LARGE $|k|$ BEHAVIOR FOR THE REFLECTION COEFFICIENT FOR DAVEY-STEWARTSON II EQUATIONS

CHRISTIAN KLEIN, JOHANNES SJÖSTRAND, AND NIKOLA STOILOV

Abstract. The study of complex geometric optics solutions to a system of d-bar equations appearing in the context of electrical impedance tomography and the scattering theory of the integrable Davey-Stewartson II equations for large values of the spectral parameter k in [\[18\]](#page-28-0) is extended to the reflection coefficient. For the case of potentials q with compact support on some domain Ω with smooth strictly convex boundary, improved asymptotic relations are provided.

CONTENTS

Date: March 29, 2022.

This work is partially supported by the ANR-FWF project ANuI - ANR-17- CE40-0035, the isite BFC project NAANoD, the EIPHI Graduate School (contract ANR-17-EURE-0002) and by the European Union Horizon 2020 research and innovation program under the Marie Sklodowska-Curie RISE 2017 grant agreement no. 778010 IPaDEGAN.

1. INTRODUCTION

This paper addresses the scattering problem for the integrable Davey-Stewartson (DS) II equation given by the Dirac system

(1.1)
$$
\begin{cases} \overline{\partial}\phi_1 = \frac{1}{2}q e^{\overline{kz} - kz}\phi_2, \\ \partial\phi_2 = \sigma \frac{1}{2}\overline{q}e^{kz - \overline{kz}}\phi_1, \quad \sigma = \pm 1, \end{cases}
$$

subject to the asymptotic conditions

(1.2)
$$
\lim_{|z| \to \infty} \phi_1 = 1, \quad \lim_{|z| \to \infty} \phi_2 = 0,
$$

where $q = q(x, y)$ is a complex-valued field, where the *spectral parameter* $k \in \mathbb{C}$ is independent of $z = x + iy$, $(x, y) \in \mathbb{R}^2$, and where

$$
\partial := \frac{1}{2} \left(\frac{\partial}{\partial x} - \mathbf{i} \frac{\partial}{\partial y} \right) \quad \text{and} \quad \bar{\partial} := \frac{1}{2} \left(\frac{\partial}{\partial x} + \mathbf{i} \frac{\partial}{\partial y} \right).
$$

The functions $\phi_i(z; k)$, $i = 1, 2$ depend on z and k, and are called complex geometric optics (CGO) solutions. Note that they need not be holomorphic in either variable.

In addition to the DS II system, the CGO solutions appear in the scattering theory of two-dimensional integrable equations as the Kadomtsev-Petviashvili and the Novikov-Veselov equation, see [\[17\]](#page-27-1) for references, in electrical impedance tomography (EIT), see [\[30,](#page-28-1) [23\]](#page-28-2), and Normal Matrix Models in Random Matrix Theory, see e.g. [\[14\]](#page-27-2). Our main interest in this paper is in the scattering data, the so-called reflection coefficient R defined by

(1.3)
$$
\overline{R} = \frac{2\sigma}{\pi} \int_{\mathbb{C}} e^{kz - \overline{k}z} \overline{q}(z) \phi_1(z; k) L(\mathrm{d}z),
$$

where $L(dz)$ is the Lebesgue measure on the complex plane.

We are in particular interested in the case that the potential q has compact support on some simply connected domain $\Omega \subset \mathbb{C}$ with a smooth boundary. This is a typical situation in EIT since the body of a patient has obviously compact support. As an example for such a situation, we study in this paper the case that q is the characteristic function of the domain Ω. In the context of DS II such a setting would correspond to a situation as in the seminal work by Gurevich and Pitaevski [\[12\]](#page-27-3) for the Korteweg-de Vries (KdV) equation. For dispersive PDEs as DS and KdV, rapid modulated oscillations are expected in the vicinity of a discontinuity of the initial data q called dispersive shock waves (DSW). A detailed study of this case for DS would allow more insight into the formation of DSWs for the DS II system.

Since it is analytically difficult to solve d-bar systems explicitly, a large number of numerical approaches has been developed. The most popular ones are based on discrete Fourier transforms applied to the solution of d-bar equations in terms of the solid Cauchy transform,

see [\[22,](#page-28-3) [23\]](#page-28-2). The first approach along these lines with an exponential decrease of the numerical error with the number of Fourier modes has been given in [\[14,](#page-27-2) [15\]](#page-27-4) for Schwartz class potentials. A similar dependence of the error on the numerical resolutions could be achieved for potentials with compact support on a disk in [\[21\]](#page-28-4). However, it is only possible to reach machine precision (here 10^{-16}) for values of $|k| \simeq 1000$. Therefore the numerical approach [\[21\]](#page-28-4) was complemented in [\[18\]](#page-28-0) with explicit asymptotic formulae for large values of $|k|$ for potentials being the characteristic function of a compact domain. These results will be extended in the present paper to allow for sharper asymptotic results and explicit expressions for the reflection coefficient.

1.1. State of the art. We briefly summarize the state of the art of the theory of the Dirac system (1.1) and the results of [\[18,](#page-28-0) [19\]](#page-28-5) to be generalized in this paper. The question of existence and uniqueness of CGO solutions to system [\(1.1\)](#page-1-1) with $\sigma = 1$ was studied in [\[3\]](#page-27-5) for Schwartz class potentials and in [\[27,](#page-28-6) [28,](#page-28-7) [29\]](#page-28-8) for potentials $q \in L^{\infty}(\mathbb{C}) \cap L^{1}(\mathbb{C})$ such that also $\hat{q} \in L^{\infty}(\mathbb{C}) \cap L^{1}(\mathbb{C})$ where \hat{q} is the Fourier transform of q (the potentials have to satisfy a smallness condition in the focusing case $\sigma = -1$). In [\[6\]](#page-27-6) this was generalized respectively to real-valued, compactly supported potentials in $L^p(\mathbb{C})$ and in [\[26\]](#page-28-9) to potentials in $H^{1,1}(\mathbb{C})$, and in [\[25\]](#page-28-10) to potentials in $L^2(\mathbb{C})$.

One application of the system (1.1) as shown in $[9, 10]$ $[9, 10]$ is that it gives both the scattering and inverse scattering map for the Davey-Stewartson II equation

(1.4)
$$
iq_t + (q_{xx} - q_{yy}) + 2\sigma(\Phi + |q|^2)q = 0, \n\Phi_{xx} + \Phi_{yy} + 2(|q|^2)_{xx} = 0,
$$

a two-dimensional nonlinear Schrödinger equation; DS is *defocusing* for $\sigma = 1$, and *focusing* for $\sigma = -1$. Note that DS systems appear in the modulational regime of many dispersive equations as for instance the water wave systems, see e.g., [\[17\]](#page-27-1) for a review on DS equations and a comprehensive list of references, and are only integrable for the choice of parameters in [\(1.4\)](#page-2-1).

The scattering data are given in terms of the reflection coefficient $R = R(k)$ in [\(1.3\)](#page-1-2). The DS II equations [\(1.4\)](#page-2-1) are completely integrable in the sense that a Lax pair exists, the first part of the Lax pair being [\(1.1\)](#page-1-1). Here ϕ_1 , ϕ_2 , q can be seen as having a dependence on the physical time t which is suppressed since it will not be studied in this paper. However, it will play a role in the second equation of the Lax pair. We

put
$$
\Theta = \begin{pmatrix} \phi_1 e^{kz} \\ \phi_2 e^{\overline{kz}} \end{pmatrix}
$$
 and get for the Lax pair

$$
(1.5) \quad \Theta_x + i\sigma_3 \Theta_y = \left(\begin{array}{cc} 0 & q \\ \overline{q} & 0 \end{array}\right) \Theta,
$$

(1.6)
$$
\Theta_t = \begin{pmatrix} i \overline{\partial}^{-1} \partial (|q|^2)/2 & -i \overline{\partial}^{-1} \partial q \\ i \overline{\partial} \overline{q} & -i \overline{\partial}^{-1} \partial (|q|^2)/2 \end{pmatrix} \Theta - \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix} \Theta_y + i \sigma_3 \Theta_{yy} ,
$$

where $\sigma_3 = \text{diag}(1, -1)$ is a Pauli matrix.

As q in [\(1.4\)](#page-2-1) evolves in time t , the reflection coefficient evolves because of [\(1.6\)](#page-3-0) by a trivial phase factor:

(1.7)
$$
R(k;t) = R(k,0)e^{4it\Re(k^2)}.
$$

The inverse scattering transform for DS II is then given by [\(1.1\)](#page-1-1) and (1.2) after replacing q by R and vice versa, the derivatives with respect to z by the corresponding derivatives with respect to k , and asymptotic conditions for $k \to \infty$ instead of $z \to \infty$, see [\[1\]](#page-27-9).

The main interest in [\[18,](#page-28-0) [19\]](#page-28-5) and the present paper is in the case when |k| is large, i.e., $h := 1/|k|$ small. We introduce the following notation:

$$
kz - \overline{kz} = i|k|\Re(z\overline{\omega}) = i|k|\langle z, \omega \rangle_{\mathbb{R}^2}, \quad \omega = \frac{2ik}{|k|},
$$

and for $u \in L^2(\mathbb{R}^2)$

$$
\widehat{\tau}_{\omega}u = e^{kz - \overline{kz}}u, \quad \widehat{\tau}_{-\omega}u = e^{\overline{kz} - kz}u, \quad E = (h\overline{\partial})^{-1}, F = (h\partial)^{-1}
$$

which leads for (1.1) to

$$
\begin{cases}\nh\overline{\partial}\phi_1 = \widehat{\tau}_{-\omega}h(q/2)\phi_2, \\
h\partial\phi_2 = \sigma\widehat{\tau}_{\omega}h(\overline{q}/2)\phi_1.\n\end{cases}
$$

See [\[18\]](#page-28-0), Section 2, for the precise choice of E, F . Looking for solutions of the form (1.10) - (1.11) below, leads to the inhomogeneous system

$$
\begin{cases} \phi_1^1-E\widehat{\tau}_{-\omega}\frac{hq}{2}\phi_2^1=Ef_1,\\ \phi_2^1-\sigma F\widehat{\tau}_{\omega}\frac{h\overline{q}}{2}\phi_1^1=Ff_2, \end{cases}
$$

with $f_1 = 0$ and $f_2 = \sigma \hat{\tau}_{\omega} h \bar{q}/2$, or

$$
(1 - K) \begin{pmatrix} \phi_1^1 \\ \phi_2^1 \end{pmatrix} = \begin{pmatrix} Ef_1 \\ Ff_2 \end{pmatrix},
$$

.

where

(1.8)
$$
\mathcal{K} = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}, \quad \begin{cases} A = E\hat{\tau}_{-\omega} \frac{hq}{2}, \\ B = \sigma F \hat{\tau}_{\omega} \frac{h\overline{q}}{2} \ (=\sigma \overline{A}) \end{cases}
$$

In $([18])$ $([18])$ $([18])$ we showed,

Proposition 1.1. Let $q \in \langle \cdot \rangle^{-2}H^s$ for some $s \in]1, 2]$ and fix $\epsilon \in]0, 1]$. Define K as in [\(1.8\)](#page-3-1). Then $\mathcal{K} = \mathcal{O}(1) : (\langle \cdot \rangle^{\epsilon} L^2)^2 \to (\langle \cdot \rangle^{\epsilon} L^2)^2$,

$$
\mathcal{K}^2 = \mathcal{O}(h^{s-1}) : (\langle \cdot \rangle^{\epsilon} L^2)^2 \to (\langle \cdot \rangle^{\epsilon} L^2)^2.
$$

For $h_0 > 0$ small enough and $0 < h \leq h_0, 1 - K : (\langle \cdot \rangle^{\epsilon} L^2)^2 \to (\langle \cdot \rangle^{\epsilon} L^2)^2$ has a uniformly bounded inverse,

(1.9)

$$
(1 - K)^{-1} = (1 - K^2)^{-1} (1 + K) = \begin{pmatrix} (1 - AB)^{-1} & 0 \\ 0 & (1 - BA)^{-1} \end{pmatrix} \begin{pmatrix} 1 & A \\ B & 1 \end{pmatrix}.
$$

When q is the characteristic function of a bounded strictly convex domain with smooth boundary, the conclusions hold with $s=\frac{3}{2}$ $\frac{3}{2}$.

