

Stewart’s Theorem revisited: suppressing the norm ± 1 hypothesis

Haojie Hong

April 6, 2022

Abstract

Let γ be an algebraic number of degree 2 and not a root of unity. In this note we show that there exists a prime ideal \mathfrak{p} of $\mathbb{Q}(\gamma)$ satisfying $\nu_{\mathfrak{p}}(\gamma^n - 1) \geq 1$, such that the rational prime p underlying \mathfrak{p} grows quicker than n .

Contents

1	Introduction	1
2	Preliminary results	2
2.1	Notation	2
2.2	Uniform explicit version of Stewart’s theorem	3
2.3	Cyclotomic polynomials and primitive divisors	3
2.4	Estimates for the arimetical functions	4
3	Proof of Theorem 1.2	4
3.1	Case (3.9)	6
3.2	Case (3.10)	6

1 Introduction

Let $P(m)$ denote the largest prime factor of integer m , with the convention $P(0) = P(\pm 1) = 1$. For any integer n , we denote the n -th cyclotomic polynomial in x by $\Phi_n(x)$ as usual.

Schinzel [8] asked if there exists any integers a, b with $ab \neq \pm 2c^2, \pm c^h$ ($h \geq 2$) such that $P(a^n - b^n) > 2n$ for all sufficiently large n . Erdős [4] conjectured that $P(2^n - 1)$ grows quicker than n .

Let u_n be the n th term of a Lucas sequence. In 2013, Stewart [10] gave a lower bound of the largest prime factor of u_n , which is of the form $n \exp(\log n / 104 \log \log n)$. What Stewart actually proved is the following, see [1, Theorem 1.1].

Theorem 1.1. *Let γ be a non-zero algebraic number, not a root of unity. Denote $\omega(\gamma)$ the number of primes \mathfrak{p} of the field $K = \mathbb{Q}(\gamma)$ with the property $\nu_{\mathfrak{p}}(\gamma) \neq 0$. Let P be the biggest element of the set*

$$\{p : p \text{ is a rational prime lying below a prime } \mathfrak{p} \text{ of } K, \text{ with } \nu_{\mathfrak{p}}(\Phi_n(\gamma)) \geq 1\}.$$

If γ satisfies one of the following conditions:

- $\gamma \in \mathbb{Q}$,
- $[\mathbb{Q}(\gamma) : \mathbb{Q}] = 2$ and $\mathcal{N}\gamma = \pm 1$.

There exists a n_0 , which is effectively computable in terms of $\omega(\gamma)$ and the discriminant of K , such that, for all $n > n_0$,

$$P > n \exp(\log n / 104 \log \log n).$$

Using this result, Stewart answered questions of Schinzel and Erdős in a wider context, see [10] for more historical details.

A totally explicit expression of n_0 in the above theorem was given in [1], which also showed that n_0 depends only on the field $K = \mathbb{Q}(\gamma)$ but not on $\omega(\gamma)$.

Let q and a be integers such that $q \geq 2$ and $|a| < 2\sqrt{q}$. Assume α and $\bar{\alpha}$ are the roots of $x^2 - ax + q$. In [2], the authors concentrate on big prime factors of $\#E(\mathbb{F}_q) = q - a + 1 = (\alpha - 1)(\bar{\alpha} - 1)$, the order of group of \mathbb{F}_q -points on a certain elliptic curve E . A Stewart-type result was proved for recurrent sequences of order 4 rather than Lucas sequence.

This article is motivated by [1] and [2], we prove the following theorem.

Theorem 1.2. *Suppose γ is an algebraic number of degree 2 and not a root of unity. Set $n_0 = \exp \exp(\max\{10^{10}, 3|D_K|\})$. Let n be a positive integer satisfying $n \geq n_0$. There exists a prime ideal \mathfrak{p} of $K = \mathbb{Q}(\gamma)$ such that $\nu_{\mathfrak{p}}(\gamma^n - 1) \geq 1$ and the underlying rational prime p of \mathfrak{p} satisfies*

$$p \geq n \exp\left(0.0001 \frac{\log n}{\log \log n}\right).$$

Note that this theorem suppresses the assumption $\mathcal{N}\gamma = \pm 1$ when $[\mathbb{Q}(\gamma) : \mathbb{Q}] = 2$ in Theorem 1.1, and it can also be seen as a generalization of Schinzel's question and Erdős' conjecture. The proof uses ingredients from [1] and [2], all of which rely heavily on lower bound for p -adic logarithmic form.

2 Preliminary results

2.1 Notation

Denote by $\log^+ = \max\{\log, 0\}$, $\log^- = \min\{\log, 0\}$, $\log^* = \max\{\log, 1\}$.

Let K be a number field of degree d . We denote by D_K the discriminant of K .

Suppose $\gamma \in K$, $h(\gamma)$ denotes the usual absolute logarithmic height of γ :

$$h(\gamma) = [K : \mathbb{Q}]^{-1} \sum_{v \in M_K} d_v \log^+ |\gamma|_v,$$

where d_v denotes the local degree. The places $v \in M_K$ are normalized to extend standard places of \mathbb{Q} .

