# MAXIMAL OPERATORS OF WALSH-NÖRLUND MEANS ON THE DYADIC HARDY SPACES

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ABSTRACT. The presented paper will be proved the necessary and sufficient conditions in order maximal operator of Walsh-Nörlund means with non-increasing weights to be bounded from the dyadic Hardy space  $H_p(\mathbb{I})$  to the space  $L_p(\mathbb{I})$ .

#### 1. INTRODUCTION

In 1992 Móricz and Siddiqi investigated the rate of the approximation by Nörlund means of Walsh-Fourier series [16]. For Nörlund means with monotone weights they gave a sufficient condition which provide Nörlund means for having convergence in  $L_p$  norm  $(1 \le p < \infty)$  and  $C_W$  norm.

The result of [16] was extended by Fridli, Manchanda and Siddiqi [3] for dyadic martingale Hardy spaces and dyadic homogeneous Banach spaces. Recently, the theorem of Móricz and Siddiqi was generalized for  $\Theta$ -means of Walsh-Fourier series in  $L_p$  spaces  $(1 \le p < \infty)$  and  $C_W$  [1].

The theorems mentioned above are related to the approximation of the Nörlund means which in turn is related to the uniformly boundedness of the corresponding operators of the Nörlund means. To study of almost everywhere convergence of Nörlund means is connected to the study of the boundedness of the maximum operators corresponding to the Nörlund means.

The first result with respect to the a.e. convergence of the Walsh-Fejér means is due to Fine [2]. Later, Schipp [18] showed that the maximal operator of the Walsh-Fejér means is of weak type (1, 1), from which the a. e. convergence follows by standard argument [15]. Schipp result implies by interpolation also the boundedness of  $\sup_n |\sigma_n^1| : L_p \to L_p$  (1 ). Thisfails to hold for <math>p = 1 but Fujii [4] proved that  $\sup_n |\sigma_n^1|$  is bounded from the dyadic Hardy space  $H_1(\mathbb{I})$  to the space  $L_1(\mathbb{I})$  (see also Simon [20]). Fujii's theorem was extended by Weisz [25]. In particular, Weisz [25] proved that the maximum operator is bounded from the Hardy space  $H_p(\mathbb{I})$  to the space

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 $L_p(\mathbb{I})$ , when p > 1/2. The essence of the condition p > 1/2 was proved by the author [9]. If the  $\{q_k\}$  is an non-decreasing sequence then it can be proved that the following inequality occurs (see Persson, Tephnadze, Wall [17]),

(1) 
$$\sup_{n} |t_{n}| \le c \sup_{n} |\sigma_{n}|,$$

where by  $t_n$  is denoted Nörlund means of Walsh-Fourier series. From (1) it follows that the maximum operator  $\sup_n |t_n|$  is bounded from the Hardy

space  $H_p(\mathbb{I})$  to the space  $L_p(\mathbb{I})$ , when p > 1/2.

The situation is different when the sequence  $\{q_k\}$  is decreasing. Let us cite the following two cases:

• say  $q_k = A_k^{\alpha - 1}, \alpha \in (0, 1)$ , where

$$A_0^{\alpha} = 1, \ A_n^{\alpha} = \frac{(\alpha + 1) \cdots (\alpha + n)}{n!}$$

then it is easy to see that  $\{q_k\}$  is decreasing and at the same time the operator  $\sup_{\alpha} |\sigma_n^{\alpha}| \quad (0 < \alpha < 1)$  is bounded from the Hardy space

 $H_p(\mathbb{I})$  to the space  $L_p(\mathbb{I})$ , when  $p > 1/(1 + \alpha)$  (see Weisz [26]);

• Assume that  $q_k = 1/k$ . Then the sequence is decreasing, but the maximum operator is not bounded from the Hardy space  $H_p(\mathbb{I})$  to the space  $L_p(\mathbb{I})$  by any  $p \in (0, 1]$  (see [11]).

Therefore, Nörlund means with non-increasing weights can be divided into two groups:

- Nörlund means with non-increasing weights, whose corresponding maximum operator is bounded from the Hardy space  $H_p(\mathbb{I})$  to the space  $L_p(\mathbb{I})$  for some  $p \in (0, 1]$ ;
- Nörlund means with non-increasing weights that are not bounded from the Hardy space  $H_p(\mathbb{I})$  to the space  $L_p(\mathbb{I})$  by any  $p \in (0, 1]$ .

The presented paper will be proved the necessary and sufficient conditions in order maximal operator of Nörlund means with non-increasing weights to be bounded from the Hardy space  $H_p(\mathbb{I})$  to the space  $L_p(\mathbb{I})$ . It also follows from the established theorem that the boundedness of maximal operator of Nörlund means with non-increasing weights from the Hardy space  $H_1(\mathbb{I})$  to the space  $L_1(\mathbb{I})$  is equivalent to the type  $(\infty, \infty)$ .

## 2. Walsh Functions

We denote the set of non-negative integers by  $\mathbb{N}$ . By a dyadic interval in  $\mathbb{I} := [0,1)$  we mean one of the form  $I(l,k) := \left[\frac{l-1}{2^k}, \frac{l}{2^k}\right)$  for some  $k \in \mathbb{N}$ ,  $0 < l \leq 2^k$ . Given  $k \in \mathbb{N}$  and  $x \in [0,1)$ , let  $I_k(x)$  denote the dyadic interval of length  $2^{-k}$  which contains the point x. We also use the notation

$$I_n := I_n(0) (n \in \mathbb{N}), \overline{I}_k(x) := \mathbb{I} \setminus I_k(x).$$
 Let  
$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)}$$

be the dyadic expansion of  $x \in \mathbb{I}$ , where  $x_n = 0$  or 1 and if x is a dyadic rational number we choose the expansion which terminates in 0's.

For any given  $n \in \mathbb{N}$  it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} \varepsilon_k(n) \, 2^k,$$

where  $\varepsilon_k(n) = 0$  or 1 for  $k \in \mathbb{N}$ . This expression will be called the binary expansion of n and the numbers  $\varepsilon_k(n)$  will be called the binary coefficients of n. Let us introduce for  $1 \leq n \in \mathbb{N}$  the notation  $|n| := \max\{j \in \mathbb{N}: \varepsilon_j(n) \neq 0\}$ , that is  $2^{|n|} \leq n < 2^{|n|+1}$ .

