ANOSOV FLOWS AND DYNAMICAL ZETA FUNCTIONS (ERRATA)

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ABSTRACT. This errata fixes a mistake in the part of [1] which proves a spectral gap for contact Anosov flows with respect to the measure of maximal entropy ([1, Section 7]). However, the first part of [1], in which it is proved that the Ruelle zeta function is meromorphic, is unaffected.

1. THE MISTAKE

Equation [1, Equation (7.14)] is wrong, since it does not take into account the factor $e^{z\tau_W\circ H_{\beta,\boldsymbol{i},W}}$ from [1, Equation (7.10)] and estimates incorrectly the norm of $\varphi_{k,\beta,\boldsymbol{i}}\circ H_{\beta,\boldsymbol{i},W}$. The correct version of [1, Equation (7.14)] is (see (6) below): let d_s be the dimension of the stable manifolds, then for each $\eta\in(0,1)$,[1]

$$(1)\quad \|e^{z\tau_W\circ H_{\beta,\boldsymbol{i},W}}\hat g_{k,\beta,\boldsymbol{i},W}\|_{\Gamma_c^{d_s,\varpi'}(\widetilde W_{\beta,\boldsymbol{i}})}\leq C_\#(r^{-1}+|z|)\frac{(kr)^{n-1}e^{-akr}}{(n-1)!}\|g\|_{\Gamma_c^{d_s,1+\eta}(W)}.$$

Unfortunately, this weaker estimate does not suffice to carry out the proof of [1, Proposition 7.5] as presented in [1] (i.e. using [1, Equation (7.16)]).

2. CORRECTION

Nevertheless, [1, Theorem 2.4] holds under a stronger assumption (namely some homogeneity), as we shall show here. In particular, it applies to small perturbations of constant curvature geodesic flows in any dimension. To simplify the argument we did not try to optimise the estimate of the size of the perturbation. Before stating the correct results we need to recall and introduce some notation.

Let $C_0, c_0>0$, and $\lambda_+(x,t),\lambda_-(x,t)>0$ be such that, for each $x\in M$ and $t>0$, $\sup_{v\in E^s(x)}\frac{\|D_x\phi_{-t}v\|}{\|v\|}\leq C_0e^{\lambda_+(x,t)}$ and $\inf_{v\in E^s(x)}\frac{\|D_x\phi_{-t}v\|}{\|v\|}\geq c_0e^{\lambda_-(x,t)}$. Also, let

$$\lambda_+(t)=\sup_{x\in M}\lambda_+(x,t)\ ;\quad \lambda_-(t)=\inf_{x\in M}\lambda_-(x,t).$$

and, for some n_0 large enough, $\lambda_+=\sup_{t\geq n_0}\frac{\lambda_+(t)}{t}$, $\lambda_-=\inf_{t\geq n_0}\frac{\lambda_-(t)}{t}$. In [1] we use the notation $\hat\varpi=\frac{2\lambda_-}{\lambda_+}$, $\varpi'=\min\{1,\hat\varpi\}$.

Next, we introduce a parameter $\vartheta>0$ which measures the homogeneity. Let $J^s\phi_t$ be the stable Jacobian of the flow. Given $n_0\in\mathbb{N}$, we assume, for all $t\geq n_0$,

$$(1)\qquad \left[\sup_{x\in M}\ln J^s\phi_t(x)-\inf_{x\in M}\ln J^s\phi_t(x)\right]\leq\vartheta\lambda_-(t)d_s.$$

With this notation we can state a correct version of [1, Theorem 2.4].

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[1]See below for the definition of $\lambda_+,\lambda_-,\varpi'$.

Theorem 1. *For any $\mathcal{C}^r, r > 2$, contact flow with*

$$\frac{\sqrt{5}-1}{2} < \hat{\varpi}\ ; \quad \vartheta < \frac{(\varpi')^2 + \varpi' - 1}{2d_s(1+\varpi')}$$

there exists $\tau_* > 0$ such that the Ruelle zeta function is analytic in $\{z \in \mathbb{C} : \Re(z) \geq h_{top}(\phi_1) - \tau_*\}$ apart from a simple pole at $z = h_{top}(\phi_1)$.

The rest of this note contains the proof of Theorem 1.

First, it suffices to prove [1, Proposition 7.5] under the hypotheses of Theorem 1, since the derivation of [1, Theorem 2.4] from [1, Proposition 7.5], holds unchanged.

By [1, Remark 7.1] we can restrict the discussion to $d_s = (d-1)/2$ forms. Since the first inequality in [1, Proposition 7.5] is correct, we need only prove the second. Also [1, equation (7.6)] is correct, hence it suffices to estimate the $\|\cdot\|_{1+\eta}$ norm of some power of $\widehat{R}_n(z)$. Indeed, by [1, equation (7.7) and the previous displayed equation], for each $z = a + ib$, $a \geq \sigma_{d_s}$, and $\eta \in [0,1]$,

$$\begin{aligned} \left\|\widehat{R}_n(z)^3 h\right\|_\eta^s \leq &\frac{C_\eta}{(a - h_{\text{top}}(\phi_1) + \lambda\eta)^n (a - h_{\text{top}}(\phi_1))^{2n}} \|h\|_\eta^s \\ &+ \frac{C_\eta}{(a - h_{\text{top}}(\phi_1))^n} \left\|\widehat{R}_n(z)^2 h\right\|_{1+\eta}^s. \end{aligned} \tag{2}$$

We will use the above equation instead of [1, equation (7.7)].