Let $\sigma : K \hookrightarrow \mathbb{C}$ be an arbitrary complex embedding of K , \mathfrak{p} be a prime ideal of the ring of integers \mathcal{O}_K . The following formulas are immediate consequences of the above definition:

$$h(\gamma) = \frac{1}{d} \left(\sum_{\sigma:K \hookrightarrow \mathbb{C}} \log^+ |\gamma^\sigma| + \sum_{\mathfrak{p}} \max\{0, -\nu_{\mathfrak{p}}(\gamma)\} \log \mathcal{N}\mathfrak{p} \right). \quad (2.1)$$

$$h(\gamma) = \frac{1}{d} \left(\sum_{\sigma:K \hookrightarrow \mathbb{C}} -\log^- |\gamma^\sigma| + \sum_{\mathfrak{p}} \max\{0, \nu_{\mathfrak{p}}(\gamma)\} \log \mathcal{N}\mathfrak{p} \right). \quad (2.2)$$

2.2 Uniform explicit version of Stewart's theorem

The following two theorems go back to Stewart, see [10, Lemma 4.3], but in present form they are Theorem 1.4 and Theorem 1.5 of [1].

Theorem 2.1. *Let γ be a non-zero algebraic number of degree d , not a root of unity. Set $p_0 = \exp(80000d(\log^*d)^2)$. Then for every prime \mathfrak{p} of the field $K = \mathbb{Q}(\gamma)$ whose absolute norm $\mathcal{N}\mathfrak{p}$ satisfies $\mathcal{N}\mathfrak{p} \geq p_0$, and every positive integer n we have*

$$\nu_{\mathfrak{p}}(\gamma^n - 1) \leq \mathcal{N}\mathfrak{p} \exp\left(-0.002d^{-1} \frac{\log \mathcal{N}\mathfrak{p}}{\log \log \mathcal{N}\mathfrak{p}}\right) h(\gamma) \log^* n.$$

Theorem 2.2. *Let γ be a non-zero algebraic number of degree 2, not a root of unity. Assume that $\mathcal{N}\gamma = \pm 1$. Set $p_0 = \exp \exp(\max\{10^8, 2|D_K|\})$, where D_K is the discriminant of the quadratic field $K = \mathbb{Q}(\gamma)$. Then for every prime \mathfrak{p} of K with underlying rational prime $p \geq p_0$, and every positive integer n we have*

$$\nu_{\mathfrak{p}}(\gamma^n - 1) \leq p \exp\left(-0.001 \frac{\log p}{\log \log p}\right) h(\gamma) \log^* n. \quad (2.3)$$

2.3 Cyclotomic polynomials and primitive divisors

The following proposition, which is about estimates of cyclotomic polynomials, goes back to Schinzel [9], but in the present form, item 1 is [3, Theorem 3.1] and item 2 is proved in [1, Proposition 8.1].

Proposition 2.3. *1. Let γ be an algebraic number. Then*

$$h(\Phi_n(\gamma)) = \varphi(n)h(\gamma) + O_1(2^{\omega(n)} \log(\pi n)),$$

where $A = O_1(B)$ means $|A| \leq B$.

2. Let γ be a complex algebraic number of degree d , non-zero and not a root of unity. Then

$$\log |\Phi_n(\gamma)| \geq -10^{14} d^5 h(\gamma) \cdot 2^{\omega(n)} \log^* n. \quad (2.4)$$

Let K be a number field of degree d and $\gamma \in K^\times$ not a root of unity. We consider the sequence $u_n = \gamma^n - 1$. Let \mathfrak{p} be a prime ideal of \mathcal{O}_K . We call \mathfrak{p} *primitive divisor* of u_n if

$$\nu_{\mathfrak{p}}(u_n) \geq 1, \quad \nu_{\mathfrak{p}}(u_k) = 0 \quad (k = 1, \dots, n-1).$$

Let us recall some basic properties of primitive divisors. Items 1 of the following proposition are well-known and easy, and item 2 is Lemma 4 of Schinzel [9]; see also [3, Lemma 4.5].

Proposition 2.4. *1. Let \mathfrak{p} be a primitive divisor of u_n . Then $\nu_{\mathfrak{p}}(\Phi_n(\gamma)) \geq 1$ and $\mathcal{N}\mathfrak{p} \equiv 1 \pmod{n}$; in particular, $\mathcal{N}\mathfrak{p} \geq n+1$.*

2. Assume that $n \geq 2^{d+1}$. Let \mathfrak{p} be not a primitive divisor of u_n . Then $\nu_{\mathfrak{p}}(\Phi_n(\gamma)) \leq \nu_{\mathfrak{p}}(n)$.

2.4 Estimates for the arimetical functions

Denote by $\varphi(n)$, $\omega(n)$, $\tau(n)$ the Euler's totient function, the number of distinct prime factors of n , the number of divisors of n , respectively.

