# ORTHOGONALITY QUESTIONS IN THE HARDY SPACE RELATED TO $\zeta\text{-}\mathbf{Z}\mathbf{E}\mathbf{R}\mathbf{O}\mathbf{S}$

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ABSTRACT. A Hardy space approach to the Nyman-Beurling and Báez-Duarte criterion for the Riemann Hypothesis (RH) was introduced recently in [19] and further developed in [14]. It states that the RH holds if and only if a particular sequence of functions  $(h_k)_{k\geq 2}$  is complete in the Hardy space  $H^2$ . This article is concerned with orthogonality questions related to the family  $(h_k)_{k\geq 2}$ . The first goal is to analyze the orthogonal complement of  $\mathcal{N} = \operatorname{span}(h_k)_{k\geq 2}$  in  $H^2$ . Unbounded Toeplitz operators on  $H^p$  spaces and de Branges-Rovnyak spaces play a central role and our results show that the size and dimension of  $\mathcal{N}^{\perp}$  reveal information on the zeros of the Riemann  $\zeta$ -function. The second goal is to show that  $(h_k)_{k\geq 2}$  possesses a complete biorthogonal sequence in  $H^2$ . We also discuss a folklore conjecture about the number of  $\zeta$ -zeros if the RH fails.

#### INTRODUCTION

A classical result of Nyman and Beurling (see [6], [20]) shows that the Riemann Hypothesis (RH) is equivalent to the completeness of  $\{\rho_{\lambda} : 0 \leq \lambda \leq 1\}$  in  $L^2(0, 1)$ where  $\rho_{\lambda}(x) = \{\lambda/x\} - \lambda\{1/x\}$  and  $\{x\}$  denotes the fractional part of x. This is furthermore equivalent to the characteristic function  $\chi_{(0,1)}$  belonging to the closed linear span of  $\{\rho_{\lambda} : 0 \leq \lambda \leq 1\}$ . Half a century later Báez-Duarte [2] strengthened this result by showing that RH is true if and only if  $\chi_{(0,1)} \in \overline{\text{span}\{\rho_{1/k} : k \geq 2\}}$ . See the expository article of Bagchi [3] and the survey of Balazard [4]. Recently the Nyman-Beurling and Báez-Duarte approaches to the RH have been explored via tools from Hardy space  $H^2$  theory [19] and other analytic function spaces [14].

For each  $k \geq 2$ , define

$$h_k(z) = \frac{1}{1-z} \log\left(\frac{1+z+\dots+z^{k-1}}{k}\right)$$

and denote by  $\mathcal{N}$  the linear span of  $\{h_k : k \geq 2\}$ . That each  $h_k$  belongs to  $H^2$  was proved in [19, Lemma 7]. One of the main results of [19] was a reformulation of Báez-Duarte's result as a completeness problem in  $H^2$ . Then in [14] the same completeness problem in the  $H^p$  spaces was shown to provide zero-free half planes for the Riemann  $\zeta$ -function. We state both results here as one.

**Theorem 1.** The RH holds if and only if  $\mathcal{N}$  is dense in  $H^2$ . If  $\mathcal{N}$  is dense in  $H^p$  for some 1 , then

$$\zeta(s) \neq 0$$
 for  $\Re(s) > \frac{1}{p}$ .

The density of  $\mathcal{N}$  in  $H^1$  gives the known zero-free region  $\Re(s) \geq 1$ .

Key words and phrases. Riemann hypothesis, Hardy space, local Dirichlet space, de Branges-Rovnyak space, Smirnov class.

#### 2 FRANCISCO CALDERARO, JUAN MANZUR, WALEED NOOR AND CHARLES SANTOS

Although  $\mathcal{N}$  is unconditionally dense in  $H^p$  for 0 (see [14, Cor. 4.6]), this $provides no new information regarding <math>\zeta$ -zeros. Also note that the RH is equivalent to  $\mathcal{N}^{\perp} = \{0\}$  in  $H^2$  by Theorem 1. This suggests the question of the size and dimension of  $\mathcal{N}^{\perp}$  and their possible relation to the  $\zeta$ -zeros. The first archetypal result addressing this is the following (See [19, Thm. 12]).

# **Theorem 2.** $\mathcal{N}^{\perp} \cap \mathcal{D}_{\delta_1} = \{0\}$ , where $\mathcal{D}_{\delta_1}$ denotes the local Dirichlet space at 1.

Since  $\mathcal{D}_{\delta_1} \subset H^2 \subset H^p$  for  $0 , Theorems 1 and 2 inspire the following question: which linear spaces of analytic functions on <math>\mathbb{D}$  with  $X_1 \subset H^2 \subset X_2$  satisfy

(1) 
$$\mathcal{N}^{\perp} \cap X_1 = \{0\}$$
 and (2)  $\mathcal{N}$  is dense in  $X_2$ ?

Interestingly, both (1) and (2) are equivalent if one takes  $X_2 = X$  a Frechet space and  $X_1 = X^*$  its Cauchy dual (see Proposition 9). In Section 2 we analyze this *orthogonality question* using tools from de Branges-Rovnyak spaces and unbounded Toeplitz operators on  $H^p$  (see Proposition 11), and in particular develop a new approach for finding zero-free half-planes for the  $\zeta$  function (see Theorem 14).

In Section 3 we deal with the biorthogonality question: Does the sequence  $(h_k)_{k\geq 2}$  posses a complete biorthogonal sequence in  $H^2$ ? (see Subsection 1.5). We affirmatively answer this question (see Theorem 15). To provide some context, Vasyunin showed in [22] that the Báez-Duarte sequence  $\{\rho_{1/k} : k \geq 2\}$  is minimal by constructing a biorthogonal sequence for it in  $L^2(0,1)$ . Whereas the sequence  $\{\rho_{1/k} : k \geq 2\}$  is not complete in  $L^2(0,1)$ , the completeness of  $(h_k)_{k\geq 2}$  in  $H^2$  is equivalent to the RH by Theorem 1. The property of completeness of a sequence is not in general inherited by its biorthogonal sequence. But it may do so in very special cases such as for sequences of complex exponentials in  $L^2(-\pi,\pi)$  (see [25]). Therefore an affirmative answer to this question (independent of RH) is intriguing.

Finally in Section 4 we discuss a folklore conjecture about the  $\zeta$ -zeros which we name the *RH failure* (RHF) conjecture: *If the RH fails, then it fails infinitely often.* More precisely, either  $\zeta$  has no nontrivial zeros outside the critical line, or it has infinitely many. The main result of this section (Theorem 21) states that

RHF conjecture  $\implies \dim(\mathcal{N}^{\perp})$  is either 0 or  $\infty$ .

We were unable to locate a reference for this conjecture in the literature. But an informative discussion of this conjecture does appear in mathoverflow.net [11].

# 1. Preliminaries

We denote by  $\mathbb{D}$  and  $\mathbb{T}$  the open unit disk and the unit circle respectively. By  $\operatorname{Hol}(\mathbb{D})$  we denote the space of holomorphic functions on  $\mathbb{D}$  and the shift operator is defined by (Sf)(z) = zf(z) for  $f \in \operatorname{Hol}(\mathbb{D})$ .

1.1. Hardy spaces and the Smirnov class. A holomorphic function f on  $\mathbb{D}$  belongs to the Hardy-Hilbert space  $H^2$  if

$$||f||_{H^2} = \sup_{0 \le r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right)^{1/2} < \infty.$$

The space  $H^2$  is a Hilbert space with the  $\ell^2$ -inner product

$$\langle f,g\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n},$$

where  $(a_n)_{n\in\mathbb{N}}$  and  $(b_n)_{n\in\mathbb{N}}$  are the Fourier coefficients for f and g respectively. For any  $f \in H^2$  and  $\zeta \in \mathbb{T}$ , the radial limit  $f^*(\zeta) := \lim_{r \to 1^-} f(r\zeta)$  exists *m*-a.e. on  $\mathbb{T}$ , where *m* denotes the normalized Lebesgue measure on  $\mathbb{T}$ . Analogously for p > 0, the Hardy space  $H^p$  consists of those holomorphic f on  $\mathbb{D}$  such that

$$||f||_{H^p}^p = \sup_{0 < r < 1} \int_{\mathbb{T}} |f(rz)|^p \, \mathrm{d}m(z) < \infty.$$

The  $H^p$  are Banach spaces for  $p \ge 1$  and complete metric spaces for 0 $and <math>H^{\infty}$  denotes the space of bounded holomorphic functions on  $\mathbb{D}$ . A function f is called a cyclic vector for the shift S in  $H^p$  if  $\operatorname{span}(S^n f)_{n\ge 0} = \mathbb{C}[z]f$  is dense in  $H^p$ . When p = 2 these cyclic vectors are commonly known as outer functions. Since  $H^{\infty} \subset H^p$  for all p > 0, the  $H^{\infty}$  outer functions are cyclic for all  $H^p$  spaces. The Smirnov class  $N^+$  consists of all holomorphic functions g/h on  $\mathbb{D}$  such that  $g,h \in H^{\infty}$  and h is an outer function. The space  $N^+$  is a topological algebra with respect to pointwise multiplication and  $g/h \in N^+$  is a unit if both g and h are outer functions. The topology on  $N^+$  can be metrized with the translation-invariant complete metric

$$d(f,g) = \int_{\mathbb{T}} \log(1+|f-g|) \,\mathrm{d}m, \qquad f,g \in N^+.$$

Similar to  $H^p$  spaces, convergence in  $N^+$  implies locally unifom convergence on  $\mathbb{D}$ and functions have radial limits *m*-a.e. on  $\mathbb{T}$ . In fact we have  $H^p \subset H^q \subset N^+$  for all  $0 < q < p \le \infty$ . Duren [10] is a classic reference for the  $H^p$  and  $N^+$  spaces.