Remark 2. *The estimate (2) can be restricted to forms proportional to the volume on the stable manifold. More precisely, given a stable manifold W, if $\{v_i^s\}_{i=1}^{d_s}$, $\{v_i^u\}_{i=1}^{d_s}$ are a base for the tangent space of W and the unstable foliation, respectively, and $\{dx_i\}_{i=1}^{2ds+1} = \{dx_i^s, dx_i^u\}_{i=1}^{d_s} \cup \{dx_0\}$ the dual base (dx_0 being the flow direction), then for all g not proportional to $w^s := dx_1^s \wedge \cdots \wedge dx_{d_s}^s$ we have*

$$\left|\int_W \langle g, \widehat{R}_n(z)^3 h\rangle\right| \leq \frac{C_\eta}{(a - h_{top}(\phi_1) + \lambda\eta)^{3n}} \|h\|_\eta^s,$$

which yields already the required estimate. Hence, from now on, by $\Gamma_c^{d_s,\alpha}(\widetilde{W}_+)$, defined in [1, Section 3.2]*, we mean the subset of forms proportional to w^s.*

Remark 3. *If $v \in \mathcal{V}^u$,*[2] *then the Lie derivative L_v acting on the above d_s forms is well defined even for Hölder vector fields. Indeed, the pushforward by the flow generated by v yields a quantity proportional to the Jacobian of the unstable holonomy which is well defined, together with its derivative along the unstable direction.*

Next, we must estimate the right hand side of (2): let $g \in \hat{\Gamma}_c^{d_s,1+\eta}$ and $h \in \Omega_{0,1}^{d_s}$,

$$\begin{aligned} &\int_{W_{\alpha,G}} \langle g, \widehat{R}_n(z)^2 h\rangle = \sum_{k,\beta,\boldsymbol{i}} \sum_{W \in \mathcal{W}_{k,\beta,\boldsymbol{i}}} \int_{\widetilde{W}} \langle \hat{g}_{k,\beta,\boldsymbol{i}}, \widehat{R}_n(z) h\rangle \\ &\hat{g}_{k,\beta,\boldsymbol{i}} = \varphi_{k,\beta,\boldsymbol{i}} \frac{(kr + \tau_W)^{n-1} J_W \phi_{kr} \circ \phi_{\tau_W}}{e^{z(kr+\tau_W)}(n-1)!} * \phi^*_{kr+\tau_W} * g \\ &\varphi_{k,\beta,\boldsymbol{i}}(x) = \psi_\beta(x) \Phi_{r,\boldsymbol{i}}(\Theta_\beta(x)) p(r^{-1}\tau_W(x)) \|V(x)\|^{-1}. \end{aligned} \tag{3}$$

[2] These are the unstable vector fields, see [1, Definition 7.2] for a precise definition.

Recall that $\tau_W : \widetilde{W} \doteq \cup_{t\in[-2r,2r]}\phi_t W \to \mathbb{R}$ is defined by $\phi_{-\tau_W(x)}(x) \in W$.[3]

Next, as in [1, Equation (7.13)], we want to "project" $\hat{g}_{k,\beta,\boldsymbol{i}}$ from $\widetilde{W}$ to some preferred manifold $\widetilde{W}_{k,\beta,\boldsymbol{i}}$. To this end, we need a refinement of [1, Lemma 7.3].

Lemma 4. *For each $\alpha \in \mathcal{A}$, $W, W' \in \Sigma_\alpha$ such that $H_{W,W'}(\widetilde{W}) \subset \widetilde{W}'_+$, $\varphi \in \Gamma_c^{d_s,q}(\widetilde{W})$, $q \in [0,1]$, supported in a ball of size r, there exists $\hat{\varphi} \in \Gamma_c^{d_s,q\varpi'}(\widetilde{W}'_+)$, $\|\hat{\varphi}\|_{\Gamma_c^{d_s,q\varpi'}(\widetilde{W}'_+)} \leq C_\# \|\varphi\|_{\Gamma_c^{d_s,q}(\widetilde{W})}$, such that for all $h \in \Omega_r^{d_s}$ we have*

$$\left| \int_{\widetilde{W}} \langle \varphi, h\rangle - \int_{\widetilde{W}'_+} \langle \hat{\varphi}, h\rangle \right| \leq C_\# r d(W,W') \, \|h\|^u \, \|\varphi\|_{\Gamma_c^{d_s,0}(\widetilde{W})}.$$

Proof. Working in appropriate coordinates we can write $\widetilde{W}'_+$ as $\{(\xi, 0, \tau)\}_{(\xi,\tau)\in\mathbb{R}^{d_s+1}}$, and $\widetilde{W}$ as $\{(\xi, \mathbb{e}, \tau)\}_{(\xi,\tau)\in\mathbb{R}^{d_s+1}}$ for $\mathbb{e} = d(W,W')e_1$.

We can describe the unstable foliation by $\mathbb{U}(\xi,\eta,\tau) = (U(\xi,\eta), \eta, \Upsilon(\xi,\eta) + \tau)$, $\mathbb{U}(\xi, 0, \tau) = (\xi, 0, \tau)$. Then the intersection between $\widetilde{W}_\upsilon = \{(\xi, \upsilon\mathbb{e}, \tau)\}_{(\xi,\tau)\in\mathbb{R}^{d_s+1}}$ and the fiber $\mathbb{U}(\xi, \cdot, \tau)$ gives the holonomy $\mathbb{H}_\upsilon(\xi,\tau) = (U(\xi,\upsilon\mathbb{e}), \upsilon\mathbb{e}, \Upsilon(\xi,\upsilon\mathbb{e}) + \tau)$. As mentioned in Remark 2, $\varphi = \bar{\varphi}\, d\xi_1 \wedge \cdots \wedge d\xi_{d_s}$, hence we can assume w.l.o.g. that $h = \bar{h}\, d\xi_1 \wedge \cdots \wedge d\xi_{d_s}$, for some function $\bar{h}$. It is then natural to define, for each $\xi \in \mathbb{R}^{d_s}$, $\tau \in \mathbb{R}$ and $\eta \in \mathbb{R}^{d_s}$,

$$\overline{\mathbb{H}}_s(\xi,\eta,\tau) = \mathbb{U}(\mathbb{U}^{-1}(\xi,\eta,\tau) + (0, s\mathbb{e}, 0)).$$

