

# Quantum Statistical Mechanics of Vortices

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## Abstract

The asymptotic partition function for quantized Abelian Higgs vortices at high temperature  $T$  is found to leading and subleading order, and from this the equation of state of the vortex gas is derived, including the first quantum correction. It is assumed that the Hamiltonian is proportional to the Laplace–Beltrami operator on the moduli space of static  $N$ -vortex solutions. The partition function is calculated using the total volume and total scalar curvature of the moduli space.

Keywords: Abelian Higgs vortices, Quantum statistical mechanics, Moduli space

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# 1 Introduction

In [1], the author studied the classical statistical mechanics of Abelian Higgs vortices at critical coupling (i.e., BPS vortices), where the static forces between vortices cancel. It is assumed that there are  $N$  vortices at temperature  $T$  on a large spherical surface of area  $A$ , where  $A > 4\pi N$ . The last inequality, due to Bradlow [2], ensures that static vortex solutions exist, and its interpretation is that each vortex occupies an area  $4\pi$ , although a vortex is not a hard disc. The  $N$ -vortex classical dynamics is then assumed to be modelled by geodesic motion through the moduli space  $\mathcal{M}$  of static  $N$ -vortex solutions [3]. The vortex number  $N$  is conserved, because it is a topological invariant. One is interested in the statistical mechanical equilibrium state in the thermodynamic limit, where  $N, A \rightarrow \infty$  and the number density  $N/A$  remains finite.

In this model of a vortex gas, the partition function reduces to a simple Gaussian integral over momenta, dependent on the temperature  $T$ , multiplied by the volume of the moduli space  $\mathcal{M}$ . Fortunately, the volume can be calculated exactly.  $\mathcal{M}$  has a Kähler metric, for which a localised expression was discovered by Samols [4]. Although this metric is not known explicitly, it is possible to find exactly the real cohomology class of the Kähler form, and from this the volume can be determined by simple algebra. The coefficient of this class (relative to a basis element for the integer cohomology) is related to the effective mass of a cluster of  $N$  coincident vortices, moving coherently around the sphere. This effective mass can be calculated using the Samols metric and symmetry arguments, and is less than  $N$  times the mass of one vortex.

From the partition function, it is straightforward to find the pressure  $P$  and hence the equation of state of the vortex gas, using standard thermodynamic formulae [5]. The result is

$$P = \frac{NT}{A - 4\pi N}. \quad (1)$$

This is the Clausius equation of state for a 2-dimensional non-ideal gas, where the gas particles each take up some area that is excluded to others, but otherwise exert no static forces on each other.

Remarkably, this is an exact equation of state, given our assumptions, illustrating that the moduli space dynamics of Abelian Higgs vortices is an elegant topic within mathematical physics. In the partition function, the indistinguishability of the vortices is inescapable, so a Gibbs factor of  $N!$  appears automatically and does not need to be introduced by hand. Hard

discs have something in common with vortices, but the equation of state for a gas of hard discs is not known exactly, and beyond the second virial coefficient, it differs from eq.(1). On the other hand, eq.(1) is very similar to the equation of state for a gas of hard rods of finite length, moving along a line [6]. This similarity has been explained through the use of T-duality by Eto *et al.* [7].

One might worry that the above equation of state is sensitive to the curvature of the background spherical surface, even though this curvature is very small in the thermodynamic limit. This was investigated by Nasir and the present author [8], who found the volume of the  $N$ -vortex moduli space  $\mathcal{M}$  for vortices on a compact surface of any genus  $g$  and area  $A$ , having any smooth Riemannian metric. As a manifold,  $\mathcal{M}$  is the symmetrised  $N$ th power of the underlying surface. Again, the Bradlow inequality  $A - 4\pi N > 0$  needs to be satisfied for  $N$ -vortex solutions to exist.