1.2. Local Dirichlet spaces. Let  $\mu$  be a finite positive Borel measure on  $\mathbb{T}$ , and let  $P\mu$  denote its Poisson integral. The generalized Dirichlet space  $\mathcal{D}_{\mu}$  consists of  $f \in H^2$  satisfying

$$\mathcal{D}_{\mu}(f) := \int_{\mathbb{D}} \left| f'(z) \right|^2 P\mu(z) dA(z) < \infty.$$

Then  $\mathcal{D}_{\mu}$  is a Hilbert space with norm  $\|f\|_{\mathcal{D}_{\mu}}^{2} := \|f\|_{2}^{2} + \mathcal{D}_{\mu}(f)$ . If  $\mu = m$ , then  $\mathcal{D}_{m}$  is the classicial Dirichlet space. If  $\mu = \delta_{\zeta}$  is the Dirac measure at  $\zeta \in \mathbb{T}$ , then  $\mathcal{D}_{\zeta} := \mathcal{D}_{\delta_{\zeta}}$  is called the *local Dirichlet space* at  $\zeta$  and in particular

(1.1) 
$$\mathcal{D}_{\delta_{\zeta}}(f) = \int_{\mathbb{D}} \left| f'(z) \right|^2 \frac{1 - \left| z^2 \right|}{\left| z - \zeta \right|^2} dA(z)$$

The recent book [18] contains a comprehensive treatment of local Dirichlet spaces and the following result establishes a criterion for their membership.

**Theorem 3.** (See [18, Thm. 7.2.1]) Let  $\zeta \in \mathbb{T}$  and  $f \in Hol(\mathbb{D})$ . Then  $\mathcal{D}_{\delta_{\zeta}}(f) < \infty$  if and only if

$$f(z) = (z - \zeta)g(z) + a$$

for some  $g \in H^2$  and  $a \in \mathbb{C}$ . In particular  $f^*(\zeta)$  exists for all  $f \in \mathcal{D}_{\zeta}$ .

So each local Dirichlet space  $\mathcal{D}_{\zeta} = (S - \zeta I)H^2 + \mathbb{C}$  is a proper subspace of  $H^2$ . We define the  $H^p$ -analogues of these spaces for p > 0 by

$$\mathcal{D}^p_{\zeta} := (S - \zeta I)H^p + \mathbb{C}$$

and note that  $\mathcal{D}_{\zeta}^2 = \mathcal{D}_{\delta_{\zeta}}$  and  $\mathcal{D}_{\zeta}^p \subsetneq \mathcal{D}_{\zeta}^q$  for q < p since  $H^p \subsetneq H^q$ . Straightforward but lengthy computations show that  $\mathcal{D}_{\zeta}^p \subsetneq H^2$  for p > 1 and  $H^2 \subsetneq \mathcal{D}_{\zeta}^p$  for 0 .

1.3. The de Branges-Rovnyak spaces. Given  $\psi \in L^{\infty}(\mathbb{T})$ , the corresponding Toeplitz operator  $T_{\psi}: H^2 \to H^2$  is defined by

$$T_{\psi}f := P_+(\psi f)$$

where  $P_+ : L^2(\mathbb{T}) \to H^2$  denotes the orthogonal projection of  $L^2(\mathbb{T})$  onto  $H^2$ . Clearly  $T_{\psi}$  is a bounded operator on  $H^2$  with  $||T_{\psi}|| \leq ||\psi||_{L^{\infty}}$ . If  $h \in H^{\infty}$ , then  $T_h$  is the operator of multiplication by h and its adjoint is  $T_{\overline{h}}$ . Given b in the closed unit ball of  $H^{\infty}$ , the *de Branges-Rovnyak* space  $\mathcal{H}(b)$  is the image of  $H^2$  under the operator  $(I - T_b T_{\overline{b}})^{1/2}$ . The general theory of  $\mathcal{H}(b)$  spaces divides into two distinct cases, according to whether b is an extreme point or a non-extreme point of the unit ball of  $H^{\infty}$ . We shall be concerned only with the non-extreme case. In this case there exists a unique outer function  $a \in H^{\infty}$  such that a(0) > 0 and  $|a^*|^2 + |b^*|^2 = 1$  a.e. on  $\mathbb{T}$ . The pair (b, a) is called a *Pythagorean pair* and the function b/a belongs to the Smirnov class  $N^+$ . That all  $N^+$  functions arise as the quotient of a pair associated to a non-extreme function was shown by Sarason [21]. The two-volume work ([12][13]) is an encyclopedic reference for these spaces.

If  $\varphi$  is a rational function in  $N^+$  the corresponding pair (b, a) is also rational (see [21, Remark. 3.2]). Constara and Ransford [8] characterized the rational pairs (b, a) for which  $\mathcal{H}(b)$  is a generalized Dirichlet space.

**Theorem 4.** (See [8, Theorem 4.1]) Let (b, a) be a rational pair and  $\mu$  a finite positive measure on  $\mathbb{T}$ . Then  $\mathcal{H}(b) = \mathcal{D}_{\mu}$  if and only if

- (1) the zeros of a on  $\mathbb{T}$  are all simple, and
- (2) the support of  $\mu$  is exactly equal to this set of zeros.

As an example, if (b, a) is the rational pair associated with the  $N^+$  function  $\varphi(z) = \frac{1}{1-\zeta}$  for  $\zeta \in \mathbb{T}$ , then  $\mathcal{H}(b) = \mathcal{D}_{\zeta}$  is a local Dirichlet space.

1.4. Unbounded Toeplitz operators on  $H^p$ . Sarason [21] demonstrated how  $\mathcal{H}(b)$  spaces appear naturally as the domains of some unbounded Toeplitz operators. Let  $\varphi$  be holomorphic in  $\mathbb{D}$  and  $T_{\varphi}$  the operator of multiplication by  $\varphi$  on the domain

(1.2) 
$$\operatorname{dom}(T_{\varphi}) = \{ f \in H^2 : \varphi f \in H^2 \}$$

Then  $T_{\varphi}$  is a closed operator, and dom $(T_{\varphi})$  is dense in  $H^2$  if and only if  $\varphi \in N^+$  (see [21, Lemma 5.2]). In this case its adjoint  $T_{\varphi}^*$  is also densely defined and closed. In fact the domain of  $T_{\varphi}^*$  is a de Branges-Rovnyak space.

**Theorem 5.** (See [21, Prop. 5.4]) Let  $\varphi$  be a nonzero function in  $N^+$  with  $\varphi = b/a$ , where (b, a) is the associated pair. Then dom $(T^*_{\varphi}) = \mathcal{H}(b)$ .

Choosing the symbol  $\varphi(z) = \frac{1}{\zeta - z}$  in Theorem 5 in conjunction with Theorem 4 gives dom $(T_{\varphi}^*) = \mathcal{D}_{\zeta}$  which played a key role in the proof of Theorem 2 (see [19]). Our goal here is to extend these ideas to  $H^p$  spaces for all p > 1. Let  $\varphi \in N^+$  and define the analytic Toeplitz operator on  $H^p$  with symbol  $\varphi$  by

$$T_{\varphi}f = \varphi f$$
, where  $f \in \operatorname{dom}_p(T_{\varphi}) := \{f \in H^p : \phi f \in H^p\}$ .

