{n} R_{\begin{pmatrix} f_i & \phi_i \\ g_i & \psi_i \end{pmatrix}} - R_{\begin{pmatrix} f_1 & \phi_1 \\ g_1 & \psi_1 \end{pmatrix}} R_{\begin{pmatrix} \Pi_{i=2}^n f_i & \phi \\ g & \Pi_{i=2}^n \psi_i \end{pmatrix}} \\
&\quad + R_{\begin{pmatrix} f_1 & \phi_1 \\ g_1 & \psi_1 \end{pmatrix}} R_{\begin{pmatrix} \Pi_{i=1}^n f_i & \phi \\ g & \Pi_{i=1}^n \psi_i \end{pmatrix}} - R_{\begin{pmatrix} \Pi_{i=1}^n f_i & \phi \\ g & \Pi_{i=1}^n \psi_i \end{pmatrix}} \\
=&R_{\begin{pmatrix} f_1 & \phi_1 \\ g_1 & \psi_1 \end{pmatrix}} \Bigg(\underbrace{\prod_{i=2}^{n} R_{\begin{pmatrix} f_i & \phi_i \\ g_i & \psi_i \end{pmatrix}} - R_{\begin{pmatrix} \Pi_{i=2}^n f_i & \phi \\ g & \Pi_{i=2}^n \psi_i \end{pmatrix}}}_{\in \mathfrak{S}\mathfrak{R}_{L^\infty}} \Bigg) \\
&\quad \underbrace{+ R_{\begin{pmatrix} f_1 & \phi_1 \\ g_1 & \psi_1 \end{pmatrix}} R_{\begin{pmatrix} \Pi_{i=2}^n f_i & \phi \\ g & \Pi_{i=2}^n \psi_i \end{pmatrix}} - R_{\begin{pmatrix} \Pi_{i=1}^n f_i & \phi \\ g & \Pi_{i=1}^n \psi_i \end{pmatrix}}}_{\in \mathfrak{S}\mathfrak{R}_{L^\infty}}.
\end{aligned}$$

By induction hypothesis,the result holds.

(4)Suppose $g, \phi \in L^\infty$. Linear combinations of operators of the form

$$\begin{aligned}
&R_{H_1} R_{H_2} \cdots R_{H_{n-1}} \Bigg(R_{H_n} R_{H_{n+1}} - R_{\begin{pmatrix} f_n f_{n+1} & \phi \\ g & \psi_n \psi_{n+1} \end{pmatrix}} \Bigg) R_{H_{n+2}} R_{H_{n+3}} \cdots R_{H_{n+k}} \\
=&R_{H_1} R_{H_2} \cdots R_{H_{n-1}} R_{H_n} R_{H_{n+1}} R_{H_{n+2}} R_{H_{n+3}} \cdots R_{H_{n+k}} \\
&- R_{H_1} R_{H_2} \cdots R_{H_{n-1}} R_{\begin{pmatrix} f_n f_{n+1} & \phi \\ g & \psi_n \psi_{n+1} \end{pmatrix}} R_{H_{n+2}} R_{H_{n+3}} \cdots R_{H_{n+k}},
\end{aligned}$$

form a dense subset of $\mathfrak{S}\mathfrak{R}_{L^\infty}$. Lemma 3.1 (1)(2) gives the result.

(5) If R is a bounded operator on L^2 and $\lim_{n\to\infty} ||R_{H_n} - R|| = 0$, then

$$\begin{aligned}
\lim_{n\to\infty} ||P_+(R_{H_n} - R)P_+|| &\le \lim_{n\to\infty} ||R_{H_n} - R|| = 0, \\
\lim_{n\to\infty} ||P_-(R_{H_n} - R)P_+|| &\le \lim_{n\to\infty} ||R_{H_n} - R|| = 0, \\
\lim_{n\to\infty} ||P_+(R_{H_n} - R)P_-|| &\le \lim_{n\to\infty} ||R_{H_n} - R|| = 0, \\
\lim_{n\to\infty} ||P_-(R_{H_n} - R)P_-|| &\le \lim_{n\to\infty} ||R_{H_n} - R|| = 0.
\end{aligned}$$

Since

$$\begin{aligned}
P_+ R_{H_n} P_+|_{H^2} &= T_{f_n}, \\
P_- R_{H_n} P_+|_{H^2} &= H_{g_n}, \\
P_+ R_{H_n} P_-|_{\overline{\bar{z}H^2}} &= H^*_{\bar{\varphi}_n}, \\
P_- R_{H_n} P_-|_{\overline{\bar{z}H^2}} &= \tilde{T}_{\psi_n},
\end{aligned}$$

and

$$\begin{aligned}
&\|T_{\bar{z}}P_+RP_+T_z - P_+RP_+\| \\
=&\|T_{\bar{z}}P_+RP_+T_z - T_{\bar{z}}T_{f_n}T_z + T_{f_n} - P_+RP_+\| \\
\leq&\|T_{\bar{z}}P_+RP_+T_z - T_{\bar{z}}T_{f_n}T_z\| + \|T_{f_n} - P_+RP_+\| \\
\leq&\|T_{\bar{z}}(P_+RP_+ - T_{f_n})T_z\| + \|T_{f_n} - P_+RP_+\| \\
\leq&\|T_{\bar{z}}\|\|P_+RP_+ - T_{f_n}\|\|T_z\| + \|T_{f_n} - P_+RP_+\| \to 0 \quad (n \to \infty).
\end{aligned}$$

it follows that $T_{\bar{z}}P_+RP_+T_z = P_+RP_+$. We have $P_+RP_+|_{H^2}$ is a Toeplitz operator, because an operator T is a Toeplitz operator if and only if $T_{\bar{z}}TT_z = T$ [5, Theorem 6]. Moreover,

$$\begin{aligned}
&\|P_-RP_+T_z - S_zP_+RP_+\| \\
=&\|P_-RP_+T_z - P_-R_{H_n}P_+T_z + S_zP_-R_{H_n}P_+ - S_zP_+RP_+\| \\
\leq&\|P_-RP_+T_z - P_-R_{H_n}P_+T_z\| + \|S_zP_-R_{H_n}P_+ - S_zP_+RP_+\| \\
\leq&\|P_-RP_+ - P_-R_{H_n}P_+\|\|T_z\| + \|S_z\|\|P_-R_nP_+ - P_+RP_+\|
\end{aligned}$$

shows that $P_-RP_+T_z = S_zP_+RP_+$. Since an operator H is a Hankel operator if and only if $HT_z = \tilde{T}_zH$ [26, Theorem 1.8], we have $P_-RP_+|_{H^2}$ is a Hankel operator. Similary, $P_+RP_-|_{\overline{zH^2}}$ is the adjoint of a Hankel operator. By $VT_\psi V = \tilde{T}_{\bar{\psi}}$, then $P_-R_{H_n}P_-|_{\overline{zH^2}}$ is a dual Toeplitz operator. Hence R is a GSIO. □

Theorem 3.2. *The sequence*

$$0 \longrightarrow \mathfrak{S}\mathfrak{R}_{L^\infty} \longrightarrow \mathfrak{R}_{L^\infty} \longrightarrow L_2^\infty(\mathbb{T}) \longrightarrow 0$$

*is a short exact sequence; that is, the quotient algebra $\mathfrak{R}_{L^\infty}/\mathfrak{S}\mathfrak{R}_{L^\infty}$ is *-isometrically isomorphic to $L^\infty \oplus L^\infty$.*

Proof. Linear combinations of operators of the form $\prod_{j=1}^m R_{\left(\begin{smallmatrix} f_i & \phi_i \\ g_i & \psi_i \end{smallmatrix}\right)}$ span a dense subset of $\mathfrak{R}_{L^\infty}$, compute

$$\prod_{j=1}^m R_{\left(\begin{smallmatrix} f_i & \phi_i \\ g_i & \psi_i \end{smallmatrix}\right)} = R_{\left(\begin{smallmatrix} \Pi_{i=1}^m f_i & 0 \\ 0 & \Pi_{i=1}^m \psi_i \end{smallmatrix}\right)} + \underbrace{\prod_{j=1}^m R_{\left(\begin{smallmatrix} f_i & \phi_i \\ g_i & \psi_i \end{smallmatrix}\right)} - R_{\left(\begin{smallmatrix} \Pi_{i=1}^m f_i & 0 \\ 0 & \Pi_{i=1}^m \psi_i \end{smallmatrix}\right)}}_{\in \mathfrak{S}\mathfrak{R}_{L^\infty}(By Lemma 3.1(3))}.$$