Let us set the *n*th  $(n \in \mathbb{N})$  Walsh-Paley function at point  $x \in \mathbb{I}$  as:

$$w_n(x) = (-1)^{\sum_{j=0}^{\infty} \varepsilon_j(n)x_j}$$

Let us denote the logical addition on  $\mathbb{I}$  by  $\dot{+}$ . That is, for any  $x, y \in \mathbb{I}$ 

$$x + y := \sum_{n=0}^{\infty} |x_n - y_n| 2^{-(n+1)}.$$

The nth Walsh-Dirichlet kernel is defined by

$$D_n\left(x\right) = \sum_{k=0}^{n-1} w_k\left(x\right).$$

Recall that [14, 19]

(2) 
$$D_{2^n}(x) = 2^n \mathbf{1}_{I_n}(x),$$

where  $\mathbf{1}_E$  is the characteristic function of the set E.

As usual, denote by  $L_1(\mathbb{I})$  the set of measurable functions defined on  $\mathbb{I}$ , for which

$$\left\|f\right\|_{1} := \int_{\mathbb{I}} \left|f\left(t\right)\right| dt < \infty.$$

Let  $f \in L_1(\mathbb{I})$ . The partial sums of the Walsh-Fourier series are defined as follows:

$$S_M(f;x) := \sum_{i=0}^{M-1} \widehat{f}(i) w_i(x),$$

where the number

$$\widehat{f}(i) = \int_{\mathbb{I}} f(t) w_i(t) dt$$

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is said to be the *i*th Walsh-Fourier coefficient of the function f. Let us set  $E_n(f;x) = S_{2^n}(f;x)$ . The maximal function is defined by

$$E^{*}(f;x) = \sup_{n \in \mathbb{N}} E_{n}(f;x).$$

### 3. Walsh-Nörlund means

Let us set  $\{q_k : k \ge 0\}$  be a sequence of non-negative numbers. We define the *n*th Nörlund mean of the Walsh-Fourier series by

(3) 
$$t_n(f;x) := \frac{1}{Q_n} \sum_{k=1}^n q_{n-k} S_k(f;x),$$

where  $Q_n := \sum_{k=0}^{n-1} q_k$   $(n \ge 1)$ . It is always assumed that  $q_0 > 0$  and  $\lim_{n\to\infty} Q_n = \infty$ . In this case, the summability method generated by the sequence  $\{q_k : k \ge 0\}$  is regular (see [16]) if and only if

(4) 
$$\lim_{n \to \infty} \frac{q_{n-1}}{Q_n} = 0.$$

The Nörlund kernels are defined by

$$F_n(t) := \frac{1}{Q_n} \sum_{k=1}^n q_{n-k} D_k(t).$$

The Fejér means and kernels are

$$\sigma_n(f,x) := \frac{1}{n} \sum_{k=1}^n S_k(f,x), \quad K_n(t) := \frac{1}{n} \sum_{k=1}^n D_k(t).$$

It is easily seen that the means  $t_n(f)$  and  $\sigma_n(f)$  can be got by convolution of f with the kernels  $F_n$  and  $K_n$ . That is,

$$t_n(f,x) = \int_G f(x + t) F_n(t) dt = (f * F_n)(x),$$
  
$$\sigma_n(f,x) = \int_G f(x + t) K_n(t) dt = (f * K_n)(x).$$

It is well-known that  $L_1$  norm of Fejér kernels are uniformly bounded, that is

(5) 
$$||K_n||_1 \le c \text{ for all } n \in \mathbb{N}.$$

Yano estimated the value of c and he have c = 2 [27]. Recently, in paper (see [23]) it was shown that the exact value of c is  $\frac{17}{15}$ .

#### 4. AUXILIARY PROPOSITIONS

In order to prove main results we need the following theorems.

**Theorem SWS**. Let  $n \in \mathbb{N}$  and  $e_j := 2^{-j-1}$ . Then

(6) 
$$K_{2^{n}}(x) = \frac{1}{2} \left( 2^{-n} D_{2^{n}}(x) + \sum_{j=0}^{n} 2^{j-n} D_{2^{n}}(x + e_{j}) \right).$$

The proof can be found in [19].

**Theorem GN1.** Let  $n = 2^{n_1} + 2^{n_2} + \dots + 2^{n_r}$  with  $n_1 > n_2 > \dots > n_r \ge 0$ . Let us set  $n^{(0)} := n$  and  $n^{(i)} := n^{(i-1)} - 2^{n_i}$   $(i = 1, \dots, r-1), n^{(r)} := 0$ . Then the following decomposition holds.

(7) 
$$F_n = \frac{w_n}{Q_n} \sum_{j=1}^r Q_{n^{(j-1)}} w_{2^{n_j}} D_{2^{n_j}} -\frac{w_n}{Q_n} \sum_{j=1}^r w_{n^{(j-1)}} w_{2^{n_j}-1} \sum_{k=1}^{2^{n_j}-1} q_{k+n^{(j)}} D_k$$
$$= :F_{n,1} + F_{n,2}.$$

**Theorem GN2.** Let  $\{q_k : k \in \mathbb{N}\}$  be a sequence of non-negative numbers. If the sequence  $\{q_k : k \in \mathbb{N}\}$  is monotone non-increasing (in sign  $q_k \downarrow$ ). Then

(8) 
$$||F_n||_1 \sim \frac{1}{Q_n} \sum_{k=1}^{|n|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| Q_{2^k}.$$

The proof of Theorems GN1 and GN2 can be found in [13]. Using Abel's transformation we have

$$\sum_{k=1}^{2^{n_j}-1} q_{k+n^{(j)}} D_k = \sum_{k=1}^{2^{n_j}-2} \left( q_{k+n^{(j)}} - q_{k+n^{(j)}+1} \right) k K_k + q_{n^{(j-1)}-1} (2^{n_j} - 1) K_{2^{n_j}-1}.$$

Thus, we get

(9) 
$$F_{n,2} = \frac{w_n}{Q_n} \sum_{j=1}^r \sum_{k=1}^{2^{n_j}-2} w_{n^{(j-1)}} w_{2^{n_j}-1} \left( q_{k+n^{(j)}} - q_{k+n^{(j)}+1} \right) k K_k$$
$$+ \frac{w_n}{Q_n} \sum_{j=1}^r w_{n^{(j-1)}} w_{2^{n_j}-1} q_{n^{(j-1)}-1} (2^{n_j} - 1) K_{2^{n_j}-1}$$
$$= :F_{n,2}^{(1)} + F_{n,2}^{(2)}.$$