These  $T_{\varphi}$  are bounded on  $H^p$  precisely when  $\varphi \in H^{\infty}$  (see the survey article [23]). For  $\varphi = \frac{b}{a} \in N^+$  with  $a, b \in H^{\infty}$  and a outer as usual, these  $T_{\varphi}$  are densely defined on  $H^p$  for p > 1. Indeed, dom<sub>p</sub>( $T_{\varphi}$ ) contains the dense subspace  $aH^p$  since a is outer and  $T_{\varphi}(aH^p) = bH^p \subset H^p$ . It follows then that the adjoint  $T_{\varphi}^*$  is well-defined on the dual  $(H^p)^* = H^q$  where  $\frac{1}{p} + \frac{1}{q} = 1$ . The domain of  $T_{\varphi}^*$  is then defined by

$$\operatorname{dom}_{q}(T_{\varphi}^{*}) := \{g \in H^{q} : \exists h \in H^{q} \ s.t \ \langle f, h \rangle = \langle \varphi f, g \rangle \ \forall f \in \operatorname{dom}_{p}(T_{\varphi}) \}$$

where  $\langle f,h \rangle := \int_{\mathbb{T}} f \overline{h} dm$  represents the  $H^p$ - $H^q$  duality. The elements in dom<sub>q</sub>( $T^*_{\varphi}$ ) can be characterized via the bounded Toeplitz operators  $T_a$  and  $T_b$  as follows.

**Lemma 6.** Given  $\varphi = \frac{b}{a} \in N^+$  as described above, a function  $g \in \text{dom}_q(T_{\varphi}^*)$  if and only if there exists an  $h \in H^q$  such that  $T_b^*g = T_a^*h$ .

*Proof.* Suppose  $g \in \text{dom}_q(T^*_{\varphi})$ . Then  $\langle f, h \rangle = \langle \varphi f, g \rangle$  for some  $h \in H^q$  and for all  $f \in aH^p \subset \text{dom}_p(T_{\varphi})$ . Writing  $f = a\tilde{f}$  for  $\tilde{f} \in H^p$ , we get

$$\langle f,h\rangle = \langle \varphi f,g\rangle \iff \langle f,h\rangle = \langle bf/a,g\rangle \iff \langle a\tilde{f},h\rangle = \langle b\tilde{f},g\rangle \; \forall \; \tilde{f} \in H^p$$

which is equivalent to  $T_a^*h = T_b^*g$ . This argument works in both directions because a is an  $H^{\infty}$  outer function and hence  $aH^p$  is dense in  $H^p$ .

We can now extend the identity  $\operatorname{dom}(T^*_{\varphi}) = \mathcal{D}_{\zeta}$  from  $H^2$  to all  $H^p$  with p > 1.

**Proposition 7.** Let  $\varphi(z) = \frac{1}{\zeta - z}$  for  $\zeta \in \mathbb{T}$  and  $T_{\varphi}$  the densely defined Toeplitz operator on  $H^p$  for any p > 1. We then have

$$\operatorname{dom}_q(T^*_{\varphi}) = \mathcal{D}^q_{\zeta}, \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

*Proof.* Choosing  $\varphi = \frac{b}{a}$  with b(z) = 1 and  $a(z) = \zeta - z$  (which is outer) in Lemma 6, we get  $g \in \text{dom}_q(T_{\varphi}^*)$  if and only if  $g = T_{\zeta-z}^*h = (\overline{\zeta}I - S^*)h$  for some  $h \in H^q$ . Therefore, it suffices to verify that  $(\overline{\zeta}I - S^*)H^q = \mathcal{D}_{\zeta}^q$ . For any  $h \in H^q$ , we have

$$(\overline{\zeta}I - S^*)h = (\zeta I - S)(-\overline{\zeta}S^*h) + \overline{\zeta}h(0) \in \mathcal{D}^q_{\zeta} := (S - \zeta I)H^q + \mathbb{C}$$

and therefore  $(\overline{\zeta}I - S^*)H^q \subset \mathcal{D}^q_{\zeta}$ . Conversely, if  $h \in H^q$  and  $c \in \mathbb{C}$  then

$$(\zeta I - S)h + c = (\overline{\zeta}I - S^*)\zeta(c - Sh) \in (\overline{\zeta}I - S^*)H^q$$

and hence  $\mathcal{D}^q_{\zeta} \subset (\overline{\zeta}I - S^*)H^q$  which concludes the proof.

1.5. Cauchy duality. Let X be a complete metrizable linear subspace of  $\operatorname{Hol}(\mathbb{D})$ . Inspired by terminology used by Malman and Seco [16], we call  $X^*$  the Cauchy dual of X if any continuous linear functional on X can be represented by the Cauchy pairing

$$\langle f,g\rangle := \lim_{r \to 1^-} \int_{\mathbb{T}} f(r\zeta) \overline{g(r\zeta)} \, \mathrm{d}m(\zeta) \,, \quad f \in X, \ g \in X^* \,.$$

If  $H^2 \subset X$  then  $X^* \subset H^2$  and vice-versa. Hence when both f and g are in  $H^2$ , the pairing above reduces to the standard inner product in  $H^2$ . Some examples of Cauchy duals for our context are listed below (see [10],[15],[24]).

- (1)  $H^p$  and  $H^q$  for p > 1 and 1/p + 1/q = 1,
- (2)  $H^1$  and BMOA (analytic functions with bounded mean oscillation on  $\mathbb{T}$ ),
- (3)  $H^p$  for  $1/2 and <math>\Lambda_{\alpha}$  (the Lipschitz class of Hol( $\mathbb{D}$ )-functions with  $\alpha$ -Hölder continuous extension to  $\mathbb{T}$ , where  $\alpha = 1/p 1$ ).
- (4)  $N^+$  and the *Gevrey class*  $\mathcal{G}$  (Hol( $\mathbb{D}$ )-functions whose Taylor coefficients satisfy  $a_n = O(e^{-c\sqrt{n}})$  for some constant c > 0).

A deep result of Davis and McCarthy [9] shows that the class  $\mathcal{G}$  coincides with the universal multipliers for all non-extreme  $\mathcal{H}(b)$  spaces. In particular  $\mathcal{G} \subset \mathcal{H}(b)$  for all non-extreme b. The concept of Cauchy duality leads to an equivalence between orthogonality and density questions involving  $\mathcal{N}$  which is explored in Section 2.

1.6. Minimality and biorthogonality. Let  $\mathcal{H}$  be a Hilbert space. Two sequences  $(e_n)_{n \in \mathbb{N}}$  and  $(f_n)_{n \in \mathbb{N}}$  in  $\mathcal{H}$  are said to be *biorthogonal* to each other if

$$\langle e_n, f_m \rangle = \delta_{nm} \quad \forall \ n, m \in \mathbb{N}$$

where  $\delta_{nm}$  is the Kronecker delta. The sequence  $(e_n)_{n \in \mathbb{N}}$  is called *minimal* if  $e_n \notin \overline{\text{span}}(e_k)_{k \neq n}$  for all  $n \in \mathbb{N}$ . The notions of biorthogonality, minimality and completeness are all related via the following well-known result.

**Proposition 8.** (see [7, Lemma 3.3.1]) Let  $(e_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{H}$ . Then,

- (i)  $(e_n)_{n \in \mathbb{N}}$  has a biorthogonal sequence if and only if  $(e_n)_{n \in \mathbb{N}}$  is minimal.
- (ii) If (e<sub>n</sub>)<sub>n∈ℕ</sub> has a biorthogonal sequence, then (e<sub>n</sub>)<sub>n∈ℕ</sub> is complete in H if and only if its biorthogonal sequence is unique.

In Section 3 we shall prove that the sequence  $(u_k)_{k\geq 2}$  defined by

$$u_k(z) = \sum_{d|k} \frac{\mu(k/d)}{k/d} (z^{d-1} - z^d)$$

forms a complete biorthogonal sequence for  $(h_k)_{k\geq 2}$  in  $H^2$ , where  $\mu$  denotes the Möbius function defined on  $\mathbb{N}$  by  $\mu(k) = (-1)^s$  if k is the product of s distinct primes, and  $\mu(k) = 0$  otherwise.