This shows that operators of the form

$$T = R_{\left(\begin{smallmatrix} f & 0 \\ 0 & \psi \end{smallmatrix}\right)} + E_0, \quad f, \psi \in L^\infty, E_0 \in \mathfrak{S}\mathfrak{R}_{L^\infty}.$$

form a dense subset of $\mathfrak{R}_{L^\infty}$. Therefore, for every operator T in $\mathfrak{R}_{L^\infty}$, there exists a sequence of operators

$$T_n = R_{\left(\begin{smallmatrix} f_n & 0 \\ 0 & \psi_n \end{smallmatrix}\right)} + E_n, \quad E_n \in \mathfrak{S}\mathfrak{R}_{L^\infty}$$

such that $\lim_{n\to\infty} ||T_n - T|| = 0$. By Lemma 3.1(1)and(4), we have

$$f_n(\xi) = \lim_{r\to 1^-} \langle T_n k_{r\xi}, k_{r\xi}\rangle.$$

and

$$|f_n(\xi) - f_m(\xi)| \leq \|T_n - T_m\|. \tag{3.2}$$

So $\{f_n(\xi)\}$ is a Cauchy sequence. Define

$$f(\xi) \triangleq \lim_{n\to\infty} f_n(\xi).$$

we then have

$$\begin{aligned}
&| \lim_{r\to 1^-} \langle Tk_{r\xi}, k_{r\xi}\rangle - f(\xi) \,| \\
=&| \lim_{r\to 1^-} \langle Tk_{r\xi}, k_{r\xi}\rangle - \lim_{r\to 1^-} \langle T_n k_{r\xi}, k_{r\xi}\rangle + \lim_{r\to 1^-} \langle T_n k_{r\xi}, k_{r\xi}\rangle - f_n(\xi) + f_n(\xi) - f(\xi)| \\
\leq& \,| \lim_{r\to 1^-} \langle Tk_{r\xi}, k_{r\xi}\rangle - \lim_{r\to 1^-} \langle T_n k_{r\xi}, k_{r\xi}\rangle \,| + |\, f_n(\xi) - f(\xi) \,| \\
\leq& \|T - T_n\| + |\, f_n(\xi) - f(\xi) \,|
\end{aligned}$$

and it follows that

$$\lim_{r\to 1^-} \langle Tk_{r\xi}, k_{r\xi}\rangle = f(\xi).$$

Similarly, define

$$\psi(\xi) \triangleq \lim_{n\to\infty} \psi_n(\xi),$$

we have

$$\lim_{r\to 1^-} \langle T\bar{z}\bar{k}_{r\xi}, \bar{z}\bar{k}_{r\xi}\rangle = \psi(\xi).$$

Using(3.2), $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$. Similarly, $\lim_{n\to\infty} \|\psi_n - \psi\|_\infty = 0$. Thus

$$\|R_{\left(\begin{smallmatrix} f_n - f & 0 \\ 0 & \psi_n - \psi \end{smallmatrix}\right)}\| \leq \|f_n - f\| + \|\psi_n - \psi\| \to 0 \quad (n \to \infty).$$

Let $E = T - R_{\left(\begin{smallmatrix} f & 0 \\ 0 & \psi \end{smallmatrix}\right)}$, we have $\lim_{n\to\infty} \|E_n - E\| = 0$, since $\mathfrak{S}\mathfrak{R}_{L^\infty}$ is closed, $E \in \mathfrak{S}\mathfrak{R}_{L^\infty}$. It follows that T have the following form

$$T = R_{\left(\begin{smallmatrix} f & 0 \\ 0 & \psi \end{smallmatrix}\right)} + E, \quad f, \psi \in L^\infty(\mathbb{T}), E \in \mathfrak{S}\mathfrak{R}_{L^\infty}.$$

Define the map $\rho : \mathfrak{R}_{L^\infty} \to L_2^\infty(\mathbb{T})$ by

$$\rho(T)(\xi) = \left(\lim_{r\to 1^-} \langle Tk_{r\xi}, k_{r\xi}\rangle, \lim_{r\to 1^-} \langle T\bar{z}\bar{k}_{r\xi}, \bar{z}\bar{k}_{r\xi}\rangle \right). \tag{3.3}$$

Recall the norm of $L_2^\infty(\mathbb{T})$, $\|(a, b)\| = max\{\|a\|_\infty, \|b\|_\infty\}$. Clearly, $\|\rho(T)\| \leq \|T\|$. The map ρ is linear,contractive, and preserves conjugation. Moreover,

$$\rho(T) = (f, \psi).$$

If $A_1, A_2 \in \mathfrak{R}_{L^\infty}$, and

$$A_1 = R_{\left(\begin{smallmatrix} f_1 & 0 \\ 0 & \psi_1 \end{smallmatrix}\right)} + E_1, \quad A_2 = R_{\left(\begin{smallmatrix} f_2 & 0 \\ 0 & \psi_2 \end{smallmatrix}\right)} + E_2, \quad E_1, E_2 \in \mathfrak{S}\mathfrak{R}_{L^\infty},$$

then

$$A_1 A_2 = R_{\left(\begin{smallmatrix} f_1 & 0 \\ 0 & \psi_1 \end{smallmatrix}\right)} R_{\left(\begin{smallmatrix} f_2 & 0 \\ 0 & \psi_2 \end{smallmatrix}\right)} + \underbrace{R_{\left(\begin{smallmatrix} f_1 & 0 \\ 0 & \psi_1 \end{smallmatrix}\right)} E_2 + E_1 R_{\left(\begin{smallmatrix} f_2 & 0 \\ 0 & \psi_2 \end{smallmatrix}\right)} + E_1 E_2}_{\in \mathfrak{S}\mathfrak{R}_{L^\infty}}.$$

Using Lemma 3.1(1) and (4), we have

$$\begin{aligned}\lim_{r\to1^-}\langle A_1A_2k_{r\xi},k_{r\xi}\rangle &= \lim_{r\to1^-}\langle R_{\left(\begin{smallmatrix} f_1 & 0\\ 0 & \psi_1\end{smallmatrix}\right)}R_{\left(\begin{smallmatrix} f_2 & 0\\ 0 & \psi_2\end{smallmatrix}\right)}k_{r\xi},k_{r\xi}\rangle\\ &= \lim_{r\to1^-}\langle R_{\left(\begin{smallmatrix} f_1f_2 & 0\\ 0 & \psi_1\psi_2\end{smallmatrix}\right)}k_{r\xi},k_{r\xi}\rangle\\ &=f_1(\xi)\cdot f_2(\xi)\\ &= \lim_{r\to1^-}\langle R_{\left(\begin{smallmatrix} f_1 & 0\\ 0 & \psi_1\end{smallmatrix}\right)}k_{r\xi},k_{r\xi}\rangle\cdot \lim_{r\to1^-}\langle R_{\left(\begin{smallmatrix} f_2 & 0\\ 0 & \psi_2\end{smallmatrix}\right)}k_{r\xi},k_{r\xi}\rangle\\ &= \lim_{r\to1^-}\langle A_1k_{r\xi},k_{r\xi}\rangle\cdot \lim_{r\to1^-}\langle A_2k_{r\xi},k_{r\xi}\rangle \quad a.e.\ on\ \mathbb{T}.\end{aligned}$$

Similarly,

$$\lim_{r\to1^-}\langle A_1A_2\bar z\bar k_{r\xi},\bar z\bar k_{r\xi}\rangle = \lim_{r\to1^-}\langle A_1\bar z\bar k_{r\xi},\bar z\bar k_{r\xi}\rangle\cdot \lim_{r\to1^-}\langle A_2\bar z\bar k_{r\xi},\bar z\bar k_{r\xi}\rangle \quad a.e.\ on\ \mathbb{T}.$$

Since the algebraic operations of $L_2^\infty(\mathbb{T})$ are all performed coordinated-wise, we have ρ is multiplicative.

By Lemma 3.1(4), we have $\mathfrak{S}\mathfrak{R}_{L^\infty}\subseteq\ker\rho$. For every $T=R(f,0,0,\psi)+E\in\ker\rho$, thus, $f=\psi=0$. Hence $\mathfrak{S}\mathfrak{R}_{L^\infty}=\ker\rho$.