The metric on  $\mathcal{M}$ , in this general situation, is a variant of the Samols metric, and is again Kähler. The Kähler 2-form for  $g \geq 1$  is a real combination of two basic, integer cocycles, whose coefficients can be calculated by considering the 2-cycle where  $N$  coincident vortices move together, and the 2-cycle where one vortex moves and the remaining  $N - 1$  are coincident and fixed. These coefficients have been confirmed by Perutz, using an independent argument [10]. The total volume of  $\mathcal{M}$  is then calculated from the  $N$ th power of the Kähler 2-form, divided by  $N!$ , using the cocycle algebra established by Macdonald for symmetrised powers of Riemann surfaces [9]. This calculation gives an explicit formula for the volume, depending only on  $g$ ,  $N$  and  $A$ . Although the volume is genus-dependent, one finds that this dependence drops out in the thermodynamic limit, and the equation of state is independent of genus.

In this paper, we will recalculate the equation of state for Abelian Higgs vortices using quantum statistical mechanics. The quantum Hamiltonian will be taken to be  $\frac{1}{2}\hbar^2$  times the Laplace–Beltrami operator on the moduli space  $\mathcal{M}$ , the quantized version of the classical kinetic energy for the vortex gas. We will need to assume that the temperature  $T$  is large and that  $\hbar$  is small, in senses to be clarified below. We can then use the asymptotic, large  $T$  expansion for the partition function [11]. The leading term depends on the volume of the moduli space  $\mathcal{M}$ , and reproduces the classical partition function. The subleading term depends on the total scalar curvature of  $\mathcal{M}$ , and is proportional to  $\hbar^2$ . This total curvature is again calculable, because  $\mathcal{M}$  is a Kähler manifold, and it has been found by Baptista [12]. The calculation combines the real cohomology class of the Kähler 2-form on  $\mathcal{M}$  with the first Chern class of (the tangent bundle of)  $\mathcal{M}$ . Both of these have known coefficients in terms of basic, integer cocycles, so the Macdonald algebra can

again be used to complete the calculation. The total volume and total scalar curvature of the  $N$ -vortex moduli space for general genus  $g$  are found to be sums of  $g + 1$  terms, but these sums simplify in the thermodynamic limit. The subleading curvature term gives the novel result of this paper, the first quantum correction to the classical partition function at high temperature, and hence the first quantum correction to the equation of state. In particular, we find the quantum correction to the second virial coefficient of the vortex gas.

## 2 $N$ -vortex partition function

We will not review here the fundamental, first order field equations for static vortex solutions in the Abelian Higgs model at critical coupling. For this, see [13, 14]. We just recall that these equations can be defined on any compact Riemann surface  $\Sigma$  of genus  $g$ , equipped with an arbitrary, smooth Riemannian metric compatible with the complex structure (i.e., having the given conformal structure).  $N$ -vortex solutions exist provided  $A > 4\pi N$ , where  $A$  is the area of  $\Sigma$ . Up to gauge transformations, there is a unique solution for any set of  $N$  (not necessarily distinct) unordered points on  $\Sigma$  – the points where the Higgs field vanishes [2, 15]. These solutions minimise the field potential energy in the  $N$ -vortex sector, and they all have the same energy, a constant multiple of  $N$ . The moduli space of solutions,  $\mathcal{M}$ , is therefore the  $N$ th symmetrised power of  $\Sigma$ .  $\mathcal{M}$  inherits the complex structure of  $\Sigma$ , and is a smooth manifold with complex dimension  $N$ .

One may assume that the dynamics of  $N$  vortices with small kinetic energy is geodesic motion through the moduli space  $\mathcal{M}$  [3]. The field kinetic energy is defined as a gauge-invariant integral over  $\Sigma$ , quadratic in the time-derivatives of the fields, and when restricted to motion tangent to  $\mathcal{M}$ , it defines the Riemannian metric on  $\mathcal{M}$ , which is actually Kähler. Using the linearised field equations and Gauss’s law, Samols reduced this integral to a compact, elegant formula for the metric, depending only on local information about the Higgs field close to each vortex [4]. The Samols metric on  $\mathcal{M}$  is smooth. Physically, this means that vortices scatter smoothly even if they are instantaneously coincident. This is because the underlying field dynamics is smooth, and vortices have no point singularities.

