ORTHOGONALITY QUESTIONS IN THE HARDY SPACE RELATED TO ζ-ZEROS

FRANCISCO CALDERARO, JUAN MANZUR, WALEED NOOR AND CHARLES SANTOS

ABSTRACT. A Hardy space approach to the Nyman-Beurling and Báez-Duarte criterion for the Riemann Hypothesis (RH) was introduced recently in [\[19\]](#page-16-0) and further developed in [\[14\]](#page-16-1). It states that the RH holds if and only if a particular sequence of functions $(h_k)_{k\geq 2}$ is complete in the Hardy space H^2 . This article is concerned with orthogonality questions related to the family $(h_k)_{k>2}$. The first goal is to analyze the orthogonal complement of $\mathcal{N} = \text{span}(h_k)_{k \geq 2}$ in H^2 . Unbounded Toeplitz operators on H^p spaces and de Branges-Rovnyak spaces play a central role and our results show that the size and dimension of \mathcal{N}^{\perp} reveal information on the zeros of the Riemann ζ-function. The second goal is to show that $(h_k)_{k\geq 2}$ possesses a complete biorthogonal sequence in H^2 . We also discuss a folklore conjecture about the number of ζ-zeros if the RH fails.

INTRODUCTION

A classical result of Nyman and Beurling (see [\[6\]](#page-16-2), [\[20\]](#page-16-3)) shows that the Riemann Hypothesis (RH) is equivalent to the completeness of $\{\rho_\lambda: 0 \leq \lambda \leq 1\}$ in $L^2(0,1)$ where $\rho_{\lambda}(x) = {\lambda/x} - {\lambda(1/x)}$ and $\{x\}$ denotes the fractional part of x. This is furthermore equivalent to the characteristic function $\chi_{(0,1)}$ belonging to the closed linear span of $\{\rho_{\lambda}: 0 \leq \lambda \leq 1\}$. Half a century later Báez-Duarte [\[2\]](#page-15-0) strenghthened this result by showing that RH is true if and only if $\chi_{(0,1)} \in \text{span}\{\rho_{1/k} : k \geq 2\}.$ See the expository article of Bagchi [\[3\]](#page-15-1) and the survey of Balazard [\[4\]](#page-15-2). Recently the Nyman-Beurling and Baez-Duarte approaches to the RH have been explored via tools from Hardy space H^2 theory [\[19\]](#page-16-0) and other analytic function spaces [\[14\]](#page-16-1).

For each $k \geq 2$, define

$$
h_k(z) = \frac{1}{1-z} \log \left(\frac{1+z+\dots+z^{k-1}}{k} \right)
$$

and denote by N the linear span of $\{h_k : k \geq 2\}$. That each h_k belongs to H^2 was proved in [\[19,](#page-16-0) Lemma 7]. One of the main results of [\[19\]](#page-16-0) was a reformulation of Báez-Duarte's result as a completeness problem in H^2 . Then in [\[14\]](#page-16-1) the same completeness problem in the H^p spaces was shown to provide zero-free half planes for the Riemann ζ-function. We state both results here as one.

Theorem 1. The RH holds if and only if N is dense in H^2 . If N is dense in H^p for some $1 < p \leq 2$, then

$$
\zeta(s) \neq 0
$$
 for $\Re(s) > \frac{1}{p}$.

The density of N in H^1 gives the known zero-free region $\Re(s) \geq 1$.

Key words and phrases. Riemann hypothesis, Hardy space, local Dirichlet space, de Branges-Rovnyak space, Smirnov class.

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Although N is unconditionally dense in H^p for $0 < p < 1$ (see [\[14,](#page-16-1) Cor. 4.6]), this provides no new information regarding ζ -zeros. Also note that the RH is equivalent to $\mathcal{N}^{\perp} = \{0\}$ in H^2 by Theorem [1.](#page-0-0) This suggests the question of the size and dimension of \mathcal{N}^{\perp} and their possible relation to the ζ -zeros. The first archetypal result addressing this is the following (See [\[19,](#page-16-0) Thm. 12]).

Theorem 2. $\mathcal{N}^{\perp} \cap \mathcal{D}_{\delta_1} = \{0\}$, where \mathcal{D}_{δ_1} denotes the local Dirichlet space at 1.

Since $\mathcal{D}_{\delta_1} \subset H^2 \subset H^p$ for $0 < p < 2$ $0 < p < 2$, Theorems [1](#page-0-0) and 2 inspire the following question: which linear spaces of analytic functions on D with $X_1 \subset H^2 \subset X_2$ satisfy

(1)
$$
\mathcal{N}^{\perp} \cap X_1 = \{0\}
$$
 and (2) \mathcal{N} is dense in X_2 ?

Interestingly, both (1) and (2) are equivalent if one takes $X_2 = X$ a Frechet space and $X_1 = X^*$ its Cauchy dual (see Proposition [9\)](#page-6-0). In Section 2 we analyze this orthogonality question using tools from de Branges-Rovnyak spaces and unbounded Toeplitz operators on H^p (see Proposition [11\)](#page-6-1), and in particular develop a new approach for finding zero-free half-planes for the ζ function (see Theorem [14\)](#page-8-0).

In Section 3 we deal with the *biorthogonality question*: Does the sequence $(h_k)_{k\geq 2}$ posses a complete biorthogonal sequence in H^2 ? (see Subsection 1.5). We affirmatively answer this question (see Theorem [15\)](#page-8-1). To provide some context, Vasyunin showed in [\[22\]](#page-16-4) that the Báez-Duarte sequence $\{\rho_{1/k} : k \geq 2\}$ is minimal by constructing a biorthogonal sequence for it in $L^2(0,1)$. Whereas the sequence $\{\rho_{1/k}: k \geq 2\}$ is not complete in $L^2(0,1)$, the completeness of $(h_k)_{k\geq 2}$ in H^2 is equivalent to the RH by Theorem [1.](#page-0-0) The property of completeness of a sequence is not in general inherited by its biorthogonal sequence. But it may do so in very special cases such as for sequences of complex exponentials in $L^2(-\pi, \pi)$ (see [\[25\]](#page-16-5)). Therefore an affirmative answer to this question (independent of RH) is intriguing.

Finally in Section 4 we discuss a folklore conjecture about the ζ-zeros which we name the RH failure (RHF) conjecture: If the RH fails, then it fails infinitely often. More precisely, either ζ has no nontrivial zeros outside the critical line, or it has infinitely many. The main result of this section (Theorem [21\)](#page-12-0) states that

RHF conjecture \implies dim(\mathcal{N}^{\perp}) is either 0 or ∞ .

We were unable to locate a reference for this conjecture in the literature. But an informative discussion of this conjecture does appear in mathoverflow.net [\[11\]](#page-16-6).

1. Preliminaries

We denote by $\mathbb D$ and $\mathbb T$ the open unit disk and the unit circle respectively. By $Hol(\mathbb{D})$ we denote the space of holomorphic functions on \mathbb{D} and the shift operator is defined by $(Sf)(z) = zf(z)$ for $f \in Hol(D)$.

1.1. Hardy spaces and the Smirnov class. A holomorphic function f on D belongs to the Hardy-Hilbert space H^2 if

$$
||f||_{H^2} = \sup_{0 \le r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right)^{1/2} < \infty.
$$

The space H^2 is a Hilbert space with the ℓ^2 -inner product

$$
\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n},
$$

where $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are the Fourier coefficients for f and g respectively. For any $f \in H^2$ and $\zeta \in \mathbb{T}$, the radial limit $f^*(\zeta) := \lim_{r \to 1^-} f(r\zeta)$ exists m-a.e. on \mathbb{T} , where m denotes the normalized Lebesgue measure on \mathbb{T} . Analogously for $p > 0$, the Hardy space H^p consists of those holomorphic f on D such that

$$
||f||_{H^p}^p = \sup_{0 < r < 1} \int_{\mathbb{T}} |f(rz)|^p \, dm(z) < \infty.
$$

The H^p are Banach spaces for $p \geq 1$ and complete metric spaces for $0 \lt p \lt 1$ and H^{∞} denotes the space of bounded holomorphic functions on \mathbb{D} . A function f is called a cyclic vector for the shift S in H^p if $\text{span}(S^n f)_{n \geq 0} = \mathbb{C}[z]f$ is dense in H^p . When $p = 2$ these cyclic vectors are commonly known as outer functions. Since $H^{\infty} \subset H^p$ for all $p > 0$, the H^{∞} outer functions are cyclic for all H^p spaces. The Smirnov class N^+ consists of all holomorphic functions g/h on $\mathbb D$ such that $g, h \in H^{\infty}$ and h is an outer function. The space N^{+} is a topological algebra with respect to pointwise multiplication and $g/h \in N^+$ is a unit if both g and h are outer functions. The topology on N^+ can be metrized with the translation-invariant complete metric

$$
d(f,g) = \int_{\mathbb{T}} \log(1 + |f - g|) \, dm, \qquad f, g \in N^+.
$$

Similar to H^p spaces, convergence in N^+ implies locally unifom convergence on $\mathbb D$ and functions have radial limits m-a.e. on T. In fact we have $H^p \subset H^q \subset N^+$ for all $0 < q < p \leq \infty$. Duren [\[10\]](#page-16-7) is a classic reference for the H^p and N^+ spaces.