We define the map

$$\begin{aligned}\widetilde{\rho}:\mathfrak{R}_{L^\infty}/\mathfrak{S}\mathfrak{R}_{L^\infty}&\longrightarrow L_2^\infty(\mathbb{T}),\\ R_{\left(\begin{smallmatrix} f & 0\\ 0 & \psi\end{smallmatrix}\right)}+\mathfrak{S}\mathfrak{R}_{L^\infty}&\longmapsto (f,\psi).\end{aligned}$$

Hence, $\widetilde{\rho}$ is a C^*-isomorphism. □

Corollary 3.3. *If $T\in\mathfrak{R}_{L^\infty}$, then $\rho(T^*T-TT^*)=(0,0)$.*

Example 3.4. In fact, $\mathfrak{R}_{L^\infty}$ is a proper subalgebra of $B(L^2(\mathbb{T}))$. We make some modification to [10, Example 4]. Let T be the operator defined by

$$Tz^n=z^{2n+1},\quad n\in\mathbb{Z}.$$

Note that

$$T^*z^n=\begin{cases} z^{\frac{n-1}{2}}, & \text{if } n \text{ is odd};\\ 0, & \text{if } n \text{ is even}.\end{cases}\tag{3.4}$$

and

$$(T^*T-TT^*)z^n=\begin{cases} 0, & \text{if } n \text{ is odd};\\ z^n, & \text{if } n \text{ is even}.\end{cases}$$

Hence T^*T-TT^* is the orthogonal projeciton onto $\text{span}\{z^{2n}\}_{n\in\mathbb{Z}}$.

$$\begin{aligned}\langle (T^*T-TT^*)k_{r\xi},k_{r\xi}\rangle &=(1-r^2)\langle (T^*T-TT^*)\sum_{i=0}^\infty (r\bar\xi)^iz^i,\sum_{j=0}^\infty(r\bar\xi)^jz^j\rangle\\ &=(1-r^2)\langle\sum_{n=0}^\infty (r\bar\xi)^{2n}z^{2n},\sum_{m=0}^\infty(r\bar\xi)^{2m}z^{2m}\rangle\\ &=\langle k_{(r\bar\xi)^2},k_{(r\bar\xi)^2}\rangle\\ &=\frac{1-r^2}{1-r^4}=\frac{1}{1+r^2}\to\frac12(r\to1^-).\end{aligned}$$

By Corollary 3.3, we have $T \notin \mathfrak{R}_{L^\infty}$.

4. C^*-ALGEBRAS $\mathfrak{R}_{C(\mathbb{T})}$ AND $\mathfrak{R}_{QC}$

Let $C(\mathbb{T})$ denote the set of continuous complex-valued functions on $\mathbb{T}$, and $C(\mathbb{T})$ is a closed subalgebra of L^∞. The set of all compact operators on $L^2(\mathbb{T})$ is denoted by $\mathcal{K}(L^2(\mathbb{T}))$.

Lemma 4.1. *The C^*-algebra $\mathfrak{R}_{C(\mathbb{T})}$ is irreducible. Furthermore, $LC(L^2(\mathbb{T})) \subset \mathfrak{R}_{C(\mathbb{T})}$.*

Proof. If $\mathfrak{R}_{C(\mathbb{T})}$ is reducible, then there exists a nontrivial orthogonal projection Q which commutes with each element of $\mathfrak{R}_{C(\mathbb{T})}$. In particular, $QR_{\left(\begin{smallmatrix} z & z \\ z & z\end{smallmatrix}\right)} = R_{\left(\begin{smallmatrix} z & z \\ z & z\end{smallmatrix}\right)}Q$ and $R_{\left(\begin{smallmatrix} z & z \\ z & z\end{smallmatrix}\right)}$ is the bilateral shift. Since the commutant of the bilateral shift is the set of all multiplications[18, 146], it follows that $Q = M_{\chi_\Delta}$, where χ_Δ is a characteristic function. Note that

$$\begin{aligned} R_{\left(\begin{smallmatrix} z & 0 \\ 0 & 0\end{smallmatrix}\right)}Q =& QR_{\left(\begin{smallmatrix} z & 0 \\ 0 & 0\end{smallmatrix}\right)}, \\ \begin{pmatrix} T_z & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T_{\chi_\Delta} & H^*_{\tilde{\chi}_\Delta} \\ H_{\chi_\Delta} & \tilde{T}_{\chi_\Delta} \end{pmatrix} =& \begin{pmatrix} T_z & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T_{\chi_\Delta} & H^*_{\tilde{\chi}_\Delta} \\ H_{\chi_\Delta} & \tilde{T}_{\chi_\Delta} \end{pmatrix}, \\ \begin{pmatrix} T_zT_{\chi_\Delta} & T_zH^*_{\tilde{\chi}_\Delta} \\ 0 & 0 \end{pmatrix} =& \begin{pmatrix} T_{\chi_\Delta}T_z & 0 \\ T_{\chi_\Delta}H_{\chi_\Delta} & 0 \end{pmatrix}, \end{aligned}$$

implies

$$T_zT_{\chi_\Delta} = T_{\chi_\Delta}T_z.$$

Since the commutant of T_z is the set of all analytic Toeplitz operators on H^2 [18, 147], it follows that χ_Δ is 0 or 1, and $Q = I$ or $Q = 0$. This contradicts our assumption. Therefore $\mathfrak{R}_{C(\mathbb{T})}$ is irreducible.

Applying the formula $I - T_zT_{\bar{z}} = 1 \otimes 1$ yields

$$R_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right)} - R_{\left(\begin{smallmatrix} z & 0 \\ 0 & 0\end{smallmatrix}\right)}R_{\left(\begin{smallmatrix} \bar{z} & 0 \\ 0 & 0\end{smallmatrix}\right)} = 1 \otimes 1.$$

where $1 \otimes 1$ is an operator of rank 1, thus $LC(L^2(\mathbb{T})) \cap \mathfrak{R}_{C(\mathbb{T})} \neq \{0\}$. By [9, 5.39], we have $LC(L^2(\mathbb{T})) \subset \mathfrak{R}_{C(\mathbb{T})}$. □

The algebra $QC \triangleq (H^\infty + C(\mathbb{T})) \cap (\overline{H^\infty + C(\mathbb{T})})$ is a closed subalgebra of $L^\infty(\mathbb{T})$ which properly contains $C(\mathbb{T})$. Let $\mathfrak{S}\mathfrak{R}_{QC}$(resp. $\mathfrak{S}\mathfrak{R}_{C(\mathbb{T})}$) be the closed ideal of $\mathfrak{R}_{QC}$(resp. $\mathfrak{R}_{C(\mathbb{T})}$) generated by operators of the form

$$R_{\left(\begin{smallmatrix} f_1 & \phi_1 \\ g_1 & \psi_1\end{smallmatrix}\right)}R_{\left(\begin{smallmatrix} f_2 & \phi_2 \\ g_2 & \psi_2\end{smallmatrix}\right)} - R_{\left(\begin{smallmatrix} f_1f_2 & \phi \\ g & \psi_1\psi_2\end{smallmatrix}\right)} \tag{4.1}$$

where$f_i, g_i, \phi_i, \psi_i, g, \phi$ are in QC(resp. $C(\mathbb{T})$)$(i = 1, 2)$.

Lemma 4.2. $\mathfrak{S}\mathfrak{R}_{C(\mathbb{T})} = \mathcal{K}(L^2(\mathbb{T}))$, *and* $\mathfrak{S}\mathfrak{R}_{QC} = \mathcal{K}(L^2(\mathbb{T}))$.