Since, $\overline{\mathbb{H}}_0(\xi,\eta,\tau) = (\xi,\eta,\tau)$ and $\overline{\mathbb{H}}_r \circ \overline{\mathbb{H}}_s = \overline{\mathbb{H}}_{r+s}$, we have just defined a flow, let $\bar{w} = (w, \mathbb{e}, \sigma)$ be the associated vector field. Note that, by construction, $\bar{w}$ is a vector field in the unstable direction. By the regularity of the holonomy (see the discussion at the beginning of [1, Appendix E]), we have $\|\bar{w}\| \leq C_\# d(W,W')$. Hence, $\hat{w} = d(W,W')^{-1}\bar{w} \in \mathcal{V}^u$.

Since $\overline{\mathbb{H}}_s^* h = \bar{h} \circ \overline{\mathbb{H}}_s \, J\overline{\mathbb{H}}_s \, d\xi_1 \wedge \cdots \wedge d\xi_{d_s}$, $J\overline{\mathbb{H}}_s$ is the Jacobian of $\overline{\mathbb{H}}_s$, we have

$$\begin{aligned}\int_{\widetilde{W}} \langle \varphi, h\rangle &= \int_{\widetilde{W}'_+} \langle \varphi, h\rangle \circ \mathbb{H}_1 \cdot J\mathbb{H}_1 = \int_{\widetilde{W}'_+} \bar{\varphi} \circ \mathbb{H}_1 \bar{h} \circ \overline{\mathbb{H}}_1 \, J\overline{\mathbb{H}}_1 \\ &= \int_{\widetilde{W}'_+} \int_0^1 \bar{\varphi} \circ \mathbb{H}_1 \frac{d}{ds}\left(\bar{h} \circ \overline{\mathbb{H}}_s \, J\overline{\mathbb{H}}_s\right) ds + \int_{\widetilde{W}'_+} \bar{\varphi} \circ \mathbb{H}_1 \bar{h}.\end{aligned}$$

Since

$$\begin{aligned}\frac{d}{ds}\left(\bar{h} \circ \overline{\mathbb{H}}_s \, J\overline{\mathbb{H}}_s\right) &= \frac{d}{ds}\langle d\xi_1 \wedge \cdots \wedge d\xi_{d_s}, \overline{\mathbb{H}}_s^* h\rangle = \langle d\xi_1 \wedge \cdots \wedge d\xi_{d_s}, \mathbb{H}_s^* L_{\bar{w}} h\rangle \\ &= \langle d\xi_1 \wedge \cdots \wedge d\xi_{d_s}, L_{\bar{w}} h\rangle \circ \overline{\mathbb{H}}_s J\overline{\mathbb{H}}_s,\end{aligned}$$

it is convenient to define,

$$\begin{aligned}\hat{\varphi} &= \bar{\varphi} \circ \mathbb{H}_1(\xi,\tau) d\xi_1 \wedge \cdots \wedge d\xi_{d_s} \\ \psi_s &= \bar{\varphi} \circ \mathbb{H}_1 \circ \overline{\mathbb{H}}_s^{-1} d\xi_1 \wedge \cdots \wedge d\xi_{d_s},\end{aligned} \tag{4}$$

[3] The point of the above equation is that it allows one to go from an integral over a strong stable manifold to integrals over weak stable manifolds. See [1, Section 3] for the necessary definitions. To compare the formulae below with [1, equations (7.9, 7.10, 7.11)] recall that the flow is contact, hence $J\phi_t = 1$, and $(-1)^{d_s(d-d_s)} = (-1)^{d_s(d_s+1)} = 1$. Also, recall that $\sum_{k\in\mathbb{Z}} p(k+t) = 1$ and $\text{supp}(p) \subset (-1, 1)$. Finally, the minus sign in front of z in [1, equation (7.10)] is a misprint and, just before [1, Equation (7.9)], the definition of τ_W has a minus sign missing due to a misprint.

which, setting $\widetilde{W}_s = \{(\xi, s\mathbb{e}, \tau)\}_{(\xi,\tau)\in\mathbb{R}^{d_s+1}}$, allows to write

$$\begin{aligned}\int_{\widetilde{W}} \langle\varphi, h\rangle &= \int_{\widetilde{W}'_+}\int_0^1 \langle\psi_s, L_{\bar{w}}h\rangle \circ \overline{\mathbb{H}}_s\, J\overline{\mathbb{H}}_s ds + \int_{\widetilde{W}'_+}\langle\hat{\varphi}, h\rangle\\ &= \int_{\widetilde{W}'_+}\int_0^1 \langle\psi_s, L_{\bar{w}}h\rangle \circ \mathbb{H}_s\, J\mathbb{H}_s ds + \int_{\widetilde{W}'_+}\langle\hat{\varphi}, h\rangle\\ &= d(W, W')\int_0^1 ds \int_{\widetilde{W}_s}\langle\psi_s, L_{\hat{w}}h\rangle + \int_{\widetilde{W}'_+}\langle\hat{\varphi}, h\rangle.\end{aligned}$$

From the above equation the Lemma follows, since $\|\psi_s\|_{\Gamma_c^{d_s,0}(\widetilde{W}_s)} \leq C_\# \|\varphi\|_{\Gamma_c^{d_s,0}(\widetilde{W})}$ and $\|\hat{\varphi}\|_{\Gamma_c^{d_s,q\varpi'}(\widetilde{W}')} \leq C_\#\|\varphi\|_{\Gamma_c^{d_s,q}(\widetilde{W})}$. The extra r comes from the size of the support of φ, and hence of ψ_s, in the flow direction. □