1.2. Main results. The main goal of this paper is to obtain improved asymptotics for the reflection coefficient, in particular for the case of potentials with compact support. To this end we solve the system [\(1.1\)](#page-1-1) for $q \in \langle \cdot \rangle^{-2} H^s$ for some $s \in]1, 2]$ for small h in the form

(1.10)
$$
\phi_1 = \phi_1^0 + \phi_1^1, \quad \phi_2 = \phi_2^0 + \phi_2^1.
$$

We start with

(1.11)
$$
\phi_1^0 = 1, \ \phi_2^0 = 0.
$$

The functions ϕ_1^1 and ϕ_2^1 should satisfy

$$
\begin{cases} h\overline{\partial}\phi_1^1-\widehat{\tau}_{-\omega}\frac{hq}{2}\phi_2^1=f_1,\\ h\partial\phi_2^1-\sigma\widehat{\tau}_{\omega}\frac{h\overline{q}}{2}\phi_1^1=f_2, \end{cases}
$$

with $f_1 = 0$ and $f_2 = \sigma \hat{\tau}_{\omega} h \overline{q}/2$ which is $\mathcal{O}(h)$ in $\langle \cdot \rangle^{-2}L^2$. We look for $\phi_j^1 \in \langle \cdot \rangle^{\epsilon} L^2$ for $\epsilon \in]0,1]$. This is equivalent to

$$
\begin{cases} \phi_1^1 - E\widehat{\tau}_{-\omega} \frac{hq}{2} \phi_2^1 = Ef_1, \\ \phi_2^1 - \sigma F \widehat{\tau}_{\omega} \frac{h\overline{q}}{2} \phi_1^1 = Ff_2, \end{cases}
$$

or

$$
(1 - \mathcal{K}) \begin{pmatrix} \phi_1^1 \\ \phi_2^1 \end{pmatrix} = \begin{pmatrix} Ef_1 \\ Ff_2 \end{pmatrix}.
$$

Here $F f_2 = \sigma F \hat{\tau}_{\omega} h \overline{q}/2 = \mathcal{O}(1)$ in $\langle \cdot \rangle^{\epsilon} L^2$ and Proposition [1.1](#page-4-3) gives us a unique solution in $({\langle \cdot \rangle}^{\epsilon}L^2)^2$ which is $\mathcal{O}(1)$ in that space. More precisely by [\(1.9\)](#page-4-4), we get

Theorem 1.2. The system (1.1) has the solution (1.10) , (1.11) with

hq

(1.12)
$$
\phi_1^1 = (1 - AB)^{-1} A \sigma F \hat{\tau}_w \frac{h \overline{q}}{2} = (1 - AB)^{-1} AB(1)
$$

$$
\phi_2^1 = (1 - BA)^{-1} \sigma F \hat{\tau}_w \frac{h \overline{q}}{2}
$$

where $\sigma F \hat{\tau}_{\omega} h \overline{q}/2 = B(1)$ and $A \sigma F \hat{\tau}_{\omega} h \overline{q}/2 = AB(1)$ are $\mathcal{O}(1)$ in $\langle \cdot \rangle^{\epsilon} L^2$ by [\(1.8\)](#page-3-1). Note that formally $\phi_1 = (1 - AB)^{-1}(1)$, $\phi_2 = \phi_2^1$.

On the way we improve the results of Prop. [1.1](#page-4-3) for potentials q being the characteristic function of a compact domain,

Proposition 1.3. If q is the characteristic function of a bounded strictly convex domain with smooth boundary, the conclusions of Prop. [1.1](#page-4-3) hold with $s=2$.

In the following we will put $\sigma = 1$ for the ease of presentation. When $\Omega \in \mathbb{C}$ is a bounded domain, let

(1.13)
$$
D_{\Omega}(z) = \frac{1}{\pi} \int_{\Omega} \frac{1}{z - w} L(dw), \quad z \in \mathbb{C}
$$

be the solution of the d-bar problem

(1.14)
$$
\begin{cases} \partial_{\bar{z}} D_{\Omega} = 1_{\Omega}, \\ D_{\Omega}(z) \to 0, \quad z \to \infty. \end{cases}
$$

The main theorem of this paper reads

Theorem 1.4. Let $\Omega \in \mathbb{C}$ be open with a strictly convex smooth boundary, and let $iu(w, k)$ be a holomorphic extension of $kw-\overline{kw}$ from $\partial\Omega$ to $neigh(\partial\Omega,\mathbb{C})$. D_{Ω} is continuous, $D_{\Omega}|_{\Omega} \in C^{\infty}(\overline{\Omega})$, $D_{\Omega}|_{\mathbb{C}\setminus\Omega} \in C^{\infty}(\mathbb{C}\setminus\Omega)$. Moreover (1.15)

$$
\overline{R} = \frac{2}{\pi} \int_{\Omega} e^{kz - \overline{k}z} L(dz) \n+ \frac{1}{4i\pi |k|^2} \left(- \int_{\tilde{\Gamma}} D_{\Omega}(w) e^{iu(w,k)} dw + \overline{\int_{\Gamma} D_{\Omega}(w) e^{-iu(w,k)} dw} \right) + \mathcal{O}(|k|^{-3} \ln |k|),
$$

for $k \in \mathbb{C}$, $|k| \gg 1$. Here Γ is the contour from Fig. [1,](#page-16-0) and Γ is defined as Γ after replacing k with $-k$.

We parametrize the positively oriented boundary $\partial\Omega$ by $\gamma(t)$, $t \in$ [0, L] and put $\varphi(t) = -(k\gamma(t) - \overline{k\gamma(t)})/|k|$ as well as $a(t) = \dot{\gamma}(t)$. We denote the critical points of φ by t_{\pm} . Applying a stationary phase approximation to reflection coefficient [\(1.15\)](#page-5-0), we obtain

Corollary 1.5. The leading order of the reflection coefficient for $|k| \gg$ 1 is given by

$$
(1.16)
$$
\n
$$
\overline{R} = \frac{i\sqrt{2\pi}}{\pi k} \sum_{t=t_{-},t_{+}} e^{-|k|\varphi(t)} \left[|k|^{-\frac{1}{2}} a(t) \varphi''(t)^{-\frac{1}{2}} + |k|^{-\frac{3}{2}} \left(\frac{a''(t)}{2} \varphi''(t)^{-\frac{3}{2}} - \left(a(t) \frac{\varphi^{(4)}(t)}{6} + a'(t) \frac{\varphi^{(3)}(t)}{2} \right) \varphi''(t)^{-\frac{5}{2}} + \frac{5}{24} a(t) \varphi^{(3)}(t)^{2} \varphi''(t)^{-\frac{7}{2}} \right) \right] + \frac{\sqrt{2\pi}}{4\pi i |k|^{2}} \sum_{t=t_{-},t_{+}} e^{-|k|\varphi(t)} |k|^{-\frac{1}{2}} \varphi''(t)^{-\frac{1}{2}} \left(-D_{\Omega}(\gamma(t)) a(t) + \overline{D_{\Omega}(\gamma(t)) \bar{a}(t)} \right) + \mathcal{O}(|k|^{-3} \ln |k|).
$$

The branches of the square roots are chosen as in Remark [4.1.](#page-25-1)

The paper is organized as follows: in section 2, we give estimates for the operator AB. In section 3 these estimates are applied to the reflection coefficient. In section 4 we provide explicit formulae via a stationary phase approximation. We consider the example of the characteristic function of the unit disk and give a partial proof of a conjecture in [\[18\]](#page-28-0) for the reflection coefficient in this case. We add some concluding remarks in section 5.

2. Estimates for the operator AB

Let $\Omega \in \mathbb{C}$ be strictly convex with smooth boundary. The central problem is to study AB where A, B are given in [\(1.8\)](#page-3-1),

$$
A = E\hat{\tau}_{-\omega} \frac{hq}{2}, \quad B = \sigma F \hat{\tau}_{\omega} \frac{h\overline{q}}{2} (= \sigma \overline{A}), \quad q = 1_{\Omega},
$$

$$
Au(z) = \frac{1}{2\pi} \int_{\Omega} \frac{1}{z - w} e^{-kw + \overline{kw}} u(w) L(dw),
$$

$$
Bu(z) = \frac{\sigma}{2\pi} \int_{\Omega} \frac{1}{\overline{z} - \overline{w}} e^{kw - \overline{kw}} u(w) L(dw),
$$

(2.1)

$$
ABu(z) = \frac{\sigma}{4\pi^2} \iint_{\Omega \times \Omega} \frac{1}{z - \zeta} e^{-k\zeta + \overline{k\zeta}} \frac{1}{\overline{\zeta} - \overline{w}} e^{kw - \overline{kw}} u(w) L(\mathrm{d}\zeta) L(\mathrm{d}w)
$$

=
$$
\int_{\Omega} K(z, w) u(w) L(\mathrm{d}w),
$$

(2.2)
$$
K(z, w) = \frac{\sigma}{4\pi^2} \iint_{\Omega} \frac{e^{k\zeta}}{(z - \zeta)} \frac{e^{-k\zeta}}{(\overline{\zeta} - \overline{w})} \frac{d\overline{\zeta} \wedge d\zeta}{2i} e^{kw - \overline{kw}},
$$

for the case of the characteristic function of Ω .

We look for functions f, \tilde{g} such that with d denoting exterior differentiation with respect to ζ ,

$$
\frac{e^{\overline{k\zeta}}}{(z-\zeta)}\frac{e^{-k\zeta}}{(\overline{\zeta}-\overline{w})}\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta=\mathrm{d}\left(e^{\overline{k\zeta}-k\zeta}(\widetilde{f}\mathrm{d}\zeta+\widetilde{g}\overline{\mathrm{d}\zeta})\right)+\ldots,
$$

i.e.

$$
\frac{e^{\overline{k\zeta}}}{(z-\zeta)\overline{(\overline{\zeta}-\overline{w})}} = \partial_{\overline{\zeta}}(e^{\overline{k\zeta}-k\zeta}\widetilde{f}) - \partial_{\zeta}(e^{\overline{k\zeta}-k\zeta}\widetilde{g}) + \dots
$$

With

$$
\widetilde{f} = \frac{f}{\overline{k}(z-\zeta)(\overline{\zeta}-\overline{w})}, \quad \widetilde{g} = \frac{g}{k(z-\zeta)(\overline{\zeta}-\overline{w})},
$$

we get

$$
\partial_{\overline{\zeta}}(e^{\overline{k\zeta}-k\zeta}\widetilde{f}) = \frac{1}{\overline{k}}\partial_{\overline{\zeta}}\left(\frac{e^{\overline{k\zeta}-k\zeta}f}{(z-\zeta)(\overline{\zeta}-\overline{w})}\right) =
$$

$$
\frac{e^{\overline{k\zeta}-k\zeta}}{(z-\zeta)(\overline{\zeta}-\overline{w})}\left(1-\frac{1}{\overline{k}(\overline{\zeta}-\overline{w})}+\frac{1}{\overline{k}}\partial_{\overline{\zeta}}\right)f - \frac{1}{\overline{k}(\overline{\zeta}-\overline{w})}\pi f(z)\delta_z(\zeta),
$$

$$
- \partial_{\zeta} (e^{\overline{k\zeta} - k\zeta} \widetilde{g}) = -\frac{1}{k} \partial_{\zeta} \left(\frac{e^{\overline{k\zeta} - k\zeta} g}{(z - \zeta)(\overline{\zeta} - \overline{w})} \right) =
$$

$$
\frac{e^{\overline{k\zeta} - k\zeta}}{(z - \zeta)(\overline{\zeta} - \overline{w})} \left(1 - \frac{1}{k(z - \zeta)} - \frac{1}{k} \partial_{\zeta} \right) g - \frac{1}{k} \frac{e^{\overline{k\zeta} - k\zeta}}{(z - \zeta)} \pi g(w) \delta_w(\zeta).
$$

Hence,

$$
(2.3) \quad \frac{\partial_{\overline{\zeta}}(e^{\overline{k\zeta}-k\zeta}\tilde{f}) - \partial_{\zeta}(e^{\overline{k\zeta}-k\zeta}\tilde{g}) =
$$
\n
$$
\frac{e^{\overline{k\zeta}-k\zeta}}{(z-\zeta)(\overline{\zeta}-\overline{w})} \left[\left(1 - \frac{1}{\overline{k(\overline{\zeta}-\overline{w})}} + \frac{1}{\overline{k}}\partial_{\overline{\zeta}} \right)f + \left(1 - \frac{1}{k(z-\zeta)} - \frac{1}{k}\partial_{\zeta}\right)g \right]
$$
\n
$$
- \frac{\pi e^{\overline{k\zeta}-k\zeta}}{\overline{k(\overline{\zeta}-\overline{w})}} f\delta_{z}(\zeta) - \frac{\pi e^{\overline{k\zeta}-k\zeta}}{k(z-\zeta)} g\delta_{w}(\zeta)
$$

We would like to have

$$
\left(1 - \frac{1}{\overline{k}(\overline{\zeta} - \overline{w})} + \frac{1}{\overline{k}}\partial_{\overline{\zeta}}\right) f(\zeta) + \left(1 - \frac{1}{k(z - \zeta)} - \frac{1}{k}\partial_{\zeta}\right) g(\zeta) = 1.
$$

We start by constructing a partition $1 = \chi_w + \chi_z + r_{w,z}$, where $r_{w,z}$ is supported in a region $|\zeta - z|, |\zeta - w| < \mathcal{O}(\frac{1}{|k|})$ $\frac{1}{|k|}$ and then solve, up to asymptotic errors,

(2.4)
$$
\left(1 - \frac{1}{\overline{k}(\overline{\zeta} - \overline{w})} + \frac{1}{\overline{k}} \partial_{\overline{\zeta}}\right) f = \chi_w,
$$

(2.5)
$$
\left(1 - \frac{1}{k(z - \zeta)} - \frac{1}{k}\partial_{\zeta}\right)g = \chi_z.
$$

Put

(2.6)
$$
d(z, w, k) = |z - w| + \frac{1}{|k|},
$$

Proposition 2.1. Let

(2.7)
$$
\widehat{d}(z, w, \zeta, k) = d(z, w, k) + \left| \zeta - \frac{z + w}{2} \right|
$$

and notice that $d(z, w, \zeta, k)$ is uniformly of the same order of magnitude as $d(z, w, k) + |\zeta - z|$ and $d(z, w, k) + |\zeta - w|$, since $|z - w| \leq d(z, w, k)$.