Lemma 1. Let  $p \in \left(\frac{1}{2}, 1\right]$ . Then

$$\int_{\mathbb{I}} \sup_{1 \le n \le 2^N} (n |K_n|)^p \le c_p 2^{N(2p-1)}.$$

Proof of Lemma 1. Let p = 1. Since (see [19])

(10) 
$$n |K_n(x)| \le c \sum_{s=0}^{|n|} 2^s K_{2^s}(x)$$

from (5) we have

$$\int_{\mathbb{T}} \sup_{1 \le n \le 2^N} \left( n \left| K_n \left( x \right) \right| \right) dx \le c \sum_{s=0}^N 2^s \int_{\mathbb{T}} K_{2^s} \left( x \right) dx \le c 2^N.$$

Let 1/2 . Applying the inequality

$$\left(\sum_{k=0}^{\infty} a_k\right)^p \le \sum_{k=0}^{\infty} a_k^p \qquad (a_k \ge 0, \quad 0$$

and (see [8])

$$\int_{\mathbb{I}} \left( 2^{s} K_{2^{s}}(x) \right)^{p} dx \le c_{p} 2^{s(2p-1)}, 1/2$$

we get

$$\int_{\mathbb{I}} \sup_{1 \le n \le 2^N} \left( n \left| K_n \left( x \right) \right| \right)^p dx \le c_p \sum_{s=0}^N \int_{\mathbb{I}} \left( 2^s K_{2^s} \left( x \right) \right)^p dx \le c_p 2^{N(2p-1)}.$$

Lemma 1 is proved.

## 5. Dyadic Hardy Spaces

The norm (or quasinorm) of the space  $L_{p}(\mathbb{I})$  is defined by

$$||f||_p := \left( \int_{\mathbb{I}} |f(x)|^p dx \right)^{1/p} \quad (0$$

In case  $p = \infty$ , by  $L_p(\mathbb{I})$  we mean  $L_{\infty}(\mathbb{I})$ , endoved with the supremum norm.

The space weak- $L_1(\mathbb{I})$  consists of all measurable functions f for which

$$\|f\|_{\operatorname{weak}-L_1(\mathbb{I})} := \sup_{\lambda>0} \lambda \left| (|f| > \lambda) \right| < +\infty.$$

Let  $f \in L_1(\mathbb{I}).$  For  $0 the Hardy space <math display="inline">H_p(\mathbb{I})$  consists all functions for which

$$||f||_{H_p} := ||E^*(f)||_p < \infty.$$

A bounded measurable function a is a p-atom, if there exists a dyadic interval I, such that

a) 
$$\int_{I} a = 0;$$
  
b)  $||a||_{\infty} \le |I|^{-1/p};$ 

c) supp  $a \subset I$ .

An operator T be called p-quasi-local if there exist a constant  $c_p > 0$  such that for every p-atom a

$$\int_{\mathbb{I}\setminus I} |Ta|^p \le c_p < \infty,$$

where I is the support of the atom. We shall need the following

**Theorem W1**. Suppose that the operator T is  $\sigma$ -sublinear and p-quasi-local for each  $0 . If T is bounded from <math>L_{\infty}(\mathbb{I})$  to  $L_{\infty}(\mathbb{I})$ , then

$$\|Tf\|_{p} \leq c_{p} \|f\|_{p} \qquad (f \in H_{p}(\mathbb{I}))$$

for every  $0 . In particular for <math>f \in L_1(\mathbb{I})$ , it holds

$$||Tf||_{weak \ L_1(\mathbb{I})} \le C ||f||_1.$$

**Theorem W2.** If a sublinear operator is bounded from  $H_{p_0}(\mathbb{I})$  to  $L_{p_0(\mathbb{I})}$  and from  $L_{p_1}(\mathbb{I})$  to  $L_{p_1}(\mathbb{I})$   $(p_0 \leq 1 < p_1 \leq \infty)$  then it is also bounded from  $H_p(\mathbb{I})$ to  $L_p(\mathbb{I})$  if  $p_0 .$ 

The proofs of Theorems W1 and W2 can be found in [24].

# 6. MAXIMAL OPERATORS OF WALSH-NÖRLUND MEANS

The goal of this paragraph is to investigate the boundedness of the maximal operators of the Walsh-Nörlund means on the dyadic Hardy spaces. More precisely, to find the necessary and sufficient conditions for the maximal operator of the Walsh-Nörlund means to be bounded from the Hardy space  $H_p(\mathbb{I})$  to the space  $L_p(\mathbb{I})$  for fixed  $p \in (0, 1]$ .

Let us first prove that if the condition

(11) 
$$\sup_{n \in \mathbb{N}} \frac{1}{Q_n} \sum_{k=1}^{|n|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| Q_{2^k} = \infty$$

is fulfilled, then the boundedness of the maximum operator from the Hardy space  $H_1(\mathbb{I})$  to the space  $L_1(\mathbb{I})$  does not occur. Moreover, we prove that the following is valid

**Theorem 1.** Let  $\{m_A : A \in \mathbb{N}\}$  be a subsequence for which the condition

$$\sup_{A \in \mathbb{N}} \frac{1}{Q_{m_A}} \sum_{k=1}^{|m_A|} |\varepsilon_k(m_A) - \varepsilon_{k+1}(m_A)| Q_{2^k} = \infty$$

holds. The operator  $t_{m_A}(f)$  is not uniformly bounded from the dyadic Hardy spaces  $H_1(\mathbb{I})$  to the space  $L_1(\mathbb{I})$ .