Let  $\{q^i : 1 \leq i \leq 2N\}$  denote real coordinates on  $\mathcal{M}$ , and  $g_{ij}(\mathbf{q})$  the Samols metric. Then the kinetic energy for motion through moduli space is

simply

$$L = \frac{1}{2}g_{ij}(\mathbf{q})\dot{q}^i\dot{q}^j. \quad (2)$$

This is the complete Lagrangian, if we drop the constant potential energy, and so it is the complete classical energy. [Note: Samols [4] has an additional explicit factor of  $\pi$  here, representing the mass of one vortex. We follow Baptista [12] in absorbing this into the metric.] The conjugate momenta are

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = g_{ij}(\mathbf{q})\dot{q}^j, \quad (3)$$

and the classical Hamiltonian, expressed in terms of the momenta, is

$$H = \frac{1}{2}g^{ij}(\mathbf{q})p_i p_j, \quad (4)$$

where  $g^{ij}(\mathbf{q})$  is the inverse metric.

We assume that the quantized vortex dynamics can be treated as the quantized, purely kinetic dynamics on the moduli space  $\mathcal{M}$ . This may be an oversimplification, because the quantum field theory may generate a non-constant potential energy on  $\mathcal{M}$  – however, we ignore this possibility. The quantized conjugate momentum operators,

$$p_i = -i\hbar\frac{\partial}{\partial q^i}, \quad (5)$$

are inserted into the classical Hamiltonian to determine the quantum Hamiltonian. There is a familiar operator-ordering problem here, resolved in the standard way by assuming that the final quantum Hamiltonian is

$$H = \frac{1}{2}\hbar^2\Delta, \quad (6)$$

where  $\Delta \equiv -\nabla^2$  is the Laplace–Beltrami operator on  $\mathcal{M}$ . Since  $\mathcal{M}$  is compact, without boundary, the spectrum of  $\Delta$  is a discrete set of non-negative eigenvalues  $\lambda$ . The partition function at temperature  $T$  is then

$$Z(T) = \sum_{\lambda} e^{-\frac{\hbar^2}{2T}\lambda}. \quad (7)$$

Let us now clarify the order of magnitude of the parameters here. In the thermodynamic limit,  $N$  is large, so the Bradlow inequality requires  $A$  also to be large. The field theory Lagrangian of the Abelian Higgs model has  $O(1)$  coefficients, so the vortex mass is  $O(1)$ . Because of the Higgs mechanism,

there is a massive photon and a massive scalar particle in the quantized field theory. Their physical masses are  $O(\hbar)$ . We assume  $\hbar \ll 1$ , so that we are in the perturbative regime of the  $N$ -vortex sector of the theory, where vortices are heavy relative to the other particles. We need the temperature  $T$  to be small enough, so that photons and scalar particles are not created in vortex collisions. Therefore  $T \ll \hbar$ . Additionally, we need the  $N$ -vortex dynamics to be close to the classical regime, so  $T$  must not be too small. The typical, small energy eigenvalue for a quantized single vortex moving on a surface of area  $A$  is  $O(\hbar^2/A)$ , so we require  $T \gg \hbar^2/A$ . These conditions can be satisfied if  $T$  is  $O(\hbar^2)$ .

We cannot calculate the partition function (7) exactly, but it has a well-known asymptotic expansion in  $\hbar^2/2T$  – this is essentially the asymptotic expansion of the trace of the heat kernel for the Laplace–Beltrami operator on  $\mathcal{M}$ . As  $\mathcal{M}$  has real dimension  $2N$ , the first two terms of this expansion are [11]

$$Z(T) = \left( \frac{2\pi\hbar^2}{T} \right)^{-N} \left( \text{Vol}(\mathcal{M}) + \frac{\hbar^2}{12T} \text{Curv}(\mathcal{M}) \right), \quad (8)$$

where

$$\text{Vol}(\mathcal{M}) = \int_{\mathcal{M}} d\text{vol}_{\mathcal{M}} \quad (9)$$

is the  $2N$ -dimensional, total volume of  $\mathcal{M}$  and

$$\text{Curv}(\mathcal{M}) = \int_{\mathcal{M}} s d\text{vol}_{\mathcal{M}} \quad (10)$$

is the total curvature, the integral over  $\mathcal{M}$  of the scalar curvature  $s$ , where all the geometrical quantities are determined from the Samols metric.