For all $(w, z, k) \in \mathbb{C}^3$ with $|k| \geq 1$, there exist $\chi_w, \chi_z \in C^{\infty}(\mathbb{C}),$ $r_{w,z} \in C_0^{\infty}(\mathbb{C})$ such that

(2.8) χw, χz, rw,z ≥ 0,

(2.9)
$$
\chi_w + \chi_z + r_{w,z} = 1,
$$

(2.10)
$$
|\zeta - z| \ge \frac{1}{\mathcal{O}(1)} \widehat{d}(z, w, \zeta, k) \text{ on } \text{supp } \chi_z,
$$

$$
|\zeta - w| \ge \frac{1}{\mathcal{O}(1)} \widehat{d}(z, w, \zeta, k) \text{ on } \text{supp } \chi_w,
$$

(2.11)
$$
r_{w,z}
$$
 has its support in $\left\{ \zeta \in \mathbb{C}; |\zeta - z|, |\zeta - w| \le \frac{3}{2|k|} \right\}$,
(2.12)

$$
\nabla^{\alpha}_{\zeta} \chi_z = \mathcal{O}\left(\hat{d}^{-|\alpha|}\right), \quad \nabla^{\alpha}_{\zeta} \chi_w = \mathcal{O}\left(\hat{d}^{-|\alpha|}\right), \quad \forall \alpha \in \mathbb{N}^2, \quad |\alpha| = |\alpha|_{l^1} = \alpha_1 + \alpha_2,
$$

(2.13)
$$
\nabla^{\alpha}_{\zeta} r_{z,w} = \mathcal{O}_{N,\alpha}(1) \widehat{d}^{-|\alpha|} (|k|\widehat{d})^{-N}, \forall \alpha \in \mathbb{N}^2, \forall N \in \mathbb{N}.
$$

These estimates are uniform with respect to w, z, k .

Proof. Put

$$
\widetilde{\chi}_z(\zeta) = (1 - \psi^0) \left(\frac{\zeta - z}{d} \right), \quad \widetilde{\chi}_w(\zeta) = (1 - \psi^0) \left(\frac{\zeta - w}{d} \right),
$$

where $\psi^0 \in C_0^{\infty}(D(0,1/3))$ is real valued with $1_{D(0,1/4)} \leq \psi^0 \leq 1_{D(0,1/3)}$. Clearly $\widetilde{\chi}_z(\zeta) + \widetilde{\chi}_z(\zeta) \geq 0$ and (2.14)

$$
\widetilde{\chi}_z(\zeta) + \widetilde{\chi}_z(\zeta) = 0 \Rightarrow \begin{cases} |\frac{\zeta - z}{d}| \leq \frac{1}{3} \\ |\frac{\zeta - w}{d}| \leq \frac{1}{3} \end{cases} \Leftrightarrow \begin{cases} |\zeta - z| \leq \frac{1}{3}|z - w| + \frac{1}{3}\frac{1}{|k|}, \\ |\zeta - w| \leq \frac{1}{3}|z - w| + \frac{1}{3}\frac{1}{|k|}. \end{cases}
$$

The last inequalities imply that $|z - w| \leq \frac{2}{3}|z - w| + \frac{2}{3|z - w|}$ $\frac{2}{3|k|}$, so

$$
|z - w| \le \frac{2}{|k|}, \quad |\zeta - z|, \quad |\zeta - w| \le \frac{1}{|k|}.
$$

We have $\nabla_{\zeta}^{\alpha} \widetilde{\chi}_{z}, \nabla_{\zeta}^{\alpha} \widetilde{\chi}_{w} = \mathcal{O}(d^{-|\alpha|})$ and

(2.15)
\n
$$
|\zeta - z| \ge \frac{1}{\mathcal{O}(1)} d(z, w, k) \text{ on } \text{supp }\widetilde{\chi}_z,
$$
\n
$$
|\zeta - w| \ge \frac{1}{\mathcal{O}(1)} d(z, w, k) \text{ on } \text{supp }\widetilde{\chi}_w,
$$

where we can replace $d(z, w, k)$ with $\hat{d}(z, w, \zeta, k)$. Moreover, since $\widetilde{\chi}(\zeta) = 1$ when $|\cdot - \zeta| \ge \frac{1}{3}d$, we have

$$
\nabla_{\zeta}^{\alpha}\widetilde{\chi}_{z} = \mathcal{O}\left((d+|z-\zeta|)^{-|\alpha|}\right), \quad \nabla_{\zeta}^{\alpha}\widetilde{\chi}_{w} = \mathcal{O}\left((d+|w-\zeta|)^{-|\alpha|}\right).
$$

Hence as noted in the statement of the proposition,

(2.16)
$$
\nabla_{\zeta}^{\alpha} \widetilde{\chi}_z, \ \nabla_{\zeta}^{\alpha} \widetilde{\chi}_w = \mathcal{O}(\widehat{d}^{-|\alpha|}).
$$

Let $\Psi^1 \in C_0^{\infty}(D(0, \frac{3}{2}))$ Let $\Psi^1 \in C_0^{\infty}(D(0, \frac{3}{2}))$ satisfy $1_{D(0,1)} \leq \Psi^1 \leq 1$ and put $\widetilde{r}_{z,w} = \Psi^1(|k|(\zeta - z))\Psi^1(|k|(\zeta - w))$. Then, by [\(2.14\)](#page-8-0) and the subsequent observation,

(2.17)
$$
f := \widetilde{\chi}_z(\zeta) + \widetilde{\chi}_w(\zeta) + \widetilde{r}_{z,w} \approx 1,
$$

and by construction the last term has its support in

$$
\left\{\zeta \in \mathbb{C}; |\zeta - z|, |\zeta - w| \leq \frac{3}{2|k|}\right\},\
$$

so that $|z-w| \leq \frac{\mathcal{O}(1)}{|k|}$, $\widehat{d}(z, w, \zeta, k) \asymp \frac{1}{|k|}$ $\frac{1}{|k|}$ on supp $\widetilde{r}_{z,w}$. It follows that $\widetilde{r}_{z,w} = \mathcal{O}_N(1)(|k|\tilde{d})^{-N}, \ \forall N \ge 0$ and more generally

(2.18)
$$
\nabla_{\zeta}^{\alpha} \widetilde{r}_{z,w} = \mathcal{O}_{N,\alpha}(1) \widehat{d}^{-|\alpha|} (|k|\widehat{d})^{-N}.
$$

From (2.16) we get

(2.19)
$$
\nabla^{\alpha}_{\zeta} f, \ \nabla^{\alpha}_{\zeta} \frac{1}{f} = \mathcal{O}(1) \widehat{d}^{-|\alpha|}.
$$

Put

(2.20)
$$
\chi_z = \tilde{\chi}_z/f, \quad \chi_w = \tilde{\chi}_w/f, \quad r_{z,w} = \tilde{r}_{z,w}/f.
$$

Then we have

(2.21)
$$
\chi_z + \chi_w + r_{z,w} = 1,
$$

and (2.8) – (2.13) follow.

We now return to the problem $(2.4) - (2.5)$ $(2.4) - (2.5)$. As a first approximate solution, we take $f^0 = \chi_w$, $g^0 = \chi_z$ and treat the other terms in the LHSs as perturbations. We then get f, g as formal Neumann series sums (2.22)

$$
f = \sum_{0}^{\infty} \left(\frac{1}{\overline{k}(\overline{\zeta} - \overline{w})} - \frac{1}{\overline{k}} \partial_{\overline{\zeta}} \right)^{\nu} \chi_w, \quad g = \sum_{0}^{\infty} \left(\frac{1}{k(z - \zeta)} + \frac{1}{k} \partial_{\zeta} \right)^{\nu} \chi_z.
$$

Recall (2.10) and (2.11) . Then

$$
\nabla^{\nu_0}_{\zeta} \frac{1}{(\overline{\zeta} - \overline{w})^{\nu_1}} \partial^{\nu_2}_{\overline{\zeta}} \chi_w = \mathcal{O}(\widehat{d}^{-(\nu_0 + \nu_1 + \nu_2)}),
$$

$$
\nabla^{\nu_0}_{\zeta} \frac{1}{(z-\zeta)^{\nu_1}} \partial^{\nu_2}_{\zeta} \chi_z = \mathcal{O}(\widehat{d}^{-(\nu_0+\nu_1+\nu_2)}).
$$

Thus

(2.23)
$$
\nabla_{\zeta}^{\nu_{0}} \left(\frac{1}{\overline{k}(\overline{\zeta} - \overline{w})} - \frac{1}{\overline{k}} \partial_{\overline{\zeta}} \right)^{\nu}_{\chi} \chi_{w} \right\} = \mathcal{O}(\widehat{d}^{-\nu_{0}}(k\widehat{d})^{-\nu}).
$$

For $N \in \mathbb{N}$ define f_N , g_N as in [\(2.22\)](#page-9-0) but with finite sums (2.24)

$$
f_N = \sum_{0}^{N} \left(\frac{1}{\overline{k}(\overline{\zeta} - \overline{w})} - \frac{1}{\overline{k}} \partial_{\overline{\zeta}} \right)^{\nu} \chi_w, \quad g_N = \sum_{0}^{N} \left(\frac{1}{k(z - \zeta)} + \frac{1}{k} \partial_{\zeta} \right)^{\nu} \chi_z.
$$

Notice that by [\(2.23\)](#page-9-1)

$$
(2.25) \t f_N, g_N = \mathcal{O}(1).
$$

Then c.f. (2.4), (2.5)
\n(2.26)
\n
$$
\left(1 - \frac{1}{\overline{k}(\overline{\zeta} - \overline{w})} + \frac{1}{\overline{k}}\partial_{\overline{\zeta}}\right) f_N = \chi_w - \left(\frac{1}{\overline{k}(\overline{\zeta} - \overline{w})} - \frac{1}{\overline{k}}\partial_{\overline{\zeta}}\right)^{N+1} \chi_w =: \chi_w - S_w^{N+1}
$$
\n(2.27)
\n
$$
\left(1 - \frac{1}{k(z - \zeta)} - \frac{1}{k}\partial_{\zeta}\right) g_N = \chi_z - \left(\frac{1}{k(z - \zeta)} + \frac{1}{k}\partial_{\zeta}\right)^{N+1} \chi_z =: \chi_z - T_z^{N+1}
$$

where

(2.28)
$$
\nabla_{\zeta}^{\nu_0} S_w^{N+1}, \ \nabla_{\zeta}^{\nu_0} T_z^{N+1} = \mathcal{O}(\hat{d}^{-\nu_0} (k\hat{d})^{-N-1}).
$$

Now, combine [\(2.26\)](#page-10-0) - [\(2.28\)](#page-10-1) with [\(2.21\)](#page-9-2) and [\(2.18\)](#page-9-3) (valid also for $r_{z,w})$ to get

$$
(2.29) \quad \left(1 - \frac{1}{\overline{k}(\overline{\zeta} - \overline{w})} + \frac{1}{\overline{k}}\partial_{\overline{\zeta}}\right)f_N + \left(1 - \frac{1}{k(z - \zeta)} - \frac{1}{k}\partial_{\zeta}\right)g_N = 1 - S_w^{N+1} - T_z^{N+1} - r_{z,w} =: 1 - r^{N+1}
$$

where

(2.30)
$$
\nabla_{\zeta}^{\nu_0} r^{N+1} = \mathcal{O}(\widehat{d}^{-\nu_0}(k\widehat{d})^{-N-1}), \quad \nu_0 \in \mathbb{N}.
$$

We use this in the discussion after [\(2.2\)](#page-6-1). With

$$
\widetilde{f}_N = \frac{f_N}{\overline{k}(z-\zeta)(\overline{\zeta}-\overline{w})}, \quad \widetilde{g}_N = \frac{g_N}{k(z-\zeta)(\overline{\zeta}-\overline{w})}
$$

we get (cf. [\(2.3\)](#page-7-2))

$$
d\left(e^{\overline{k\zeta}-k\zeta}(\widetilde{f}_Nd\zeta+\widetilde{g}_N\overline{d\zeta})\right)=
$$

$$
\left[\frac{e^{\overline{k\zeta}-k\zeta}}{(z-\zeta)(\overline{\zeta}-\overline{w})}-\frac{e^{\overline{k\zeta}-k\zeta}r^{N+1}}{(z-\zeta)(\overline{\zeta}-\overline{w})}-\frac{\pi e^{\overline{k\zeta}-k\zeta}f_N\delta_z(\zeta)}{\overline{k(\zeta-\overline{w})}}-\frac{\pi e^{\overline{k\zeta}-k\zeta}g_N\delta_w(\zeta)}{k(z-\zeta)}\right]\overline{d\zeta}\wedge d\zeta.
$$