Proof. If $f_i, g_i, \phi_i, \psi_i, g, \phi \in C(\mathbb{T})$(resp. QC)(i=1,2), an easy computation shows that

$$R_{\begin{pmatrix} f_1 & \phi_1 \\ g_1 & \psi_1 \end{pmatrix}} R_{\begin{pmatrix} f_2 & \phi_2 \\ g_2 & \psi_2 \end{pmatrix}} - R_{\begin{pmatrix} f_1 f_2 & \phi \\ g & \psi_1\psi_2 \end{pmatrix}}$$
$$= \begin{pmatrix} T_{f_1}T_{f_2} + H^*_{\bar{\phi}_1}H_{g_2} - T_{f_1f_2} & T_{f_1}H^*_{\bar{\phi}_2} + H^*_{\bar{\phi}_1}\tilde{T}_{\psi_2} - H^*_{\bar{\phi}} \\ H_{g_1}T_{f_2} + \tilde{T}_{\psi_1}H_{g_2} - H_g & H_{g_1}H^*_{\bar{\phi}_2} + \tilde{T}_{\psi_1}\tilde{T}_{\psi_2} - \tilde{T}_{\psi_1\psi_2} \end{pmatrix}$$
$$= \begin{pmatrix} H^*_{\bar{\phi}_1}H_{g_2} - H^*_{\bar{f}_1}H_{f_2} & T_{f_1}H^*_{\bar{\phi}_2} + H^*_{\bar{\phi}_1}\tilde{T}_{\psi_2} - H^*_{\bar{\phi}} \\ H_{g_1}T_{f_2} + \tilde{T}_{\psi_1}H_{g_2} - H_g & H_{g_1}H^*_{\bar{\phi}_2} - H_{\psi_1}H^*_{\bar{\psi}_2} \end{pmatrix}.$$

The second equality follows form the formulas $T_{ab} - T_aT_b = H^*_{\bar{a}}H_b$ and $\tilde{T}_{ab} - \tilde{T}_a\tilde{T}_b = H_aH^*_{\bar{b}}$. Since the Hankel operator H_φ is compact if and only if $\varphi \in H^\infty + C(\mathbb{T})$ by[26, p.27], it follows that

$$R_{\begin{pmatrix} f_1 & \phi_1 \\ g_1 & \psi_1 \end{pmatrix}} R_{\begin{pmatrix} f_2 & \phi_2 \\ g_2 & \psi_2 \end{pmatrix}} - R_{\begin{pmatrix} f_1 f_2 & \phi \\ g & \psi_1\psi_2 \end{pmatrix}}$$

is compact, and $\mathfrak{S}\mathfrak{R}_{C(\mathbb{T})} \subset \mathcal{K}(L^2(\mathbb{T}))$(resp.$\mathfrak{S}\mathfrak{R}_{QC)} \subset \mathcal{K}(L^2(\mathbb{T}))$. On the other hand, $LC(L^2(\mathbb{T}))$ contains no proper closed ideal. Hence, $\mathfrak{S}\mathfrak{R}_{C(\mathbb{T})} = LC(L^2(\mathbb{T}))$ (resp.$\mathfrak{S}\mathfrak{R}_{QC)} = \mathcal{K}(L^2(\mathbb{T}))$). □

Corollary 4.3. *For every $T \in \mathfrak{R}_{L^\infty}$, we have*

$$\|\rho(T)\| \le \|T\|_e.$$

In particular, if $H = \begin{pmatrix} f & \phi \\ g & f \end{pmatrix}$, then

$$max\{\|f\|_\infty, \|\psi\|_\infty\} \le \|R_H\|_e.$$

Proof. If $T \in \mathfrak{R}_{L^\infty}$, by Theorem 3.2, we have

$$\inf_{A \in \mathfrak{S}\mathfrak{R}_{L^\infty}} \|T + A\| = \|\rho(T)\|.$$

On the other hand, $\mathcal{K} \subset \mathfrak{S}\mathfrak{R}_{L^\infty}$ by Lemma 4.2. Therefore,

$$\inf_{A \in \mathfrak{S}\mathfrak{R}_{L^\infty}} \|T + A\| \le \inf_{K \in \mathcal{K}(L^2(\mathbb{T}))} \|T + K\| = \|T\|_e.$$

Use Theorem 3.2 again,

$$\|\rho(R_H)\| = max\{\|f\|_\infty, \|\psi\|_\infty\}.$$

□

If T is a bounded linear operator on Hilbert space H, $\sigma_e(T)$ denotes the essential spectrum of T. For $\varphi \in L^\infty$, $Ran_{ess}\varphi$ denotes the essential range of φ. If E is a subset of complex plane $\mathbb{C}$, the convex hull of E will be denoted by coE. Combining Theorem 3.2 and Lemma 4.2, we get the following result.

Corollary 4.4. *There exists a *-homomorphism ζ from the quotient algebra $\mathfrak{R}_{L^\infty}/\mathcal{K}$ onto $L^\infty_2(\mathbb{T})$ such that the diagram*

$$\begin{array}{ccc} \mathfrak{R}_{L^\infty} & \xrightarrow{\ \pi\ } & \mathfrak{R}_{L^\infty}/\mathcal{K}(L^2(\mathbb{T})) \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\zeta} & \\ & L_2^\infty(\mathbb{T}) & \end{array}$$

commutes. Moreover,

(1) *For every* $T \in \mathfrak{R}_{L^\infty}$*, if* T *is Fredholm, then* $\rho(T)$ *is invertible in* $L_2^\infty(\mathbb{T})$*;*
(2) $\mathrm{Ran}_{ess} f \cup \mathrm{Ran}_{ess} \psi \subset \sigma_e(R_H)$.

Recall the spectral inclusion theorem of Toeplitz operator[9],

$$Ran_{ess} f \subset \sigma_e(T_f) \subset \sigma(T_f) \subset coRan_{ess} f. \tag{4.2}$$

Corollary 4.4 give the first inclusion similar to (4.2), the next theorem will show the third inclusion similar to (4.2).

Proposition 4.5. *Let* $H = \begin{pmatrix} f_1 & \phi \\ g & f_2 \end{pmatrix} \in L_\infty^{2\times 2}(\mathbb{T})$*. If we define*

$$\mathcal{G}_i = co\mathrm{Ran}_{ess} f_i \cup \{\lambda \notin \mathrm{Ran}_{ess} f_i : d_i(\lambda) \leq \delta \|(f_i - \lambda)^{-1}\|_\infty\}$$

where

$$d_i(\lambda) = (1 - dist((f_i - \lambda)/|f_i - \lambda|, H^\infty)^2)^{1/2}, \quad \delta = \min\{dist(\bar{\phi}, H^\infty), dist(g, H^\infty)\}$$

for $n = 1, 2,$ *then*

$$\sigma(R_H) \subset \mathcal{G}_1 \cup \mathcal{G}_2.$$

Proof. Suppose $\lambda \in \rho(T_{f_1}) \cap \rho(\tilde{T}_{f_2})$, we have

$$\begin{aligned} R_H - \lambda I_{L^2} &= \begin{pmatrix} T_{f_1-\lambda} & H^*_{\bar{\phi}} \\ H_g & \tilde{T}_{f_2-\lambda} \end{pmatrix} \\ &= \begin{pmatrix} T_{f_1-\lambda} & 0 \\ 0 & \tilde{T}_{f_2-\lambda} \end{pmatrix} + \begin{pmatrix} 0 & H^*_{\bar{\phi}} \\ H_g & 0 \end{pmatrix} \\ &= \begin{pmatrix} T_{f_1-\lambda} & 0 \\ 0 & \tilde{T}_{f_2-\lambda} \end{pmatrix} \left(I_{L^2} + \begin{pmatrix} 0 & T^{-1}_{f_1-\lambda} H^*_{\bar{\phi}} \\ \tilde{T}^{-1}_{f_2-\lambda} H_g & 0 \end{pmatrix} \right). \end{aligned}$$

If $\|H_{\bar{\phi}}\| < \|T^{-1}_{f_1-\lambda}\|^{-1}$ and $\|H_g\| < \|\tilde{T}^{-1}_{f_2-\lambda}\|^{-1}$, then

$$\left\| \begin{pmatrix} 0 & T^{-1}_{f_1-\lambda} H^*_{\bar{\phi}} \\ \tilde{T}^{-1}_{f_2-\lambda} H_g & 0 \end{pmatrix} \right\| = \max\{\|T^{-1}_{f_1-\lambda} H^*_{\bar{\phi}}\|, \|\tilde{T}^{-1}_{f_2-\lambda} H_g\|\} < 1,$$

and $\lambda \in \rho(R_H)$. This mean that

$$\{\lambda \in \rho(T_{f_1}) : \|H_{\bar{\phi}}\| < \|T^{-1}_{f_1-\lambda}\|^{-1}\} \cap \{\lambda \in \rho(\tilde{T}_{f_2}) : \|H_g\| < \|\tilde{T}^{-1}_{f_2-\lambda}\|^{-1}\} \subset \rho(R_H)$$

or

$$\begin{aligned} \sigma(R_H) \subset & \sigma(T_{f_1}) \cup \{\lambda \in \rho(T_{f_1}) : \|T^{-1}_{f_1-\lambda}\|^{-1} \leq \|H_{\bar{\phi}}\|\} \\ & \cup \sigma(\tilde{T}_{f_2}) \cup \{\lambda \in \rho(\tilde{T}_{f_2}) : \|\tilde{T}^{-1}_{f_2-\lambda}\|^{-1} \leq \|H_g\|\}. \end{aligned} \tag{4.3}$$