Proof of Theorem 1. Set

$$f_A := D_{2^{|m_A|+1}} - D_{2^{|m_A|}}.$$

Then it is easy to see that

$$\sup_{n \in \mathbb{N}} |S_{2^n}(f_A)| = D_{2^{|m_A|}}$$

and consequently,

$$\|f_A\|_{H_1} = \left\|\sup_{n\in\mathbb{N}} |S_{2^n}(f_A)|\right\|_1 = \left\|D_{2^{|m_A|}}\right\|_1 = 1.$$

 $\operatorname{Set}$ 

$$m_A = 2^{|m_A|} + q_A,$$

where

$$q_A := \sum_{j=0}^{|m_A|-1} \varepsilon_j(m_A) \, 2^j.$$

Then we can write

$$t_{m_A}(f_A) = \frac{1}{Q_{m_A}} \sum_{k=2^{|m_A|}+1}^{2^{|m_A|}+q_A-1} q_{m_A-k} S_k(f_A).$$

It is easy to see that

$$S_k(f_A) = S_k \left( D_{2^{|m_A|+1}} - D_{2^{|m_A|}} \right)$$
  
=  $S_{2^{|m_A|+1}}(D_k) - S_k \left( D_{2^{|m_A|}} \right)$   
=  $D_k - D_{2^{|m_A|}}, 2^{|m_A|} < k \le m_A.$ 

Hence, we have

$$t_{m_A}(f_A) = \frac{1}{Q_{m_A}} \sum_{k=2^{|m_A|}+1}^{2^{|m_A|}+q_A} q_{m_A-k} \left( D_k - D_{2^{|m_A|}} \right)$$
$$= \frac{1}{Q_{m_A}} \sum_{k=1}^{q_A} q_{q_A-k} \left( D_{k+2^{|m_A|}} - D_{2^{|m_A|}} \right)$$
$$= \frac{w_{2^{|m_A|}}}{Q_{m_A}} \sum_{k=1}^{q_A} q_{q_A-k} D_k.$$

From the condition of Theorem 1 and by (8) we conclude that

$$\sup_{A \in \mathbb{N}} \|t_{m_A}(f_A)\|_1 = \sup_{A \in \mathbb{N}} \|F_{q_A}\|_1 = \infty.$$

Theorem 1 is proved.

Now, we prove that the maximal operator of Walsh-Nörlund means with non-increasing weights can not be bounded from the Hardy space  $H_{1/2}(\mathbb{I})$  to the space  $L_{1/2}(\mathbb{I})$ . Based on the interpolation Theorem W2, the maximum operator of Walsh-Nörlund means with non-increasing weights can not be bounded from the Hardy space  $H_p(\mathbb{I})$  to the space  $L_p(\mathbb{I})$  when p < 1/2 (see Persson, Tephnadze, Wall [17]).

**Theorem 2.** The maximal operator of Walsh-Nörlund means with nonincreasing weights can not be bounded from the Hardy space  $H_{1/2}(\mathbb{I})$  to the space  $L_{1/2}(\mathbb{I})$ .

Proof of Theorem 2. Set

$$f_n := D_{2^{n+1}} - D_{2^n}.$$

Then it is easy to see that

$$\sup_{m\in\mathbb{N}}\left|S_{2^{m}}\left(f_{n}\right)\right|=D_{2^{n}}$$

and consequently,

(12) 
$$||f_n||_{H_p} = \left\| \sup_{m \in \mathbb{N}} |S_{2^m}(f_n)| \right\|_p = ||D_{2^n}||_p = 2^{n(1-1/p)}.$$

Let s < n. Then we can write

$$t_{2^{n}+2^{s}}(f_{n}) = \frac{1}{Q_{2^{n}+2^{s}}} \sum_{j=1}^{2^{n}+2^{s}} q_{2^{n}+2^{s}-j} S_{j}(f_{n})$$

$$= \frac{1}{Q_{2^{n}+2^{s}}} \sum_{j=2^{n}+1}^{2^{n}+2^{s}} q_{2^{n}+2^{s}-j} S_{j}(D_{2^{n}+1} - D_{2^{n}})$$

$$= \frac{1}{Q_{2^{n}+2^{s}}} \sum_{j=2^{n}+1}^{2^{n}+2^{s}} q_{2^{n}+2^{s}-j}(S_{2^{n+1}}(D_{j}) - S_{j}(D_{2^{n}}))$$

$$= \frac{1}{Q_{2^{n}+2^{s}}} \sum_{j=2^{n}+1}^{2^{n}+2^{s}} q_{2^{n}+2^{s}-j}(D_{j} - D_{2^{n}})$$

$$= \frac{1}{Q_{2^{n}+2^{s}}} \sum_{j=1}^{2^{s}} q_{2^{s}-j}(D_{j+2^{n}} - D_{2^{n}})$$

$$= \frac{w_{2^{n}}}{Q_{2^{n}+2^{s}}} \sum_{j=1}^{2^{s}} q_{2^{s}-j}D_{j}.$$

Consequently,

(13)  

$$\int_{\mathbb{I}} \left( \sup_{1 \le s < n} |t_{2^{n}+2^{s}}(f_{n})| \right)^{p} \\
\geq \sum_{s=0}^{n-1} \int_{I_{s} \setminus I_{s+1}} |t_{2^{n}+2^{s}}(f_{n})|^{p} \\
= \sum_{s=0}^{n-1} \int_{I_{s} \setminus I_{s+1}} \frac{1}{Q_{2^{n}+2^{s}}^{p}} \left| \sum_{j=1}^{2^{s}} q_{2^{s}-j} D_{j} \right|^{p} \\
= \sum_{s=0}^{n-1} \frac{1}{2^{s+1}Q_{2^{n}+2^{s}}^{p}} \left| \sum_{j=1}^{2^{s}} q_{2^{s}-j} J_{j} \right|^{p}.$$

Since  $q_k$  is non-increasing we can write

$$Q_{2^{n}} = \sum_{k=0}^{2^{n-1}-1} q_{k} + \sum_{k=2^{n-1}}^{2^{n-1}-1} q_{k}$$
$$\leq 2\sum_{k=0}^{2^{n-1}-1} q_{k} = 2Q_{2^{n-1}}$$
$$\leq \cdots \leq 2^{n-s}Q_{2^{s}}$$

 $\quad \text{and} \quad$ 

(14) 
$$\frac{Q_{2^s}}{2^s} \ge \frac{Q_{2^n}}{2^n} \, (s \le n) \, .$$

Combine (13) and (14) we get (p = 1/2)

$$\begin{split} & \int_{\mathbb{I}} \left( \sup_{1 \le s < n} |t_{2^{n} + 2^{s}} (f_{n})| \right)^{1/2} \\ \ge & \sum_{s=0}^{n-1} \frac{1}{2^{s+1} Q_{2^{n} + 2^{s}}^{1/2}} \left| \sum_{j=2^{s-1}+1}^{2^{s}} q_{2^{s} - j} j \right|^{1/2} \\ \ge & c \sum_{s=0}^{n-1} \frac{2^{s/2} Q_{2^{s}}^{1/2}}{2^{s} Q_{2^{n}}^{1/2}} \\ \ge & c \sum_{s=0}^{n-1} \frac{1}{2^{s/2}} \left( \frac{2^{s}}{2^{n}} \right)^{1/2} \\ = & \frac{cn}{2^{n/2}}. \end{split}$$

Hence,

$$\frac{\|t^*(f_n)\|_{1/2}^{1/2}}{\|f_n\|_{H_{1/2}}^{1/2}} \ge cn \to \infty$$

as  $n \to \infty$ . Theorem 2 is proved.