We use this in [\(2.2\)](#page-6-1) and apply Stokes' formula:

$$
(2.31)
$$
\n
$$
K(z, w) = \frac{\sigma e^{kw - \overline{kw}}}{(2\pi)^2 2i} \left[\underbrace{\iint_{\Omega} \frac{e^{\overline{k\zeta} - k\zeta}}{(z - \zeta)(\overline{\zeta} - \overline{w})} r^{N+1} \overline{d\zeta} \wedge d\zeta}_{=:K_1(z, w)} + \underbrace{\frac{2i\pi e^{\overline{kz} - kz}}{\overline{k(\overline{z} - \overline{w})}} f_N(z, w, z) 1_{\Omega}(z) + \underbrace{\frac{2i\pi e^{\overline{kw} - kw}}{k(z - w)} g_N(z, w, w) 1_{\Omega}(w)}_{=:K_3(z, w)} + \underbrace{\int_{\partial\Omega} \frac{e^{\overline{k\zeta} - k\zeta} f_N}{\overline{k(\overline{\zeta} - \overline{w})(z - \zeta)}} d\zeta}_{=:K_4(z, w)} + \underbrace{\int_{\partial\Omega} \frac{e^{\overline{k\zeta} - k\zeta} g_N}{k(\overline{\zeta} - \overline{w})(z - \zeta)}}_{=:K_5(z, w)} d\zeta}_{=:K_5(z, w)}
$$

The factor in front of the big bracket is bounded, so it will suffice to estimate the norms between various weighted L^p spaces of the operators

(2.32)
$$
A_j u(z) = \int_{\Omega} K_j(z, w) u(w) L(dw)
$$

with integral kernel K_j . Recalling (2.30) , we get

$$
K_1(z, w) = \frac{\mathcal{O}(1)}{|k|^{N+1}} \int_{\Omega} \frac{1}{|z - \zeta| |w - \zeta|} \frac{1}{\left(|z - w| + |\zeta - \frac{z + w}{2}| + \frac{1}{|k|} \right)^{N+1}} L(d\zeta).
$$

Consider separately the integrals over $\Omega_1 = \{ \zeta \in \Omega; |\zeta - w| \geq |\zeta - z| \}$ and $\Omega_2 = \{ \zeta \in \Omega; |\zeta - w| < |\zeta - z| \}.$ The two integrals can be handled similarly, and we only need to consider the first case $|\zeta - w| \ge |\zeta - z|$. Here $|\zeta - w| \ge \frac{1}{2} |z - w|$ and the corresponding integral is

$$
\leq \frac{\mathcal{O}(1)}{|z-w|} \frac{1}{|k|^{N+1}} \int_{\Omega_1} \frac{1}{|z-\zeta|} \frac{1}{\left(|z-w|+|\zeta-z|+\frac{1}{|k|}\right)^{N+1}} L(d\zeta)
$$

$$
\leq \frac{\mathcal{O}(1)}{|z-w|} \frac{1}{|k|^{N+1}} \int_{\mathbb{C}} \frac{1}{|\zeta|} \frac{1}{\left(|\zeta|+\lambda\right)^{N+1}} L(d\zeta),
$$

where $\lambda := |z - w| + 1/|k|$. Putting

$$
\zeta = \lambda \widetilde{\zeta}, \quad L(d\zeta) = \lambda^2 L(d\widetilde{\zeta}),
$$

gives the upper bound

$$
= \frac{\mathcal{O}(1)}{|z-w|} \frac{1}{|k|^{N+1}} \frac{\lambda^2}{\lambda^{N+2}} \int_{\mathbb{C}} \frac{1}{|\tilde{\zeta}|} \frac{1}{(|\tilde{\zeta}|+1)} \frac{L(d\zeta)}{M} = \mathcal{O}(1) \frac{1}{|k||z-w| \left(|k|\lambda\right)^N} = \frac{\mathcal{O}(1)}{|k||z-w|} \frac{1}{\left(|k||z-w|+1\right)^N} = \mathcal{O}(1) f_{k,N}(z-w).
$$

Here

$$
f_{k,N}(z) = \frac{1}{|k||z|(1+|k||z|)^N},
$$

and

$$
\int f_{k,N}(z)L(dz) = \int \frac{1}{|k||z|(1+|k||z|)^N} L(dz) = \frac{1}{|k|^2} \int \frac{1}{|\tilde{z}|(1+|\tilde{z}|)^N} L(d\tilde{z}) = \mathcal{O}(1)\frac{1}{|k|^2}.
$$

We deduce that K_1 is bounded by an L^1 convolution kernel, hence

$$
A_1 = \mathcal{O}(1)\frac{1}{|k|^2} : L^p \to L^p, \quad 1 \le p \le \infty.
$$

By [\(2.25\)](#page-10-3) we have $K_2(z, w)$, $K_3(z, w) = \frac{\mathcal{O}(1)}{|k||z-w|}$ and it follows (here we integrate over Ω in [\(2.32\)](#page-11-0)) that for every bounded set $V \subset \mathbb{C}$,

(2.33)
$$
1_V A_2, 1_V A_3 = \mathcal{O}(|k|^{-1}): L^p \to L^p, 1 \le p \le \infty.
$$

In fact, $1/|z|$ is integrable on every bounded set.

We next estimate the contribution to A_2 , A_3 from $|z| \gg 1$. For $j = 2, 3$ let $C_j = 1_{\mathbb{C}\backslash \text{neigh}} \overline{\Omega} A_j$. Then

$$
C_j u(z) = \int_{\Omega} \widetilde{K}_j(z, w) u(w) L(dw), \quad |\widetilde{K}_j(z, w)| \leq \frac{1}{\langle z \rangle |k|},
$$

 $|C_j u(z)| \leq \mathcal{O}(1)\langle z \rangle^{-1} |k|^{-1} ||u||_{L^1(\Omega)} \leq \mathcal{O}(1)\langle z \rangle^{-1} |k|^{-1} ||u||_{L^p(\Omega)}, \ 1 \leq p \leq \infty.$ Here

$$
\int \langle z \rangle^{-q} L(dz) < \infty \text{ if } q > 2,
$$

so

$$
C_j = \mathcal{O}(1/|k|) : L^p \to L^q \text{ if } 1 \le p \le \infty, \ q > 2.
$$

If $q \leq 2$, $\epsilon > 0$. Then,

$$
\|\langle \cdot \rangle^{-1-\epsilon}\|_{L^q}^q = \int \langle z \rangle^{-(1+\epsilon)q} L(dz) < \infty
$$

iff $(1+\epsilon)q > 2$, i.e. iff $\epsilon > 2/q-1$. When $q = 2$, this amounts to $\epsilon > 0$. We conclude that

$$
C_j = \mathcal{O}_{\epsilon}(1/|k|) : L^p \to \langle \cdot \rangle^{\epsilon} L^q \text{ when } 1 \le p \le \infty, \ 1 \le q \le 2, \ \epsilon > \frac{2}{q} - 1.
$$

Hence for $j = 2, 3$:

$$
A_j = \mathcal{O}(1/|k|) : \begin{cases} L^q \to L^q, \text{ when } q > 2, \\ L^q \to \langle \cdot \rangle^{\epsilon} L^q, \text{ when } 1 \le q \le 2, \epsilon > 2/q - 1. \end{cases}
$$

We next estimate A_4 , A_5 with kernels K_4 , K_5 in [\(2.31\)](#page-11-1). It suffices to treat A_4, K_4 since the expression for K_5 is very similar. Let

 $\Gamma_z = \{ \zeta \in \partial \Omega; |\zeta - z| \ge \frac{1}{2} |w - z| \}, \quad \Gamma_w = \{ \zeta \in \partial \Omega; |\zeta - w| \ge \frac{1}{2} |z - w| \},$ so that $\partial\Omega \subset \Gamma_z \cup \Gamma_w$. Then (2.34)

$$
\begin{split} |K_4(z,w)| &\leq \frac{\mathcal{O}(1)}{|k|}\left(\int_{\Gamma_z}\frac{|d\zeta|}{|z-\zeta||\zeta-w|}+\int_{\Gamma_w}\frac{|d\zeta|}{|z-\zeta||\zeta-w|}\right)\\ &\leq \frac{\mathcal{O}(1)}{|k|}\left(\frac{1}{|w-z|}\int_{\partial\Omega}\frac{|d\zeta|}{|\zeta-w|}+\frac{1}{|w-z|}\int_{\partial\Omega}\frac{|d\zeta|}{|z-\zeta|}\right)\\ &\leq \frac{\mathcal{O}(1)}{|k|}\left(\frac{1}{|w-z|}G(w)+\frac{1}{|w-z|}G(z)\right), \end{split}
$$

where

(2.35)
$$
G(z) = \begin{cases} 1 + |\ln d(\partial \Omega, z)|, & z \in \text{neigh}(\overline{\Omega}, \mathbb{C}), \\ 1/\langle z \rangle, & z \in \mathbb{C} \setminus \text{neigh}(\overline{\Omega}, \mathbb{C}), \end{cases}
$$

and $d(\partial\Omega, w)$ denotes the distance between $\partial\Omega$ and w.

 $|\cdot - z|^{-\alpha}$ and G^{β} are integrable on any bounded set when $\beta \geq 0$, $0 < \alpha < 2$. Choose $\alpha = 3/2$ and write $\frac{|G(w)|}{|w-z|}$ as the geometric mean $\begin{pmatrix} 1 \end{pmatrix}$ $|w-z|$ $\frac{3}{2}$) $\frac{3}{3}$ ($G(w)^3$) $\frac{1}{3}$. Using that geometric means are bounded by the arithmetic ones, we get

$$
\frac{1}{|w-z|}G(w) \le \frac{2}{3}\frac{1}{|w-z|^{3/2}} + \frac{1}{3}G(w)^3.
$$

Using this and the corresponding estimate with $G(z)$ in [\(2.34\)](#page-13-0) we get (2.36)

$$
1_{\text{neigh}(\overline{\Omega})}(z)|K_j(z,w)|1_{\Omega}(w) \leq \frac{\mathcal{O}(1)}{|k|} \frac{1}{|z-w|^{\frac{3}{2}}} + \frac{\mathcal{O}(1)}{|k|} G(w)^3 + \frac{\mathcal{O}(1)}{|k|} G(z)^3,
$$

when $j = 4$. Clearly the same estimate holds when $j = 5$. The first term is an L^1 -convolution kernel (neglecting a region $|z| \gg 1$ and recalling that $w \in \Omega$, gives rise to an operator

$$
\mathcal{O}(1/|k|): L^p(\Omega) \to L^p(\text{neigh }(\overline{\Omega})), \quad p \in [1, \infty]
$$

for any bounded neighborhood of $\overline{\Omega}$. By the Hölder inequality and the fact that $G^{\alpha} \in L^1$, $\forall \alpha > 0$, we see that the second term gives rise to an operator

$$
\mathcal{O}(1/|k|): L^p \to L^{\infty}, \ \forall p > 1.
$$

Similarly the third term gives rise to an operator

$$
\mathcal{O}(1/|k|): L^1(\Omega) \to L^q(\text{neigh }(\overline{\Omega})), \quad q \in [1, \infty[
$$

for any bounded neighborhood of $\overline{\Omega}$. Recalling again that we work on a bounded subset of $\mathbb C$ where $L^p \subset L^q$ for $1 \le q \le p \le \infty$ we conclude that

$$
(2.37) \quad 1_{\text{neigh}(\overline{\Omega})}A_4, \ 1_{\text{neigh}(\overline{\Omega})}A_5 = \mathcal{O}(1/|k|): L^p \to L^p, \quad 1 < p < \infty.
$$

For $z \in \mathbb{C} \setminus$ neigh $(\overline{\Omega})$, $w \in \Omega$, [\(2.34\)](#page-13-0) and [\(2.35\)](#page-13-1) give

$$
|K_4(z, w)| \le \frac{\mathcal{O}(1)}{|k|\langle z\rangle}(1+|\ln d(w, \partial\Omega)|).
$$

This is the same estimate as for K_2, K_3 except that the $1_{\Omega}(w)$, belonging to all $L^{p'}$ with $1 \leq p' \leq +\infty$, is replaced by $(1 + |\ln(w, \partial \Omega)|)1_{\Omega}(w)$ belonging to all $L^{p'}$ with $1 \leq p' < +\infty$. The estimates for $1_{\mathbb{C}\setminus \text{neigh}}(\overline{\Omega})A_j$, $j = 2, 3$ extend to $1_{\mathbb{C}\backslash \text{neigh}} \overline{\Omega} A_4$, $1_{\mathbb{C}\backslash \text{neigh}} \overline{\Omega} A_5 : L^p \to L^q$, for $1 \leq p \leq$ ∞ : For $j = 4, 5$,

$$
1_{\mathbb{C}\backslash\mathrm{neigh}\,(\overline{\Omega})}A_j=\mathcal{O}(1/|k|):\begin{cases} L^p\to L^q,\quad 12,\\ L^p\to \langle\cdot\rangle^\epsilon L^q,\ 1\frac{2}{q}-1.\end{cases}
$$

Hence with [\(2.37\)](#page-14-1)

$$
A_j=\mathcal O(1/|k|):\begin{cases}L^q\to L^q, &2\frac{2}{q}-1.\end{cases}
$$

Combining the estimates for A_j , $1 \le j \le 5$, we get

Theorem 2.2.