1.2. Local Dirichlet spaces. Let μ be a finite positive Borel measure on \mathbb{T} , and let $P\mu$ denote its Poisson integral. The *generalized Dirichlet space* \mathcal{D}_{μ} consists of $f \in H^2$ satisfying

$$
\mathcal{D}_{\mu}(f) := \int_{\mathbb{D}} |f'(z)|^2 P \mu(z) dA(z) < \infty.
$$

Then \mathcal{D}_{μ} is a Hilbert space with norm $||f||_{\mathcal{I}}^2$ $\mathcal{Q}_{\mu}^{2} := ||f||_{2}^{2} + \mathcal{D}_{\mu}(f).$ If $\mu = m$, then \mathcal{D}_m is the classicial Dirichlet space. If $\mu = \delta_{\zeta}$ is the Dirac measure at $\zeta \in \mathbb{T}$, then $\mathcal{D}_{\zeta} := \mathcal{D}_{\delta_{\zeta}}$ is called the *local Dirichlet space* at ζ and in particular

(1.1)
$$
\mathcal{D}_{\delta_{\zeta}}(f) = \int_{\mathbb{D}} |f'(z)|^2 \frac{1 - |z^2|}{|z - \zeta|^2} dA(z).
$$

The recent book [\[18\]](#page-16-8) contains a comprehensive treatment of local Dirichlet spaces and the following result establishes a criterion for their membership.

Theorem 3. (See [\[18,](#page-16-8) Thm. 7.2.1]) Let $\zeta \in \mathbb{T}$ and $f \in Hol(\mathbb{D})$. Then $\mathcal{D}_{\delta_{\zeta}}(f) < \infty$ if and only if

$$
f(z) = (z - \zeta)g(z) + a
$$

for some $g \in H^2$ and $a \in \mathbb{C}$. In particular $f^*(\zeta)$ exists for all $f \in \mathcal{D}_{\zeta}$.

So each local Dirichlet space $\mathcal{D}_{\zeta} = (S - \zeta I)H^2 + \mathbb{C}$ is a proper subspace of H^2 . We define the H^p -analogues of these spaces for $p > 0$ by

$$
\mathcal{D}_{\zeta}^{p} := (S - \zeta I)H^{p} + \mathbb{C}
$$

and note that $\mathcal{D}_{\zeta}^2 = \mathcal{D}_{\delta_{\zeta}}$ and $\mathcal{D}_{\zeta}^p \subsetneq \mathcal{D}_{\zeta}^q$ for $q < p$ since $H^p \subsetneq H^q$. Straightforward but lengthy computations show that $\mathcal{D}_{\zeta}^{\rho} \subsetneq H^2$ for $p > 1$ and $H^2 \subsetneq \mathcal{D}_{\zeta}^p$ for $0 < p < \frac{2}{3}$. 1.3. The de Branges-Rovnyak spaces. Given $\psi \in L^{\infty}(\mathbb{T})$, the corresponding Toeplitz operator $T_{\psi}: H^2 \to H^2$ is defined by

$$
T_{\psi}f := P_{+}(\psi f)
$$

where $P_+ : L^2(\mathbb{T}) \to H^2$ denotes the orthogonal projection of $L^2(\mathbb{T})$ onto H^2 . Clearly T_{ψ} is a bounded operator on H^2 with $||T_{\psi}|| \le ||\psi||_{L^{\infty}}$. If $h \in H^{\infty}$, then T_h is the operator of multiplication by h and its adjoint is $T_{\overline{h}}$. Given b in the closed unit ball of H^{∞} , the *de Branges-Rovnyak* space $\mathcal{H}(b)$ is the image of H^2 under the operator $(I - T_b T_{\overline{b}})^{1/2}$. The general theory of $\mathcal{H}(b)$ spaces divides into two distinct cases, according to whether b is an extreme point or a non-extreme point of the unit ball of H^{∞} . We shall be concerned only with the non-extreme case. In this case there exists a unique outer function $a \in H^{\infty}$ such that $a(0) > 0$ and $|a^*|^2 + |b^*|^2 = 1$ a.e. on \mathbb{T} . The pair (b, a) is called a *Pythagorean pair* and the function b/a belongs to the Smirnov class N^+ . That all N^+ functions arise as the quotient of a pair associated to a non-extreme function was shown by Sarason [\[21\]](#page-16-9). The two-volume work ([\[12\]](#page-16-10)[\[13\]](#page-16-11)) is an encyclopedic reference for these spaces.

If φ is a rational function in N^+ the corresponding pair (b, a) is also rational (see [\[21,](#page-16-9) Remark. 3.2]). Constara and Ransford [\[8\]](#page-16-12) characterized the rational pairs (b, a) for which $\mathcal{H}(b)$ is a generalized Dirichlet space.

Theorem 4. (See [\[8,](#page-16-12) Theorem 4.1]) Let (b, a) be a rational pair and μ a finite positive measure on \mathbb{T} . Then $\mathcal{H}(b) = \mathcal{D}_{\mu}$ if and only if

- (1) the zeros of a on $\mathbb T$ are all simple, and
- (2) the support of μ is exactly equal to this set of zeros.

As an example, if (b, a) is the rational pair associated with the N^+ function $\varphi(z) = \frac{1}{1-\zeta}$ for $\zeta \in \mathbb{T}$, then $\mathcal{H}(b) = \mathcal{D}_{\zeta}$ is a local Dirichlet space.

1.4. Unbounded Toeplitz operators on H^p . Sarason [\[21\]](#page-16-9) demonstrated how $\mathcal{H}(b)$ spaces appear naturally as the domains of some unbounded Toeplitz operators. Let φ be holomorphic in $\mathbb D$ and T_{φ} the operator of multiplication by φ on the domain

(1.2)
$$
\text{dom}(T_{\varphi}) = \{ f \in H^2 : \varphi f \in H^2 \}.
$$

Then T_{φ} is a closed operator, and $dom(T_{\varphi})$ is dense in H^2 if and only if $\varphi \in N^+$ (see [\[21,](#page-16-9) Lemma 5.2]). In this case its adjoint T^*_{φ} is also densely defined and closed. In fact the domain of T^*_{φ} is a de Branges-Rovnyak space.

Theorem 5. (See [\[21,](#page-16-9) Prop. 5.4]) Let φ be a nonzero function in N^+ with $\varphi = b/a$, where (b, a) is the associated pair. Then $dom(T^*_{\varphi}) = \mathcal{H}(b)$.

Choosing the symbol $\varphi(z) = \frac{1}{\zeta-z}$ in Theorem [5](#page-3-0) in conjunction with Theorem [4](#page-3-1) gives $\text{dom}(T^*_\varphi) = \mathcal{D}_\zeta$ which played a key role in the proof of Theorem [2](#page-1-0) (see [\[19\]](#page-16-0)). Our goal here is to extend these ideas to H^p spaces for all $p > 1$. Let $\varphi \in N^+$ and define the analytic Toeplitz operator on H^p with symbol φ by

 $T_{\varphi}f = \varphi f$, where $f \in \text{dom}_p(T_{\varphi}) := \{ f \in H^p : \phi f \in H^p \}.$

These T_{φ} are bounded on H^p precisely when $\varphi \in H^{\infty}$ (see the survey article [\[23\]](#page-16-13)). For $\varphi = \frac{b}{a} \in N^+$ with $a, b \in H^\infty$ and a outer as usual, these T_φ are densely defined on H^p for $p > 1$. Indeed, $\text{dom}_p(T_\varphi)$ contains the dense subspace aH^p since a is outer and $T_{\varphi}(aH^p) = bH^p \subset H^p$. It follows then that the adjoint T_{φ}^* is well-defined on the dual $(H^p)^* = H^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. The domain of T^*_{φ} is then defined by

$$
\mathrm{dom}_q(T^*_\varphi) := \{ g \in H^q : \exists \ h \in H^q \ s.t \ \langle f, h \rangle = \langle \varphi f, g \rangle \ \ \forall \ f \in \mathrm{dom}_p(T_\varphi) \}
$$

where $\langle f, h \rangle := \int_{\mathbb{T}} f \overline{h} dm$ represents the $H^p \text{-} H^q$ duality. The elements in $\text{dom}_q(T^*_\varphi)$ can be characterized via the bounded Toeplitz operators T_a and T_b as follows.