1.7. The zeta kernels. Let  $X \subset \operatorname{Hol}(\mathbb{D})$  be a topological vector space where the monomials  $(z^k)_{k \in \mathbb{N}}$  form a Schauder basis. It was shown in [14] that for each  $s \in \mathbb{C} \setminus \{0\}$  a linear functional  $\Lambda^{(s)}$  can be defined on X by assigning

$$\Lambda^{(s)}(1) = -\frac{1}{s}, \quad \Lambda^{(s)}(z^k) = -\frac{1}{s} \left( (k+1)^{1-s} - k^{1-s} \right) \quad (k \ge 1).$$

In particular  $\Lambda^{(s)}$  is bounded on  $H^p$  for  $1 if <math>\Re s > 1/p$  and on  $H^1$  if  $\Re s \geq 1$ (see [14, Prop. 4.7]). So there exist functions  $\kappa_s \in H^q$  with 1/p + 1/q = 1 such that  $\Lambda^{(s)}(f) = \langle f, \kappa_s \rangle$ . The function  $\kappa_s$  will be called the zeta kernel at s and

(1.3) 
$$\kappa_s(z) = \sum_{k=0}^{\infty} \phi_k(\bar{s}) z^k \text{ where } \phi_k(s) := \Lambda^{(s)}(z^k).$$

The name comes from their relation to  $h_k$  and  $\zeta$  via the important identity

(1.4) 
$$\Lambda^{(s)}(h_k) = \langle h_k, \kappa_s \rangle = -\frac{\zeta(s)}{s} (k^{1-s} - 1) \quad \forall \quad \Re s > 1/2, \ k \ge 2.$$

The identity (1.4) appears in [14] but we provide an alternate proof in the appendix for the sake of completeness. It is important to mention that the definition of  $h_k$  in [14] has an additional factor of 1/k which has been adjusted in (1.4) accordingly. The zeta kernels are used in Chapters 3 and play a key role in Chapter 4.

#### 2. The orthogonality question

The objective of this section is to develop a framework for proving when

$$(2.1) \qquad \qquad \mathcal{N}^{\perp} \cap L = \{0\}$$

for topological vector spaces  $L \subset H^2$ . Since  $\mathcal{N}^{\perp} = \{0\}$  is equivalent to the RH by Theorem 1, one may also ask if solutions to (2.1) can lead to new zero-free half-planes for the  $\zeta$ -function. We start by showing that Cauchy duality serves as a bridge between this orthogonality question and completeness questions.

**Proposition 9.** Let X be a topological linear space with  $H^2 \subset X \subset Hol(\mathbb{D})$ , where the inclusions are continuous. If  $\mathcal{N}$  is dense in X, then

$$\mathcal{N}^{\perp} \cap X^* = \{0\}$$

where  $X^* \subset H^2$  is the Cauchy dual of X. The converse holds when X is Fréchet.

*Proof.* First note that since both  $\mathcal{N}^{\perp}$  and  $X^*$  are subspaces of  $H^2$ , their intersection above makes sense. Let  $\langle f, g \rangle$  denote the Cauchy pairing for  $f \in X$  and  $g \in X^*$  and recall that this pairing becomes the usual  $H^2$ -inner product  $\langle f, g \rangle_{H^2}$  when  $f, g \in H^2$ (see Subsection 1.5). Therefore if  $\mathcal{N}$  is dense in X, then

$$g \in \mathcal{N}^{\perp} \cap X^* \implies \langle f, g \rangle = \langle f, g \rangle_{H^2} = 0 \text{ for all } f \in \mathcal{N}$$

which implies that g must be identically zero in  $X^*$ . Conversely if X is additionally a Fréchet space, then we have access to the Hahn-Banach Theorem. Indeed if  $\mathcal{N}$  is not dense in X, then there exists a non-zero  $g \in X^*$  such that  $\langle f, g \rangle = 0 \ \forall f \in \mathcal{N}$ . This implies that g is  $H^2$ -orthogonal to  $\mathcal{N}$  and hence  $g \in \mathcal{N}^{\perp} \cap X^* \neq \{0\}$ .  $\Box$ 

Since  $\mathcal{N}$  is dense in  $H^p$  for 0 (see [14, Cor. 4.6]), it is also dense $in <math>N^+$  since  $H^p \subset N^+$  for all p > 0. Therefore it follows by Proposition 9 that  $\mathcal{N}^{\perp} \cap L = \{0\}$  if L is the Lipschitz class  $\Lambda_{\alpha}$  (1/2 < p < 1 and  $\alpha = 1/p - 1$ ) or the Gevrey class  $\mathcal{G}$  (see Subsection 1.5). However to obtain new zero-free half-planes for  $\zeta$ , we need  $L \subset H^2$  to be large enough to contain some  $H^q$  space for  $q \geq 2$ .

**Corollary 10.** If  $\mathcal{N}^{\perp} \cap H^q = \{0\}$  for some  $q \geq 2$ , then  $\zeta(s) \neq 0$  for  $\Re s > 1/p$ , where 1/p + 1/q = 1. If  $\mathcal{N}^{\perp} \cap BMOA = \{0\}$ , then  $\zeta(s) \neq 0$  for  $\Re s \geq 1$ .

*Proof.* Notice that  $H^q$  is the Cauchy dual of  $H^p$  and  $q \ge 2$  implies that 1 . $In this range the <math>H^p$  are Banach spaces, and in particular Fréchet spaces. Similarly the BMOA space is the Cauchy dual of  $H^1$ . Therefore the result follows by the converse in Proposition 9 and by Theorem 1.

The next result relates Toeplitz operators on  $H^p$  and the orthogonality question. Recall that if  $\varphi \in N^+$  is a unit, then  $1/\varphi \in N^+$  and hence both  $T_{\varphi}$  and its inverse  $T_{1/\varphi}$  are densely defined Toeplitz operators on  $H^p$  for p > 1 (see Subsection 1.4).

**Proposition 11.** Let  $\varphi$  be a unit in  $N^+$  with  $T_{\varphi}$  the Toeplitz operator on  $H^p$  for some p > 1. If  $T_{\varphi}\mathcal{N} = \varphi \mathcal{N}$  is dense in  $H^p$ , then

$$\mathcal{N}^{\perp} \cap \text{dom}_q(T^*_{1/\varphi}) = \{0\}, \quad where \ \frac{1}{p} + \frac{1}{q} = 1.$$

The Hilbertian case p = 2 gives  $\mathcal{N}^{\perp} \cap \mathcal{H}(b) = \{0\}$  where (b, a) is the Pythagorean pair associated with  $1/\varphi \in N^+$ .

*Proof.* Let  $g \in \mathcal{N}^{\perp} \cap \text{dom}_q(T^*_{1/\varphi})$ . Since both g and  $h_k$  belong to  $H^2$  for all  $k \geq 2$ , the Cauchy duality  $\langle h_k, g \rangle$  coincides with the  $H^2$ -inner product  $\langle h_k, g \rangle_2$ . Hence

$$\langle \varphi h_k, T_{1/\omega}^* g \rangle = \langle h_k, g \rangle = \langle h_k, g \rangle_2 = 0 \quad \forall \ k \in \mathbb{N}$$

which implies that  $T_{1/\varphi}^*g = 0$  in  $H^q$  by the density of  $\varphi \mathcal{N}$  in  $H^p$ . Since  $1/\varphi = b/a$ is a unit in  $N^+$  (as the inverse of  $\varphi$ ), both a and b are  $H^\infty$  outer functions in  $H^p$ . So  $T_{1/\varphi}(aH^p) = bH^p$  shows that  $T_{1/\varphi}$  has dense range in  $H^p$  since b is outer. So  $T_{1/\varphi}^*$  is injective and therefore g = 0 in  $H^q$ . This concludes the general case. The Hilbertian case p = 2 now follows by Theorem 5.

We shall derive two non-trivial applications of this result. The first one extends Theorem 2 to all local Dirichlet spaces  $\mathcal{D}_1^p$  with p > 1. We note that the classical  $\mathcal{D}_{\delta_1}$  is just  $\mathcal{D}_1^2$  which is strictly smaller than  $\mathcal{D}_1^p$  for  $p \in (1, 2)$  (see Subsection 1.2). We will need with the following approximation result from [14].

**Lemma 12.** Let  $\mu$  the Möbius function. Then

$$\sum_{k=2}^{n} \frac{\mu(k)}{k} (I-S)h_k \to 1-z$$

in the  $H^p$  norm for all p > 0.

We observe that  $I - S = T_{\varphi}$  where  $\varphi(z) = 1 - z$  is an  $H^{\infty}$  outer function since  $\mathbb{C}[z]\varphi$  is dense in  $H^2$ . In particular  $\varphi$  is a unit in  $N^+$ . Define operators on  $H^p$  by

(2.2) 
$$(W_n)f(z) = (1+z+\dots+z^{n-1})f(z^n) = \frac{1-z^n}{1-z}f(z^n)$$

and  $(T_n)f(z) = f(z^n)$  for  $n \ge 1$  and  $f \in H^p$ . The multiplicative semigroup of operators  $(W_n)_{n\ge 1}$  was introduced in [19] and is the main object of study in [17]. They are bounded on  $H^p$  for p > 1 (see [14, Cor. 4.6]). We shall need the identities

$$W_n \mathcal{N} \subset \mathcal{N}$$
 and  $T_n (I - S) = (I - S) W_n$ 

for  $k, n \ge 1$  which appear in [19, p. 249]. We are ready for the first application.