(2.38)
$$
AB = \mathcal{O}(1/|k|) : \begin{cases} L^q \to L^q, & 2 < q < +\infty, \\ L^q \to \langle \cdot \rangle^{\epsilon} L^q, & 1 < q \le 2, \epsilon > \frac{2}{q} - 1. \end{cases}
$$

3. Back to the reflection coefficient

Recall [\(1.3\)](#page-1-2)

(3.1)
$$
\overline{R} = \frac{2\sigma}{\pi} \int_{\mathbb{C}} e^{kz - \overline{k}z} \overline{q}(z) \phi_1(z; k) L(\mathrm{d}z),
$$

where $\phi_1 = 1 + \phi_1^1$ and we assume $q = 1_\Omega$ where $\Omega \Subset \mathbb{C}$ is open with strictly convex smooth boundary. ϕ_1^1 is given by [\(1.12\)](#page-4-5),

.

(3.2)
$$
\phi_1^1 = (1 - AB)^{-1}AB(1) = AB(1) + (AB)^2(1) + \dots
$$

and A, B are given in [\(1.8\)](#page-3-1), now with $q = 1_{\Omega}$

(3.3)
$$
\begin{cases} A = E\widehat{\tau}_{-\omega} \frac{h}{2} 1_{\Omega}, \\ B = \sigma F \widehat{\tau}_{\omega} \frac{h}{2} 1_{\Omega} \ (= \sigma \overline{A}) \end{cases}
$$

By Theorem [2.2](#page-14-2) we know that $1_{\Omega}AB = \mathcal{O}(1/|k|) : L^2(\Omega) \to L^2(\Omega)$ and we shall frequently use that $A = A \circ 1_{\Omega}$, $B = B \circ 1_{\Omega}$. Combining $(3.1), (3.2),$ $(3.1), (3.2),$ $(3.1), (3.2),$ $(3.1), (3.2),$ we get (assuming $\sigma = 1$ for simplicity)

(3.4)
$$
\overline{R} = \frac{2}{\pi} \sum_{\nu=0}^{N-1} \int_{\Omega} e^{kz - \overline{k}z} (AB)^{\nu} (1) L(dz) + \mathcal{O}(|k|^{-N}),
$$

for every $N = 1, 2, \dots$ Here

$$
\left| \int_{\Omega} e^{kz - \overline{kz}} (AB)^{\nu} (1) L(dz) \right| \le \text{vol}(\Omega)^{1/2} ||(AB)^{\nu} (1)||_{L^2(\Omega)} \le C(C/|k|)^{\nu}
$$

and we shall see that this estimate can be improved by using more information about $A = A_k$ from Section 5 in [\[18\]](#page-28-0) in the case when $\partial\Omega$ is analytic. In remark [3.3](#page-22-2) we explain how to extend the discussion to the case when $\partial\Omega$ is merely smooth.

First, recall from [\(3.3\)](#page-14-5) and the explicit formula for the fundamental solution of $\overline{\partial}$ appearing in E, that

(3.5)
$$
Au(z) = \frac{1}{2\pi} \int_{\Omega} e^{-kw + \overline{kw}} \frac{1}{z - w} u(w) L(dw),
$$

or

(3.6)
$$
A = A_k = A_0 \circ e^{-k \cdot + k},
$$

where

(3.7)
$$
A_0 u(z) = \frac{1}{2\pi} \int_{\Omega} \frac{1}{z - w} u(w) L(dw).
$$

 A_0 is anti-symmetric for the standard bilinear scalar product on L^2 ; $A_0^{\text{t}} = -A_0$, so the transpose of A_k is given by

(3.8)
$$
A_k^{\dagger} = -e^{-k \cdot + \overline{k} \cdot} A_0 = -e^{-k \cdot + \overline{k} \cdot} A_{-k} e^{-k \cdot + \overline{k} \cdot},
$$

and B_k is the complex conjugate of A_k (here $\sigma = 1$ for simplicity):

$$
B_k = \overline{A_k}.
$$

In [\[18\]](#page-28-0) we studied the function

(3.9)
$$
f(z,k) = 2\pi A_k(1_\Omega)(z) = \int_{\Omega} e^{-kw + \overline{kw}} \frac{1}{z - w} L(dw),
$$

when $\partial\Omega$ is analytic.

By (5.32), (5.31) in [\[18\]](#page-28-0), we have (3.10) $f(z, k) = \frac{1}{2i}$ $2ik$ $F(z)+(\pi/\overline{k})\left(e^{-iu(z,k)}(1_{\Omega_{-}}(z)-1_{\Omega_{+}}(z))+e^{-kz+\overline{kz}}1_{\Omega}(z)\right),$

where

(3.11)
$$
F(z) = F_{\Gamma}(z) = \int_{\Gamma} \frac{1}{z - w} e^{-iu(w,k)} dw.
$$

Here the deformation Γ of $\partial\Omega$ and the domains Ω_{\pm} are defined in [\[18,](#page-28-0) Section 5], see Fig. [1.](#page-16-0) Further $-iu(\cdot, k)$ is the holomorphic extension of $-k \cdot +\overline{k}$ from $\partial\Omega$ to a neighborhood.

FIGURE 1. Real analytic strictly convex boundary $\partial\Omega$ (solid) of some domain Ω and the deformed contour Γ (dashed) for this example.

In (5.69), (5.70) in [\[18\]](#page-28-0) we have seen that (3.12) $F(z) = \mathcal{O}(1)(|z - w_+(k)||k|^{1/2} + 1)^{-1} + \mathcal{O}(1)(|z - w_-(k)||k|^{1/2} + 1)^{-1},$

where $w_+(k)$, $w_-(k) \in \partial\Omega$ are the North and South poles, determined by the fact that the interior unit normal of $\partial\Omega$ at $w_{\pm}(k)$ is of the form $c_{\pm}\omega$ for some c_{\pm} with $\pm c_{\pm} > 0$. Here $\omega \in \mathbb{R}^2 \simeq \mathbb{C}$ is determined by $kz - \overline{kz} = i|k|\Re(z\overline{\omega})$ for all $z \in z$.

We deduced that

(3.13)
$$
||F||_{L^{2}(\Omega)} = \mathcal{O}(1) \frac{(\ln |k|)^{1/2}}{|k|^{1/2}},
$$

(3.14)
$$
||F/(2i\overline{k})||_{L^{2}(\Omega)} = \mathcal{O}(1)\frac{(\ln |k|)^{1/2}}{|k|^{3/2}},
$$

and using [\(3.12\)](#page-16-1) we now add the observation that

(3.15)
$$
||F/(2i\overline{k})||_{L^{1}(\Omega)} = \mathcal{O}(1)\frac{1}{|k|^{3/2}}.
$$

We next estimate $(\pi/\overline{k})e^{-iu(\cdot,k)}1_{\Omega_{+}}$, appearing in [\(3.10\)](#page-15-0). (Notice that $e^{-iu(\cdot,k)}1_{\Omega_{-}}$ is absent, since we restrict the attention to Ω and $\Omega_{-} \cap \Omega =$ \emptyset .) In (5.75) in [\[18\]](#page-28-0) we found that

(3.16)
$$
\left\| (\pi/\overline{k}) e^{-iu(\cdot,k)} 1_{\Omega_+} \right\|_{L^2(\Omega)} = \frac{\mathcal{O}(1)(\ln |k|)^{1/2}}{|k|^{3/2}},
$$

and we shall now apply the same procedure to the L^1 -norm. We restrict the attention to the contribution to the L^1 -norm from a neighborhood of one of the poles, say $w_{+}(k)$. (Away from such neighborhoods, the estimates are simpler and lead to a stronger conclusion.) In suitable coordinates $\mu = t + is$ we have

$$
(\pi/\overline{k})e^{-iu(\cdot,k)}1_{\Omega_+} = \mathcal{O}(k^{-1})e^{-|k|ts/C}, \ 0 \le t \le 1/\mathcal{O}(1), \ 0 \le s \le t.
$$

(Away from a neighborhood of $\{w_+(k), w_-(k)\}\$ we have the same estimate, now for $1/\mathcal{O}(1) \le t \le \mathcal{O}(1)$, $0 \le s \le 1/\mathcal{O}(1)$.) The contribution to the L¹-norm of $(\pi/\overline{k})e^{-iu(\cdot,k)}1_{\Omega_+}$ is

$$
\mathcal{O}(|k|^{-1}) \int_0^1 \int_0^t e^{-ts|k|/C} ds dt = \mathcal{O}(|k|^{-1}) \int_0^1 \frac{1}{t|k|} \left(1 - e^{-t^2|k|/C}\right) dt
$$

=
$$
\frac{\mathcal{O}(1)}{|k|} \left(\int_0^{|k|^{-1/2}} t dt + \int_{|k|^{-1/2}}^1 \frac{1}{t|k|} dt \right) = \frac{\mathcal{O}(1) \ln |k|}{|k|^2}.
$$

The contribution from $\overline{\Omega} \neq \{w_+, w_-\}$ to the L¹-norm is $\mathcal{O}(1)|k|^{-2}$. Hence

(3.17)
$$
\|(\pi/\overline{k})e^{-iu(\cdot,k)}1_{\Omega_+}\|_{L^1(\Omega)} = \frac{\mathcal{O}(1)\ln|k|}{|k|^2}.
$$

Combining [\(3.10\)](#page-15-0), [\(3.14\)](#page-17-0), [\(3.16\)](#page-17-1), we get as in [\[18\]](#page-28-0) that

(3.18)
$$
A_k(1_\Omega) = \frac{1}{2\overline{k}}e^{-kz+\overline{k}z}1_\Omega + r(z,k),
$$

i.e.,

(3.19)
$$
r = \frac{F}{4\pi i\overline{k}} - \frac{1}{2\overline{k}}e^{-iu}1_{\Omega_+} \text{ in } \Omega,
$$

where

(3.20)
$$
r(\cdot, k) = \mathcal{O}(1)|k|^{-3/2}(\ln |k|)^{1/2} \text{ in } L^2(\Omega).
$$

Using [\(3.15\)](#page-17-2), [\(3.17\)](#page-17-3) instead of [\(3.14\)](#page-17-0), [\(3.16\)](#page-17-1), we get

(3.21)
$$
||r(\cdot,k)||_{L^1(\Omega)} = \mathcal{O}(1)|k|^{-3/2}.
$$

We now return to the expansion [\(3.4\)](#page-15-1) for \overline{R} , and start with the term for $\nu = 1$. Let

$$
\langle u|v\rangle = \int_{\Omega} u(z)v(z)L(dz)
$$

denote the bilinear L^2 scalar product. We get with $A = A_k$, $B = B_k$ if nothing else is indicated,

$$
\frac{2}{\pi} \int_{\Omega} e^{kz - \overline{kz}} AB(1_{\Omega})(z) L(dz) = \frac{2}{\pi} \langle AB(1_{\Omega})|e^{k - \overline{k}} 1_{\Omega} \rangle
$$

$$
= \frac{2}{\pi} \langle B(1_{\Omega})|A^{\dagger}e^{k - \overline{k}} (1_{\Omega}) \rangle = -\frac{2}{\pi} \langle B_k(1_{\Omega})|e^{-k + \overline{k}} A_{-k}(1_{\Omega}) \rangle,
$$

where we used [\(3.8\)](#page-15-2) in the last step. Applying [\(3.18\)](#page-17-4) to $A_{-k}(1_{\Omega})$ and the fact that $B_k(1_\Omega) = \overline{A_k(1_\Omega)}$, we get

$$
(3.22) \frac{2}{\pi} \int_{\Omega} e^{kz - \overline{kz}} AB(1_{\Omega})(z) L(dz)
$$

= $-\frac{2}{\pi} \langle \frac{1}{2k} e^{-\overline{k} + k \cdot 1} 1_{\Omega} + \overline{r(\cdot, k)} | e^{-k + \overline{k} \cdot 1} (-\frac{1}{2\overline{k}} e^{k - \overline{k} \cdot 1} 1_{\Omega} + r(\cdot, -k)) \rangle$
= $\frac{2}{\pi} \frac{1}{4|k|^2} \int_{\Omega} e^{kz - \overline{kz}} L(dz) - \frac{2}{\pi} \int_{\Omega} \frac{1}{2k} r(z, -k) L(dz)$
+ $\frac{2}{\pi} \int_{\Omega} \overline{r(z, k)} \frac{1}{2\overline{k}} L(dz) - \frac{2}{\pi} \int_{\Omega} e^{-kz + \overline{kz}} \overline{r(z, k)} r(z, -k) L(dz).$

As we have already seen, the integral in the first term in the last member is $\mathcal{O}(|k|^{-3/2})$, so this term is $\mathcal{O}(|k|^{-7/2})$. By [\(3.20\)](#page-17-5) the last term in [\(3.22\)](#page-18-0) is $\mathcal{O}(|k|^{-3} \ln |k|)$. Thus (3.22) gives

(3.23)
\n
$$
\frac{2}{\pi} \int_{\Omega} e^{kz - \overline{k}z} AB(1_{\Omega})(z) L(dz)
$$
\n
$$
= -\frac{2}{\pi} \int_{\Omega} \frac{1}{2k} r(z, -k) L(dz) + \frac{2}{\pi} \int_{\Omega} \overline{r(z, k)} \frac{1}{2\overline{k}} L(dz) + \mathcal{O}(|k|^{-3} \ln |k|).
$$
\n(3.23)

[\(3.21\)](#page-17-6) now yields

(3.24)
$$
\frac{2}{\pi} \int_{\Omega} e^{kz - \overline{kz}} AB(1_{\Omega})(z) L(dz) = \mathcal{O}(|k|^{-5/2}).
$$