Repeat the above reasoning for R_H^*, we have

$$\begin{aligned}\sigma(R_H^*) \subset &\sigma(T_{\bar{f}_1}) \cup \{\lambda \in \rho(T_{\bar{f}_1}) : \|T_{\bar{f}_1-\lambda}^{-1}\|^{-1} \leq \|H_g\|\} \\ &\cup \sigma(\tilde{T}_{\bar{f}_2}) \cup \{\lambda \in \rho(\tilde{T}_{\bar{f}_2}) : \|\tilde{T}_{\bar{f}_2-\lambda}^{-1}\|^{-1} \leq \|H_{\bar{\phi}}\|\}.\end{aligned}$$

Taking conjugates, we get

$$\begin{aligned}\sigma(R_H) \subset &\sigma(T_{f_1}) \cup \{\bar{\lambda} \in \rho(T_{\bar{f}_1}) : \|T_{\bar{f}_1-\bar{\lambda}}^{-1}\|^{-1} \leq \|H_g\|\} \\ &\cup \sigma(\tilde{T}_{\bar{f}_2}) \cup \{\bar{\lambda} \in \rho(\tilde{T}_{\bar{f}_2}) : \|\tilde{T}_{\bar{f}_2-\bar{\lambda}}^{-1}\|^{-1} \leq \|H_{\bar{\phi}}\|\}.\end{aligned}$$

Since $\|T_{\bar{f}_1-\bar{\lambda}}^{-1}\| = \|(T_{f_1-\lambda}^{-1})^*\| = \|(T_{f_1-\lambda}^{-1})\|$ and $\|\tilde{T}_{\bar{f}_2-\bar{\lambda}}^{-1}\| = \|(\tilde{T}_{f_2-\lambda}^{-1})^*\| = \|(\tilde{T}_{f_2-\lambda}^{-1})\|$, it follows that

$$\begin{aligned}\sigma(R_H) \subset &\sigma(T_{f_1}) \cup \{\lambda \in \rho(T_{f_1}) : \|T_{f_1-\lambda}^{-1}\|^{-1} \leq \|H_g\|\} \\ &\cup \sigma(\tilde{T}_{f_2}) \cup \{\lambda \in \rho(\tilde{T}_{f_2}) : \|\tilde{T}_{f_2-\lambda}^{-1}\|^{-1} \leq \|H_{\bar{\phi}}\|\}.\end{aligned} \tag{4.4}$$

According the norm of Hankel operator ([26, Theorem 1.4]), we have $\|H_{\bar{\phi}}\| = dist(\bar{\phi}, H^\infty)$ and $\|H_g\| = dist(g, H^\infty)$. Let $\delta = \min\{dist(\bar{\phi}, H^\infty), dist(g, H^\infty)\}$. We combine (4.3) and (4.4). Thus

$$\begin{aligned}\sigma(R_H) \subset &\sigma(T_{f_1}) \cup \{\lambda \in \rho(T_{f_1}) : \|T_{f_1-\lambda}^{-1}\|^{-1} \leq \delta\} \\ &\cap \sigma(\tilde{T}_{f_2}) \cap \{\lambda \in \rho(\tilde{T}_{f_2}) : \|\tilde{T}_{f_2-\lambda}^{-1}\|^{-1} \leq \delta\|\}.\end{aligned}$$

Since $\tilde{T}_{f_2}$ and $T_{f_2}^*$ are anti-unitary, $\sigma(\tilde{T}_{f_2}) = \sigma(T_{f_2})$ and $\|\tilde{T}_{f_2-\lambda}^{-1}\| = \|T_{f_2-\lambda}^{-1}\|$. Using the (4.2) and norm estimation of the inverse of Toeplitz operator[25, page.125.]

$$\frac{(1 - dist(\varphi/|\varphi|, H^\infty)^2)^{1/2}}{\|\varphi^{-1}\|} \leq \|T_\varphi^{-1}\|^{-1},$$

we have

$$\begin{aligned}\{\lambda \in \rho(T_{f_1}) : \|T_{f_1-\lambda}^{-1}\|^{-1} \leq \delta\} \subset &\{\lambda \notin Ran_{ess} f_1 : d_1(\lambda) \leq \delta \|(f_1 - \lambda)^{-1}\|_\infty\}, \\ \{\lambda \in \rho(T_{f_2}) : \|T_{f_2-\lambda}^{-1}\|^{-1} \leq \delta\} \subset &\{\lambda \notin Ran_{ess} f_2 : d_2(\lambda) \leq \delta \|(f_2 - \lambda)^{-1}\|_\infty\}, \\ \sigma(T_{f_1}) \subset &coRan_{ess} f_1, \\ and \quad \sigma(T_{f_2}) \subset &coRan_{ess} f_2,\end{aligned}$$

where $d_i(\lambda) = (1 - dist((f_i - \lambda)/|f_i - \lambda|, H^\infty)^2)^{1/2}, i = 1, 2.$ □

Theorem 4.6. *The sequence*

$$0 \longrightarrow \mathcal{K}(L^2(\mathbb{T})) \longrightarrow \mathfrak{R}_{C(\mathbb{T})} \longrightarrow C_2(\mathbb{T}) \longrightarrow 0$$

*is a short exact sequence; that is, the quotient algebra $\mathfrak{R}_{C(\mathbb{T})}/\mathcal{K}$ is *-isometrically isomorphic to $C_2(\mathbb{T})$.*

Proof. Using the proof of Theorem 3.2 and Lemma 4.2 , for every operator $T \in \mathfrak{R}_{C(\mathbb{T})}$ have the following form

$$T = R_{\left(\begin{smallmatrix} f & 0 \\ 0 & \psi \end{smallmatrix}\right)} + K, \quad f, \psi \in C(\mathbb{T}), K \in \mathcal{K}. \tag{4.5}$$

The map $\tilde{\rho}$ defined in (3.4) is *-isometrically isomorphic from $\mathfrak{R}_{C(\mathbb{T})}/\mathcal{K}(L^2(\mathbb{T}))$ to $C_2(\mathbb{T})$. □

Remark 4.7. In fact, the previous theorem can be extend to the algebra QC. The sequence

$$0 \longrightarrow \mathcal{K}(L^2(\mathbb{T})) \longrightarrow \mathfrak{R}_{QC} \longrightarrow QC_2 \longrightarrow 0$$

is a short exact sequence. The proof is similar in spirit to Theorem 4.6.

Corollary 4.8. *For every $T \in \mathfrak{R}_{QC}$, we have*

$$\|\rho(T)\| = \|T\|_e.$$

In particular, if $H = \begin{pmatrix} f & \phi \\ g & f \end{pmatrix} \in QC_{2\times 2}$, then

$$max\{\|f\|_\infty, \|\psi\|_\infty\} = \|R_H\|_e.$$

Corollary 4.9. *If $H = \begin{pmatrix} f & \phi \\ g & f \end{pmatrix} \in QC_{2\times 2}$, then $\sigma_e(R_H) = \mathrm{Ran}_{ess} f \cup \mathrm{Ran}_{ess}\psi$. Moreover, R_H is Fredholm if and only if f and ψ are invertible in QC.*

Remark 4.10. If $H = \begin{pmatrix} f & \phi \\ g & f \end{pmatrix} \in C(\mathbb{T})_{2\times 2}$, then $\sigma_e(R_H) = f(\mathbb{T}) \cup \psi(\mathbb{T})$.