Lemma 6. Given $\varphi = \frac{b}{a} \in N^+$ as described above, a function $g \in \text{dom}_q(T^*_{\varphi})$ if and only if there exists an $h \in H^q$ such that $T_b^* g = T_a^* h$.

Proof. Suppose $g \in \text{dom}_q(T^*_\varphi)$. Then $\langle f, h \rangle = \langle \varphi f, g \rangle$ for some $h \in H^q$ and for all $f \in aH^p \subset \text{dom}_p(T_\varphi)$. Writing $f = a\tilde{f}$ for $\tilde{f} \in H^p$, we get

$$
\langle f, h \rangle = \langle \varphi f, g \rangle \Leftrightarrow \langle f, h \rangle = \langle bf/a, g \rangle \Leftrightarrow \langle a\tilde{f}, h \rangle = \langle b\tilde{f}, g \rangle \ \forall \ \tilde{f} \in H^p
$$

which is equivalent to $T_a^*h = T_b^*g$. This argument works in both directions because a is an H^{∞} outer function and hence aH^p is dense in H^p . — Процессиональные просто производства и продага в собстановки производства и производства и производства и
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We can now extend the identity $dom(T^*_{\varphi}) = \mathcal{D}_{\zeta}$ from H^2 to all H^p with $p > 1$.

Proposition 7. Let $\varphi(z) = \frac{1}{\zeta - z}$ for $\zeta \in \mathbb{T}$ and T_{φ} the densely defined Toeplitz operator on H^p for any $p > 1$. We then have

$$
\text{dom}_q(T^*_\varphi) = \mathcal{D}^q_\zeta, \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.
$$

Proof. Choosing $\varphi = \frac{b}{a}$ with $b(z) = 1$ and $a(z) = \zeta - z$ (which is outer) in Lemma [6,](#page-4-0) we get $g \in \text{dom}_q(T^*_\varphi)$ if and only if $g = T^*_{\zeta - z}h = (\overline{\zeta}I - S^*)h$ for some $h \in H^q$. Therefore, it suffices to verify that $(\overline{\zeta}I - S^*)H^q = \mathcal{D}_{\zeta}^q$. For any $h \in H^q$, we have

$$
(\overline{\zeta}I - S^*)h = (\zeta I - S)(-\overline{\zeta}S^*h) + \overline{\zeta}h(0) \in \mathcal{D}_{\zeta}^q := (S - \zeta I)H^q + \mathbb{C}
$$

and therefore $(\overline{\zeta}I - S^*)H^q \subset \mathcal{D}_{\zeta}^q$. Conversely, if $h \in H^q$ and $c \in \mathbb{C}$ then

$$
(\zeta I - S)h + c = (\overline{\zeta}I - S^*)\zeta(c - Sh) \in (\overline{\zeta}I - S^*)H^q
$$

and hence $\mathcal{D}_{\zeta}^{q} \subset (\overline{\zeta}I - S^*)H^q$ which concludes the proof.

1.5. Cauchy duality. Let X be a complete metrizable linear subspace of Hol($\mathbb D$). Inspired by terminology used by Malman and Seco [\[16\]](#page-16-14), we call X^* the Cauchy dual of X if any continuous linear functional on X can be represented by the Cauchy pairing

$$
\langle f, g \rangle := \lim_{r \to 1^{-}} \int_{\mathbb{T}} f(r\zeta) \overline{g(r\zeta)} \, dm(\zeta), \quad f \in X, \ g \in X^*.
$$

If $H^2 \subset X$ then $X^* \subset H^2$ and vice-versa. Hence when both f and g are in H^2 , the pairing above reduces to the standard inner product in H^2 . Some examples of Cauchy duals for our context are listed below (see [\[10\]](#page-16-7),[\[15\]](#page-16-15),[\[24\]](#page-16-16)).

- (1) H^p and H^q for $p > 1$ and $1/p + 1/q = 1$,
- (2) H^1 and BMOA (analytic functions with bounded mean oscillation on \mathbb{T}),
- (3) H^p for $1/2 < p < 1$ and Λ_α (the Lipschitz class of Hol(D)-functions with α -Hölder continuous extension to T, where $\alpha = 1/p - 1$.
- (4) N^+ and the Gevrey class G (Hol(D)-functions whose Taylor coefficients satisfy $a_n = O(e^{-c\sqrt{n}})$ for some constant $c > 0$).

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A deep result of Davis and McCarthy [\[9\]](#page-16-17) shows that the class $\mathcal G$ coincides with the universal multipliers for all non-extreme $\mathcal{H}(b)$ spaces. In particular $\mathcal{G} \subset \mathcal{H}(b)$ for all non-extreme b. The concept of Cauchy duality leads to an equivalence between orthogonality and density questions involving N which is explored in Section [2.](#page-6-2)

1.6. Minimality and biorthogonality. Let $\mathcal H$ be a Hilbert space. Two sequences $(e_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ in H are said to be *biorthogonal* to each other if

$$
\langle e_n, f_m \rangle = \delta_{nm} \quad \forall \ n, m \in \mathbb{N}
$$

where δ_{nm} is the Kronecker delta. The sequence $(e_n)_{n\in\mathbb{N}}$ is called minimal if $e_n \notin \overline{\text{span}}(e_k)_{k\neq n}$ for all $n \in \mathbb{N}$. The notions of biorthogonality, minimality and completeness are all related via the following well-known result.

Proposition 8. (see [\[7,](#page-16-18) Lemma 3.3.1]) Let $(e_n)_{n \in \mathbb{N}}$ be a sequence in H. Then,

- (i) $(e_n)_{n\in\mathbb{N}}$ has a biorthogonal sequence if and only if $(e_n)_{n\in\mathbb{N}}$ is minimal.
- (ii) If $(e_n)_{n\in\mathbb{N}}$ has a biorthogonal sequence, then $(e_n)_{n\in\mathbb{N}}$ is complete in H if and only if its biorthogonal sequence is unique.

In Section 3 we shall prove that the sequence $(u_k)_{k\geq 2}$ defined by

$$
u_k(z) = \sum_{d|k} \frac{\mu(k/d)}{k/d} (z^{d-1} - z^d)
$$

forms a complete biorthogonal sequence for $(h_k)_{k\geq 2}$ in H^2 , where μ denotes the Möbius function defined on N by $\mu(k) = (-1)^s$ if k is the product of s distinct primes, and $\mu(k) = 0$ otherwise.

1.7. The zeta kernels. Let $X \subset Hol(\mathbb{D})$ be a topological vector space where the monomials $(z^k)_{k\in\mathbb{N}}$ form a Schauder basis. It was shown in [\[14\]](#page-16-1) that for each $s \in \mathbb{C} \setminus \{0\}$ a linear functional $\Lambda^{(s)}$ can be defined on X by assigning

$$
\Lambda^{(s)}(1) = -\frac{1}{s}, \quad \Lambda^{(s)}(z^k) = -\frac{1}{s} \left((k+1)^{1-s} - k^{1-s} \right) \quad (k \ge 1).
$$

In particular $\Lambda^{(s)}$ is bounded on H^p for $1 < p \le 2$ if $\Re s > 1/p$ and on H^1 if $\Re s \ge 1$ (see [\[14,](#page-16-1) Prop. 4.7]). So there exist functions $\kappa_s \in H^q$ with $1/p + 1/q = 1$ such that $\Lambda^{(s)}(f) = \langle f, \kappa_s \rangle$. The function κ_s will be called the zeta kernel at s and

(1.3)
$$
\kappa_s(z) = \sum_{k=0}^{\infty} \phi_k(\bar{s}) z^k \text{ where } \phi_k(s) := \Lambda^{(s)}(z^k).
$$

The name comes from their relation to h_k and ζ via the important identity

$$
(1.4) \qquad \Lambda^{(s)}(h_k) = \langle h_k, \kappa_s \rangle = -\frac{\zeta(s)}{s} (k^{1-s} - 1) \quad \forall \quad \Re s > 1/2, \ k \ge 2.
$$

The identity [\(1.4\)](#page-5-0) appears in [\[14\]](#page-16-1) but we provide an alternate proof in the appendix for the sake of completeness. It is important to mention that the definition of h_k in [\[14\]](#page-16-1) has an additional factor of $1/k$ which has been adjusted in [\(1.4\)](#page-5-0) accordingly. The zeta kernels are used in Chapters 3 and play a key role in Chapter 4.