Before studying the leading asymptotics of the integrals in the left hand side of (3.23) , we shall gain a power of k in the estimate of the general term in [\(3.4\)](#page-15-1) for $\nu \geq 2$:

$$
(3.25)
$$

\n
$$
\frac{2}{\pi} \int_{\Omega} e^{kz - \overline{kz}} (AB)^{\nu} (1_{\Omega})(z) L(dz) = \frac{2}{\pi} \int_{\Omega} e^{kz - \overline{kz}} A(BA)^{\nu - 1} B(1_{\Omega})(z) L(dz)
$$

\n
$$
= -\frac{2}{\pi} \langle (BA)^{\nu - 1} B(1_{\Omega}) | e^{-k + \overline{k} \cdot} A_{-k}(1_{\Omega}) \rangle = \mathcal{O}(1) |k|^{1 - \nu} |k|^{-1} |k|^{-1}
$$

\n
$$
= \mathcal{O}(|k|^{-\nu - 1}) = \mathcal{O}(|k|^{-3}),
$$

since $(BA)^{\nu-1} = \mathcal{O}(|k|^{1-\nu}) : L^2(\Omega) \to L^2(\Omega)$ and $B(1_{\Omega}), A_{-k}(1_{\Omega}) =$ $\mathcal{O}(1/|k|)$ in $L^2(\Omega)$. Combining [\(3.4\)](#page-15-1), [\(3.24\)](#page-18-2), [\(3.25\)](#page-18-3), we get

$$
(3.26)\quad \overline{R} = \frac{2}{\pi} \int_{\Omega} e^{kz - \overline{k}z} L(dz) + \frac{2}{\pi} \int_{\Omega} e^{kz - \overline{k}z} AB(1_{\Omega}) L(dz) + \mathcal{O}(|k|^{-3}),
$$

and in particular,

(3.27)
$$
\overline{R} = \frac{2}{\pi} \int_{\Omega} e^{kz - \overline{kz}} L(dz) + \mathcal{O}(|k|^{-5/2}).
$$

We next study the second term in the right hand side of [\(3.26\)](#page-19-0), starting from (3.23) . By (3.17) and (3.19) we have

(3.28)
$$
r = \frac{F}{4\pi i\overline{k}} + \frac{\mathcal{O}(1)\ln|k|}{|k|^2} \text{ in } L^1(\Omega),
$$

where $F = F(z, k)$. Using this in [\(3.23\)](#page-18-1), we get

$$
(3.29) \frac{2}{\pi} \int_{\Omega} e^{kz - \overline{kz}} AB(1_{\Omega})(z) L(dz) =
$$

$$
\frac{2}{\pi} \int_{\Omega} \frac{1}{2k} \frac{F(z, -k)}{4i\pi \overline{k}} L(dz) + \frac{2}{\pi} \int_{\Omega} \frac{\overline{F}(z, k)}{-4i\pi k} \frac{1}{2\overline{k}} L(dz) + \mathcal{O}(|k|^{-3} \ln |k|)
$$

$$
= \frac{1}{4i\pi^2 |k|^2} \int_{\Omega} (F(z, -k) - \overline{F}(z, k)) L(dz) + \mathcal{O}(|k|^{-3} \ln |k|).
$$

Here we recall [\(3.11\)](#page-15-3) for $F(z, k) = F_{\Gamma}(z)$, where $\Gamma = \Gamma(k)$ is a deformation of $\partial\Omega$ passing through the poles $w_+(k)$, $w_-(k)$, situated outside $\overline{\Omega}$ near the boundary segment Γ_{-} from $w_{+}(k)$ to $w_{-}(k)$ and inside Ω near the boundary segment Γ_+ from $w_-(k)$ to $w_+(k)$ (when following the boundary with the positive orientation). This choice is given by the method of steepest descent for $e^{-iu(x)}$. When replacing k with $-k$, we have $e^{-iu(z,-k)} = e^{iu(z,k)}$ and $w_{\pm}(-k) = w_{\mp}(k)$. Correspondingly, Γ should be replaced by a contour Γ which is a deformation of $\partial\Omega$ inwards near the segment $\Gamma_$ from $w_+(k)$ to $w_-(k)$ and outwards near the segment Γ_+ from $w_-(k)$ to $w_+(k)$.

We get

(3.30)
$$
\int_{\Omega} F(z, -k)L(dz) = \int_{\Omega} \int_{\tilde{\Gamma}} \frac{1}{z - w} e^{iu(w, k)} dw L(dz)
$$

$$
= \int_{\tilde{\Gamma}} \int_{\Omega} \frac{1}{z - w} L(dz) e^{iu(w, k)} dw = -\pi \int_{\tilde{\Gamma}} D_{\Omega}(w) e^{iu(w, k)} dw,
$$

using that $\frac{1}{z-w}e^{iu(w,k)}$ is integrable on $\Omega \times \tilde{\Gamma}$ for the measure $L(dz)|dw|$. Here

(3.31)
$$
D_{\Omega}(z) = \frac{1}{\pi} \int_{\Omega} \frac{1}{z - w} L(dw)
$$

is the solution to the $\overline{\partial} \text{-problem:}$

(3.32)
$$
\begin{cases} \partial_{\overline{z}} D_{\Omega} = 1_{\Omega}, \\ D_{\Omega}(z) \to 0, \ z \to \infty. \end{cases}
$$

Similarly,

(3.33)
$$
- \int_{\Omega} \overline{F(z,k)} L(dz) = \overline{\int_{\Omega} \int_{\Gamma} \frac{1}{w - z} e^{-iu(w,k)} dw L(dz)} = \overline{\int_{\Gamma} \int_{\Omega} \frac{1}{w - z} L(dz) e^{-iu(w,k)} dw} = \pi \overline{\int_{\Gamma} D_{\Omega}(w) e^{-iu(w,k)} dw}.
$$

Using [\(3.30\)](#page-19-1), [\(3.33\)](#page-20-0) in [\(3.29\)](#page-19-2), we get

$$
(3.34) \frac{2}{\pi} \int_{\Omega} e^{kz - \overline{kz}} AB(1_{\Omega})(z) L(dz) =
$$

$$
\frac{1}{4i\pi |k|^2} \left(- \int_{\widetilde{\Gamma}} D_{\Omega}(w) e^{iu(w,k)} dw + \overline{\int_{\Gamma} D_{\Omega}(w) e^{-iu(w,k)} dw} \right) + \mathcal{O}(|k|^{-3} \ln |k|).
$$

Remark 3.1. It is not obvious whether the error term in (3.34) is optimal or a consequence of the applied technique to prove the result.

Proposition 3.2. $D_{\Omega} \in C(\mathbb{C})$ and the restrictions of this function to the open sets Ω and $\mathbb{C}\setminus\overline{\Omega}$ extend to functions in $C^{\infty}(\overline{\Omega})$ and $C^{\infty}(\mathbf{C}\setminus\Omega)$ respectively.

Proof. The continuity of D_{Ω} is clear. We first look at $D_{\Omega}(z)$ in $\mathbb{C}\setminus\overline{\Omega}$. Here $D_{\Omega}(z)$ is holomorphic and

$$
(3.35)
$$

$$
\partial_z D_{\Omega}(z) = \frac{1}{\pi} \int_{\Omega} \partial_z \left(\frac{1}{z - w} \right) L(dw)
$$

=
$$
- \frac{1}{\pi} \int_{\Omega} \partial_w \left(\frac{1}{z - w} \right) \frac{d\overline{w} \wedge dw}{2i} = \frac{1}{2\pi i} \int_{\Omega} d_w \left(\frac{1}{z - w} d\overline{w} \right)
$$

By Stokes' formula,

(3.36)
$$
\partial_z D_{\Omega}(z) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{1}{z - w} d\overline{w} = \frac{1}{2\pi i} \int_0^L \frac{1}{z - \gamma(t)} \overline{\dot{\gamma}(t)} dt
$$

where $\gamma : [0, L] \ni t \mapsto \gamma(t) \in \partial \Omega$ is a smooth positively oriented parametrization of $\partial\Omega$.

It follows that

(3.37)
$$
\partial_z D_{\Omega}(z) = \mathcal{O}(1) |\ln d(z, \partial \Omega)|, \ z \in (\mathbb{C} \setminus \overline{\Omega}) \cap \mathrm{neigh}(\partial \Omega),
$$

where $d(z, \partial \Omega)$ denotes the distance from z to $\partial \Omega$.

.

For $n = 1, 2, \dots$, we apply ∂_z^n to [\(3.36\)](#page-20-2):

$$
(3.38) \quad \partial_z^{n+1} D_{\Omega}(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \partial_z^n \frac{1}{z - w} d\overline{w}
$$

=
$$
\frac{1}{2\pi i} \int_{\partial\Omega} (-\partial_w)^n \frac{1}{z - w} d\overline{w} = \frac{1}{2\pi i} \int_0^L (-\dot{\gamma}^{-1} \partial_t)^n \left(\frac{1}{z - \gamma(t)}\right) \overline{\gamma} dt
$$

=
$$
\frac{1}{2\pi i} \int_0^L \frac{1}{z - \gamma(t)} \underbrace{(\partial_t \circ \dot{\gamma}^{-1})^n}_{\mathcal{O}(1)} (\overline{\gamma}) dt = \mathcal{O}(1) |\ln d(z, \partial\Omega)|,
$$

still for $z \in \text{neigh } (\partial \Omega) \setminus \overline{\Omega}$. Integrating these estimates, we see that $\partial_z^n D_{\Omega}(z)$ is bounded on $\mathbb{C} \setminus \overline{\Omega}$ for $n \in \Omega$ (which we already knew to be valid away from a neighborhood of $\partial\Omega$) and hence that $D_{\Omega} \in C^{\infty}(\mathbb{C}\backslash\Omega)$.

If instead of D_{Ω} we look at

(3.39)
$$
D_{\Omega,\psi}(z) = \frac{1}{\pi} \int_{\Omega} \frac{1}{z - w} \psi(w) L(dw)
$$

for some $\psi \in C^{\infty}(\overline{\Omega})$ with $\psi = 1$ near $\partial \Omega$, we still have $D_{\Omega, \psi} \in C^{\infty}(\mathbb{C} \setminus \overline{\Omega})$ Ω). Indeed, we get very much as in (3.35) ,

$$
\partial_z D_{\Omega,\psi}(z) = \frac{1}{\pi} \int_{\Omega} \partial_z \left(\frac{1}{z - w} \right) \psi(w) L(dw)
$$

= $-\frac{1}{\pi} \iint_{\Omega} \partial_w \left(\frac{1}{z - w} \right) \psi(w) \frac{d\overline{w} \wedge dw}{2i}$
= $\frac{1}{2\pi i} \int_{\Omega} d_w \left(\frac{\psi(w)}{z - w} d\overline{w} \right) + \frac{1}{\pi} \iint_{\Omega} \frac{\partial_w \psi}{z - w} \frac{d\overline{w} \wedge dw}{2i},$

where the last integral belongs to $C^{\infty}(\mathbb{C}\backslash\Omega)$ and the second last integral is equal to $\frac{1}{2\pi i} \int_{\partial \Omega}$ 1 $\frac{1}{z-w}d\bar{w}$ and is also in $C^{\infty}(\mathbb{C}\setminus\Omega)$ as we have just seen.

We finally show that $D_{\Omega}(z)$, $z \in \Omega$, extends to a function in $C^{\infty}(\Omega)$. Let $\chi \in C_0^{\infty}(\mathbb{C})$ be equal to 1 near $\overline{\Omega}$. Then $f(z) = \frac{1}{\pi} \int_{\mathbb{C}} (z-w)^{-1} \chi(w) L(dw)$ solves $\partial_{\overline{z}} f = \chi$ and belongs to $C^{\infty}(\mathbb{C})$. For $z \in \Omega$ we have

$$
D_{\Omega}(z) = f(z) - \frac{1}{\pi} \int_{\mathbb{C} \setminus \Omega} \frac{1}{z - w} \chi(w) L(dw).
$$

As in the remark above about $D_{\Omega,\psi}$ in $\mathbb{C}\backslash\overline{\Omega}$ we see that $\frac{1}{\pi}\int$ $\mathbb{C}\backslash\Omega \ \frac{1}{z-1}$ $\frac{1}{z-w}\chi(w)L(dw)$ belongs to $C^{\infty}(\Omega)$. Hence $D_{\Omega}(z)$ extends from Ω to a smooth function on $\overline{\Omega}$.

This completes the proof of Theorem [1.4](#page-5-1) in the case when $\partial\Omega$ is analytic.