Definition 4.11. Let f is an invertible function in $C(\mathbb{T})$, the winding number of f about the origin is defined by

$$\sharp(f) = \frac{1}{2\pi i}\int_{f(\mathbb{T})} \frac{dz}{z}.$$

Definition 4.12. Let T be a bounded linear operator on Hilbert space H, a bounded linear operator B on H is called the regularizer of T if $BT-I$ and $TB-I$ are compact. If T is Fredholm, the difference $\mathrm{ind}\, T = \dim\ker T - \dim\ker T^*$ is call the index of T.

Corollary 4.13. *If T is Fredholm operator in $\mathfrak{R}_{C(\mathbb{T})}$, then*

(1) $R_{\begin{pmatrix} f_0^{-1} & 0 \\ 0 & \psi_0^{-1} \end{pmatrix}}$ is a regularizer of T;
(2) $ind(T) = \sharp(\psi_0) - \sharp(f_0)$,

where $f_0(\xi) = \lim_{r\to 1^-}\langle Tk_{r\xi}, k_{r\xi}\rangle, \psi_0(\xi) = \lim_{r\to 1^-}\langle T\bar{z}\bar{k}_{r\xi}, \bar{z}\bar{k}_{r\xi}\rangle$.

In particular, if $H = \begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} \in C(\mathbb{T})_{2\times 2}$ and R_H is a Fredholm operator, then

$$ind(R_H) = \sharp(\psi) - \sharp(f).$$

Proof. If $T \in \mathfrak{R}_{C(\mathbb{T})}$, by the formula (4.5), we have

$$T = R_{\begin{pmatrix} f_0 & 0 \\ 0 & \psi_0 \end{pmatrix}} + K, \quad f_0, \psi_0 \in C(\mathbb{T}), \quad K \in \mathcal{K}(L^2(\mathbb{T})).$$

where $f_0(\xi) = \lim_{r\to 1^-}\langle Tk_{r\xi}, k_{r\xi}\rangle$, $\psi_0(\xi) = \lim_{r\to 1^-}\langle T\bar{z}\bar{k}_{r\xi}, \bar{z}\bar{k}_{r\xi}\rangle$, and hence f and ψ are invertible in $C(\mathbb{T})$ by the remark 4.10. A calculation shows that

$$\begin{aligned} R_{\begin{pmatrix} f_0^{-1} & 0 \\ 0 & \psi_0^{-1}\end{pmatrix}} T =& R_{\begin{pmatrix} f_0^{-1} & 0 \\ 0 & \psi_0^{-1}\end{pmatrix}} R_{\begin{pmatrix} f_0 & 0 \\ 0 & \psi_0\end{pmatrix}} + R_{\begin{pmatrix} f_0 & 0 \\ 0 & \psi_0\end{pmatrix}} K \\ =& \begin{pmatrix} T_{f_0^{-1}} & 0 \\ 0 & \tilde{T}_{\psi_0^{-1}}\end{pmatrix} \begin{pmatrix} T_{f_0} & 0 \\ 0 & \tilde{T}_{\psi_0}\end{pmatrix} + R_{\begin{pmatrix} f_0 & 0 \\ 0 & \psi_0\end{pmatrix}} K \\ =& \begin{pmatrix} T_{f_0^{-1}}T_{f_0} & 0 \\ 0 & \tilde{T}_{\psi_0^{-1}}\tilde{T}_{\psi_0}\end{pmatrix} + R_{\begin{pmatrix} f_0 & 0 \\ 0 & \psi_0\end{pmatrix}} K \\ =& \begin{pmatrix} I_{H^2} - H^*_{\overline{f_0^{-1}}} H_{f_0} & 0 \\ 0 & I_{\overline{zH^2}} - H_{\psi_0^{-1}} H^*_{\bar{\psi}_0}\end{pmatrix} + R_{\begin{pmatrix} f_0 & 0 \\ 0 & \psi_0\end{pmatrix}} K \\ =& I + \begin{pmatrix} -H^*_{\overline{f_0^{-1}}} H_{f_0} & 0 \\ 0 & -H_{\psi_0^{-1}} H^*_{\bar{\psi}_0}\end{pmatrix} + R_{\begin{pmatrix} f_0 & 0 \\ 0 & \psi_0\end{pmatrix}} K. \end{aligned}$$

Since the Hankel operator H_φ is compact if and only if $\varphi \in H^\infty + C(\mathbb{T})$ by [26, p.27], we have $H^*_{\overline{f_0^{-1}}} H_{f_0}$ and $H_{\psi_0^{-1}} H^*_{\bar{\psi}_0}$ are compact, so $R_{\begin{pmatrix} f_0^{-1} & 0 \\ 0 & \psi_0^{-1}\end{pmatrix}} T - I$ is compact, similarly, $TR_{\begin{pmatrix} f_0^{-1} & 0 \\ 0 & \psi_0^{-1}\end{pmatrix}} - I$ is compact.

Since the Fredholm index is stable under compact operator perturbations[2, p.98], it follows that

$$\begin{aligned} \mathrm{ind}(T) &= \mathrm{ind}(R_{\begin{pmatrix} f_0 & 0 \\ 0 & \psi_0\end{pmatrix}} + K) \\ &= \mathrm{ind} R_{\begin{pmatrix} f_0 & 0 \\ 0 & \psi_0\end{pmatrix}} \\ &= \mathrm{ind} \begin{pmatrix} T_{f_0} & 0 \\ 0 & \tilde{T}_{\psi_0}\end{pmatrix} \\ &= \mathrm{ind}\ T_{f_0} + \mathrm{ind}\ \tilde{T}_{\psi_0}. \end{aligned}$$

Note that

$$\begin{aligned} \mathrm{ind}\ \tilde{T}_{\psi_0} &= \dim\ker(\tilde{T}_{\psi_0}) - \dim\ker(\tilde{T}^*_{\psi_0}) \\ &= \dim\ker(VT^*_{\psi_0}V) - \dim\ker(VT_{\psi_0}V) \\ &= \dim\ker(T^*_{\psi_0}) - \dim\ker(T_{\psi_0}) \\ &= -\mathrm{ind}\ T_{\psi_0}. \end{aligned}$$

By the theorem [9, 7,26], we have $\mathrm{ind}\ T_{f_0} = -\sharp(f_0)$ and $\mathrm{ind}\ \tilde{T}_{\psi_0} = \sharp(\psi_0)$. Therefore, $\mathrm{ind}(T) = \sharp(\psi_0) - \sharp(f_0)$. □

Corollary 4.14. *If* $H = \begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} \in C(\mathbb{T})_{2\times 2}$, *then* R_H *is invertible if and only if the following conditions hold:*

(1) f *and* ϕ *are invertible,*
(2) $\sharp(\psi) = \sharp(f)$, *and*

(3) either $\mathrm{Ker}(R_H) = \{0\}$ *or* $\mathrm{Ker}(R_H^*) = \{0\}$.

Proof. By Corollary 4.4, we have $Ran_{ess} f \cup Ran_{ess}\psi \subset \sigma(R_H)$. Suppose that R_H is invertible, then

(a) f and ϕ are invertible;
(b) $\mathrm{Ker}(R_H) = \{0\}$ and $\mathrm{Ker}(R_H^*) = \{0\}$.

It follows that ind $(R_H) = 0$. By Corollary 4.13, we have $\sharp(\psi) = \sharp(f)$;

On the other hand, if R_H is Fredholm, then R_H is invertible if and only if

(i) $\mathrm{ind}(R_H) = 0$;
(ii) either $\mathrm{Ker}(R_H) = \{0\}$ or $\mathrm{Ker}(R_H^*) = \{0\}$.

By Remark 4.10, R_H is Fredholm if and only if f and ψ are invertible, hence the result follows. □

Remark 4.15. There exist some examples showing that both of $\mathrm{Ker}(R_H)$ and $\mathrm{Ker}(R_H^*)$ are nontrivial. For example, if u and θ are nonconstant inner functions, then

$$\bar{z}\overline{(H^2 \ominus \theta H^2)} \subseteq \mathrm{Ker}\, R_{\left(\begin{smallmatrix} u & 0 \\ 0 & \theta \end{smallmatrix}\right)}$$

and

$$H^2 \ominus uH^2 \subseteq \mathrm{Ker}\, R^*_{\left(\begin{smallmatrix} u & 0 \\ 0 & \theta \end{smallmatrix}\right)}.$$

Let Δ is a proper subset of $\mathbb{T}$ and has positive measure, χ_Δ is the characteristic function of Δ, we have $R_{\left(\begin{smallmatrix} \chi_\Delta & \chi_\Delta \\ \chi_\Delta & \chi_\Delta \end{smallmatrix}\right)} = M_{\chi_\Delta}$ and $\dim\ker M_{\chi_\Delta} = \dim\ker M^*_{\chi_\Delta} = \infty$.