2. The orthogonality question

The objective of this section is to develop a framework for proving when

$$
(2.1) \t\t \mathcal{N}^{\perp} \cap L = \{0\}
$$

for topological vector spaces $L \subset H^2$. Since $\mathcal{N}^{\perp} = \{0\}$ is equivalent to the RH by Theorem [1,](#page-0-0) one may also ask if solutions to [\(2.1\)](#page-6-3) can lead to new zero-free half-planes for the ζ -function. We start by showing that Cauchy duality serves as a bridge between this orthogonality question and completeness questions.

Proposition 9. Let X be a topological linear space with $H^2 \subset X \subset Hol(\mathbb{D})$, where the inclusions are continuous. If N is dense in X, then

$$
\mathcal{N}^{\perp} \cap X^* = \{0\}
$$

where $X^* \subset H^2$ is the Cauchy dual of X. The converse holds when X is Fréchet.

Proof. First note that since both \mathcal{N}^{\perp} and X^* are subspaces of H^2 , their intersection above makes sense. Let $\langle f, g \rangle$ denote the Cauchy pairing for $f \in X$ and $g \in X^*$ and recall that this pairing becomes the usual H^2 -inner product $\langle f, g \rangle_{H^2}$ when $f, g \in H^2$ (see Subsection 1.5). Therefore if $\mathcal N$ is dense in X, then

$$
g \in \mathcal{N}^{\perp} \cap X^* \implies \langle f, g \rangle = \langle f, g \rangle_{H^2} = 0 \text{ for all } f \in \mathcal{N}
$$

which implies that g must be identically zero in X^* . Conversely if X is additionally a Fréchet space, then we have access to the Hahn-Banach Theorem. Indeed if $\mathcal N$ is not dense in X, then there exists a non-zero $g \in X^*$ such that $\langle f, g \rangle = 0 \ \forall f \in \mathcal{N}$. This implies that g is H²-orthogonal to N and hence $g \in \mathcal{N}^{\perp} \cap X^* \neq \{0\}.$

Since N is dense in H^p for $0 < p < 1$ (see [\[14,](#page-16-1) Cor. 4.6]), it is also dense in N^+ since $H^p \subset N^+$ for all $p > 0$. Therefore it follows by Proposition [9](#page-6-0) that $\mathcal{N}^{\perp} \cap L = \{0\}$ if L is the Lipschitz class Λ_{α} $(1/2 < p < 1$ and $\alpha = 1/p - 1)$ or the Gevrey class G (see Subsection 1.5). However to obtain new zero-free half-planes for ζ , we need $L \subset H^2$ to be large enough to contain some H^q space for $q \geq 2$.

Corollary 10. If $\mathcal{N}^{\perp} \cap H^q = \{0\}$ for some $q \geq 2$, then $\zeta(s) \neq 0$ for $\Re s > 1/p$, where $1/p + 1/q = 1$. If $\mathcal{N}^{\perp} \cap \text{BMOA} = \{0\}$, then $\zeta(s) \neq 0$ for $\Re s \geq 1$.

Proof. Notice that H^q is the Cauchy dual of H^p and $q \ge 2$ implies that $1 < p \le 2$. In this range the H^p are Banach spaces, and in particular Fréchet spaces. Similarly the BMOA space is the Cauchy dual of H^1 . Therefore the result follows by the converse in Proposition [9](#page-6-0) and by Theorem [1.](#page-0-0)

The next result relates Toeplitz operators on H^p and the orthogonality question. Recall that if $\varphi \in N^+$ is a unit, then $1/\varphi \in N^+$ and hence both T_φ and its inverse $T_{1/\varphi}$ are densely defined Toeplitz operators on H^p for $p>1$ (see Subsection 1.4).

Proposition 11. Let φ be a unit in N^+ with T_{φ} the Toeplitz operator on H^p for some $p > 1$. If $T_{\varphi} \mathcal{N} = \varphi \mathcal{N}$ is dense in H^p , then

$$
\mathcal{N}^{\perp} \cap \text{dom}_q(T^*_{1/\varphi}) = \{0\}, \quad where \ \frac{1}{p} + \frac{1}{q} = 1.
$$

The Hilbertian case $p = 2$ gives $\mathcal{N}^{\perp} \cap \mathcal{H}(b) = \{0\}$ where (b, a) is the Pythagorean pair associated with $1/\varphi \in N^+$.

Proof. Let $g \in \mathcal{N}^{\perp} \cap \text{dom}_{q}(T^{*}_{1/\varphi})$. Since both g and h_{k} belong to H^{2} for all $k \geq 2$, the Cauchy duality $\langle h_k, g \rangle$ coincides with the H^2 -inner product $\langle h_k, g \rangle$ ₂. Hence

$$
\langle \varphi h_k, T^*_{1/\varphi} g \rangle = \langle h_k, g \rangle = \langle h_k, g \rangle_2 = 0 \quad \forall \ k \in \mathbb{N}
$$

which implies that $T^*_{1/\varphi}g = 0$ in H^q by the density of $\varphi \mathcal{N}$ in H^p . Since $1/\varphi = b/a$ is a unit in N^+ (as the inverse of φ), both a and b are H^{∞} outer functions in H^p . So $T_{1/\varphi}(aH^p) = bH^p$ shows that $T_{1/\varphi}$ has dense range in H^p since b is outer. So $T_{1/\varphi}^*$ is injective and therefore $g=0$ in H^q . This concludes the general case. The Hilbertian case $p = 2$ now follows by Theorem [5.](#page-3-0)

We shall derive two non-trivial applications of this result. The first one extends Theorem [2](#page-1-0) to all local Dirichlet spaces \mathcal{D}_1^p with $p > 1$. We note that the classical \mathcal{D}_{δ_1} is just \mathcal{D}_1^2 which is strictly smaller than \mathcal{D}_1^p for $p \in (1,2)$ (see Subsection 1.2). We will need with the following approximation result from [\[14\]](#page-16-1).

Lemma 12. Let μ the Möbius function. Then

$$
\sum_{k=2}^{n} \frac{\mu(k)}{k} (I - S) h_k \to 1 - z
$$

in the H^p norm for all $p > 0$.

We observe that $I - S = T_{\varphi}$ where $\varphi(z) = 1 - z$ is an H^{∞} outer function since $\mathbb{C}[z]\varphi$ is dense in H^2 . In particular φ is a unit in N^+ . Define operators on H^p by

(2.2)
$$
(W_n)f(z) = (1 + z + \dots + z^{n-1})f(z^n) = \frac{1 - z^n}{1 - z}f(z^n)
$$

and $(T_n)f(z) = f(z^n)$ for $n \ge 1$ and $f \in H^p$. The multiplicative semigroup of operators $(W_n)_{n\geq 1}$ was introduced in [\[19\]](#page-16-0) and is the main object of study in [\[17\]](#page-16-19). They are bounded on H^p for $p > 1$ (see [\[14,](#page-16-1) Cor. 4.6]). We shall need the identities

$$
W_n \mathcal{N} \subset \mathcal{N}
$$
 and $T_n(I - S) = (I - S)W_n$

for $k, n \ge 1$ which appear in [\[19,](#page-16-0) p. 249]. We are ready for the first application.

Theorem 13. We have $\mathcal{N}^{\perp} \cap \mathcal{D}_{1}^{p} = \{0\}$ for all $p > 1$.