**Theorem 13.** We have  $\mathcal{N}^{\perp} \cap \mathcal{D}_1^p = \{0\}$  for all p > 1.

Proof. Let  $\varphi(z) = 1 - z$ . By Propositions 7 and 11 we only need to prove that  $\varphi \mathcal{N}$  is dense in  $H^p$  for p > 1. First note that Lemma 12 implies that  $\varphi$  belongs to the  $H^p$ -closure of  $\varphi \mathcal{N} = (I - S)\mathcal{N}$ . This in turn implies that  $T_n\varphi$  belongs to the  $H^p$ -closure of  $T_n(\varphi \mathcal{N}) = \varphi W_n \mathcal{N} \subset \varphi \mathcal{N}$  for all  $n \ge 1$ . So in particular span $(T_n\varphi)_{n\ge 1} \subset \operatorname{clos}_{H^p}(\varphi \mathcal{N})$ . Now  $\operatorname{span}(T_n\varphi)_{n\ge 1} = \operatorname{span}(1-z^n)_{n\ge 1} = \mathbb{C}[z]\varphi$  which is dense in  $H^p$  for all p > 1 because  $\varphi$  is an  $H^{\infty}$  outer function. This proves that  $\operatorname{clos}_{H^p}(\varphi \mathcal{N}) = H^p$  and concludes the proof.

Our second application of Proposition 11 utilizes recent discoveries in  $\mathcal{H}(b)$ -space theory to obtain zero-free half-planes for  $\zeta$ . In view of Corollary 10, we would like to know when the  $H^p$  and BMOA spaces are contained in some  $\mathcal{H}(b)$  for  $\varphi = b/a \in N^+$ . Fortuitously for us, these problems were completely solved recently in a preprint by Malman and Seco [16]. They show that  $H^{\tilde{p}} \subset \mathcal{H}(b)$  for  $\tilde{p} \in (2, \infty)$  if and only if  $\varphi \in H^p$  where  $p = \frac{2\tilde{p}}{\tilde{p}-2} \in (2,\infty)$ , and also that  $H^{\infty} \subset \text{BMOA} \subset \mathcal{H}(b)$  if and only if  $\varphi \in H^2$ . By definition we always have  $\mathcal{H}(b) \subset H^2$ , and  $\mathcal{H}(b) = H^2$  precisely when  $\varphi \in H^{\infty}$ . Therefore it makes sense to allow the values p = 2 and  $p = \infty$ . **Theorem 14.** Suppose  $\varphi$  is a unit in  $N^+$  such that  $1/\varphi \in H^p$  for some  $p \in (2, \infty)$ . If  $\varphi \mathcal{N}$  is dense in  $H^2$ , then  $\zeta(s) \neq 0$  for  $\Re s > \frac{1}{2} + \frac{1}{p}$ . The case p = 2 gives the Prime Number Theorem ( $\Re s \geq 1$ ) and  $p = \infty$  gives the RH ( $\Re s > 1/2$ ).

*Proof.* By the results of Malman and Seco [16] mentioned above,  $1/\varphi$  belongs to  $H^2$  or to  $H^\infty$  precisely when  $\mathcal{H}(b)$  contains BMOA or  $\mathcal{H}(b) = H^2$  respectively, where  $1/\varphi = b/a$  and (b, a) the associated Pythagorean pair. Hence the cases  $p = 2, \infty$  follow by Corollary 10 and Proposition 11. For the case when  $1/\varphi \in H^p$  for  $p \in (2, \infty)$ , we have  $H^{\tilde{p}} \subset \mathcal{H}(b)$  where  $p = \frac{2\tilde{p}}{\tilde{p}-2}$  (again by Malman and Seco) or equivalently  $\tilde{p} = \frac{2p}{p-2}$ . If  $1/\tilde{p} + 1/\tilde{q} = 1$ , then we see that  $\tilde{q} = \frac{2p}{(p+2)}$  and hence  $1/\tilde{q} = 1/2 + 1/p$ . The result again follows by Corollary 10 and Proposition 11.  $\Box$ 

An important distinction between Theorem 1 and Theorem 14 is that in the former one must solve density problems in  $H^p$  spaces that are non-Hilbertian, while in the latter the density problems are always in  $H^2$ . The following simple examples of  $\varphi$  satisfy the hypothesis of Theorem 14. Let  $\varphi(z) = (1-z)^{\alpha}$  for some  $0 < \alpha < 1/2$ . Then  $\varphi$  is an  $H^{\infty}$  outer function and hence a unit in  $N^+$  with the property that  $1/\varphi \in H^p$  for some  $2 . It follows that the density of <math>\varphi \mathcal{N}$  in  $H^2$  would give a new zero-free half-plane for  $\zeta$ .

#### 3. The biorthogonality question

Define the sequence of polynomials  $\{u_k : k \ge 2\}$  by

(3.1) 
$$u_k(z) = \sum_{d|k} \frac{\mu(k/d)}{k/d} (z^{d-1} - z^d),$$

where  $\mu$  denotes the Möbius function and d|k denotes d divides k. The main goal of this section is to prove the following theorem.

**Theorem 15.**  $\{u_k : k \ge 2\}$  is complete and biorthogonal to  $\{h_k : k \ge 2\}$  in  $H^2$ .

Balazard [5] noted that with the additional vector  $u_1(z) = 1 - z$  the sequence  $\{u_k : k \ge 1\}$  is complete. However it is no longer minimal following Theorem 15. We first make a key observation. Note that  $u_k = (I - S)v_k$ , where

(3.2) 
$$v_k(z) = \sum_{d|k} \frac{\mu(k/d)}{k/d} z^{d-1}.$$

It follows that  $\langle h_k, u_j \rangle = \langle (I - S^*)h_k, v_j \rangle$ . Hence to show that  $\{u_k : k \ge 2\}$  and  $\{h_k : k \ge 2\}$  are biorthogonal, it is suffices to show that

$$\langle (I - S^*)h_k, v_j \rangle = \delta_{kj}.$$

The proof of Theorem 15 will be divided into four steps.

**Step 1**. Calculate the Fourier coefficients of  $(I - S^*)h_k$ .

**Step 2**. Prove  $\{u_k : k \ge 2\}$  is biorthogonal to  $\{h_k : k \ge 2\}$ .

**Step 3.** Characterize all sequences biorthogonal to  $\{v_k : k \ge 2\}$ .

**Step 4**. Show that  $\{h_k : k \ge 2\}$  is the unique biorthogonal sequence for  $\{u_k : k \ge 2\}$ .

The **Step 4** implies the completeness of  $\{u_k : k \ge 2\}$  in  $H^2$  by Proposition 8.

**Step 1**. We first calculate the Fourier coefficients of  $(I - S^*)h_k$ .

Lemma 16. We have

$$(I - S^*)h_k(z) = \sum_{n=0}^{\infty} B_k(n+1)z^n$$

for all  $k \geq 2$  where

$$B_k(n) = \begin{cases} \frac{k}{n} - \frac{1}{n}, & k|n\\ -\frac{1}{n}, & k \not|n \end{cases}$$

*Proof.* Note that if  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ , then

$$S^*f(z) = \sum_{n=0}^{\infty} a_{n+1} z^n.$$

Let  $c_n(k)$  be the Fourier coefficients of  $h_k$ , i.e.,

$$h_k(z) = \sum_{n=0}^{\infty} c_n(k) z^n$$

Then, the *n*-th Fourier coefficient of  $(I - S^*)h_k$  is  $c_n(k) - c_{n+1}(k)$ . The coefficients  $c_n(k)$  are calculated in [19, p. 249]:

(3.3) 
$$c_n(k) = H(n) - H\left(\frac{n}{k}\right) - \log k,$$

where  $H(x) := \sum_{n \le x} \frac{1}{n}$  for x > 0 and H(0) = 0. It follows from (3.3) that

$$c_{n-1}(k) - c_n(k) = H(n-1) - H\left(\frac{n-1}{k}\right) - \log k - \left(H(n) - H\left(\frac{n}{k}\right) - \log k\right)$$
$$= -\frac{1}{n} + \sum_{\frac{n-1}{k} < m \le \frac{n}{k}} \frac{1}{m}.$$

Note that if there is some  $m \in \mathbb{N}$  such that  $\frac{n-1}{k} < m \leq \frac{n}{k}$ , then  $mk \leq n < mk+1$ , so that n = mk. Therefore, the sum above is non-zero if and only if k|n. Then,

(3.4) 
$$B_k(n) = c_{n-1}(k) - c_n(k) = \begin{cases} -\frac{1}{n}, & k \not| n \\ \frac{k}{n} - \frac{1}{n}, & k | n \end{cases}$$

Step 2. We are now able to prove the first part of Theorem 15.