Finally, we are ready to formulate a basic problem: to say  $\{q_k\}$  is a non-increasing and positive sequence. It is known that the operator  $t^*(f)$  is bounded from  $L_{\infty}(\mathbb{I})$  to  $L_{\infty}(\mathbb{I})$  and  $p \in (1/2, 1]$ . Find the necessary and sufficient conditions for the  $\{q_k\}$  sequence in order for the maximum operator  $t^*(f)$  to be bounded from the Hardy space  $H_p(\mathbb{I})$  to the space  $L_p(\mathbb{I})$ .

The paper will provide a complete answer to the given question. The following theorem is true.

**Theorem 3.** Let  $\{q_k\}$  be a non-increasing and positive sequence. It is known that the operator  $t^*(f)$  is bounded from  $L_{\infty}(\mathbb{I})$  to  $L_{\infty}(\mathbb{I})$  and  $p \in (1/2, 1]$ . In order for the given operator to be bounded from the Hardy space  $H_p(\mathbb{I})$  to the space  $L_p(\mathbb{I})$  it is necessary and sufficient that

$$\sup_{N} \frac{2^{N(1-p)}}{Q_{2^{N}}^{p}} \sum_{j=1}^{N} Q_{2^{j}}^{p} 2^{j(p-1)} < \infty.$$

*Proof of Theorem 3.* Necessity. We assume that

(15) 
$$\sup_{N} \frac{2^{N(1-p)}}{Q_{2^{N}}^{p}} \sum_{j=1}^{N} Q_{2^{j}}^{p} 2^{j(p-1)} = \infty.$$

From (12) and (13) we have

$$\begin{aligned} & \frac{\|t^*\left(f_n\right)\|_p^p}{\|f_n\|_{H_p}^p} \\ \geq & c_p 2^{n(1-p)} \int\limits_{\mathbb{I}} \left( \sup_{1 \le s < n} |t_{2^n + 2^s}\left(f_n\right)| \right)^p \\ \geq & c_p 2^{n(1-p)} \sum_{s=0}^{n-1} \frac{1}{2^{s+1} Q_{2^n + 2^s}^p} \left| \sum_{j=1}^{2^s} q_{2^s - jj} \right|^p \\ \geq & c_p \frac{2^{n(1-p)}}{Q_{2^n}^p} \sum_{s=0}^{n-1} \frac{1}{2^s} \left| \sum_{j=2^{s-1}+1}^{2^s} q_{2^s - jj} \right|^p \\ \geq & c_p \frac{2^{n(1-p)}}{Q_{2^n}^p} \sum_{s=0}^{n-1} 2^{s(p-1)} Q_{2^s}^p. \end{aligned}$$

Then from (15) we get

$$\sup_{n \in \mathbb{N}} \frac{\left\| t^* \left( f_n \right) \right\|_p^p}{\left\| f_n \right\|_{H_p}^p} = \infty$$

and consequently, the operator  $t^*$  is not bounded from the Hardy space  $H_p(\mathbb{I})$  to the space  $L_p(\mathbb{I})$ .

**Sufficiency.** We suppose that  $f \in H_p(\mathbb{I})$ . Let function a be an  $H_p$  atom. It means that either a is constant or there is an interval  $I_N(u)$  such that  $\operatorname{supp}(a) \subset I_N(u)$ ,  $||a||_{\infty} \leq 2^{N/p}$  and  $\int a = 0$ . Without lost of generality we can suppose that u = 0. Consequently, for any function g which is  $\mathcal{A}_N$ -measurable we have that  $\int ag = 0$ . We prove that the operator  $\sup_{n>N} (f * F_n)(x)$  is  $H_p$ -quasi local. That is,

(16) 
$$\int_{\overline{I}_N} \left( \sup_{n>N} |a * F_n| \right)^p \le c_p.$$

Let  $x \in \overline{I}_N$ . Then from (7) we can write

(17) 
$$|a * F_n| = \left| \int_{\mathbb{I}} a(t) F_n(x + t) dt \right| \le 2^{N/p} \int_{I_N} |F_n(x + t)| dt$$
  
$$= 2^{N/p} \int_{I_N} |F_{n,1}(x + t)| dt + 2^{N/p} \int_{I_N} |F_{n,2}(x + t)| dt$$

We have

$$\int_{I_N} |F_{n,1} (x + t)| dt$$

$$\leq \frac{1}{Q_n} \sum_{j=1, n_j > N}^r Q_{n^{(j-1)}} \int_{I_N} D_{2^{n_j}} (x + t) dt$$

$$+ \frac{1}{Q_n} \sum_{j=1, n_j \le N}^r Q_{n^{(j-1)}} \int_{I_N} D_{2^{n_j}} (x + t) dt$$

Since  $t \in I_N$  and  $x \notin I_N$  we have that  $x \dotplus t \notin I_N$  and consequently by (2) we get  $D_{2^{n_j}}(x \dotplus t) = 0$  for  $n_j > N$ . On the other hand,  $\int_{I_N} D_{2^{n_j}}(x \dotplus t) dt =$ 

 $\frac{1}{2^{N}}D_{2^{n_{j}}}\left(x\right)$  for  $n_{j}\leq N.$  Hence, we obtain

$$\begin{split} & \int_{I_N} |F_{n,1} \left( x \dotplus t \right)| \, dt \\ & \leq \quad \frac{1}{Q_n} \sum_{j=1, n_j \leq N}^r Q_{n^{(j-1)}} \int_{I_N} D_{2^{n_j}} \left( x \dotplus t \right) dt \\ & = \quad \frac{1}{2^N Q_n} \sum_{j=1, n_j \leq N}^r Q_{n^{(j-1)}} D_{2^{n_j}} \left( x \right) \\ & \leq \quad \frac{1}{2^N Q_{2^N}} \sum_{j=1}^N Q_{2^j} D_{2^j} \left( x \right). \end{split}$$