Proof. Let $\varphi(z) = 1 - z$. By Propositions [7](#page-4-1) and [11](#page-6-1) we only need to prove that $\varphi \mathcal{N}$ is dense in H^p for $p > 1$. First note that Lemma [12](#page-7-0) implies that φ belongs to the H^p-closure of $\varphi \mathcal{N} = (I - S)\mathcal{N}$. This in turn implies that $T_n\varphi$ belongs to the H^p-closure of $T_n(\varphi \mathcal{N}) = \varphi W_n \mathcal{N} \subset \varphi \mathcal{N}$ for all $n \geq 1$. So in particular $\text{span}(T_n\varphi)_{n\geq 1} \subset \text{clos}_{H^p}(\varphi \mathcal{N}).$ Now $\text{span}(T_n\varphi)_{n\geq 1} = \text{span}(1-z^n)_{n\geq 1} = \mathbb{C}[z]\varphi$ which is dense in H^p for all $p > 1$ because φ is an H^{∞} outer function. This proves that $\cos_{HP}(\varphi \mathcal{N}) = H^p$ and concludes the proof.

Our second application of Proposition [11](#page-6-1) utilizes recent discoveries in $\mathcal{H}(b)$ -space theory to obtain zero-free half-planes for ζ . In view of Corollary [10,](#page-6-4) we would like to know when the H^p and BMOA spaces are contained in some $\mathcal{H}(b)$ for $\varphi = b/a \in N^+$. Fortuitously for us, these problems were completely solved recently in a preprint by Malman and Seco [\[16\]](#page-16-14). They show that $H^{\tilde{p}} \subset \mathcal{H}(b)$ for $\tilde{p} \in (2,\infty)$ if and only if $\varphi \in H^p$ where $p = \frac{2\tilde{p}}{\tilde{p}-2} \in (2,\infty)$, and also that $H^\infty \subset \text{BMOA} \subset \mathcal{H}(b)$ if and only if $\varphi \in H^2$. By definition we always have $\mathcal{H}(b) \subset H^2$, and $\mathcal{H}(b) = H^2$ precisely when $\varphi \in H^{\infty}$. Therefore it makes sense to allow the values $p = 2$ and $p = \infty$.

Theorem 14. Suppose φ is a unit in N^+ such that $1/\varphi \in H^p$ for some $p \in (2,\infty)$. If $\varphi \mathcal{N}$ is dense in H^2 , then $\zeta(s) \neq 0$ for $\Re s > \frac{1}{2} + \frac{1}{p}$. The case $p = 2$ gives the Prime Number Theorem ($\Re s \ge 1$) and $p = \infty$ gives the RH ($\Re s > 1/2$).

Proof. By the results of Malman and Seco [\[16\]](#page-16-14) mentioned above, $1/\varphi$ belongs to H^2 or to H^{∞} precisely when $\mathcal{H}(b)$ contains BMOA or $\mathcal{H}(b) = H^2$ respectively, where $1/\varphi = b/a$ and (b,a) the associated Pythagorean pair. Hence the cases $p = 2, \infty$ follow by Corollary [10](#page-6-4) and Proposition [11.](#page-6-1) For the case when $1/\varphi \in H^p$ for $p \in (2,\infty)$, we have $H^{\tilde{p}} \subset \mathcal{H}(b)$ where $p = \frac{2\tilde{p}}{\tilde{p}-2}$ (again by Malman and Seco) or equivalently $\tilde{p} = \frac{2p}{p-2}$. If $1/\tilde{p} + 1/\tilde{q} = 1$, then we see that $\tilde{q} = \frac{2p}{(p+2)}$ and hence $1/\tilde{q} = 1/2 + 1/p$. The result again follows by Corollary [10](#page-6-4) and Proposition [11.](#page-6-1) \Box

An important distinction between Theorem [1](#page-0-0) and Theorem [14](#page-8-0) is that in the former one must solve density problems in H^p spaces that are non-Hilbertian, while in the latter the density problems are always in H^2 . The following simple examples of φ satisfy the hypothesis of Theorem [14.](#page-8-0) Let $\varphi(z) = (1-z)^{\alpha}$ for some $0 < \alpha < 1/2$. Then φ is an H^{∞} outer function and hence a unit in N^{+} with the property that $1/\varphi \in H^p$ for some $2 < p < 1/\alpha$. It follows that the density of $\varphi \mathcal{N}$ in H^2 would give a new zero-free half-plane for ζ .

3. The biorthogonality question

Define the sequence of polynomials $\{u_k : k \geq 2\}$ by

(3.1)
$$
u_k(z) = \sum_{d|k} \frac{\mu(k/d)}{k/d} (z^{d-1} - z^d),
$$

where μ denotes the Möbius function and $d|k$ denotes d divides k. The main goal of this section is to prove the following theorem.

Theorem 15. $\{u_k : k \geq 2\}$ is complete and biorthogonal to $\{h_k : k \geq 2\}$ in H^2 .

Balazard [\[5\]](#page-16-20) noted that with the additional vector $u_1(z) = 1 - z$ the sequence ${u_k : k \ge 1}$ is complete. However it is no longer minimal following Theorem [15.](#page-8-1) We first make a key observation. Note that $u_k = (I - S)v_k$, where

(3.2)
$$
v_k(z) = \sum_{d|k} \frac{\mu(k/d)}{k/d} z^{d-1}.
$$

It follows that $\langle h_k, u_j \rangle = \langle (I - S^*)h_k, v_j \rangle$. Hence to show that $\{u_k : k \geq 2\}$ and ${h_k : k \geq 2}$ are biorthogonal, it is suffices to show that

$$
\langle (I - S^*)h_k, v_j \rangle = \delta_{kj}.
$$

The proof of Theorem [15](#page-8-1) will be divided into four steps.

Step 1. Calculate the Fourier coefficients of $(I - S^*)h_k$.

Step 2. Prove $\{u_k : k \geq 2\}$ is biorthogonal to $\{h_k : k \geq 2\}$.

Step 3. Characterize all sequences biorthogonal to $\{v_k : k \geq 2\}$.

Step 4. Show that $\{h_k : k \geq 2\}$ is the unique biorthogonal sequence for $\{u_k : k \geq 2\}$.

The **Step 4** implies the completeness of $\{u_k : k \geq 2\}$ in H^2 by Proposition [8.](#page-5-1)

Step 1. We first calculate the Fourier coefficients of $(I - S^*)h_k$.

Lemma 16. We have

$$
(I - S^*)h_k(z) = \sum_{n=0}^{\infty} B_k(n+1)z^n
$$

for all $k \geq 2$ where

$$
B_k(n) = \begin{cases} \frac{k}{n} - \frac{1}{n}, & k|n\\ -\frac{1}{n}, & k\nmid n \end{cases}
$$

.

Proof. Note that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$
S^*f(z) = \sum_{n=0}^{\infty} a_{n+1} z^n.
$$

Let $c_n(k)$ be the Fourier coefficients of h_k , i.e.,

$$
h_k(z) = \sum_{n=0}^{\infty} c_n(k) z^n.
$$

Then, the *n*-th Fourier coefficient of $(I - S^*)h_k$ is $c_n(k) - c_{n+1}(k)$. The coefficients $c_n(k)$ are calculated in [\[19,](#page-16-0) p. 249]:

(3.3)
$$
c_n(k) = H(n) - H\left(\frac{n}{k}\right) - \log k,
$$

where $H(x) := \sum_{n \le x} \frac{1}{n}$ for $x > 0$ and $H(0) = 0$. It follows from [\(3.3\)](#page-9-0) that

$$
c_{n-1}(k) - c_n(k) = H(n-1) - H\left(\frac{n-1}{k}\right) - \log k - \left(H(n) - H\left(\frac{n}{k}\right) - \log k\right)
$$

$$
= -\frac{1}{n} + \sum_{\frac{n-1}{k} < m \le \frac{n}{k}} \frac{1}{m}.
$$

Note that if there is some $m \in \mathbb{N}$ such that $\frac{n-1}{k} < m \leq \frac{n}{k}$, then $mk \leq n < mk + 1$, so that $n = mk$. Therefore, the sum above is non-zero if and only if $k|n$. Then,

(3.4)
$$
B_k(n) = c_{n-1}(k) - c_n(k) = \begin{cases} -\frac{1}{n}, & k/n \\ \frac{k}{n} - \frac{1}{n}, & k|n \end{cases}
$$

Step 2. We are now able to prove the first part of Theorem [15.](#page-8-1)

Theorem 17. $\{u_k : k \geq 2\}$ is biorthogonal to $\{h_k : k \geq 2\}$.