Next, following verbatim [1], and using Lemma 4 (with $q = 1$), we obtain the equivalent of [1, Equation (7.13)]: for each $g \in \Gamma_c^{d_s,1+\eta}(W_{\alpha,G})$,

$$\begin{aligned}\int_{W_{\alpha,G}}\langle g, \widehat{R}_n(z)^2 h\rangle = &\sum_{k,\beta,\boldsymbol{i}}\sum_{W\in\mathcal{W}_{k,\beta,\boldsymbol{i}}}\int_{\widetilde{W}_{\beta,\boldsymbol{i}}}\langle\hat{g}_{k,\beta,\boldsymbol{i},W}, \widehat{R}_n(z)h\rangle\\ &+\sum_k \mathcal{O}\left(\frac{r^2(kr)^{n-1}\left\|\widehat{R}_n(z)h\right\|^u \|g\|_{\Gamma_c^{d_s,1+\eta}(W_{\alpha,G})}}{(n-1)!e^{(a-\sigma_{d_s})kr}}\right),\end{aligned}\tag{5}$$

where $\hat{g}_{k,\beta,\boldsymbol{i},W}$ is given by (4), with $\mathbb{H}_1 = \mathbb{H}_{\widetilde{W}_{\beta,\boldsymbol{i}},\widetilde{W}}$ being the holonomy between $\widetilde{W}_{\beta,\boldsymbol{i}}$ and $\widetilde{W}$. Note that equation (4) was implicitly used (but missing) in [1].

Next, we slightly depart from [1] insofar as we simplify immediately the expression of $\hat{g}_{k,\beta,\boldsymbol{i},W}$ instead of doing it during the proof of [1, Lemma 7.10].

By (3) we can write

$$\hat{g}_{k,\beta,\boldsymbol{i}} = \varphi_{k,\beta,\boldsymbol{i}}e^{-z\tau_W}\check{g}_{k,\beta,\boldsymbol{i}}.$$

Hence, setting $\varphi_{k,\beta,\boldsymbol{i},W} = \varphi_{k,\beta,\boldsymbol{i}} \circ \mathbb{H}_{\widetilde{W}_{\beta,\boldsymbol{i}},\widetilde{W}}$, by the first line of (4) we have

$$\hat{g}_{k,\beta,\boldsymbol{i},W} = \varphi_{k,\beta,\boldsymbol{i},W}e^{-z\tau_W\circ\mathbb{H}_{\widetilde{W}_{\beta,\boldsymbol{i}},\widetilde{W}}}\check{g}_{k,\beta,\boldsymbol{i},W},$$

where, recalling [1, Equation (7.12)],

$$\begin{aligned}&\|\check{g}_{k,\beta,\boldsymbol{i},W}\|_{\Gamma^{\varpi'}(\widetilde{W}_{\beta,\boldsymbol{i}})} \leq C_\#\frac{(kr)^{n-1}e^{-akr}}{(n-1)!}\|g\|_{\Gamma_c^{d_s,1+\eta}(W)}\\ &\|\varphi_{k,\beta,\boldsymbol{i},W}\|_{\Gamma^{\varpi'}(\widetilde{W}_{\beta,\boldsymbol{i}})} \leq C_\# r^{-1}.\end{aligned}\tag{6}$$

Hence, setting $\mathbb{c}_{k,\beta,\boldsymbol{i},W} = \check{g}_{k,\beta,\boldsymbol{i},W}(x^{\boldsymbol{i}})$,

$$\|\check{g}_{k,\beta,\boldsymbol{i},W} - \mathbb{c}_{k,\beta,\boldsymbol{i},W}\|_{\Gamma_c^0(\widetilde{W}_{\beta,\boldsymbol{i}})} \leq C_\# r^{\varpi'}\frac{(kr)^{n-1}e^{-akr}}{(n-1)!}\|g\|_{\Gamma_c^{d_s,1+\eta}(W)}.\tag{7}$$

Next, setting $\Delta_W^*(\xi) = \tau_W \circ \mathbb{H}_{\widetilde{W}_{\beta,\boldsymbol{i}},\widetilde{W}}(\xi) - \xi_{2d_s+1}$ and $w_W(\xi) = \mathbb{H}_{\widetilde{W}_{\beta,\boldsymbol{i}},\widetilde{W}}(\xi) - \xi$,[4] we have that [1, Equations (7.27) and (7.30)] implies, for all $\zeta = (\tilde{\zeta}, 0)$ with $\|\tilde{\zeta}\| \leq r$,

$$\begin{aligned}\left\|\Delta_W^*(\xi+\zeta) - \Delta_W^*(\xi) - d\alpha_0(w_W(x^{\boldsymbol{i}}),\zeta)\right\| &\leq C_\# r^{2-\varpi'}\|w_W(x^{\boldsymbol{i}})\|^{\varpi'}\|\zeta\|^{\varpi'}\\ &\leq C_\# r^{2+\varpi'}.\end{aligned}\tag{8}$$

[4] Here, again, we are computing using some appropriate coordinates.