Consequently, from the condition of the theorem we get

(18) 
$$\int_{\overline{I}_{N}} \sup_{n>N} \left( 2^{N/p} \int_{I_{N}} |F_{n,1}(x + t)| dt \right)^{p} dx$$
$$\leq \frac{c_{p} 2^{N}}{2^{N_{p}} Q_{2^{N}}^{p}} \sum_{j=1}^{N} Q_{2^{j}}^{p} \int_{\overline{I}_{N}} D_{2^{j}}^{p}(x) dx$$
$$= \frac{c_{p} 2^{N(1-p)}}{Q_{2^{N}}^{p}} \sum_{j=1}^{N} Q_{2^{j}}^{p} 2^{j(p-1)} \leq c_{p} < \infty.$$

From (9) we get

(19) 
$$\int_{I_N} |F_{n,2} (x + t)| dt$$
$$\leq \int_{I_N} \left| F_{n,2}^{(1)} (x + t) \right| dt + \int_{I_N} \left| F_{n,2}^{(2)} (x + t) \right| dt.$$

We can write

$$\begin{split} &\frac{1}{Q_n} \sum_{j=1}^r \sum_{k=1}^{2^{n_j-1}} \left( q_{k+n^{(j)}} - q_{k+n^{(j)}+1} \right) k \left| K_k \right| \\ &= \frac{1}{Q_n} \sum_{j=1}^r \sum_{m=1}^{n_j} \sum_{k=2^{m-1}}^{2^{m-1}} \left( q_{k+n^{(j)}} - q_{k+n^{(j)}+1} \right) k \left| K_k \right| \\ &= \frac{1}{Q_n} \sum_{j=1}^r \sum_{m=1}^{n_{j+1}} \sum_{k=2^{m-1}}^{2^{m-1}} \left( q_{k+n^{(j)}} - q_{k+n^{(j)}+1} \right) k \left| K_k \right| \\ &+ \frac{1}{Q_n} \sum_{j=1}^r \sum_{m=n_{j+1}+1}^{n_j} \sum_{k=2^{m-1}}^{2^{m-1}} \left( q_{k+n^{(j)}} - q_{k+n^{(j)}+1} \right) k \left| K_k \right| \\ &\leq \frac{1}{Q_n} \sum_{j=1}^r \sum_{m=1}^{n_{j+1}} \sup_{2^{m-1} \leq k < 2^m} \left( k \left| K_k \right| \right) \sum_{k=2^{m-1}}^{2^{m-1}} \left( q_{k+n^{(j)}} - q_{k+n^{(j)}+1} \right) \\ &+ \frac{1}{Q_n} \sum_{j=1}^r \sum_{m=n_{j+1}+1}^{n_j} \sup_{2^{m-1} \leq k < 2^m} \left( k \left| K_k \right| \right) \sum_{k=2^{m-1}}^{2^{m-1}} \left( q_{k+n^{(j)}} - q_{k+n^{(j)}+1} \right) \\ &= \frac{1}{Q_n} \sum_{j=1}^r \sum_{m=1}^{n_{j+1}} \sup_{2^{m-1} \leq k < 2^m} \left( k \left| K_k \right| \right) \left( q_{2^{m-1}+n^{(j)}} - q_{2^m+n^{(j)}} \right) \end{split}$$

$$\begin{aligned} &+ \frac{1}{Q_n} \sum_{j=1}^r \sum_{m=n_{j+1}+1}^{n_j} \sup_{2^{m-1} \le k < 2^m} \left( k \left| K_k \right| \right) \left( q_{2^{m-1}+n^{(j)}} - q_{2^m+n^{(j)}} \right) \\ &\leq \quad \frac{1}{Q_n} \sum_{j=1}^r q_{2^{n_{j+1}}} \sum_{m=1}^{n_{j+1}} \sup_{2^{m-1} \le k < 2^m} \left( k \left| K_k \right| \right) \\ &+ \frac{1}{Q_n} \sum_{j=1}^r \sum_{m=n_{j+1}+1}^{n_j} q_{2^{m-1}} \sup_{2^{m-1} \le k < 2^m} \left( k \left| K_k \right| \right) \\ &\leq \quad \frac{1}{Q_n} \sum_{j=1}^{n_1} q_{2^j} \sum_{m=1}^j \sup_{2^{m-1} \le k < 2^m} \left( k \left| K_k \right| \right) \\ &+ \frac{1}{Q_n} \sum_{m=1}^{n_1} q_{2^{m-1}} \sup_{2^{m-1} \le k < 2^m} \left( k \left| K_k \right| \right) \\ &\leq \quad \frac{2}{Q_n} \sum_{j=1}^{n_1} q_{2^{j-1}} \sum_{m=1}^j \sup_{2^{m-1} \le k < 2^m} \left( k \left| K_k \right| \right). \end{aligned}$$

Consequently, we have

$$\int_{I_N} \left| F_{n,2}^{(1)}(x + t) \right| dt$$

$$\leq \frac{2}{2^N Q_n} \sum_{j=1}^N q_{2^{j-1}} \sum_{m=1}^j \sup_{2^{m-1} \le k < 2^m} \left( k \left| K_k(x) \right| \right) \\
+ \frac{2}{2^N Q_n} \sum_{j=N+1}^{n_1} q_{2^{j-1}} \sum_{m=1}^N \sup_{2^{m-1} \le k < 2^m} \left( k \left| K_k(x) \right| \right) \\
+ \frac{2}{Q_n} \sum_{j=N+1}^{n_1} q_{2^{j-1}} \sum_{m=N+1}^j \int_{I_N} \sup_{2^{m-1} \le k < 2^m} \left( k \left| K_k(x + t) \right| \right) dt \\
: = J_1 + J_2 + J_3.$$