Proof. By Step 1 it suffices to prove that

$$
\sum_{d|j} B_k(d) \frac{\mu(j/d)}{j/d} = \langle (I - S^*)h_k, v_j \rangle = \delta_{kj}, \qquad \forall k, j \ge 2.
$$

There are two cases:

(i) k/j . Then, k/d for every d/j , therefore

$$
\sum_{d|j} B_k(d) \frac{\mu(j/d)}{j/d} = \sum_{d|j} -\frac{1}{d} \frac{\mu(j/d)}{j/d} = -\frac{1}{j} \sum_{d|j} \mu(j/d) = -\frac{1}{j} \left\lfloor \frac{1}{j} \right\rfloor = 0,
$$

since $j \ge 2$ and by the basic relation $\sum_{d|k} \mu(d) = \lfloor 1/k \rfloor$.

(ii)
$$
k|j
$$
. Let $q = \frac{j}{k}$. Then
\n
$$
\sum_{d|j} B_k(d) \frac{\mu(j/d)}{j/d} = \sum_{\substack{d|j \\ k \nmid d}} -\frac{1}{d} \frac{\mu(j/d)}{j/d} + \sum_{\substack{d|j \\ k|d}} \left(\frac{k}{d} - \frac{1}{d}\right) \frac{\mu(j/d)}{j/d}.
$$

The last sum is summing over those d that satisfy $k|d|j$. However, $k|d \iff d = mk$ for some $m \in \mathbb{N}$. Since $j = qk$, it follows that $d|j \iff m|q$. Hence, the last sum can be written as

$$
\sum_{\substack{d|j\\k|d}} \left(\frac{k}{d} - \frac{1}{d}\right) \frac{\mu(j/d)}{j/d} = \sum_{\substack{d|j\\k|d}} -\frac{1}{d} \frac{\mu(j/d)}{j/d} + \sum_{m|q} \frac{1}{m} \frac{\mu(q/m)}{q/m}.
$$

Therefore,

$$
\sum_{d|j} B_k(d) \frac{\mu(j/d)}{j/d} = \sum_{d|j} -\frac{1}{d} \frac{\mu(j/d)}{j/d} + \sum_{m|q} \frac{1}{m} \frac{\mu(q/m)}{q/m}
$$

=
$$
-\frac{1}{j} \sum_{d|j} \mu(j/d) + \frac{1}{q} \sum_{m|q} \mu(q/m) = -\frac{1}{j} \left[\frac{1}{j} \right] + \frac{1}{q} \left[\frac{1}{q} \right].
$$

Since $j \geq 2$, the first term is always 0. On the other hand, the second term equals 1 if $q = 1$ and equals 0 otherwise. Finally note that $q = 1 \iff k = j$, and hence that $\langle h_k, u_j \rangle = \delta_{kj}$ for all $k, j \geq 2$.

Step 3. We next characterize all sequences in H^2 biorthogonal to $\{v_k : k \geq 2\}$.

Lemma 18. A sequence $\{f_k : k \geq 2\} \subset H^2$ is biorthogonal to $\{v_k : k \geq 2\}$ if and only if there exists a sequence $(c_k)_{k\geq 2} \in \mathbb{C}^{\mathbb{N}}$ such that

$$
f_k(z) = \sum_{n=0}^{\infty} A_k(n+1) z^n, \quad \forall k \ge 2,
$$

where the sequence $(A_k(n))_{n\geq 1}$ for each $k \geq 2$ is defined by

$$
A_k(n) = \begin{cases} \frac{c_k}{n} + \frac{k}{n}, & k|n\\ \frac{c_k}{n}, & k\not|n \end{cases}
$$

.

Proof. Let $\{f_k : k \geq 2\} \subset H^2$ be a sequence biorthogonal to $\{v_k : k \geq 2\}$ and $A_k : \mathbb{N} \to \mathbb{C}$ be the arithmetical functions that satisfy

$$
f_k(z) = \sum_{n=0}^{\infty} A_k(n+1)z^n, \quad \forall k \ge 2.
$$

Since the coefficients of v_i are real, the biorthogonality condition becomes

(3.5)
$$
\sum_{d|j} A_k(d) \frac{\mu(j/d)}{j/d} = \langle f_k, v_j \rangle = \delta_{kj}, \quad \forall k, j \ge 2.
$$

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Let $I_k, \nu : \mathbb{N} \to \mathbb{C}$ be arithmetic functions defined by $I_k(n) = \delta_{kn}$ and $\nu(n) = \frac{\mu(n)}{n}$. Then [\(3.5\)](#page-10-0) is equivalent to

(3.6)
$$
\forall k \geq 2, \exists c_k \in \mathbb{C} \text{ such that } A_k * \nu = c_k I_1 + I_k,
$$

where $*$ denotes the Dirichlet product (see [\[1,](#page-15-3) Section 2.6]). Indeed, [\(3.5\)](#page-10-0) doesn't impose any restriction on $A_k * \nu(1)$, since it only need to hold for $j \geq 2$, hence $c_k = A_k * \nu(1)$ is free, so [\(3.5\)](#page-10-0) and [\(3.6\)](#page-11-0) are indeed equivalent. Notice that

$$
\sum_{d|k} \frac{\mu(k/d)}{k/d} \frac{1}{d} = \frac{1}{k} \left\lfloor \frac{1}{k} \right\rfloor = I_1(k),
$$

i.e., $\nu^{-1}(n) = \frac{1}{n}$, since I_1 is the unity with respect to $*$. Moreover

$$
I_k * \nu^{-1}(n) = \sum_{d|n} \delta_{kd} \frac{1}{n/d} = \begin{cases} \frac{k}{n}, & k|n\\ 0, & k\not|n \end{cases}
$$

.

Therefore [\(3.6\)](#page-11-0) is equivalent to the statement that

$$
\forall k \ge 2, \exists c_k \in \mathbb{C} \text{ such that } A_k(n) = c_k \nu^{-1}(n) + I_k * \nu^{-1}(n)
$$

$$
= \begin{cases} \frac{c_k}{n} + \frac{k}{n}, & k|n\\ \frac{c_k}{n}, & k\nmid n \end{cases}.
$$

Hence the biorthogonality condition [\(3.5\)](#page-10-0) is equivalent to the condition above as desired. Finally $f_k \in H^2$ since its coefficient sequence A_k clearly belongs to ℓ^2 . \Box

Step 4. In this final step we show that $\{u_k : k \geq 2\}$ is complete in H^2 by proving that $\{h_k : k \geq 2\}$ is uniquely biorthogonal to $\{u_k : k \geq 2\}$ in H^2 by Proposition [8.](#page-5-1) To do so, recall that $u_k = (I - S)v_k$ (see [\(3.2\)](#page-8-2)) implies

$$
\langle \phi_k, u_j \rangle = \langle (I - S^*) \phi_k, v_j \rangle
$$

for any sequence $\{\phi_k : k \geq 2\}$ in H^2 . This implies that $I - S^*$ maps sequences biorthogonal to $\{u_k : k \geq 2\}$ onto sequences biorthogonal to $\{v_k : k \geq 2\}$ in the image of $I - S^*$. This correspondece is one-to-one since $I - S^*$ is injective on H^2 . Therefore it is enough to prove that $((I - S^*)h_k)_{k \geq 2}$ is the unique sequence in the image of $I - S^*$ that is biorthogonal to $\{v_k : k \geq 2\}.$

Lemma 19. A sequence $\{f_k : k \geq 2\} \subset (I - S^*)H^2$ is biorthogonal to $\{v_k : k \geq 2\}$ if and only if

(3.7)
$$
f_k(z) = \sum_{n=0}^{\infty} B_k(n+1)z^n = (I - S^*)h_k,
$$

where B_k are the sequences defined in Lemma [16.](#page-9-1)

Proof. Let $\{f_k : k \geq 2\} \subset (I - S^*)H^2$ be a sequence biorthogonal to $\{v_k : k \geq 2\}$ and let $\varphi_k \in H^2$ such that $f_k = (I - S^*)\varphi_k$. If $(b_k(n))_{n \geq 0}$ are the Maclaurin coefficients of φ_k , then

$$
f_k(z) = \sum_{n=0}^{\infty} (b_k(n) - b_k(n+1))z^n.
$$

It then follows by Lemma [18](#page-10-1) that for each $k\geq 2,$ there exists a $c_k\in \mathbb{C}$ such that

$$
b_k(n-1) - b_k(n) = A_k(n) = \begin{cases} \frac{k}{n} + \frac{c_k}{n}, & k|n \\ \frac{c_k}{n}, & k \neq n \end{cases}, \quad \forall n \ge 1.
$$