5. INVERTIBLE AND FREDHOLM OF GISO

In this section, we found that GSIOs and singular integral operators with 2×2 matrix symbol are equivalent after extension.

Definition 5.1. [3] Let T and S are bounded operator on Hilbert space $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. The operators T and S are called equivalent after extension, written $T \overset{*}{\sim} S$, if there exist Hilbert spaces Z and W such that $T \oplus I_Z$ and $S \oplus I_W$ are equivalent operators. This means that there exist invertible bounded linear operators E and F such that

$$\begin{pmatrix} T & 0 \\ 0 & I_Z \end{pmatrix} = E \begin{pmatrix} S & 0 \\ 0 & I_W \end{pmatrix} F.$$

The relation $\overset{*}{\sim}$ is reflexive, symmetric and transitive.

Theorem 5.2. [3] *If* $T \overset{*}{\sim} S$, *then* T *is invertible(Fredholm) if and only if* S *is invertible (Fredholm).*

Let

$$A = \begin{pmatrix} f & 0 \\ g & -1 \end{pmatrix}, B = \begin{pmatrix} \varphi & -1 \\ \psi & 0 \end{pmatrix},$$

where $f, g, \phi, \psi \in L^{\infty}(\mathbb{T})$. Write the Cauchy singular integral operators with 2×2 matrix symbol

$$A\mathbb{P}_+ + B\mathbb{P}_- = \begin{pmatrix} f & 0 \\ g & -1 \end{pmatrix}\begin{pmatrix} P_+ & 0 \\ 0 & P_+ \end{pmatrix} + \begin{pmatrix} \varphi & -1 \\ \psi & 0 \end{pmatrix}\begin{pmatrix} P_- & 0 \\ 0 & P_- \end{pmatrix} : L_2^2(\mathbb{T}) \to L_2^2(\mathbb{T}). \tag{5.1}$$

Theorem 5.3. *Let* $H = \begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} \in L^{\infty}_{2\times 2}(\mathbb{T})$, $R_H \overset{*}{\sim} A\mathbb{P}_+ + B\mathbb{P}_-$.

Proof. Let $H_1 = \begin{pmatrix} g & \psi \\ f & \phi \end{pmatrix}$, an easy computation shows that

$$\begin{aligned}
&\begin{pmatrix} P_+ & P_- \\ P_- & P_+ \end{pmatrix}(A\mathbb{P}_+ + B\mathbb{P}_-)\begin{pmatrix} I_{L^2} & 0 \\ R_{H_1} & -I_{L^2} \end{pmatrix} \\
=&\begin{pmatrix} P_+ & P_- \\ P_- & P_+ \end{pmatrix}\begin{pmatrix} fP_+ + \varphi P_- & -P_- \\ gP_+ + \psi P_- & -P_+ \end{pmatrix}\begin{pmatrix} I_{L^2} & 0 \\ R_{H_1} & -I_{L^2} \end{pmatrix} \\
=&\begin{pmatrix} P_+fP_+ + P_+\phi P_- + P_-gP_+ + P_-\psi P_- & 0 \\ P_-fP_+ + P_-\psi P_- + P_+gP_+ + P_+\psi P_- & -I_{L^2} \end{pmatrix}\begin{pmatrix} I_{L^2} & 0 \\ R_{H_1} & -I_{L^2} \end{pmatrix} \\
=&\begin{pmatrix} R_H & 0 \\ R_{H_1} & -I_{L^2} \end{pmatrix}\begin{pmatrix} I_{L^2} & 0 \\ R_{H_1} & -I_{L^2} \end{pmatrix} \\
=&\begin{pmatrix} R_H & 0 \\ 0 & I_{L^2} \end{pmatrix}.
\end{aligned}$$

The operators $\begin{pmatrix} P_+ & P_- \\ P_- & P_+ \end{pmatrix}$ and $\begin{pmatrix} I_{L^2} & 0 \\ R_{H_1} & -I_{L^2} \end{pmatrix}$ are invertible, and

$$\begin{pmatrix} P_+ & P_- \\ P_- & P_+ \end{pmatrix}^{-1} = \begin{pmatrix} P_+ & P_- \\ P_- & P_+ \end{pmatrix},$$

$$\begin{pmatrix} I_{L^2} & 0 \\ R_{H_1} & -I_{L^2} \end{pmatrix}^{-1} = \begin{pmatrix} I_{L^2} & 0 \\ R_{H_1} & -I_{L^2} \end{pmatrix}.$$

Hence the operators R_H and $A\mathbb{P}_+ + B\mathbb{P}_-$ are equivalent after extension. □

If f and ψ are invertible, then A and B are invertible and

$$A^{-1} = \begin{pmatrix} f^{-1} & 0 \\ f^{-1}g & -1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 0 & \psi^{-1} \\ -1 & \phi\psi^{-1} \end{pmatrix},$$

In this case

$$\begin{aligned}
&A\mathbb{P}_+ + B\mathbb{P}_- \\
=&B(B^{-1}A\mathbb{P}_+ + \mathbb{P}_-) \\
=&B(\mathbb{P}_+B^{-1}A\mathbb{P}_+ + \mathbb{P}_+B^{-1}A\mathbb{P}_+\mathbb{P}_-B^{-1}A\mathbb{P}_+ + \mathbb{P}_-B^{-1}A\mathbb{P}_+ + \mathbb{P}_-) \\
=&B(\mathbb{P}_+B^{-1}A\mathbb{P}_+(I + \mathbb{P}_-B^{-1}A\mathbb{P}_+) + \mathbb{P}_-(\mathbb{P}_-B^{-1}A\mathbb{P}_+ + I)) \\
=&B(\mathbb{P}_+B^{-1}A\mathbb{P}_+ + \mathbb{P}_-)(\mathbb{P}_-B^{-1}A\mathbb{P}_+ + I)
\end{aligned}$$

where $I + \mathbb{P}_-B^{-1}A\mathbb{P}_+$ is invertible on , the inverse is $I - \mathbb{P}_-B^{-1}A\mathbb{P}_+$. This implies

$$A\mathbb{P}_+ + B\mathbb{P}_- \sim \mathbb{P}_+B^{-1}A\mathbb{P}_+ + \mathbb{P}_-. \tag{5.2}$$

Moreover, under the decomposition $L^2_2(\mathbb{T}) = H^2_2(\mathbb{T}) \oplus (H^2(\mathbb{T}))^\perp_2$, we have

$$\mathbb{P}_+ B^{-1} A \mathbb{P}_+ + \mathbb{P}_- = \begin{pmatrix} T_{B^{-1}A} & 0 \\ 0 & I_{(H^2(\mathbb{T}))^\perp_2} \end{pmatrix},$$

where $T_{B^{-1}A}$ is a block Toeplitz operator on $H^2_2(\mathbb{T})$ and

$$B^{-1}A = \begin{pmatrix} g\psi^{-1} & -\psi^{-1} \\ g\phi\psi^{-1} - f & -\phi\psi^{-1} \end{pmatrix}, \tag{5.3}$$

$\det B^{-1}A = -f\psi^{-1}$. Hence,

$$\mathbb{P}_+ B^{-1} A \mathbb{P}_+ + \mathbb{P}_- \overset{*}{\sim} T_{B^{-1}A}. \tag{5.4}$$

Similarly,

$$A\mathbb{P}_+ + B\mathbb{P}_- = A(\mathbb{P}_+ + \mathbb{P}_- A^{-1} B \mathbb{P}_-)(\mathbb{P}_+ A^{-1} B \mathbb{P}_- + I)$$

This implies

$$A\mathbb{P}_+ + B\mathbb{P}_- \sim \mathbb{P}_+ + \mathbb{P}_- A^{-1} B \mathbb{P}_-.$$

and

$$\mathbb{P}_+ + \mathbb{P}_- A^{-1} B \mathbb{P}_- \overset{*}{\sim} \mathbb{J} T_{(A^{-1}B)^*} \mathbb{J}.$$

where $\mathbb{J}(f, f)^T = (Jf, Jf)^T = (\bar{z}\bar{f}, \bar{z}\bar{f})^T$ for $f \in L^2(\mathbb{T})$.