The exponentials in the integrals in [\(3.34\)](#page-20-1) behave like Gaussians peaked at $w_{\pm}(k)$, and hence the integrals are $\mathcal{O}(|k|^{-1/2})$ and the contribution from outside any fixed neighborhood of $\{w_+(k), w_-(k)\}\$ is exponentially small. The proposition implies that $D_{\Omega}(w)$ is a Lipschitz function and therefore we modify the integrals by $\mathcal{O}(|k|^{-1})$ only, if we

replace $D_{\Omega}(w)$ in a neighborhood of $w_{\pm}(k)$ by $D_{\Omega}(w_{\pm}(k))$. Thus for instance

$$
(3.40) \quad \int_{\tilde{\Gamma}} D_{\Omega}(w)e^{iu(w,k)}dw = D_{\Omega}(w_{+}(k)) \int_{\tilde{\Gamma} \cap \text{neigh}(w_{+}(k))} e^{iu(w,k)}dw + D_{\Omega}(w_{-}(k)) \int_{\tilde{\Gamma} \cap \text{neigh}(w_{-}(k))} e^{iu(w,k)}dw + \mathcal{O}(k^{-1}).
$$

Remark 3.3. We now drop the analyticity assumption and assume that $\Omega \in \mathbb{C}$ is open with smooth boundary and strictly convex. We have used the analyticity assumption in (3.10) , (3.11) , where iu is the holomorphic extension to a neighbourhood of $\partial\Omega$ of the function kw – kw, $w \in \partial \Omega$. In the merely smooth case we let $iu(w, k)$ denote an almost holomorphic extension of $\partial\Omega \ni w \mapsto kw - \overline{kw}$ and define F as in [\(3.11\)](#page-15-3), using the modified function u. Stokes' formula now produces a small error term to be added to (3.11) and as in (2.18) in [\[19\]](#page-28-5) and the subsequent discussion, we get

$$
f(z,k) = \frac{1}{2i\bar{k}}F(z) + \frac{\pi}{\bar{k}} \left(e^{-iu(z,k)} (1_{\Omega_{-}}(z) - 1_{\Omega_{+}}(z)) + e^{-kz + \overline{kz}} 1_{\Omega}(z) \right) + \mathcal{O}(\langle z \rangle^{-1} |k|^{-\infty}).
$$

The discussion after (3.11) goes through with only a minor change: In the formula [\(3.19\)](#page-17-7) for r we have to add a remainder $\mathcal{O}(|k|^{-\infty})$. But this does not affect the subsequent estimates, and we get Theorem [1.4](#page-5-1) also in the more general case of a smooth boundary.

4. Stationary phase approximation

To compute the leading orders in $1/|k|$ of the reflection coefficient, we apply a standard stationary phase approximation. Since higher order terms in this approximation are needed here, we briefly summarize some facts on the approach.

4.1. Two term stationary phase expansion. We have in mind $\int_{\partial\Omega}e^{-kz+\overline{kz}}$ after a change of contour from $\partial\Omega$ to Γ and choosing a parametrisation. Let $\varphi \in C^{\infty}(\text{neigh}(0, \mathbb{R}))$ satisfy $\varphi(0) = 0$, $\varphi'(0) = 0$, $\Re \varphi''(0) > 0, a \in C^\infty(\text{neigh}(0, \mathbb{R}))$. Consider

$$
I(\varphi, a; h) = h^{-\frac{1}{2}} \int_V e^{-\varphi(t)/h} a(t) dt, \quad V = \text{neigh}(0, \mathbb{R}).
$$

We already know that $I \sim I_0 + I_1 h + \ldots$, where $I_0 = \frac{\sqrt{a}}{\sqrt{a}}$ $\frac{\sqrt{2\pi}}{2\sqrt{2}}$ $\frac{\sqrt{2\pi}}{\varphi''(0)}$ with the natural choice of the branch of the square root and our problem is to compute I_1h . Write $\varphi(t) = \varphi_2 t^2 + \psi(t)$, where $\psi(t) = \mathcal{O}(t^3)$. Put

$$
\varphi(t;s) = \varphi_2 t^2 + s\psi, \quad 0 \le s \le 1.
$$

Clearly $I(\varphi, a; h)$ is smooth in s with $\partial_s^k I(\varphi_s, a; h) = \mathcal{O}(h^{k/2}), k =$ $0, 1, 2, \ldots$, hence by a limited Taylor expansion

$$
I(\varphi, a; h) = I(\varphi, a; h)|_{s=0} + (\partial_s)I(\varphi, a; h)|_{s=0} + \frac{1}{2}(\partial_s^2)I(\varphi, a; h)|_{s=0} + \mathcal{O}(h^{3/2})
$$
\n(4.1)

$$
= I(\varphi_2 t^2, a; h) - I(\varphi_2 t^2, a\psi/h; h) + \frac{1}{2}I(\varphi_2 t^2, a(\psi/h)^2; h) + \mathcal{O}(h^{3/2}).
$$

Indeed

(4.2)
$$
I(\varphi, \mathcal{O}(t^n); h) = \mathcal{O}(h^{\frac{n}{2}}).
$$

Using again [\(4.2\)](#page-23-1) we can replace a, $a \frac{\Psi}{h}$ $\frac{\Psi}{h}$ and $a(\frac{\Psi}{h})$ $(\frac{\Psi}{h})^2$ by their limited Taylor sums modulo $\mathcal{O}(t^3)$, $\mathcal{O}(\frac{t^5}{h})$ $\frac{t^5}{h}$) and $\mathcal{O}(\frac{t^7}{h^2})$ respectively.

Writing $\psi \sim \varphi_3 t^3 + \varphi_4 t^4 + \ldots, \quad a \sim a_0 + a_1 t + a_2 t^2 + \ldots$ and using also that $I(\varphi_2 t^2, t^n; h) = 0$ when n is odd (up to an exponentially small error if V is not symmetric around $t = 0$, we get

$$
I(\varphi_2 t^2, a; h) = I(\varphi_2 t^2, a_0, h) + I(\varphi_2 t^2, a_2 t^2, h) + \mathcal{O}(h^2)
$$

= $a_0 I(\varphi_2 t^2, 1; 1) + ha_2 I(\varphi_2 t^2, t^2; 1) + \mathcal{O}(h^2),$

and

$$
I(\varphi_2 t^2, \frac{a\psi}{h}; h) = I(\varphi_2 t^2, (a_0 + a_1 t) \frac{\varphi_3 t^3 + \varphi_4 t^4}{h}; h) + \mathcal{O}(h^{3/2})
$$

= $I(\varphi_2 t^2, \frac{(a_0 \varphi_4 + a_1 \varphi_3)t^4}{h}; h) + \mathcal{O}(h^{3/2})$
= $h(a_0 \varphi_4 + a_1 \varphi_3)I(\varphi_2 t^2, t^4; 1) + \mathcal{O}(h^{3/2}),$

as well as

$$
I(\varphi_2 t^2, a(\frac{\psi}{h})^2; h) = I(\varphi_2 t^2, \frac{a_0 \varphi_3^2 t^6}{h^2}, h) + \mathcal{O}(h^{3/2})
$$

= $ha_0 \varphi_3^2 I(\varphi_2 t^2, t^6; 1) + \mathcal{O}(h^{3/2}).$

We now have integrals with quadratic exponent and polynomial amplitudes and up to exponentially small corrections from now on, we integrate over $\mathbb R$ instead of V .

4.2. Reduction to the case of an exact quadratic. Now we consider the reduction to the case of a quadratic exponential,

$$
I(\varphi_0, t^{2n}; 1) = \int e^{-\varphi_2 t^2} t^{2n} dt.
$$

Reparametrise $t^2 = \frac{\tau^2}{2\omega}$ $\frac{\tau^2}{2\varphi_2},\,t=\frac{1}{(2\varphi_2)}$ $\frac{1}{(2\varphi_2)^{1/2}}\tau$, and $dt = \frac{1}{\sqrt{2a}}$ $\frac{1}{2\varphi_2}d\tau=\frac{1}{\sqrt{\varphi^{\prime}}}$ $\frac{1}{\varphi''(0)}d\tau$

$$
I(\varphi_2 t^2, t^{2n}; 1) = (2\varphi_2)^{-n-\frac{1}{2}} I(\frac{\tau^2}{2}; \tau^{2n}; 1) = (\varphi''(0))^{-n-\frac{1}{2}} I(\frac{\tau^2}{2}, \tau^{2n}; 1).
$$

Integrate by parts when $n \geq 1$:

$$
I(\tau^2/2, \tau^{2n}; 1) = \int_{\mathbb{R}} e^{-\tau^2/2} \tau^{2n} d\tau = -\int_{\mathbb{R}} \partial_\tau (e^{-\tau^2/2}) \tau^{2n-1} d\tau
$$

=
$$
\int e^{-\tau^2/2} (2n-1) \tau^{2(n-1)} d\tau = (2n-1)I(\tau^2/2, \tau^{2(n-1)}; 1).
$$

In particular

$$
I(\tau^2/2, \tau^2; 1) = I(\tau^2/2, 1; 1) = \sqrt{2\pi},
$$

\n
$$
I(\tau^2/2, \tau^4; 1) = 3I(\tau^2/2, 1; 1) = 3\sqrt{2\pi},
$$

\n
$$
I(\tau^2/2, \tau^6; 1) = 15I(\tau^2/2, 1; 1) = 15\sqrt{2\pi}.
$$

We combine the different identities:

(4.3)
\n
$$
I(\varphi, a; h) = I(\varphi_2 t^2, a; h) - I(\varphi_2 t^2, a \frac{\psi}{h}; h) + \frac{1}{2} I(\varphi_2 t^2, a(\frac{\psi}{h})^2; h) + \mathcal{O}(h^{\frac{3}{2}})
$$
\n
$$
= a_0 I(\varphi_2 t^2, 1; 1) + h a_2 I(\varphi_2 t^2, t^2; 1) - h(a_0 \varphi_4 + a_1 \varphi_3) I(\varphi_2 t^2, t^4; 1)
$$
\n
$$
+ \frac{1}{2} h a_0 \varphi_3^2 I(\varphi_2 t^2, t^6; 1) + \mathcal{O}(h^{\frac{3}{2}})
$$
\n
$$
= \sqrt{2\pi} a(0) \varphi''(0)^{-\frac{1}{2}} + \sqrt{2\pi} h \left[\frac{1}{2} a''(0) \varphi''(0)^{-\frac{3}{2}}
$$
\n
$$
- \left(\frac{1}{8} a(0) \varphi^{(4)}(0) + \frac{1}{2} a'(0) \varphi^{(3)}(0) \right) \varphi''(0)^{-\frac{5}{2}}
$$
\n
$$
+ \frac{5}{24} a(0) \varphi^{(3)}(0)^2 \varphi''(0)^{-\frac{7}{2}} \right] + \mathcal{O}(h^{\frac{3}{2}}).
$$

Here we recall that $\varphi_3 = \varphi^{(3)}(0)/6$ and $\varphi_4 = \varphi^{(4)}(0)/24$, $a_0 = a(0)$, $a_1 = a'(0), a_2 = \frac{1}{2}$ $\frac{1}{2}a''(0)$. We know that $I(\varphi, a; h)$ has an asymptotic expansion in integer powers of h , so the remainder can be improved to $\mathcal{O}(h^2)$. In view of the application to $\int_{\partial\Omega}e^{-kz+\overline{kz}}dz$, we try to express the result in terms of $\varphi/h = \Phi$, and it then seems convenient to replace $I(a,\varphi;h)$ above by $J(a,\Phi) := h^{\frac{1}{2}}I(a,\varphi;h) = \int_V e^{-\Phi(t)}a(t)dt$. From [\(4.3\)](#page-24-0) we get using $\varphi = h\Phi$,

$$
\frac{1}{\sqrt{2\pi}}J(\varphi,a) = a(0)\Phi''(0)^{-\frac{1}{2}} + \frac{a''(0)}{2}\Phi''(0)^{-\frac{3}{2}} \n- \left(a(0)\frac{\Phi^{(4)}(0)}{8} + a'(0)\frac{\Phi^{(3)}(0)}{2}\right)\Phi''(0)^{-\frac{5}{2}} \n+ \frac{5}{24}a(0)\Phi^{(3)}(0)^{2}\Phi''(0)^{-\frac{7}{2}} + \mathcal{O}(h^{2}).
$$

Notice that the first term in the final expression is homogeneous of degree $-\frac{1}{2}$ $\frac{1}{2}$ in Φ , while the following one is homogeneous of degree $-\frac{3}{2}$ $\frac{3}{2}$.

4.3. Stationary phase approximation for the reflection coefficient. We now apply the above results to the reflection coefficient (we only discuss the analytic case here, see Remark [3.3](#page-22-2) for a generalization to the smooth case). We also assume in the following that Φ does not vanish at the stationary points.

In application to $\int_{\partial\Omega}e^{kz-\overline{k}z}$, let $[0,L[\ni t \mapsto \gamma(t)]$ parametrize the boundary, so $\int_{\partial\Omega} e^{kz - \overline{kz}} dz = \int_0^L e^{-\Phi(t)} a(t) dt$, with

(4.5)
$$
\Phi(t) = -(k\gamma(t) - \overline{k\gamma(t)}), \quad a(t) = \dot{\gamma}(t).
$$

Apply [\(4.4\)](#page-24-1) with $\mathcal{O}(h^2) = \mathcal{O}(|k|^{-2})$.