Baptista has calculated the total volume and total curvature of the moduli space for a number of types of vortex – see Theorem 5.1 in ref.[12]. We only need the results for the Abelian Higgs model, and to match conventions and notation, we set  $\tau = 1$ ,  $e^2 = \frac{1}{2}$ ,  $n = 1$ ,  $d = N$  and  $\text{Vol}\Sigma = A$  in Baptista’s formulae. Baptista’s integer  $r$  equals  $N$ .

For general genus  $g$ , the total volume is

$$\text{Vol}(\mathcal{M}) = \pi^N \sum_{i=0}^g \frac{g!}{i!(N-i)!(g-i)!} (4\pi)^i (A - 4\pi N)^{N-i}, \quad (11)$$

in agreement with ref.[8]. In the thermodynamic limit, with  $g$  fixed and  $N \gg g$ , we can replace  $(N-i)!$  by  $N!/N^i$ . Then

$$\text{Vol}(\mathcal{M}) = \frac{\pi^N}{N!} (A - 4\pi N)^N \sum_{i=0}^g \frac{g!}{i!(g-i)!} \left( \frac{4\pi N}{A - 4\pi N} \right)^i \quad (12)$$

which is just a finite binomial series, so

$$\begin{aligned}\text{Vol}(\mathcal{M}) &= \frac{\pi^N}{N!} (A - 4\pi N)^N \left(1 + \frac{4\pi N}{A - 4\pi N}\right)^g \\ &= \frac{\pi^N}{N!} (A - 4\pi N)^{N-g} A^g.\end{aligned}\quad (13)$$

The total curvature is [12]

$$\text{Curv}(\mathcal{M}) = 2\pi^N \sum_{i=0}^g \frac{g!(N+1-2g+i)}{i!(N-1-i)!(g-i)!} (4\pi)^i (A - 4\pi N)^{N-1-i}.\quad (14)$$

In the thermodynamic limit, we can replace  $N+1-2g+i$  by  $N$  and  $(N-1-i)!$  by  $N!/N^{1+i}$ . The sum again simplifies to a binomial series, and the result is

$$\text{Curv}(\mathcal{M}) = \frac{2N^2\pi^N}{N!} (A - 4\pi N)^{N-g-1} A^g.\quad (15)$$

The two-term partition function (8) is therefore

$$Z(T) = \left(\frac{T}{2\hbar^2}\right)^N \frac{1}{N!} (A - 4\pi N)^{N-g} A^g \left[1 + \frac{\hbar^2 N^2}{6T(A - 4\pi N)}\right].\quad (16)$$

The first term in the square bracket gives the partition function of classical statistical mechanics, agreeing with refs.[1, 8], and the second, curvature term is the leading quantum correction.

It is worthwhile to understand heuristically the magnitude of the total scalar curvature. Roughly speaking, it has two contributions. Firstly, there is the generic region of the moduli space where all vortices are well separated. For a single vortex on a surface  $\Sigma$  of area  $A$ , the local curvature is  $O(1/A)$ , so the scalar curvature for  $N$  vortices is  $O(N/A)$ , as this involves a trace. The integration region has volume  $O(A^N/N!)$ , so the total curvature of the generic region is  $O(A^{N-1}/(N-1)!)$ . This part can be positive, negative or zero. Secondly, there is the region where one pair of vortices is close together, and the remaining  $N-2$  vortices are well separated. The curvature of the moduli space for a vortex pair, when they are close together, is  $O(1)$  and positive, as shown by Samols [4], whereas the remaining vortices contribute curvature  $O((N-2)/A)$ . At low density, the  $O(1)$  pair curvature dominates. The volume of the region of the moduli space where one pair is close together is  $O(A^{N-1}/(N-2)!)$ , because the pair acts like a double vortex that is distinct from the remaining  $N-2$  vortices. The total curvature from this region is therefore  $O(A^{N-1}/(N-2)!)$  and positive. This dominates, by a factor

of order  $N$ , the contribution from the generic region where all vortices are well separated, and is the order of magnitude for the total scalar curvature, agreeing with the result (15) to leading order in  $N/A$ .