We impose, for ς small enough,

$$|b| \le r^{-2-\varpi'+\varsigma}\,, \tag{9}$$

and define

$$\begin{aligned}
\mathfrak{g}_{k,\beta,\boldsymbol{i}} &= \sum_{W\in\mathcal{W}_{k,\beta,\boldsymbol{i}}} \hat{g}_{k,\beta,\boldsymbol{i},W}\\
\mathfrak{g}_{k,\beta,\boldsymbol{i},W}(\xi) &= \varphi_{k,\beta,\boldsymbol{i},W}(\xi) e^{-zd\alpha_0(w_W(x^{\boldsymbol{i}}),\xi-x^{\boldsymbol{i}})} e^{-z(\Delta^*_W(x^{\boldsymbol{i}})+\xi_{2d_s+1})}\mathfrak{C}_{k,\beta,\boldsymbol{i},W}\\
&\doteq \varphi_{k,\beta,\boldsymbol{i},W}(\xi) e^{-zd\alpha_0(w_W(x^{\boldsymbol{i}}),\xi-x^{\boldsymbol{i}})}\hat{\mathfrak{C}}_{k,\beta,\boldsymbol{i},W}(\xi_{2d_s+1})\\
\mathfrak{g}^*_{k,\beta,\boldsymbol{i}} &\doteq \sum_{W\in\mathcal{W}_{k,\beta,\boldsymbol{i}}} \mathfrak{g}_{k,\beta,\boldsymbol{i},W}\,.
\end{aligned} \tag{10}$$

Letting $D_{k,\beta,\boldsymbol{i}} = \frac{(kr)^{n-1}e^{-ark}}{(n-1)!}\#\mathcal{W}_{k,\beta,\boldsymbol{i}}$ and recalling (8), (9) and [1, Equation (7.30)], we can write, for $\varsigma \le \varpi'$,

$$\begin{aligned}
&\left\|\mathfrak{g}_{k,\beta,\boldsymbol{i}} - \mathfrak{g}^*_{k,\beta,\boldsymbol{i}}\right\|_{\Gamma^0_c(\widetilde{W}_{\beta,\boldsymbol{i}})} \le C_\# r^\varsigma D_{k,\beta,\boldsymbol{i}} \|g\|_{\Gamma^{d_s,1+\eta}}\\
&\|\mathfrak{g}_{k,\beta,\boldsymbol{i}}\|_{\Gamma^{\varpi'}_c(\widetilde{W}_{\beta,\boldsymbol{i}})} + \left\|\mathfrak{g}^*_{k,\beta,\boldsymbol{i}}\right\|_{\Gamma^{\varpi'}_c(\widetilde{W}_{\beta,\boldsymbol{i}})} \le C_\# \left\{\frac{1}{r} + \frac{|b|}{r^{-2+\varpi'}}\right\} D_{k,\beta,\boldsymbol{i}} \|g\|_{\Gamma^{d_s,1+\eta}}.
\end{aligned} \tag{11}$$

Note that, by the definition of $\widehat{R}(z)$ in [1, Section 7.1], [1, Equation (4.11)], [1, Equation (4.17)] and the related notation, for all $n \ge c_\star \ln r^{-1}$, with $c_\star$ large enough,

$$\begin{aligned}
\sum_{k,\beta,\boldsymbol{i}} \left| \int_{\widetilde{W}_{\beta,\boldsymbol{i}}} \langle [\mathfrak{g}_{k,\beta,\boldsymbol{i}} - \mathfrak{g}^*_{k,\beta,\boldsymbol{i}}], \widehat{R}_n(z) h\rangle \right| &= \Bigg| \int_{c_a n}^\infty dt e^{-zt} \frac{t^{n-1}}{(n-1)!}\\
&\quad\times \sum_{k,\beta,\boldsymbol{i}} \sum_{\substack{\beta'\in\mathcal{A}\\ k'\in\widetilde{K}_\beta}} \int_{\widetilde{W}_{\beta',G'_k}} J_{W_{\beta',G'_k}} \phi_t \langle *\phi_t^* * [\mathfrak{g}_{k,\beta,\boldsymbol{i}} - \mathfrak{g}^*_{k,\beta,\boldsymbol{i}}], h\rangle \Bigg|\\
&\le C_\# \sum_{k,\beta,\boldsymbol{i}} \int_{c_a n}^\infty dt e^{-at} \frac{t^{n-1}}{(n-1)!} \sum_{k,\beta,\boldsymbol{i}} \sum_{\substack{\beta'\in\mathcal{A}\\ k'\in\widetilde{K}_\beta}} r^{1+\varsigma} D_{k,\beta,\boldsymbol{i}} \|g\|_{\Gamma_c^{d_s,1+\eta}} \|h\|^s_\eta\\
&\le C_\# \int_{c_a n}^\infty dt \int_{c_a n}^\infty ds\, e^{(h_{\rm top}(\phi_1)-a)(t+s)} \frac{t^{n-1}}{(n-1)!}\frac{s^{n-1}}{(n-1)!} r^\varsigma \|g\|_{\Gamma_c^{d_s,1+\eta}} \|h\|^s_\eta\\
&\le C_\# (a - h_{\rm top}(\phi_1))^{-2n} r^\varsigma \|g\|_{\Gamma_c^{d_s,1+\eta}} \|h\|^s_\eta.
\end{aligned} \tag{12}$$

Hence, by (5), (12) and [1, Equation (7.6)], we can write

$$\begin{aligned}
\int_{W_{\alpha,G}} \langle g, \widehat{R}_n(z)^2 h\rangle &= \sum_{k,\beta,\boldsymbol{i}} \int_{\widetilde{W}_{\beta,\boldsymbol{i}}} \langle \mathfrak{g}^*_{k,\beta,\boldsymbol{i}}, \widehat{R}_n(z) h\rangle + \mathcal{O}\left(\frac{r^\varsigma \|g\|_{\Gamma_c^{d_s,1+\eta}}}{(a - h_{\rm top}(\phi_1))^{2n}} \|h\|^s_\eta \right)\\
&\quad + \mathcal{O}\left(\frac{\|h\|^u_\eta \|g\|_{\Gamma_c^{d_s,1+\eta}}}{(a - h_{\rm top}(\phi_1) + \bar\lambda)^n (a - h_{\rm top}(\phi_1))^n} \right).
\end{aligned} \tag{13}$$