Since  $x \dotplus t \notin I_N$  by (10) and (6), we have (m > N)

$$\int_{I_N} \sup_{2^{m-1} \le k < 2^m} \left( k \left| K_k \left( x \dotplus t \right) \right| \right) dt$$

$$\leq \sum_{s=0}^m 2^s \int_{I_N} K_{2^s} \left( x \dotplus t \right) dt$$

$$= \frac{1}{2^N} \sum_{s=0}^N 2^s K_{2^s} \left( x \right) + \sum_{s=N+1}^m 2^s \int_{I_N} K_{2^s} \left( x \dotplus t \right) dt$$

$$\leq \frac{1}{2^N} \sum_{s=0}^N 2^s K_{2^s} \left( x \right)$$

$$+ \sum_{s=N+1}^{m} \sum_{l=0}^{s} 2^{l} \int_{I_{N}} D_{2^{s}} \left( x \dotplus t \dotplus e_{l} \right) dt$$

$$\leq \frac{1}{2^{N}} \sum_{s=0}^{N} 2^{s} K_{2^{s}} \left( x \right) + \frac{2^{m}}{2^{N}} \sum_{l=0}^{N-1} 2^{l} \mathbf{1}_{I_{N}(e_{l})} \left( x \right).$$

Consequently,

$$J_{3} \leq \frac{2}{2^{N}Q_{n}} \sum_{j=N+1}^{n_{1}} q_{2^{j-1}} (j-N) \sum_{s=0}^{N} 2^{s} K_{2^{s}} (x) + \frac{2}{2^{N}Q_{n}} \sum_{j=N+1}^{n_{1}} q_{2^{j-1}} 2^{j} \sum_{l=0}^{N-1} 2^{l} \mathbf{1}_{I_{N}(e_{l})} (x) = \frac{2}{2^{2N}Q_{n}} \sum_{j=N+1}^{n_{1}} q_{2^{j-1}} 2^{j} \frac{(j-N)}{2^{j-N}} \sum_{s=0}^{N} 2^{s} K_{2^{s}} (x) + \frac{2}{2^{N}Q_{n}} \sum_{j=N+1}^{n_{1}} q_{2^{j-1}} 2^{j} \sum_{l=0}^{N-1} 2^{l} \mathbf{1}_{I_{N}(e_{l})} (x) .$$

Since

$$Q_n \ge \sum_{j=1}^{2^{n_1}-1} q_j = \sum_{r=1}^{n_1} \sum_{j=2^{r-1}}^{2^r-1} q_j \ge \sum_{r=1}^{n_1} q_{2^r} 2^{r-1}.$$

We can write

(20) 
$$J_3 \leq \frac{c}{2^{2N}} \sum_{s=0}^N 2^s K_{2^s}(x) + \frac{c}{2^N} \sum_{l=0}^{N-1} 2^l \mathbf{1}_{I_N(e_l)}(x) ,$$

(21) 
$$J_{2} \leq \frac{c}{2^{2N}Q_{n}} \sum_{j=N+1}^{n_{1}} 2^{j} q_{2^{j-1}} \sum_{m=1}^{N} \sup_{2^{m-1} \leq k < 2^{m}} \left( k \left| K_{k} \left( x \right) \right| \right) \\ \leq \frac{c}{2^{2N}} \sum_{m=1}^{N} \sup_{2^{m-1} \leq k < 2^{m}} \left( k \left| K_{k} \left( x \right) \right| \right).$$

By Lemma 1 and from the condition of theorem we can write

(22) 
$$\int_{\overline{I}_{N}} \sup_{n>N} \left( 2^{N/p} \int_{I_{N}} \left| F_{n,2}^{(1)} \left( x + t \right) \right| dt \right)^{p} dx$$
$$\leq \frac{c_{p} 2^{N(1-p)}}{Q_{n}^{p}} \sum_{j=1}^{N} q_{2^{j-1}}^{p} \sum_{m=1}^{j} \int_{\overline{I}_{N}} \sup_{2^{m-1} \leq k < 2^{m}} \left( k \left| K_{k} \left( x \right) \right| \right)^{p} dx$$
$$+ \frac{c_{p} 2^{N}}{2^{2pN}} \sum_{m=1}^{N} \int_{\overline{I}_{N}} \sup_{2^{m-1} \leq k < 2^{m}} \left( k \left| K_{k} \left( x \right) \right| \right)^{p} dx$$
$$+ c_{p} 2^{N(1-2p)} \sum_{s=0}^{N-1} \int_{\overline{I}_{N}} \left( 2^{s} K_{2^{s}} \left( x \right) \right)^{p} dx$$

$$\begin{split} + c_p 2^{N(1-p)} \sum_{l=0}^{N-1} 2^{lp} \int_{\overline{I}_N} \mathbf{1}_{I_N(e_l)} (x) \, dx \\ &\leq \frac{c_p 2^{N(1-p)}}{Q_n^p} \sum_{j=1}^N q_{2j-1}^p 2^{j(2p-1)} \\ &+ c_p 2^{N(1-2p)} \sum_{s=0}^{N-1} 2^{s(2p-1)} \\ &+ c_p \frac{2^{N(1-p)}}{2^N} \sum_{l=0}^{N-1} 2^{lp} \\ &\leq \frac{c_p 2^{N(1-p)}}{Q_n^p} \sum_{j=1}^N \left( q_{2j-1} 2^j \right)^p 2^{j(p-1)} + c_p \\ &\leq c_p \sup_{N \in \mathbb{N}} \frac{2^{N(1-p)}}{Q_{2N}^p} \sum_{j=1}^N Q_{2j}^p 2^{j(p-1)} + c_p \\ &\leq c_p < \infty. \end{split}$$

Analogously, we can prove that

(23) 
$$\int_{\overline{I}_N} \sup_{n>N} \left( 2^{N/p} \int_{I_N} \left| F_{n,2}^{(2)} \left( x \dotplus t \right) \right| dt \right)^p dx \le c_p < \infty.$$

Combine (17), (18), (??), (22) and (23) we complete the proof of Theorem 3.  $\hfill \Box$ 

For p = 1, Theorem 3 implies that the following two conditions are equivalent:

• The maximal operator  $t^*$  is bounded from the dyadic Hardy space  $H_1(\mathbb{I})$  to the space  $L_1(\mathbb{I})$ ;

• 
$$\sup_{N\in\mathbb{N}}\frac{1}{Q_{2^N}}\sum_{j=1}^N Q_{2^j}<\infty.$$

On the other hand, in [13] it is proved that the following two conditions are equivalent:

- The maximal operator  $t^*$  is bounded from the space  $L_{\infty}(\mathbb{I})$  to the space  $L_{\infty}(\mathbb{I})$ ;
- $\sup_{N\in\mathbb{N}}\frac{1}{Q_{2^N}}\sum_{j=1}^NQ_{2^j}<\infty.$

Hence, we can conclude that the following.