By induction, we obtain

$$
b_k(n) = b_k(0) - \sum_{j=1}^n A_k(j) = b_k(0) - \sum_{j \le n} \frac{c_k}{j} - \sum_{\substack{j \le n \\ k \mid j}} \frac{k}{j}
$$

= $b_k(0) - c_k \sum_{j \le n} \frac{1}{j} - \sum_{m \le n/k} \frac{1}{m} = b_k(0) - c_k H(n) - H\left(\frac{n}{k}\right),$

where H is the same function used in [\(3.3\)](#page-9-0). Since $\varphi_k \in H^2$, we get $(b_k(n))_n \in \ell^2$ and hence $\lim_{n\to\infty} b_k(n) = 0$. So $(c_kH(n) + H(n/k))_n$ converges. Using Euler summation, one gets (see [\[19\]](#page-16-0))

(3.8)
$$
H(x) = \log x + \gamma + O\left(\frac{1}{x}\right),
$$

where γ is Euler-Mascheroni constant. Therefore,

$$
c_k H(n) + H\left(\frac{n}{k}\right) = c_k \log n + c_k \gamma + \log n - \log k + \gamma + O\left(\frac{k}{n}\right)
$$

$$
= (c_k + 1) \log n + (c_k + 1)\gamma - \log k + O\left(\frac{k}{n}\right),
$$

which converges as $n \to \infty$ if and only if $c_k = -1$. Hence $c_k = -1$ for all $k \geq 2$. In that case $A_k = B_k$ and we obtain [\(3.7\)](#page-11-1). The converse is equivalent to Theorem [17](#page-9-2) by the remarks at the start of Step 4. \Box

As a consequence of Theorem [1](#page-0-0) and Proposition [8](#page-5-1) the RH holds if and only if $(u_j)_{j\geq 2}$ is the unique sequence in H^2 that is biorthogonal to $(h_k)_{k\geq 2}$. On the other hand the next result shows what happens if some ζ-zero violates the RH.

Corollary 20. If $\zeta(s_0) = 0$ for some $1/2 < \Re s_0 < 1$, then

$$
\langle h_k, u_j + \kappa_{s_0} \rangle = \delta_{kj} \quad \forall \quad k, j \ge 2
$$

where κ_{s_0} is the zeta kernel at s_0 . So $(u_j + \kappa_{s_0})_{j \geq 2}$ is also biorthogonal to $(h_k)_{k \geq 2}$. *Proof.* This follows by [\(1.4\)](#page-5-0) and Theorem [17](#page-9-2) since $\langle h_k, \kappa_{s_0} \rangle = 0$ for all $k \geq 2$. \Box

4. The RH-failure conjecture

The RH-failure (RHF) conjecture states that if the RH is false, then $\zeta(s) = 0$ for infinitely many $s \in \mathbb{C}$ with $1/2 < \Re s < 1$. Our goal is to prove the following.

Theorem 21. The RHF conjecture implies that $\dim(\mathcal{N}^{\perp})$ is either 0 or ∞ .

Let $\mathcal{K} := \{\kappa_s : \Re s > 1/2\}$ denote the family of zeta kernels. If $\zeta(s) = 0$ for some $\Re s > 1/2$, then $\langle h_k, \kappa_s \rangle = 0$ for all $k \geq 2$ by [\(1.4\)](#page-5-0) and hence $\kappa_s \in \mathcal{N}^\perp$. So the RHF conjecture implies that $\mathcal{N}^{\perp} \cap \mathcal{K}$ is either empty (by Theorem [1\)](#page-0-0) or has infinitely many elements. Therefore Theorem [21](#page-12-0) follows if we show that K is linearly independent in H^2 . We first show that elements of $\mathcal K$ are common eigenvectors for the adjoints of operators $(W_n)_{n\geq 1}$ defined in (2.2) . For $f \in H^2$ and $n \in \mathbb{N}$, we have

$$
W_n^* f(z) = \sum_{k=0}^{\infty} [\hat{f}(nk) + \hat{f}(nk+1) + \dots + \hat{f}(nk+n-1)] z^k
$$

where $\hat{f}(n)$ denotes the *n*-th Fourier coefficient of f. This formula first appeared in [\[17\]](#page-16-19). It is possible to describe the common eigenvectors of $(W_n^*)_{n\geq 1}$ completely.

Proposition 22. A non-zero $f \in H^2$ is a common eigenvector for $(W_n^*)_{n \geq 1}$ if and only if there exists a multiplicative sequence $(\lambda_n)_{n\geq 1}$ with $(\lambda_{n+1} - \lambda_n)_{n\geq 1} \in \ell^2$ and

(4.1)
$$
\hat{f}(n) = (\lambda_{n+1} - \lambda_n)\hat{f}(0) \quad \forall \quad n \ge 1.
$$

Moreover $W_n^* f = \lambda_n f$ for all $n \geq 1$.

By a multiplicative sequence $(\lambda_n)_{n\geq 1}$ we mean that $\lambda_n \lambda_m = \lambda_{nm}$ and $\lambda_1 = 1$. Similarly one can see that $W_n W_m = W_{nm}$ and $W_1 = I$ by [\(2.2\)](#page-7-1).

Proof. Let $W_n^* f = \lambda_n f$ for $n \ge 1$ and some sequence $(\lambda_n)_{n \ge 1}$. Since $W_1^* = I$ and $W_{nm}^* = W_n^* W_m^*$ it follows that $(\lambda_n)_{n \geq 1}$ is multiplicative. Furthermore

$$
\lambda_n \hat{f}(k) = \langle W_n^* f, z^k \rangle = \langle f, W_n z^k \rangle = \left\langle f, \sum_{j=0}^{n-1} z^{nk+j} \right\rangle = \sum_{j=0}^{n-1} \hat{f}(nk+j).
$$

which gives $\hat{f}(n) = \lambda_{n+1} \hat{f}(0) - \lambda_n \hat{f}(0)$ for all $n \ge 1$ and hence $(\lambda_{n+1} - \lambda_n)_{n \ge 1} \in \ell^2$. Conversely suppose f is a non-zero function satisfying (4.1) for some multiplicative $(\lambda_n)_{n\geq 1}$ with $(\lambda_{n+1} - \lambda_n)_{n\geq 1} \in \ell^2$. Normalizing by supposing $\hat{f}(0) = 1$, we get

$$
(W_n^* f)(z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{n-1} \hat{f}(nk+j) \right) z^k = \sum_{j=0}^{n-1} \hat{f}(j) + \sum_{k=1}^{\infty} \left(\sum_{j=0}^{n-1} \hat{f}(nk+j) \right) z^k
$$

$$
= \lambda_n + \sum_{k=1}^{\infty} (\lambda_{nk+n} - \lambda_{nk}) z^k = \lambda_n \left(1 + \sum_{k=1}^{\infty} (\lambda_{k+1} - \lambda_k) z^k \right) = \lambda_n f(z)
$$

for all $n \geq 2$. So $f \in H^2$ is a common eigenvector for $(W_n^*)_{n \geq 1}$.

Choosing $\lambda_k = k^{1-\bar{s}}$ and $\hat{f}(0) = -1/\bar{s}$ in Proposition [22](#page-13-1) for any fixed $\Re s > 1/2$ shows that each $\kappa_s \in \mathcal{K}$ is a common eigenvector for $(W_n^*)_{n \geq 1}$ (see [\(1.3\)](#page-5-2)) with

(4.2)
$$
W_n^* \kappa_s = n^{1-\bar{s}} f \quad \forall \ n \ge 1.
$$

We want to prove that for any finite subset $\{\kappa_{s_1},\ldots,\kappa_{s_\ell}\}\subset \mathcal{K}$ there exists some W_n^* such that the corresponding eigenvalues are all distinct. This will give us the linear independence of every finite subset of K and hence of K itself. First suppose that the real parts of s_1, \ldots, s_ℓ are all distinct. Since $|n^{1-\bar{s}}| = n^{1-\Re s}$ it follows that the eigenvalues of W_n^* (for all $n > 1$) corresponding to $\kappa_{s_1}, \ldots, \kappa_{s_\ell}$ are all distinct. If the real parts of s_1, \ldots, s_ℓ are not all distinct, then we need the following result.