To estimate the integral on the right hand side of (13) we define, similarly to [1]:

$$\begin{aligned}
\mathfrak{G}^*_{k,\beta,\boldsymbol{i},A} &\doteq \sum_{W\in\mathcal{W}_{k,\beta,\boldsymbol{i}}} \sum_{W'\in A_{k,\beta,\boldsymbol{i}}(W)} \langle \mathfrak{g}_{k,\beta,\boldsymbol{i},W}, \mathfrak{g}_{k,\beta,\boldsymbol{i},W'}\rangle\\
\mathfrak{G}^*_{k,\beta,\boldsymbol{i},B} &\doteq \sum_{W\in\mathcal{W}_{k,\beta,\boldsymbol{i}}} \sum_{W'\in B_{k,\beta,\boldsymbol{i}}(W)} \langle \mathfrak{g}_{k,\beta,\boldsymbol{i},W}, \mathfrak{g}_{k,\beta,\boldsymbol{i},W'}\rangle.
\end{aligned}$$

To conclude, we need Lemmata 5 and 6 which are refinements of [1, Lemma 7.9] and [1, Lemma 7.10], respectively. The proof of Lemma 6 follows closely [1, Lemma 7.10], but it applies the same logic to different objects. Conversely, Lemma 5 differs from [1, Lemma 7.9] as we take advantage of our new homogeneity hypothesis (1).

Lemma 5. *If $c_a \geq n_0$ and $C_\# |b|^{-\frac{\bar\lambda C_1 C_6}{2ea_0}} \leq \varrho \leq c_\# r^{\frac{1+\varsigma/d_s}{1-\vartheta}}$, for some $\varsigma > 0$, and $\vartheta \in (0,1)$, we have*

$$\|\mathfrak{G}_{k,\beta,\boldsymbol{i},A}\|_\infty \leq C_\# D^2_{k,\beta,\boldsymbol{i}} r^\varsigma \|g\|^2_{\Gamma_c^{d_s,1+\eta}(W)}. \tag{14}$$

Proof. The lower bound and the fact that the upper bound is bounded by the ratio between the volume of $\phi_t(D^u_r(W))$ and $\phi_t(D^u_\varrho(W))$ is proven exactly as in [1, Lemma 7.9]. The novelty here consists in a different estimate of such a ratio.
Let $t_0 \in \mathbb{N}$ be such that $e^{\lambda_-(t_0)}\varrho = 1$. Then, for each $x \in D^u_\varrho(W)$, let $B(x)$ be an unstable disc of diameter 1 and centred at $\phi_{t_0}(x)$, clearly $B(x) \subset \phi_{t_0}(D^u_{2\varrho}(W))$. Thus we can cover $\phi_{t_0}(D^u_\varrho(W))$ with $N_\varrho = C_\# |\phi_{t_0}(D^u_\varrho(W)|$ discs. On the other hand, arguing analogously, we can find $N_r = C_\# |\phi_{t_0}(D^u_r(W)|$ disjoint unstable discs of diameter 1 contained in $\phi_{t_0}(D^u_r(W)$.

By (1) and since the flow is contact, setting $J^u_- := \inf_{x\in M} J^u\phi_{t_0}(x)$, we have

$$\frac{N_r}{N_\varrho} \geq c_\# \frac{|\phi_{t_0}(D^u_r(W)|}{|\phi_{t_0}(D^u_\varrho(W)|} \geq c_\# \frac{J^u_- r^{d_s}}{J^u_- e^{\vartheta\lambda_-(t_0)d_s}\varrho^{d_s}} = c_\# \left(r\varrho^{\vartheta-1}\right)^{d_s} \geq C_\# r^{-\varsigma}.$$

On the other hand by [1, Lemmata C.1, C.3] we have that all the discs of radius one grow under the dynamics at the same rate (given by the topological entropy), hence for all $t \geq t_0$, we have the required estimate

$$\frac{|\phi_t(D^u_r(W)|}{|\phi_t(D^u_\varrho(W)|} \geq C_\# r^{-\varsigma}.$$

□

Lemma 6. *For $|b| \leq r^{-2-\varpi'+\varsigma}$ we have*

$$\left|\int_{\widetilde W_{\beta,\boldsymbol{i}}} \mathfrak{G}^*_{k,\beta,\boldsymbol{i},B}\right| \leq C_\# |b|^{-\varpi'} \varrho^{-\varpi'} r^{d_s} D^2_{k,\beta,\boldsymbol{i}} \|g\|^2_{\Gamma_c^{d_s,1+\eta}(W)}. \tag{15}$$

Proof. For future convenience let us set $(\eta^s_{W,W',\boldsymbol{i}}, \eta^u_{W,W',\boldsymbol{i}}, \eta^d_{W,W',\boldsymbol{i}}) = \eta_{W,W',\boldsymbol{i}} = w_W(x^{\boldsymbol{i}}) - w_{W'}(x^{\boldsymbol{i}})$ and $\eta^+_{W,W',\boldsymbol{i}} = w_W(x^{\boldsymbol{i}}) + w_{W'}(x^{\boldsymbol{i}})$. By assumption $\|\eta_{W,W',\boldsymbol{i}}\| \geq \varrho$. Also, it is convenient to work in coordinates (ξ,η,τ), $\xi,\eta \in \mathbb{R}^{d_s}$, in which $x_{\boldsymbol{i}} = 0$ and $W_{\beta,\boldsymbol{i}} \subset \{(\xi,0) : \xi \in \mathbb{R}^{d_s}\}$ and $d\alpha_0 = \sum_{i=1}^{d_s} d\xi_i \wedge d\eta_i$. We must estimate

$$\begin{aligned}\int_{\widetilde W_{\beta,\boldsymbol{i}}} \langle \mathfrak{g}_{k,\beta,\boldsymbol{i},W}, \mathfrak{g}_{k,\beta,\boldsymbol{i},W'}\rangle &= \int_{\widetilde W_{\beta,\boldsymbol{i}}} \varphi_{k,\beta,\boldsymbol{i},W}(\xi)\varphi_{k,\beta,\boldsymbol{i},W'}(\xi)\\ &\quad \times \hat{\mathfrak{c}}_{k,\beta,\boldsymbol{i},W}(\tau)\overline{\hat{\mathfrak{c}}_{k,\beta,\boldsymbol{i},W'}(\tau)} e^{-ibd\alpha_0(\eta_{W,W',\boldsymbol{i}},\xi-x^{\boldsymbol{i}})} e^{-ad\alpha_0(\eta^+_{W,W',\boldsymbol{i}},\xi-x^{\boldsymbol{i}})}.\end{aligned} \tag{16}$$