Recall the invertibility and Fredholm of Toeplitz operators with matrix-symbols via Wiener-Hopf factorization.

Definition 5.4. A representation of the form $F = F_- D F_+$ is called Winer-Hopf factorization of the invertible matrix function $F \in L^\infty_{N\times N}(\mathbb{T})$ if $D = diag(z^{\kappa_j})_{j=1}^N$ with $\kappa_j \in \mathbb{Z}$, and if F_- and F_+ satisfy the following conditions:

1. $F_+, F_+^{-1} \in H^2_{N\times N}(\mathbb{T})$, $F_-, F_-^{-1} \in \overline{H^2_{N\times N}(\mathbb{T})}$,
2. The operator $F_+^{-1}\mathbb{P}_+ F_+$ is defined on the linear space of all $\mathbb{C}^N$-valued trigonometric polynomials, can be extended to a bounded operator on $H^2_N(\mathbb{T})$.

Theorem 5.5. [30] *Let $F \in L^\infty_{N\times N}(\mathbb{T})$. Then T_F is invertible(resp. Fredholm) if and only if F admits a Wiener-Hopf factorization.$F = F_- F_+$(resp. $F = F_- D F_+$).*

If T_a is Fredholm, then

$$\dim \operatorname{Ker} T_a = -\sum_{\kappa_j < 0} \kappa_j, \quad \dim \operatorname{Coker} T_a = \sum_{\kappa_j > 0} \kappa_j.$$

Theorem 5.6. *If $H = \begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} \in L^\infty_{2\times 2}(\mathbb{T})$, then R_H is invertible (resp. Fredholm) if and only if f and ψ are invertible in $L^\infty(\mathbb{T})$ and $\begin{pmatrix} g\psi^{-1} & -\psi^{-1} \\ g\phi\psi^{-1} - f & -\phi\psi^{-1} \end{pmatrix}$ admit a Winer-Hopf factorization $F_- F_+$(resp.$F_- D F_+$).*

If R_H is Fredholm, then

$$\dim \operatorname{Ker} R_H = -\sum_{k_j < 0} k_j, \quad \dim \operatorname{Ker} R_H^* = \sum_{k_j > 0} k_j.$$

Proof. If R_H is invertible or Fredholm, by Corollary 4.4, we have f and ψ are invertible in $L^\infty(\mathbb{T})$. Since the relation $\stackrel{*}{\sim}$ is transitive, combining Theorem 5.3, (5.2) and (5.4), it follows that $R_H \stackrel{*}{\sim} T_{B^{-1}A}$. Using Theorem 5.2 and Theorem 5.5, we get the result. □

6. Applications

6.1. **The Spectral Inclusion Theorem.** In the theory of Toeplitz operator, the spectrum of T_ϕ always includes the essential range of ϕ. Corollary 4.4 shows that

$$Ran_{ess}f \cup Ran_{ess}\psi \subset \sigma(R(f, g, \phi, \psi)).$$

Hence, for the bounded singular integral operator $R_{\alpha,\beta}$, we have

$$Ran_{ess}\alpha \cup Ran_{ess}\beta \subset \sigma(R_{\alpha,\beta}),$$

for the bounded dual truncated Toeplitz operator D_ϕ, we have

$$Ran_{ess}\phi \subset \sigma(D_\phi);$$

for the bounded Foguel-Hankel operator $\begin{pmatrix} T_z^* & X \\ 0 & T_z \end{pmatrix}$, we have

$$\mathbb{T} \subset \sigma \begin{pmatrix} T_z^* & X \\ 0 & S \end{pmatrix}.$$

Moreover, for every constant λ, we have

$$\lambda I - \begin{pmatrix} T_z^* & X \\ 0 & T_z \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \lambda I - T_z \end{pmatrix} \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda I - T_z^* & 0 \\ 0 & I \end{pmatrix}.$$

Note that

$$\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}$$

is always invertible and

$$\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}.$$

If both of $\lambda I - T_z$ and $\lambda I - T_z^*$ are invertible, then $\lambda I - \begin{pmatrix} T_z^* & X \\ 0 & T_z \end{pmatrix}$ is invertible. Therefore,

$$\sigma \begin{pmatrix} T_z^* & X \\ 0 & T_z \end{pmatrix} \subset \sigma(T_z) = \overline{\mathbb{D}}.$$

6.2. **Essential spectrum.** The essential spectrum of Toeplitz operator with continous symbol equals the essential range of the symbol. Corollary 4.9 shows that If $f, g, \phi, \psi \in C(\mathbb{T})$, then

$$\sigma_e(R(f, g, \phi, \psi)) = f(\mathbb{T}) \cup \psi(\mathbb{T}).$$

Hence, for bounded singular integral operator, if $\alpha, \beta \in C(\mathbb{T})$, then

$$\sigma_e(R_{\alpha,\beta}) = \alpha(\mathbb{T}) \cup \beta(\mathbb{T}).$$

For bounded dual truncated Toeplitz operator, if $\varphi \in C(\mathbb{T})$, then

$$\sigma_e(D_\varphi) = \varphi(\mathbb{T}).$$

For bounded Foguel-Hankel operator, if $X = \Gamma_\phi$ and $\phi \in H^\infty + C(\mathbb{T})$, then

$$\sigma_e \begin{pmatrix} T_z^* & X \\ 0 & T_z \end{pmatrix} = \mathbb{T}.$$

6.3. **Special cases.** We consider one of operators T_f and S_ψ is invertible. In particular, suppose $S_\psi = I$, Suppose that $\lambda \notin Ran_{ess} f \cup \{1\}$. Now

$$\begin{pmatrix} T_{f-\lambda} & H_{\bar{\phi}}^* \\ H_g & I-\lambda \end{pmatrix} = \begin{pmatrix} I & H_{\bar{\phi}}^* \\ 0 & \frac{1}{1-\lambda} I \end{pmatrix} \begin{pmatrix} T_{f-\lambda} - H_{\bar{\phi}}^* H_g & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ (1-\lambda) H_g & I \end{pmatrix}.$$

Since $\begin{pmatrix} I & H_{\bar{\phi}}^* \\ 0 & \frac{1}{1-\lambda} I \end{pmatrix}$ and $\begin{pmatrix} I & 0 \\ (1-\lambda) H_g & I \end{pmatrix}$ are always invertible, or $\begin{pmatrix} I & H_{\bar{\phi}}^* \\ 0 & \frac{1}{1-\lambda} I \end{pmatrix}^{-1} = \begin{pmatrix} I & -(1-\lambda) H_{\bar{\phi}}^* \\ 0 & (1-\lambda) I \end{pmatrix}$ and $\begin{pmatrix} I & 0 \\ (1-\lambda) H_g & I \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -(1-\lambda) H_g & I \end{pmatrix}$, it follows that

$$\begin{pmatrix} T_{f-\lambda} & H_{\bar{\phi}}^* \\ H_g & I-\lambda \end{pmatrix}$$

is invertible if and only if

$$\begin{pmatrix} T_{f-\lambda} - H_{\bar{\phi}}^* H_g & 0 \\ 0 & I \end{pmatrix}$$

is invertible. Therefore, we have

$$\sigma(R(f, g, \phi, 1)) = \sigma(T_f - H_{\bar{\phi}}^* H_g) \cup Ran_{ess} f \cup \{1\}.$$

Since $T_f - H_{\bar{\phi}}^* H_g = T_f - T_{\phi g} + T_\phi T_g$, we have

$$\lim_{r \to 1^-} \langle T_f - H_{\bar{\phi}}^* H_g k_{r\xi}, k_{r\xi} \rangle = f(\xi) \quad a.e. \text{ on } \mathbb{T}.$$

By Corollary 4.4, we have $Ran_{ess} f \subset \sigma(T_f - H_{\bar{\phi}}^* H_g)$. Hence,

$$\sigma(R(f, g, \phi, 1)) = \sigma(T_f - H_{\bar{\phi}}^* H_g) \cup \{1\}.$$

Similarly,

$$\sigma(R(1, g, \phi, \psi)) = \sigma(S_\psi - H_g H_{\bar{\phi}}^*) \cup \{1\}.$$

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