Lemma 23. Given distinct $a_1, \ldots, a_n \in \mathbb{R}$, at most finitely many primes p have the property that there exists a pair a_i, a_j with $1 \leq i < j \leq n$ such that

$$
(4.3) \t\t\t (a_i - a_j) \log p \in 2\pi \mathbb{Z}.
$$

Proof. Suppose there are infinitely many primes that satisfy [\(4.3\)](#page-13-2). For each such prime p there exists some $1 \leq i < j \leq n$ and $k \in \mathbb{Z} \setminus \{0\}$ such that

$$
(a_i - a_j) \log p = 2\pi k \implies \frac{2\pi k}{\log p} = a_i - a_j.
$$

But since there are only finitely many numbers $a_i - a_j$ with $i < j$, and none of which equal 0, there must exist distinct primes p, q and $k_1, k_2 \in \mathbb{Z} \setminus \{0\}$ such that

$$
\frac{2\pi k_1}{\log p} = a_i - a_j = \frac{2\pi k_2}{\log q} \implies k_2 \log p = k_1 \log q \neq 0.
$$

for some pair $i < j$. In particular, $p^{k_2} = q^{k_1} \neq 1$, which is a contradiction.

The following result then completes the proof of Theorem [21.](#page-12-0)

Proposition 24. The family of zeta kernels K is linearly independent.

Proof. Let $\{\kappa_{s_1},\ldots,\kappa_{s_\ell}\}\subset \mathcal{K}$ be a finite subset. The case when the real parts of s_1, \ldots, s_ℓ are all distinct was already dealt with. Suppose some of the s_1, \ldots, s_ℓ have the same real parts. So $\{s_1, \ldots, s_\ell\}$ is the finite disjoint union of sets of the form $A_r := \{s_i : \Re s_i = r, i = 1, \ldots, \ell\}$ for $r \in \mathbb{R}$. It is enough to prove that the family $\{\kappa_s : s \in A_r\}$ is linearly independent when A_r has more than one element. Since s_1, \ldots, s_ℓ are distinct complex numbers, the imaginary parts of elements in A_r , which we denote by a_1, \ldots, a_n , must all be distinct. Applying Lemma [23](#page-13-3) to a_1, \ldots, a_n shows that there exist infinitely many primes q such that

(4.4)
$$
(a_i - a_j) \log q \notin 2\pi \mathbb{Z}, \quad \forall \ 1 \leq i < j \leq n.
$$

For such a prime q, we claim that the $\{\kappa_s : s \in A_r\}$ are W_q^* -eigenvectors with distinct eigenvalues. To see this first note that $W_q^* \kappa_s = q^{1-\bar{s}} \kappa_s$ by [\(4.2\)](#page-13-4) and

$$
q^{1-\bar{s}}=e^{(1-\bar{s})\log q}=e^{(1-r)\log q}e^{i\text{Im}(s)\log q}\ \ \forall\ s\in A_r.
$$

But Im(s) for $s \in A_r$ are precisely the real numbers a_1, \ldots, a_n . Therefore the eigenvalues $q^{1-\bar{s}}$ for $s \in A_r$ are all distinct by [\(4.4\)](#page-14-0) and hence $\{\kappa_s : s \in A_r\}$ and therefore all of K is linearly independent.

5. Appendix

Denote by \mathbb{C}_{ρ} the half-plane $\{s \in \mathbb{C} : \Re s > \rho\}$. In this appendix we provide an alternate proof for the fundamental relation

(5.1)
$$
\langle h_k, \kappa_s \rangle = -\frac{\zeta(s)}{s} (k^{1-s} - 1) \ \ \forall \ s \in \mathbb{C}_{1/2}, \ k \ge 2.
$$

We first prove that [\(5.1\)](#page-14-1) holds for all $s \in \mathbb{C}_1$. We then prove that the function $s \mapsto \langle h_k, \kappa_s \rangle$ has an analytic continuation to \mathbb{C}_0 for each $k \geq 2$. Since the right side of [\(5.1\)](#page-14-1) is already analytic for $s \in \mathbb{C} \setminus \{0\}$, the result then follows by analytic continuation. Recall from Subsection 1.7 that

$$
\kappa_s(z) = \sum_{n=0}^{\infty} \phi_n(\bar{s}) z^n
$$
 where $\phi_n(s) = -\frac{1}{s} ((n+1)^{1-s} - n^{1-s}).$

Lemma 25. The identity [\(5.1\)](#page-14-1) holds for $s \in \mathbb{C}_1$.

Proof. Let $(c_n(k))_n$ be the Fourier coefficients of h_k . Since $\overline{\phi_n(\overline{s})} = \phi_n(s)$, we have

$$
\langle h_k, \kappa_s \rangle = \sum_{n=0}^{\infty} c_n(k) \overline{\phi_n(\overline{s})} = \sum_{n=0}^{\infty} c_n(k) \phi_n(s)
$$

=
$$
\lim_{N \to \infty} \left(-\frac{c_0(k)}{s} - \frac{1}{s} \sum_{n=1}^{N} c_n(k) \left((n+1)^{1-s} - n^{1-s} \right) \right)
$$

=
$$
\lim_{N \to \infty} \left(-\frac{1}{s} \sum_{n=0}^{N} c_n(k) (n+1)^{1-s} + \frac{1}{s} \sum_{n=1}^{N} c_n(k) n^{1-s} \right)
$$

=
$$
\lim_{N \to \infty} \left(-\frac{1}{s} \sum_{n=1}^{N} (c_{n-1}(k) - c_n(k)) n^{1-s} - \frac{1}{s} c_N(k) (N+1)^{1-s} \right).
$$

Since $c_n(k) = O(k/n)$ (see [\[19,](#page-16-0) p. 249]), we have $c_N(k)(N+1) = O(1)$. Furthermore $(N+1)^{-s} \to 0$ for $\Re(s) > 0$ and therefore we get

$$
\langle h_k, \kappa_s \rangle = -\frac{1}{s} \lim_{N \to \infty} \left(\sum_{n=1}^N (c_{n-1}(k) - c_n(k)) n^{1-s} \right)
$$

$$
\stackrel{(3.4)}{=} -\frac{1}{s} \lim_{N \to \infty} \left(\sum_{n=1}^N -\frac{1}{n} n^{1-s} + \sum_{\substack{n=1 \\ k|n}}^N \frac{k}{n} n^{1-s} \right)
$$

$$
= -\frac{1}{s} \lim_{N \to \infty} \left(\sum_{n=1}^N n^{-s} + \sum_{m=1}^{\lfloor \frac{N}{k} \rfloor} \frac{1}{m} (mk)^{1-s} \right)
$$

$$
= -\frac{1}{s} \lim_{N \to \infty} \left(-\sum_{n=1}^N n^{-s} + k^{1-s} \sum_{m=1}^{\lfloor \frac{N}{k} \rfloor} \frac{1}{m} m^{1-s} \right)
$$

$$
\stackrel{(*)}{=} -\frac{1}{s} (-\zeta(s)) - \frac{k^{1-s}}{s} \zeta(s) = -\frac{\zeta(s)}{s} (k^{1-s} - 1),
$$

where in (*) we split the limit in two and use the definition of ζ for $\Re(s) > 1$. \Box

The inner product $\langle h_k, \kappa_s \rangle$ defined for $s \in \mathbb{C}_{1/2}$ also makes sense for $s \in \mathbb{C}_0$.

Lemma 26. The function $\Phi_k : \mathbb{C}_{1/2} \to \mathbb{C}$ defined by

(5.2)
$$
\Phi_k(s) := \langle h_k, \kappa_s \rangle = \sum_{n=0}^{\infty} c_n(k) \phi_n(s).
$$

has an analytic continuation to \mathbb{C}_0 for each $k \geq 2$.

Proof. Since each ϕ_n is holomorphic in \mathbb{C}_0 , it is sufficient to prove that the series in [\(5.2\)](#page-15-4) converges uniformly in every half-plane \mathbb{C}_{ρ} for $\rho > 0$. Note that

$$
|\phi_n(s)| = \frac{|1-s|}{|s|} \left| \int_n^{n+1} y^{-s} dy \right| \le \frac{|1-s|}{|s|} n^{-\Re s} = O(n^{-\rho})
$$

for $s \in \mathbb{C}_{\rho}$ with $\rho > 0$. Also $c_n(k) = O(k/n)$ for each $k \geq 2$, and hence we get $c_n(k)\phi_n(s) = O(n^{-1-\rho})$ for $s \in \mathbb{C}_{\rho}$. So Φ_k converges uniformly in \mathbb{C}_{ρ} for $\rho > 0$. \Box

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