**Theorem 17.**  $\{u_k : k \ge 2\}$  is biorthogonal to  $\{h_k : k \ge 2\}$ .

*Proof.* By **Step 1** it suffices to prove that

$$\sum_{d|j} B_k(d) \frac{\mu(j/d)}{j/d} = \langle (I - S^*)h_k, v_j \rangle = \delta_{kj}, \qquad \forall k, j \ge 2.$$

There are two cases:

(i)  $k \not\mid j$ . Then,  $k \not\mid d$  for every  $d \mid j$ , therefore

$$\sum_{d|j} B_k(d) \frac{\mu(j/d)}{j/d} = \sum_{d|j} -\frac{1}{d} \frac{\mu(j/d)}{j/d} = -\frac{1}{j} \sum_{d|j} \mu(j/d) = -\frac{1}{j} \left\lfloor \frac{1}{j} \right\rfloor = 0,$$

since  $j \ge 2$  and by the basic relation  $\sum_{d|k} \mu(d) = \lfloor 1/k \rfloor$ . (ii) k|j. Let  $q = \frac{j}{k}$ . Then

(ii) 
$$k|j$$
. Let  $q = \frac{j}{k}$ . Then

$$\sum_{d|j} B_k(d) \frac{\mu(j/d)}{j/d} = \sum_{\substack{d|j \\ k \nmid d}} -\frac{1}{d} \frac{\mu(j/d)}{j/d} + \sum_{\substack{d|j \\ k \mid d}} \left(\frac{k}{d} - \frac{1}{d}\right) \frac{\mu(j/d)}{j/d}.$$

The last sum is summing over those d that satisfy k|d|j. However,  $k|d \iff d = mk$ for some  $m \in \mathbb{N}$ . Since j = qk, it follows that  $d|j \iff m|q$ . Hence, the last sum can be written as

$$\sum_{\substack{d|j\\k|d}} \left(\frac{k}{d} - \frac{1}{d}\right) \frac{\mu(j/d)}{j/d} = \sum_{\substack{d|j\\k|d}} -\frac{1}{d} \frac{\mu(j/d)}{j/d} + \sum_{m|q} \frac{1}{m} \frac{\mu(q/m)}{q/m}$$

Therefore,

$$\sum_{d|j} B_k(d) \frac{\mu(j/d)}{j/d} = \sum_{d|j} -\frac{1}{d} \frac{\mu(j/d)}{j/d} + \sum_{m|q} \frac{1}{m} \frac{\mu(q/m)}{q/m}$$
$$= -\frac{1}{j} \sum_{d|j} \mu(j/d) + \frac{1}{q} \sum_{m|q} \mu(q/m) = -\frac{1}{j} \left\lfloor \frac{1}{j} \right\rfloor + \frac{1}{q} \left\lfloor \frac{1}{q} \right\rfloor.$$

Since  $j \ge 2$ , the first term is always 0. On the other hand, the second term equals 1 if q = 1 and equals 0 otherwise. Finally note that  $q = 1 \iff k = j$ , and hence that  $\langle h_k, u_j \rangle = \delta_{kj}$  for all  $k, j \ge 2$ . 

**Step 3**. We next characterize all sequences in  $H^2$  biorthogonal to  $\{v_k : k \ge 2\}$ .

**Lemma 18.** A sequence  $\{f_k : k \ge 2\} \subset H^2$  is biorthogonal to  $\{v_k : k \ge 2\}$  if and only if there exists a sequence  $(c_k)_{k\ge 2} \in \mathbb{C}^{\mathbb{N}}$  such that

$$f_k(z) = \sum_{n=0}^{\infty} A_k(n+1) \ z^n, \quad \forall k \ge 2,$$

where the sequence  $(A_k(n))_{n\geq 1}$  for each  $k\geq 2$  is defined by

$$A_k(n) = \begin{cases} \frac{c_k}{n} + \frac{k}{n}, & k|n\\ \frac{c_k}{n}, & k \not|n \end{cases}$$

*Proof.* Let  $\{f_k : k \geq 2\} \subset H^2$  be a sequence biorthogonal to  $\{v_k : k \geq 2\}$  and  $A_k : \mathbb{N} \to \mathbb{C}$  be the arithmetical functions that satisfy

$$f_k(z) = \sum_{n=0}^{\infty} A_k(n+1)z^n, \quad \forall k \ge 2.$$

Since the coefficients of  $v_i$  are real, the biorthogonality condition becomes

(3.5) 
$$\sum_{d|j} A_k(d) \frac{\mu(j/d)}{j/d} = \langle f_k, v_j \rangle = \delta_{kj}, \quad \forall k, j \ge 2.$$

Let  $I_k, \nu : \mathbb{N} \to \mathbb{C}$  be arithmetic functions defined by  $I_k(n) = \delta_{kn}$  and  $\nu(n) = \frac{\mu(n)}{n}$ . Then (3.5) is equivalent to

(3.6) 
$$\forall k \ge 2, \exists c_k \in \mathbb{C} \text{ such that } A_k * \nu = c_k I_1 + I_k,$$

where \* denotes the Dirichlet product (see [1, Section 2.6]). Indeed, (3.5) doesn't impose any restriction on  $A_k * \nu(1)$ , since it only need to hold for  $j \ge 2$ , hence  $c_k = A_k * \nu(1)$  is free, so (3.5) and (3.6) are indeed equivalent. Notice that

$$\sum_{d|k} \frac{\mu(k/d)}{k/d} \frac{1}{d} = \frac{1}{k} \left\lfloor \frac{1}{k} \right\rfloor = I_1(k),$$

i.e.,  $\nu^{-1}(n) = \frac{1}{n}$ , since  $I_1$  is the unity with respect to \*. Moreover

$$I_k * \nu^{-1}(n) = \sum_{d|n} \delta_{kd} \frac{1}{n/d} = \begin{cases} \frac{k}{n}, & k|n\\ 0, & k \not n \end{cases}$$

Therefore (3.6) is equivalent to the statement that

$$\forall k \ge 2, \exists c_k \in \mathbb{C} \text{ such that } A_k(n) = c_k \nu^{-1}(n) + I_k * \nu^{-1}(n)$$
$$= \begin{cases} \frac{c_k}{n} + \frac{k}{n}, & k|n\\ \frac{c_k}{n}, & k \not\mid n \end{cases}.$$

Hence the biorthogonality condition (3.5) is equivalent to the condition above as desired. Finally  $f_k \in H^2$  since its coefficient sequence  $A_k$  clearly belongs to  $\ell^2$ .  $\Box$ 

**Step 4.** In this final step we show that  $\{u_k : k \ge 2\}$  is complete in  $H^2$  by proving that  $\{h_k : k \ge 2\}$  is uniquely biorthogonal to  $\{u_k : k \ge 2\}$  in  $H^2$  by Proposition 8. To do so, recall that  $u_k = (I - S)v_k$  (see (3.2)) implies

$$\langle \phi_k, u_j \rangle = \langle (I - S^*) \phi_k, v_j \rangle$$

for any sequence  $\{\phi_k : k \geq 2\}$  in  $H^2$ . This implies that  $I - S^*$  maps sequences biorthogonal to  $\{u_k : k \geq 2\}$  onto sequences biorthogonal to  $\{v_k : k \geq 2\}$  in the image of  $I - S^*$ . This correspondece is one-to-one since  $I - S^*$  is injective on  $H^2$ . Therefore it is enough to prove that  $((I - S^*)h_k)_{k\geq 2}$  is the unique sequence in the image of  $I - S^*$  that is biorthogonal to  $\{v_k : k \geq 2\}$ .

**Lemma 19.** A sequence  $\{f_k : k \ge 2\} \subset (I - S^*)H^2$  is biorthogonal to  $\{v_k : k \ge 2\}$  if and only if

(3.7) 
$$f_k(z) = \sum_{n=0}^{\infty} B_k(n+1) z^n = (I - S^*) h_k,$$

where  $B_k$  are the sequences defined in Lemma 16.