As in [1, Section 7.2] we choose $y_{W,W'} = (-\eta^u_{W,W',\boldsymbol{i}}\|\eta^u_{W,W',\boldsymbol{i}}\|^{-1}, 0, 0)$ which implies that $d\alpha_0(\eta_{W,W',\boldsymbol{i}}, y_{W,W'}) = \|\eta^u_{W,W',\boldsymbol{i}}\|$, and $\langle y_{W,W'}, e_{2d_s+1}\rangle = 0$. Also, let $\Sigma_W = \{(\xi,\tau) \in \mathbb{R}^{d_s+1} \mid \langle \xi, \eta^u_{W,W',\boldsymbol{i}}\rangle = 0\}$ and

$$\begin{aligned}A(\xi,s,\tau) =& \varphi_{k,\beta,\boldsymbol{i},W}((\xi,0,\tau)+sy_{W,W'})\varphi_{k,\beta,\boldsymbol{i},W'}((\xi,0,\tau)+sy_{W,W'})\\ &\times e^{-ad\alpha_0(\eta^+_{W,W',\boldsymbol{i}},(\xi,s,0,\tau)+sy_{W,W'})}.\end{aligned}$$

Then, we can write

$$\int_{\widetilde{W}_{\beta,\boldsymbol{i}}} \langle \mathfrak{g}_{k,\beta,\boldsymbol{i},W}, \mathfrak{g}_{k,\beta,\boldsymbol{i},W'} \rangle = \int_{\Sigma_W} d\tau d\xi \, \hat{\mathfrak{c}}_{k,\beta,\boldsymbol{i},W}(\tau) \overline{\hat{\mathfrak{c}}_{k,\beta,\boldsymbol{i},W'}(\tau)} \int_{-c_\# r}^{c_\# r} ds A(\xi, s, \tau) e^{-ib\|\eta^u_{W,W',\boldsymbol{i}}\|s}.$$

Note that, by (6), $\|A(\xi,\cdot,\tau)\|_{\mathcal{C}^{\varpi'}} \leq C_\# r^{-1}$, hence (as in [1, Lemma 7.10])

$$\left| \int_{-c_\# r}^{c_\# r} ds A(\xi, s) e^{-ib\|\eta^u_{W,W',\boldsymbol{i}}\|s} \right| \leq C_\# |b|^{-\varpi'} \varrho^{-\varpi'}. \qquad \square$$

Here our strategy departs from [1] as we control directly where $\mathfrak{g}^*_{k,\beta,\boldsymbol{i}}$ is large.

Let $\Omega = \{x \in W_{\beta,\boldsymbol{i}} \ : \ \|\mathfrak{g}^*_{k,\beta,\boldsymbol{i}}(x)\| \geq 4C_\flat |r|^{\varsigma/2} D_{k,\beta,\boldsymbol{i}}\}$ and $\Omega_1 = \{x \in W_{\beta,\boldsymbol{i}} \ : \ \mathfrak{G}^*_{k,\beta,\boldsymbol{i},B} \geq C_\flat |r|^\varsigma D^2_{k,\beta,\boldsymbol{i}}\}$, while $\widetilde{\Omega}$ and $\widetilde{\Omega}_1$ are r thickenings in the flow direction. Note that if $x \in \Omega$, then $\phi_t(x) \in \widetilde{\Omega}_1$ for all $t \leq C_\# r$. By Lemma 5, choosing $C_\flat$ large, we have $\Omega \subset \Omega_1$. By Chebychev inequality, Lemma 6 implies

$$|\widetilde{\Omega}| = \int_{\widetilde{W}_{\beta,\boldsymbol{i}}} \mathbb{1}_{\Omega_1} \leq C_\# \int_{\widetilde{W}_{\beta,\boldsymbol{i}}} \mathfrak{G}^*_{k,\beta,\boldsymbol{i},B} |r|^{-\varsigma} D^{-2}_{k,\beta,\boldsymbol{i}} \leq C_\# (|b|\varrho)^{-\varpi'} r^{d_s - \varsigma}.$$

Thus

$$|\Omega| \leq C_\# |b|^{-\varpi'} \varrho^{-\varpi'} r^{d_s - 1 - \varsigma}. \tag{17}$$

If $x \in \Omega$ then, by (10), (11), we have that $\|\mathfrak{g}^*_{k,\beta,\boldsymbol{i}}(y)\| \geq C_\flat |r|^{\varsigma/2} D_{k,\beta,\boldsymbol{i}}$ provided

$$|y - x|^{\varpi'} r^{-1} + |y - x||b| r \leq C_\# r^{\varsigma/2}.$$

The above holds for

$$|y - x| \leq C_\# \min\{r^{\frac{1+\varsigma/2}{\varpi'}}, |b|^{-1} r^{-1+\frac{\varsigma}{2}}\} =: \rho. \tag{18}$$

We are finally ready to prove the stated Theorem.