### 3 Free energy and equation of state

We now use the trick, discussed by Landau and Lifshitz [5] (Sect. 72), of assuming temporarily that the density  $N/A$  is small and also that the total number of vortices  $N$  is relatively small, so that  $N^2/A$  is small. The effects of two-vortex interactions are still correctly accounted for, so extensive thermodynamic quantities and the equation of state can be scaled up to large  $N$  at the end.

The free energy of the vortex gas is  $F = -T \log Z$ , where  $Z$  is given by the expression (16). We can make the approximation  $\log(1+x) \simeq x$  for the terms in square brackets, because of the above-mentioned trick, and we also use the Stirling approximation  $\log N! = N \log N - N$ . The free energy then becomes

$$F = -T \left\{ N \log \left( \frac{T}{2\hbar^2} \right) - N \log N + N + (N-g) \log(A - 4\pi N) + g \log A + \frac{\hbar^2 N^2}{6T(A - 4\pi N)} \right\}. \quad (17)$$

Compared with the extensive terms proportional to  $N$ , the  $g \log A$  term is negligible, and can be replaced by  $g \log(A - 4\pi N)$ .  $F$  then has the form

$$F = -T \left\{ N \log \left( \frac{T}{2\hbar^2} \right) - N \log N + N + N \log(A - 4\pi N) + \frac{\hbar^2 N^2}{6T(A - 4\pi N)} \right\}, \quad (18)$$

which is independent of the genus  $g$ .

We are mainly interested in the pressure of the gas. This is

$$P = -\frac{\partial F}{\partial A} = \frac{NT}{A - 4\pi N} - \frac{\hbar^2 N^2}{6(A - 4\pi N)^2}, \quad (19)$$

which can be rearranged into the equation of state

$$\left( P + \frac{\hbar^2 N^2}{6(A - 4\pi N)^2} \right) (A - 4\pi N) = NT. \quad (20)$$



If we drop the quantum correction proportional to  $\hbar^2$ , this agrees with the Clausius equation of state found using classical statistical mechanics. The quantum correction converts the equation of state to the van der Waals form, if we approximate  $A - 4\pi N$  by  $A$  in the correction term.

The low-density expansion for the pressure is

$$P \simeq \frac{NT}{A} \left( 1 + \left( 4\pi - \frac{\hbar^2}{6T} \right) \frac{N}{A} \right), \quad (21)$$

so the second virial coefficient is  $B(T) = 4\pi - \frac{\hbar^2}{6T}$ . The quantum correction therefore results in a decrease of  $B(T)$  from its classical value as  $T$  decreases from infinity. However, the validity of the truncated asymptotic expansion for  $Z(T)$  requires that  $\frac{\hbar^2}{T} \ll 1$ , so we cannot conclude without further work that  $B(T)$  changes sign at some temperature  $T$  of order  $\hbar^2$ . It would be interesting if such a sign change did occur – the interpretation would be that the effective repulsive interaction of vortices, due to their finite area, becomes an effective attraction at low temperature due to their bosonic nature.

## 4 Conclusions

We have calculated the first quantum correction to the free energy and equation of state of a gas of critically-coupled Abelian Higgs vortices – BPS vortices – using knowledge of the volume and total scalar curvature of the moduli space of static  $N$ -vortex solutions. The equation of state has a van der Waals form. The results are valid when the temperature is high enough so that  $N$ -vortex dynamics is approximately classical, but not so high that massive photons and scalar particles are created. It would be interesting to extend these results to other types of BPS vortices. This should be possible using Baptista’s rather general formulae for the moduli space volume and total scalar curvature [12]. It would also be interesting to determine the thermodynamic properties of vortex gases at lower temperatures. This will require a more precise understanding of the energy spectrum of quantum states on the  $N$ -vortex moduli space.

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