*Proof.* Let  $\{f_k : k \geq 2\} \subset (I - S^*)H^2$  be a sequence biorthogonal to  $\{v_k : k \geq 2\}$ and let  $\varphi_k \in H^2$  such that  $f_k = (I - S^*)\varphi_k$ . If  $(b_k(n))_{n\geq 0}$  are the Maclaurin coefficients of  $\varphi_k$ , then

$$f_k(z) = \sum_{n=0}^{\infty} (b_k(n) - b_k(n+1))z^n$$

It then follows by Lemma 18 that for each  $k \geq 2$ , there exists a  $c_k \in \mathbb{C}$  such that

$$b_k(n-1) - b_k(n) = A_k(n) = \begin{cases} \frac{k}{n} + \frac{c_k}{n}, & k|n \\ \frac{c_k}{n}, & k \not|n \end{cases}, \quad \forall n \ge 1.$$

By induction, we obtain

$$b_k(n) = b_k(0) - \sum_{j=1}^n A_k(j) = b_k(0) - \sum_{j \le n} \frac{c_k}{j} - \sum_{\substack{j \le n \\ k \mid j}} \frac{k}{j}$$
$$= b_k(0) - c_k \sum_{j \le n} \frac{1}{j} - \sum_{\substack{m \le n/k}} \frac{1}{m} = b_k(0) - c_k H(n) - H\left(\frac{n}{k}\right)$$

where H is the same function used in (3.3). Since  $\varphi_k \in H^2$ , we get  $(b_k(n))_n \in \ell^2$ and hence  $\lim_{n\to\infty} b_k(n) = 0$ . So  $(c_k H(n) + H(n/k))_n$  converges. Using Euler summation, one gets (see [19])

(3.8) 
$$H(x) = \log x + \gamma + O\left(\frac{1}{x}\right).$$

where  $\gamma$  is Euler-Mascheroni constant. Therefore,

$$c_k H(n) + H\left(\frac{n}{k}\right) = c_k \log n + c_k \gamma + \log n - \log k + \gamma + O\left(\frac{k}{n}\right)$$
$$= (c_k + 1) \log n + (c_k + 1)\gamma - \log k + O\left(\frac{k}{n}\right),$$

which converges as  $n \to \infty$  if and only if  $c_k = -1$ . Hence  $c_k = -1$  for all  $k \ge 2$ . In that case  $A_k = B_k$  and we obtain (3.7). The converse is equivalent to Theorem 17 by the remarks at the start of Step 4.

As a consequence of Theorem 1 and Proposition 8 the RH holds if and only if  $(u_j)_{j\geq 2}$  is the unique sequence in  $H^2$  that is biorthogonal to  $(h_k)_{k\geq 2}$ . On the other hand the next result shows what happens if some  $\zeta$ -zero violates the RH.

**Corollary 20.** If  $\zeta(s_0) = 0$  for some  $1/2 < \Re s_0 < 1$ , then

$$\langle h_k, u_j + \kappa_{s_0} \rangle = \delta_{kj} \quad \forall \quad k, j \ge 2$$

where  $\kappa_{s_0}$  is the zeta kernel at  $s_0$ . So  $(u_j + \kappa_{s_0})_{j \ge 2}$  is also biorthogonal to  $(h_k)_{k \ge 2}$ . Proof. This follows by (1.4) and Theorem 17 since  $\langle h_k, \kappa_{s_0} \rangle = 0$  for all  $k \ge 2$ .  $\Box$ 

## 4. The RH-failure conjecture

The RH-failure (RHF) conjecture states that if the RH is false, then  $\zeta(s) = 0$  for infinitely many  $s \in \mathbb{C}$  with  $1/2 < \Re s < 1$ . Our goal is to prove the following.

**Theorem 21.** The RHF conjecture implies that  $\dim(\mathcal{N}^{\perp})$  is either 0 or  $\infty$ .

Let  $\mathcal{K} := \{\kappa_s : \Re s > 1/2\}$  denote the family of zeta kernels. If  $\zeta(s) = 0$  for some  $\Re s > 1/2$ , then  $\langle h_k, \kappa_s \rangle = 0$  for all  $k \ge 2$  by (1.4) and hence  $\kappa_s \in \mathcal{N}^{\perp}$ . So the RHF conjecture implies that  $\mathcal{N}^{\perp} \cap \mathcal{K}$  is either empty (by Theorem 1) or has infinitely many elements. Therefore Theorem 21 follows if we show that  $\mathcal{K}$  is linearly independent in  $H^2$ . We first show that elements of  $\mathcal{K}$  are common eigenvectors for the adjoints of operators  $(W_n)_{n>1}$  defined in (2.2). For  $f \in H^2$  and  $n \in \mathbb{N}$ , we have

$$W_n^* f(z) = \sum_{k=0}^{\infty} [\hat{f}(nk) + \hat{f}(nk+1) + \ldots + \hat{f}(nk+n-1)] z^k$$

where  $\hat{f}(n)$  denotes the *n*-th Fourier coefficient of f. This formula first appeared in [17]. It is possible to describe the common eigenvectors of  $(W_n^*)_{n\geq 1}$  completely.

**Proposition 22.** A non-zero  $f \in H^2$  is a common eigenvector for  $(W_n^*)_{n>1}$  if and only if there exists a multiplicative sequence  $(\lambda_n)_{n\geq 1}$  with  $(\lambda_{n+1} - \lambda_n)_{n\geq 1} \in \ell^2$  and

(4.1) 
$$\hat{f}(n) = (\lambda_{n+1} - \lambda_n)\hat{f}(0) \quad \forall \quad n \ge 1.$$

Moreover  $W_n^* f = \lambda_n f$  for all  $n \ge 1$ .

By a multiplicative sequence  $(\lambda_n)_{n\geq 1}$  we mean that  $\lambda_n\lambda_m = \lambda_{nm}$  and  $\lambda_1 = 1$ . Similarly one can see that  $W_n W_m = W_{nm}$  and  $W_1 = I$  by (2.2).

*Proof.* Let  $W_n^* f = \lambda_n f$  for  $n \ge 1$  and some sequence  $(\lambda_n)_{n\ge 1}$ . Since  $W_1^* = I$  and  $W_{nm}^* = W_n^* W_m^*$  it follows that  $(\lambda_n)_{n \ge 1}$  is multiplicative. Furthermore

$$\lambda_n \hat{f}(k) = \langle W_n^* f, z^k \rangle = \langle f, W_n z^k \rangle = \left\langle f, \sum_{j=0}^{n-1} z^{nk+j} \right\rangle = \sum_{j=0}^{n-1} \hat{f}(nk+j).$$

which gives  $\hat{f}(n) = \lambda_{n+1} \hat{f}(0) - \lambda_n \hat{f}(0)$  for all  $n \ge 1$  and hence  $(\lambda_{n+1} - \lambda_n)_{n \ge 1} \in \ell^2$ . Conversely suppose f is a non-zero function satisfying (4.1) for some multiplicative  $(\lambda_n)_{n\geq 1}$  with  $(\lambda_{n+1} - \lambda_n)_{n\geq 1} \in \ell^2$ . Normalizing by supposing f(0) = 1, we get

$$(W_n^*f)(z) = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{n-1} \hat{f}(nk+j) \right) z^k = \sum_{j=0}^{n-1} \hat{f}(j) + \sum_{k=1}^{\infty} \left( \sum_{j=0}^{n-1} \hat{f}(nk+j) \right) z^k$$
$$= \lambda_n + \sum_{k=1}^{\infty} (\lambda_{nk+n} - \lambda_{nk}) z^k = \lambda_n \left( 1 + \sum_{k=1}^{\infty} (\lambda_{k+1} - \lambda_k) z^k \right) = \lambda_n f(z)$$
or all  $n \ge 2$ . So  $f \in H^2$  is a common eigenvector for  $(W_n^*)_{n\ge 1}$ .

for all  $n \ge 2$ . So  $f \in H^2$  is a common eigenvector for  $(W_n^*)_{n \ge 1}$ .

Choosing  $\lambda_k = k^{1-\bar{s}}$  and  $\hat{f}(0) = -1/\bar{s}$  in Proposition 22 for any fixed  $\Re s > 1/2$ shows that each  $\kappa_s \in \mathcal{K}$  is a common eigenvector for  $(W_n^*)_{n>1}$  (see (1.3)) with

(4.2) 
$$W_n^* \kappa_s = n^{1-\bar{s}} f \quad \forall \ n \ge 1$$

We want to prove that for any finite subset  $\{\kappa_{s_1}, \ldots, \kappa_{s_\ell}\} \subset \mathcal{K}$  there exists some  $W_n^*$  such that the corresponding eigenvalues are all distinct. This will give us the linear independence of every finite subset of  $\mathcal{K}$  and hence of  $\mathcal{K}$  itself. First suppose that the real parts of  $s_1, \ldots, s_\ell$  are all distinct. Since  $|n^{1-\bar{s}}| = n^{1-\Re s}$  it follows that the eigenvalues of  $W_n^*$  (for all n > 1) corresponding to  $\kappa_{s_1}, \ldots, \kappa_{s_\ell}$  are all distinct. If the real parts of  $s_1, \ldots, s_\ell$  are *not* all distinct, then we need the following result.