Remark 4.1. It remains to choose the correct branches of $(\Phi''(t))^{\frac{1}{2}}$ at $t = t_{+}$. We adapt the notation of [\[18\]](#page-28-0), $iu(z, k) = kz - \overline{kz}$, $z \in \partial \Omega$. We have $\Phi(t) = -iu(\gamma(t), k) = -iU(t, k)$ which is purely imaginary with two non degenerate critical points at $t = t_+, t_-$ corresponding to the poles $w_+, w_-\$ respectively. By contour deformation we see that the stationary phase approximation is still valid with the branch of $(\Phi''(t_+))^{1/2}$ obtained as the limit of $(F_{\pm,\epsilon})^{1/2}$, where $F_{\pm,\epsilon}$ is a sequence converging to $\Phi''(t_{\pm})$ when $\epsilon \searrow 0$ with the property $\Re F_{\pm,\epsilon} > 0$. We get $(\Phi''(t_{\pm}))^{1/2} = e^{\mp i\pi/4} |U''(t_{\pm},k)|^{1/2}.$

We get for (3.40)
\n(4.6)
\n
$$
\int_{\tilde{\Gamma}} D_{\Omega}(w)e^{iu(w,k)} dw = \sqrt{2\pi} \sum_{t=t_+,t_-} e^{-\Phi(t)} D_{\Omega}(\gamma(t)) a(t) (\Phi''(t))^{-\frac{1}{2}} + \mathcal{O}(|k|^{-1}),
$$

where Φ and a are defined in [\(4.5\)](#page-25-2), and where the signs of the roots are chosen as detailed in Remark [4.1.](#page-25-1)

The leading term in the reflection coefficient [\(1.15\)](#page-5-0) is due to the term $\int_{\Omega} e^{kz-\overline{kz}}L(dz)$. We apply Stokes' theorem as before to write this in the form of an integral over $\partial\Omega$ and apply a stationary phase approximation,

$$
(4.7)
$$
\n
$$
\frac{i}{2k} \int_{\partial \Omega} e^{kz - \overline{kz}} dz = \frac{i\sqrt{2\pi}}{2\overline{k}} \sum_{t=t_+,t_-} e^{-\Phi(t)} \left(a(t) (\Phi''(t))^{-\frac{1}{2}} + \frac{a''(t)}{2} (\Phi''(t))^{-\frac{3}{2}} \right. \\
\left. - \left(\frac{a(t)}{8} \Phi^{(4)}(t) + \frac{1}{2} a'(t) \Phi^{(3)}(t) \right) (\Phi''(t))^{-\frac{5}{2}} \right. \\
\left. + \frac{5}{24} a(t) \Phi^{(3)}(t)^2 (\Phi''(t))^{-\frac{7}{2}} \right) + \mathcal{O}(|k|^{-3}).
$$

Again Φ and a are defined in [\(4.5\)](#page-25-2), and the sign of the roots are chosen as explained in Remark [4.1.](#page-25-1) With [\(4.6\)](#page-25-3) and [\(4.7\)](#page-25-4) we get for the reflection coefficient [\(1.15\)](#page-5-0) relation [\(1.16\)](#page-5-2).

4.4. Example: Characteristic function of the unit disk. In general, we cannot compute explicitly $D_{\Omega}(z)$, but in the special case of the unit disc, we have

$$
D_{D(0,1)}(z) = \begin{cases} \overline{z}, |z| \le 1, \\ 1/z, |z| \ge 1. \end{cases}
$$

For the stationary phase approximation, we parametrize $\partial\Omega$ via $\gamma(t) = e^{it}, t \in [0, 2\pi]$. Writing $k = |k|e^{i\vartheta}, \vartheta \in \mathbb{R}$, we have $\Phi =$ $-2i|k|\sin(t+\vartheta)$ and $a=ie^{it}$ in [\(4.5\)](#page-25-2). The critical points are $t^{\pm}=\pm\frac{\pi}{2}-\frac{\pi}{2}$ ϑ (the relation to the previously introduced t_{\pm} is $t^{\pm} := t_{\mp}$). We have $\Phi(t^{\pm}) = -\Phi''(t^{\pm}) = \Phi^{(4)}(t^{\pm}) = \pm 2i|k|$, whereas $\Phi'(t^{\pm}) = \Phi^{(3)}(t^{\pm}) = 0$ and $a(t^{\pm}) = -a''(t^{\pm}) = \mp e^{-i\vartheta}$. This implies for the right hand side of [\(4.7](#page-25-4) (4.8)

$$
\frac{i}{2k} \int_{\partial \Omega} e^{kz - \overline{kz}} dz = \frac{\sqrt{\pi}}{|k|^{3/2}} \left(\sin(2|k| - \pi/4) + \frac{3}{16|k|} \cos(2|k| - \pi/4) \right) + \mathcal{O}(|k|^{-7/2}).
$$

Similarly we get for [\(4.6\)](#page-25-3)

(4.9)
$$
\int_{\tilde{\Gamma}} D_{\Omega}(w) e^{iu(w,k)} dw = \sqrt{\frac{\pi}{|k|}} 2i \cos(2|k| - \pi/4) + \mathcal{O}(|k|^{-1}).
$$

Thus we get for the leading terms of the reflection coefficient [\(1.16\)](#page-5-2) the result conjectured in [\[18\]](#page-28-0) (note that the formula for $R/2$ was given there),

(4.10)
$$
R \approx \frac{2}{\sqrt{\pi |k|^3}} \left(\sin(2|k| - \pi/4) - \frac{5}{16|k|} \cos(2|k| - \pi/4) \right).
$$

Note that the conjectured error term is smaller than what is proven in this paper.

5. Conclusion

In this paper, we have presented asymptotic relations for large $|k|$ for the solutions to the Dirac system [\(1.1\)](#page-1-1) subject to the asymptotic conditions [\(1.2\)](#page-1-3). Previous results for potentials being the characteristic function of a compact domain with smooth convex boundary have been improved and extended to the reflection coefficient, the scattering data in the context of an integrable systems approach to the DS II equation. The results are now extended to $\mathcal{O}(|k|^{-5/2})$ which makes it possible to apply these formulae to complement numerical computations in order to get the reflection coefficient for all $k \in \mathbb{C}$ with the same precision as discussed in [\[18\]](#page-28-0). This allows to treat the reflection coefficient with a hybrid approach combining numerical and analytical results.

An interesting question in the context of EIT would be to extend the results of this paper to a compact domain with cavities. Since in applications to the human body, the organs of a patient are of essentially

constant conductivity, this corresponds to a situations of a domain with compact support and cavities all of which have smooth compact boundaries. The boundary data at the cavities are a consequence of the conductivity in the interior. It will be the subject of further work to adapt the present formulae to this case. An interesting question to be addressed is also to find the optimal error term in Theorem [1.4.](#page-5-1)

REFERENCES

- [1] M.J. Ablowitz, A.S. Fokas, On the inverse scattering transform of multidimensional nonlinear evolution equations related to first order systems in the plane, J. Math Phys. 25 no 8 (1984), 2494-2505.
- [2] Assainova, O., Klein, C., McLaughlin, K. D. and Miller, P. D., A Study of the Direct Spectral Transform for the Defocusing Davey-Stewartson II Equation the Semiclassical Limit. Comm. Pure Appl. Math., 72: 1474-1547 (2019).
- [3] R. Beals and R. Coifman, Multidimensional inverse scattering and nonlinear PDE Proc. Symp. Pure Math. (Providence: American Mathematical Society) 43, 45-70 (1985)
- [4] R.M. Brown, Estimates for the scattering map associated with a twodimensional first-order system. J. Nonlinear Sci. 11, no. 6, 459-471 (2001)
- [5] R. Brown and P. Perry, Soliton solutions and their (in)stability for the focusing Davey-Stewartson II equation, Nonlinearity 31(9) 4290 doi.org/10.1088/1361- 6544/aacc46 (2018)
- [6] R.M. Brown, G.A. Uhlmann, Communications in partial differential equations 22 (5-6), 1009-1027 (1997)
- [7] A.P. Calderón, On inverse boundary value problem. Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980) pp 65-73 (Soc. Brasil. Mat.)
- [8] M. Dimassi, J. Sjöstrand, Spectral asymptotics in the semi-classical limit, London Math. Soc. Lecture Notes Series 269, Cambridge University Press 1999.
- [9] A.S. Fokas, On the Inverse Scattering of First Order Systems in the Plane Related to Nonlinear Multidimensional Equations, Phys. Rev. Lett. 51, 3-6 (1983)
- [10] A.S. Fokas and M.J. Ablowitz, On a Method of Solution for a Class of Multi-Dimensional Nonlinear Evolution Equations, Phys. Rev. Lett. 51, 7-10 (1983)
- [11] C. Kenig, J. Sjöstrand, G. Uhlmann. The Calderón problem with partial data. Annals of Mathematics 165 (2007), 567-591.
- [12] A. G. Gurevich, L. P. Pitaevskii, Non stationary structure of a collisionless shock waves, JEPT Letters 17 (1973), 193-195.
- [13] L. Hörmander, An introduction to complex analysis in several variables, Third edition, North-Holland Mathematical Library 7 (North-Holland Publishing Co., Amsterdam, 1990)
- [14] C. Klein and K. McLaughlin, Spectral approach to D-bar problems, Comm. Pure Appl. Math., DOI: 10.1002/cpa.21684 (2017)
- [15] C. Klein, K. McLaughlin and N. Stoilov, Spectral approach to semi-classical dbar problems with Schwartz class potentials, Physica D: Nonlinear Phenomena DOI: 10.1016/j.physd.2019.05.006 (2019)
- [16] C. Klein and K. Roidot, Numerical Study of the semiclassical limit of the Davey-Stewartson II equations, Nonlinearity 27, 2177-2214 (2014).
- [17] C. Klein and J.-C. Saut, Nonlinear dispersive equations — Inverse Scattering and PDE methods, Applied Mathematical Sciences 209 (Springer, 2002)
- [18] C. Klein, J. Sjöstrand, N. Stoilov, Large |k| behavior of complex geometric optics solutions to d-bar problems, accepted for publication in Comm. Pure Appl. Maths. <https://arxiv.org/abs/2009.06909>.
- [19] C. Klein, J. Sjöstrand, N. Stoilov, Large $|k|$ behavior of d-bar problems for domains with a smooth boundary, '"Partial Differential Equations, Spectral Theory, and Mathematical Physics: The Ari Laptev Anniversary Volume' (edited by Pavel Exner, Rupert L. Frank, Fritz Gesztesy, Helge Holden and Timo Weidl), EMS Press, https://arxiv.org/abs/2009.06909
- [20] C. Klein and N. Stoilov, A numerical study of blow-up mechanisms for Davey-Stewartson II systems, Stud. Appl. Math., DOI : 10.1111/sapm.12214 (2018)
- [21] C. Klein and N. Stoilov, Numerical scattering for the defocusing Davey-Stewartson II equation for initial data with compact support, Nonlinearity 32 (2019) 4258-4280
- [22] K. Knudsen, J. L. Mueller, S. Siltanen. Numerical solution method for the d-bar equation in the plane. J. Comput. Phys. 198 no. 2, 500-517 (2004).
- [23] J.L. Mueller and S. Siltanen. Linear and Nonlinear Inverse Problems with Practical Applications, SIAM, 2012.
- [24] P. Muller, D. Isaacson, J. Newell, and G. Saulnier. A Finite Difference Solver for the D-bar Equation. Proceedings of the 15th International Conference on Biomedical Applications of Electrical Impedance Tomography, Gananoque, Canada, 2014.
- [25] A. I. Nachman, I. Regev, and D. I. Tataru, A nonlinear Plancherel theorem with applications to global well-posedness for the defocusing Davey-Stewartson equation and to the inverse boundary value problem of Calderon, Invent. Math. 220, 395–451 (2020).
- [26] P. Perry. Global well-posedness and long-time asymptotics for the defocussing Davey-Stewartson II equation in $H^{1,1}(\mathbb{R}^2)$. J. Spectr. Theory 6 (2016), no. 3, pp. 429–481.
- [27] L.Y. Sung, An inverse scattering transform for the Davey-Stewartson equations. I, J. Math. Anal. Appl. 183 (1) (1994), 121-154.
- [28] L.Y. Sung, An inverse scattering transform for the Davey-Stewartson equations. II, J. Math. Anal. Appl. 183 (2) (1994), 289-325.
- [29] L.Y. Sung, An inverse scattering transform for the Davey-Stewartson equations. III, J. Math. Anal. Appl. 183 , 477-494 (1994)
- [30] G. Uhlmann. Electrical impedance tomography and Calderón's problem. Inverse Problems, 25(12):123011, 2009.

INSTITUT DE MATHÉMATIQUES DE BOURGOGNE, UMR 5584, UNIVERSITÉ DE BOURGOGNE-FRANCHE-COMTÉ, 9 AVENUE ALAIN SAVARY, 21078 DIJON CEDEX, France, Institut Universitaire de France, E-mail Christian.Klein@ubourgogne.fr

INSTITUT DE MATHÉMATIQUES DE BOURGOGNE, UMR 5584, UNIVERSITÉ DE Bourgogne-Franche-Comt´e, 9 avenue Alain Savary, 21078 Dijon Cedex, France, E-mail Johannes.Sjostrand@u-bourgogne.fr

INSTITUT DE MATHÉMATIQUES DE BOURGOGNE, UMR 5584, UNIVERSITÉ de Bourgogne-Franche-Comt´e, 9 avenue Alain Savary, 21078 Dijon Cedex, France, and Laboratoire Jaques-Louis Lions, UMR 7598, Faculte des Sciences er Ingenière, Sorbonne Université, 4 Place Jussieu 75005 Paris, France, E-mail Nikola.Stoilov@ljll.math.upmc.fr