Proof of Theorem 1. Let $t_0 > 0$ be such that $e^{\lambda_-(t_0)}\rho = 1$. Then, recalling (1),

$$|\phi_{-t_0}(\Omega)| \leq e^{(J^s_-(t_0) + \lambda_-(t_0) d_s \vartheta)} |\Omega|$$
$$|\phi_{-t_0}(W_{\beta,\boldsymbol{i}})| \geq e^{J^s_-(t_0)} r^{d_s}.$$

It follows that if we cover $\phi_{-t_0}(W_{\beta,\boldsymbol{i}})$ by discs of radius 1, then, recalling (17), for each disc that intersects $\phi_{-t_0}(\Omega)$ there are at least

$$K = \frac{r^{d_s}}{e^{\lambda_-(t_0) d_s \vartheta} |\Omega|} \geq c_\# \rho^{\vartheta d_s} b^{\varpi'} \varrho^{\varpi'} r^{1+\varsigma}$$

discs that are disjoint from $\phi_{-t_0}(\Omega)$. Indeed, if a disc intersects $\phi_{-t_0}(\Omega)$, then a disc twice its radius must have a fixed proportion of its volume belonging to $\phi_{-t_0}(\Omega)$. We chose $\varrho = c_\# r^{\frac{1+\varsigma/d_s}{1-\vartheta}}$ (so Lemma 5 applies), $|b| = r^{-2-\varpi'+\varsigma}$ (so Lemma 6 applies). Accordingly, K can be larger than one only if $(\varpi')^2 + \varpi' - 1 > 0$, but then, choosing ς small enough, (18) implies $\rho = C_\# r^{1+\varpi'-\frac{\varsigma}{2}}$, which implies

$$K \geq r^{-\varsigma}$$

provided $\vartheta < \frac{(\varpi')^2 + \varpi' - 1}{2d_s(1+\varpi')}$ and ς is small enough. Again by [1, Lemmata C.1, C.3] this ratio persists under iteration. Hence, for each $t \geq t_0$,

$$|\phi_{-t}(\Omega)| \leq C_\# r^\varsigma |\phi_{-t}(W_{\beta,\boldsymbol{i}})|.$$

If $n \geq C_1 \ln |b|$, for C_1 and b large enough, by [1, Equation (7.12)], $c_a n \geq t_0$ and

$$\|J_W \phi_{-t_0} * \phi^*_{-t_0} * \mathfrak{g}^*_{k,\beta,\boldsymbol{i}}\|_{\Gamma_c^{d_s,\varpi'}(W)} \leq C_\# \|\mathfrak{g}^*_{k,\beta,\boldsymbol{i}} \mathbb{1}_{\phi_{t_0}(W)}\|_{\Gamma_c^{d_s,0}(W_{\beta,\boldsymbol{i}})}.$$

Hence, for each $k' \geq c_a n$, we can decompose $\phi_{-k'} W_{\beta,\boldsymbol{i}} = \cup_{W \in \mathcal{W}_{k'}} W$, $W \in \Sigma^s$ (see [1, Definition 7.2]). We write $\mathcal{W}_{k'} = \mathcal{W}^+_{k'} \cup \mathcal{W}^-_{k'}$, where $W \in \mathcal{W}^-_{k'}$ if $\phi_{k'}(W) \cap \Omega = \emptyset$, while if $W \in \mathcal{W}^+_{k'}$, then $\phi_{k'}(W) \subset \Omega_1$. By the previous discussion, and [1, Lemmata C.1, C.3], we have $\sharp \mathcal{W}^+_{k'} \leq C_\# r^\varsigma \sharp \mathcal{W}_{k'}$. Since,

$$\left| \int_{\widetilde{W}_{\beta,\boldsymbol{i}}} \langle \mathfrak{g}^*_{k,\beta,\boldsymbol{i}}, \widehat{R}_n(z) h \rangle \right| \leq \sum_{k'} \sum_{W \in \mathcal{W}_{k'}} \left| \int_{\widetilde{W}} \langle \boldsymbol{\varphi}_{k',W}, h \rangle \right|$$

with $\|\boldsymbol{\varphi}_{k',W}\|_{\Gamma_c^{d_s,\varpi'}(W)} \leq C_\# \frac{(k'r)^{n-1} e^{ak'r}}{(n-1)!} \|\mathfrak{g}^*_{k,\beta,\boldsymbol{i}} \mathbb{1}_{\phi_{k'}(W)}\|_{\Gamma_c^{d_s,0}(W_{\beta,\boldsymbol{i}})}$, it follows,

$$\begin{aligned}
\sum_{k,\beta,\boldsymbol{i}} \left| \int_{\widetilde{W}_{\beta,\boldsymbol{i}}} \langle \mathfrak{g}^*_{k,\beta,\boldsymbol{i}}, \widehat{R}_n(z) h \rangle \right| &\leq C_\# \sum_{k,\beta,\boldsymbol{i},k'} \frac{(k'r)^{n-1} \left[\sharp \mathcal{W}^-_{k'} |r|^{\varsigma/2} + \sharp \mathcal{W}^+_{k'}\right] D_{k,\beta,\boldsymbol{i}}}{e^{-ak'r}(n-1)!} \|h\|^*_{\varpi'} \\
&\leq C_\# r^{\varsigma/2} \sum_{k,k'} \frac{(kr)^{n-1} e^{-ark}}{(n-1)!} \frac{(k'r)^{n-1} |\phi_{k+k'}(W_{\alpha,G})|}{e^{-ak'r}(n-1)!} \|h\|^*_{\varpi'} \\
&\leq C_\# r^{\varsigma/2} (a - h_{\text{top}}(\phi_1))^{-2n} \|h\|^*_{\varpi'}.
\end{aligned}$$

Using the above inequality in (13) provides an estimate of $\|\widehat{R}_n(z)^2 h\|^s_{1+\eta}$ which, substituted into (2), yields [1, Proposition 7.5].
Theorem 1 follows then as in [1, Theorem 2.4]. □

References

[1] Giulietti, P.; Liverani, C.; Pollicott, M. *Anosov flows and dynamical zeta functions.* Ann. of Math. (2) **178** (2013), no. 2, 687–773.

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