**Theorem 4.** The following three conditions are equivalent:

- The maximal operator  $t^*(f)$  is bounded from  $L_{\infty}(\mathbb{I})$  to  $L_{\infty}(\mathbb{I})$ ;
- The maximal operator  $t^*(f)$  is bounded from  $H_1(\mathbb{I})$  to  $L_1(\mathbb{I})$ ;
- $\sup_{n\in\mathbb{N}}\frac{1}{Q_n}\sum_{k=1}^{|n|}Q_{2^k}<\infty.$

# 7. Applications to various summability methods

Since, the Nörlund mean is a generalization of many other well-know means with wide range of literature, in the last section we give applications of our results.

*Example* 1. Fejér means: Let  $q_j = 1$ . then it is easy to see that  $Q_j \sim j$  and we have

$$\frac{2^{N(1-p)}}{Q_{2^N}^p} \sum_{j=1}^N Q_{2^j}^p 2^{j(p-1)} = \frac{2^{N(1-p)}}{2^{Np}} \sum_{j=1}^N 2^{jp} 2^{j(p-1)}$$
$$= 2^{N(1-2p)} \sum_{j=1}^N 2^{j(2p-1)}.$$

Hence,

$$\left(\sup_{N\in\mathbb{N}}\frac{2^{N(1-p)}}{Q_{2^{N}}^{p}}\sum_{j=1}^{N}Q_{2^{j}}^{p}2^{j(p-1)}<\infty\right)\iff (p>1/2)$$

and we have that the following two conditions are equivalent:

- The maximal operator  $\sup_{n \in \mathbb{N}} |\sigma_n(f)|$  is bounded from the dyadic Hardy space  $H_p(\mathbb{I})$  to the space  $L_p(\mathbb{I})$ ;
- p > 1/2.

Under the condition p > 1/2, the boundedness of maximal operator  $\sup_{n \in \mathbb{N}} |\sigma_n(f)|$  was proved by Weisz [25], and the essence of condition p > 1/2 was proved by the author [9].

*Example 2.*  $(C, \alpha)$ -means: Let  $q_j := A_j^{\alpha-1}, \alpha \in (0, 1)$ . It is easy to see that  $Q_j \sim j^{\alpha}$ . Since

$$\frac{2^{N(1-p)}}{Q_{2N}^{p}} \sum_{j=1}^{N} Q_{2j}^{p} 2^{j(p-1)} = \frac{2^{N(1-p)}}{Q_{2N}^{p}} \sum_{j=1}^{N} 2^{jp\alpha} 2^{j(p-1)}$$
$$= \frac{2^{N(1-p)}}{Q_{2N}^{p}} \sum_{j=1}^{N} 2^{j(p(\alpha+1)-1)}$$

we conclude that

$$\sup_{N \in \mathbb{N}} \frac{2^{N(1-p)}}{Q_{2^N}^p} \sum_{j=1}^N Q_{2^j}^p 2^{j(p-1)} < \infty \iff \left(p > \frac{1}{1+\alpha}\right).$$

Consequently, we have the following two conditions are equivalent:

- The maximal operator  $\sup_{n \in \mathbb{N}} |\sigma_n^{\alpha}(f)|$  is bounded from the dyadic Hardy space  $H_p(\mathbb{I})$  to the space  $L_p(\mathbb{I})$ ;
- $p > \frac{1}{1+\alpha}$ .

Under the condition  $p > \frac{1}{1+\alpha}$ , the boundedness of maximal operator  $\sup_{n \in \mathbb{N}} |\sigma_n^{\alpha}(f)|$  was proved by Weisz [25], and the importance of condition  $p > \frac{1}{1+\alpha}$  was proved by the author [10].

*Example* 3. Let  $q_j := j^{\alpha-1}, \alpha \in [0, 1)$ . First, we consider the case when  $\alpha = 0$ . Then the Nörlund means coincide to the Nörlund logarithmic means

$$t_n(f;x) := \frac{1}{Q_n} \sum_{k=1}^{n-1} \frac{S_k(f;x)}{n-k}.$$

Nörlund's logarithmic means with respect to the trigonometric system was studied by Tkebuchava [21, 22]. The convergence and divergence of this means with respect to the Walsh systems was discussed in [5, 6, 7, 12]. Since

$$\sup_{n \in \mathbb{N}} \frac{1}{Q_n} \sum_{k=1}^{|n|} Q_{2^k} \sim \sup_{n \in \mathbb{N}} \frac{|n|^2}{\log(n+1)} \sim \sup_{n \in \mathbb{N}} \log(n+1) = \infty$$

from Theorem we conclude that the maximal operator  $\sup_{n \in \mathbb{N}} |t_n(f)|$  is not bounded from  $H_1(\mathbb{I})$  to  $L_1(\mathbb{I})$  and consequently, by interpolation theorem can not be bounded from  $H_p(\mathbb{I})$  to  $L_p(\mathbb{I})$ , when p < 1.

Now, we suppose that  $\alpha \in (0, 1)$ . It is easy to see that

$$\frac{2^{N(1-p)}}{Q_{2^N}^p} \sum_{j=1}^N Q_{2j}^p 2^{j(p-1)}$$
$$= \frac{2^{N(1-p)}}{Q_{2^N}^p} \sum_{j=1}^N 2^{j(p(\alpha+1)-1)}$$

and

(24) 
$$\lim_{n \to \infty} \frac{1}{n^{\alpha}} \sum_{k=1}^{n} (n-k)^{\alpha-1} S_k(f;x) = f(x) \text{ for a. e. } x \in \mathbb{I}.$$

we have the following two conditions are equivalent:

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The maximal operator sup 1/n<sup>α</sup> |∑<sub>k=1</sub><sup>n</sup> (n − k)<sup>α−1</sup> S<sub>k</sub>(f; x)| is bounded from the dyadic Hardy space H<sub>p</sub>(I) to the space L<sub>p</sub>(I);
p > 1/(1+α).

#### 8. Declaration

The author declare that he has not conflict of interest.

### 9. Data Availability

The author confirm that he does not use any data.

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