**Lemma 23.** Given distinct  $a_1, \ldots, a_n \in \mathbb{R}$ , at most finitely many primes p have the property that there exists a pair  $a_i, a_j$  with  $1 \le i < j \le n$  such that

$$(4.3) (a_i - a_j) \log p \in 2\pi\mathbb{Z}.$$

*Proof.* Suppose there are infinitely many primes that satisfy (4.3). For each such prime p there exists some  $1 \leq i < j \leq n$  and  $k \in \mathbb{Z} \setminus \{0\}$  such that

$$(a_i - a_j)\log p = 2\pi k \implies \frac{2\pi k}{\log p} = a_i - a_j.$$

But since there are only finitely many numbers  $a_i - a_j$  with i < j, and none of which equal 0, there must exist distinct primes p, q and  $k_1, k_2 \in \mathbb{Z} \setminus \{0\}$  such that

$$\frac{2\pi k_1}{\log p} = a_i - a_j = \frac{2\pi k_2}{\log q} \implies k_2 \log p = k_1 \log q \neq 0.$$

for some pair i < j. In particular,  $p^{k_2} = q^{k_1} \neq 1$ , which is a contradiction.

The following result then completes the proof of Theorem 21.

# **Proposition 24.** The family of zeta kernels $\mathcal{K}$ is linearly independent.

*Proof.* Let  $\{\kappa_{s_1}, \ldots, \kappa_{s_\ell}\} \subset \mathcal{K}$  be a finite subset. The case when the real parts of  $s_1, \ldots, s_\ell$  are all distinct was already dealt with. Suppose some of the  $s_1, \ldots, s_\ell$  have the same real parts. So  $\{s_1, \ldots, s_\ell\}$  is the finite disjoint union of sets of the form  $A_r := \{s_i : \Re s_i = r, i = 1, \ldots, \ell\}$  for  $r \in \mathbb{R}$ . It is enough to prove that the family  $\{\kappa_s : s \in A_r\}$  is linearly independent when  $A_r$  has more than one element. Since  $s_1, \ldots, s_\ell$  are distinct complex numbers, the imaginary parts of elements in  $A_r$ , which we denote by  $a_1, \ldots, a_n$ , must all be distinct. Applying Lemma 23 to  $a_1, \ldots, a_n$  shows that there exist infinitely many primes q such that

(4.4) 
$$(a_i - a_j) \log q \notin 2\pi \mathbb{Z}, \quad \forall \ 1 \le i < j \le n.$$

For such a prime q, we claim that the  $\{\kappa_s : s \in A_r\}$  are  $W_q^*$ -eigenvectors with distinct eigenvalues. To see this first note that  $W_q^*\kappa_s = q^{1-\bar{s}}\kappa_s$  by (4.2) and

$$q^{1-\bar{s}} = e^{(1-\bar{s})\log q} = e^{(1-r)\log q} e^{i\operatorname{Im}(s)\log q} \quad \forall \ s \in A_r.$$

But Im(s) for  $s \in A_r$  are precisely the real numbers  $a_1, \ldots, a_n$ . Therefore the eigenvalues  $q^{1-\bar{s}}$  for  $s \in A_r$  are all distinct by (4.4) and hence  $\{\kappa_s : s \in A_r\}$  and therefore all of  $\mathcal{K}$  is linearly independent.

## 5. Appendix

Denote by  $\mathbb{C}_{\rho}$  the half-plane  $\{s \in \mathbb{C} : \Re s > \rho\}$ . In this appendix we provide an alternate proof for the fundamental relation

(5.1) 
$$\langle h_k, \kappa_s \rangle = -\frac{\zeta(s)}{s} (k^{1-s} - 1) \quad \forall \ s \in \mathbb{C}_{1/2}, \ k \ge 2$$

We first prove that (5.1) holds for all  $s \in \mathbb{C}_1$ . We then prove that the function  $s \mapsto \langle h_k, \kappa_s \rangle$  has an analytic continuation to  $\mathbb{C}_0$  for each  $k \geq 2$ . Since the right side of (5.1) is already analytic for  $s \in \mathbb{C} \setminus \{0\}$ , the result then follows by analytic continuation. Recall from Subsection 1.7 that

$$\kappa_s(z) = \sum_{n=0}^{\infty} \phi_n(\bar{s}) z^n \text{ where } \phi_n(s) = -\frac{1}{s} \left( (n+1)^{1-s} - n^{1-s} \right).$$

**Lemma 25.** The identity (5.1) holds for  $s \in \mathbb{C}_1$ .

*Proof.* Let  $(c_n(k))_n$  be the Fourier coefficients of  $h_k$ . Since  $\overline{\phi_n(\overline{s})} = \phi_n(s)$ , we have

$$\begin{aligned} \langle h_k, \kappa_s \rangle &= \sum_{n=0}^{\infty} c_n(k) \overline{\phi_n(\overline{s})} = \sum_{n=0}^{\infty} c_n(k) \phi_n(s) \\ &= \lim_{N \to \infty} \left( -\frac{c_0(k)}{s} - \frac{1}{s} \sum_{n=1}^{N} c_n(k) \left( (n+1)^{1-s} - n^{1-s} \right) \right) \\ &= \lim_{N \to \infty} \left( -\frac{1}{s} \sum_{n=0}^{N} c_n(k) (n+1)^{1-s} + \frac{1}{s} \sum_{n=1}^{N} c_n(k) n^{1-s} \right) \\ &= \lim_{N \to \infty} \left( -\frac{1}{s} \sum_{n=1}^{N} (c_{n-1}(k) - c_n(k)) n^{1-s} - \frac{1}{s} c_N(k) (N+1)^{1-s} \right). \end{aligned}$$

Since  $c_n(k) = O(k/n)$  (see [19, p. 249]), we have  $c_N(k)(N+1) = O(1)$ . Furthermore  $(N+1)^{-s} \to 0$  for  $\Re(s) > 0$  and therefore we get

$$\langle h_k, \kappa_s \rangle = -\frac{1}{s} \lim_{N \to \infty} \left( \sum_{n=1}^N (c_{n-1}(k) - c_n(k)) n^{1-s} \right)$$

$$\stackrel{(3.4)}{=} -\frac{1}{s} \lim_{N \to \infty} \left( \sum_{n=1}^N -\frac{1}{n} n^{1-s} + \sum_{\substack{n=1\\k \mid n}}^N \frac{k}{n} n^{1-s} \right)$$

$$= -\frac{1}{s} \lim_{N \to \infty} \left( \sum_{n=1}^N n^{-s} + \sum_{m=1}^{\lfloor \frac{N}{k} \rfloor} \frac{1}{m} (mk)^{1-s} \right)$$

$$= -\frac{1}{s} \lim_{N \to \infty} \left( -\sum_{n=1}^N n^{-s} + k^{1-s} \sum_{m=1}^{\lfloor \frac{N}{k} \rfloor} \frac{1}{m} m^{1-s} \right)$$

$$\stackrel{(*)}{=} -\frac{1}{s} (-\zeta(s)) - \frac{k^{1-s}}{s} \zeta(s) = -\frac{\zeta(s)}{s} (k^{1-s} - 1)$$

where in (\*) we split the limit in two and use the definition of  $\zeta$  for  $\Re(s) > 1$ .  $\Box$ 

The inner product  $\langle h_k, \kappa_s \rangle$  defined for  $s \in \mathbb{C}_{1/2}$  also makes sense for  $s \in \mathbb{C}_0$ . Lemma 26. The function  $\Phi_k : \mathbb{C}_{1/2} \to \mathbb{C}$  defined by

(5.2) 
$$\Phi_k(s) := \langle h_k, \kappa_s \rangle = \sum_{n=0}^{\infty} c_n(k) \phi_n(s).$$

has an analytic continuation to  $\mathbb{C}_0$  for each  $k \geq 2$ .

*Proof.* Since each  $\phi_n$  is holomorphic in  $\mathbb{C}_0$ , it is sufficient to prove that the series in (5.2) converges uniformly in every half-plane  $\mathbb{C}_{\rho}$  for  $\rho > 0$ . Note that

$$|\phi_n(s)| = \frac{|1-s|}{|s|} \left| \int_n^{n+1} y^{-s} dy \right| \le \frac{|1-s|}{|s|} n^{-\Re s} = O(n^{-\rho})$$

for  $s \in \mathbb{C}_{\rho}$  with  $\rho > 0$ . Also  $c_n(k) = O(k/n)$  for each  $k \ge 2$ , and hence we get  $c_n(k)\phi_n(s) = O(n^{-1-\rho})$  for  $s \in \mathbb{C}_{\rho}$ . So  $\Phi_k$  converges uniformly in  $\mathbb{C}_{\rho}$  for  $\rho > 0$ .  $\Box$ 

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