We will use the following bounds for these arithmetic functions:

$$\varphi(n) \geq 0.5 \frac{n}{\log \log n} \quad (n \geq 10^{20}), \quad (2.5)$$

$$\omega(n) \leq 1.4 \frac{\log n}{\log \log n} \quad (n \geq 3), \quad (2.6)$$

$$\tau(n) \leq \exp\left(1.1 \frac{\log n}{\log \log n}\right) \quad (n \geq 3). \quad (2.7)$$

See [7, Theorem 15], [6, Théorème 11], [5, Theorem 1].

3 Proof of Theorem 1.2

Let P be the biggest element of the set

$$\{p: p \text{ is a rational prime lying below a prime } \mathfrak{p} \text{ of } K, \text{ with } \nu_{\mathfrak{p}}(\Phi_n(\gamma)) \geq 1\}.$$

It is sufficient to show that

$$P > n \exp\left(0.0001 \frac{\log n}{\log \log n}\right) \quad (3.1)$$

One may assume $P \leq n^2$, since otherwise there is nothing to prove.

By (2.2),

$$2h(\Phi_n(\gamma)) = -\log^- |\Phi_n(\gamma)| - \log^- |\Phi_n(\gamma^\sigma)| + \sum_{\mathfrak{p}} \max\{0, \nu_{\mathfrak{p}}(\Phi_n(\gamma))\} \log \mathcal{N}\mathfrak{p}, \quad (3.2)$$

where σ is the generator of $\text{Gal}(K/\mathbb{Q})$.

We use item 2 of Proposition 2.3,

$$-\log^- |\Phi_n(\gamma)| - \log^- |\Phi_n(\gamma^\sigma)| \leq 2^6 \cdot 10^{14} h(\gamma) \cdot 2^{\omega(n)} \log n. \quad (3.3)$$

We split the sum in (3.2):

$$\sum_{\mathfrak{p}} \max\{0, \nu_{\mathfrak{p}}(\Phi_n(\gamma))\} \log \mathcal{N}\mathfrak{p} = \sum_{\substack{\mathfrak{p} \text{ primi-} \\ \text{tive}}} + \sum_{\substack{\mathfrak{p} \text{ non-} \\ \text{primitive}}} = \Sigma_{\mathfrak{p}} + \Sigma_{\text{np}},$$

where \mathfrak{p} primitive, \mathfrak{p} non-primitive means those prime ideals \mathfrak{p} which are primitive, non-primitive divisors of $\gamma^n - 1$, respectively. By item 2 of Proposition 2.4,

$$\Sigma_{\text{np}} \leq \sum_{\mathfrak{p}} \nu_{\mathfrak{p}}(n) \log \mathcal{N}\mathfrak{p} \leq 2 \log n. \quad (3.4)$$

Thus

$$h(\Phi_n(\gamma)) \leq 10^{16} h(\gamma) \cdot 2^{\omega(n)} \log n + \Sigma_{\mathfrak{p}}/2 + \log n. \quad (3.5)$$

On the other hand, by item 1 of Proposition 2.3,

$$h(\Phi_n(\gamma)) \geq \varphi(n) h(\gamma) - 2^{\omega(n)} \log(\pi n). \quad (3.6)$$

Combining (3.5) and (3.6), we have

$$\Sigma_{\mathfrak{p}}/2 \geq \varphi(n) h(\gamma) - 2^{\omega(n)} \log(\pi n) - 10^{16} h(\gamma) 2^{\omega(n)} \log n - \log n. \quad (3.7)$$

Inequalities (2.5), (2.6) and our assumption $n \geq n_0 \geq 10^{10}$ imply that the right-hand side of (3.7) is bounded from below by $0.4\varphi(n)h(\gamma)$. Thus we get the lower bound of $\Sigma_{\mathfrak{p}}$

$$\Sigma_{\mathfrak{p}} \geq 0.8\varphi(n)h(\gamma). \quad (3.8)$$

Now primes may have residue degree 1 or 2. Denote by

$$\Sigma_{\mathfrak{p}1} := \sum_{\substack{\mathfrak{p} \text{ primitive} \\ f_{\mathfrak{p}}=1}} \max\{0, \nu_{\mathfrak{p}}(\Phi_n(\gamma))\} \log \mathcal{N}\mathfrak{p}$$

and

$$\Sigma_{\mathfrak{p}2} := \sum_{\substack{\mathfrak{p} \text{ primitive} \\ f_{\mathfrak{p}}=2}} \max\{0, \nu_{\mathfrak{p}}(\Phi_n(\gamma))\} \log \mathcal{N}\mathfrak{p}.$$

We have either

$$\Sigma_{\mathfrak{p}1} \geq 0.4\varphi(n)h(\gamma), \quad (3.9)$$

or

$$\Sigma_{\mathfrak{p}2} \geq 0.4\varphi(n)h(\gamma). \quad (3.10)$$

3.1 Case (3.9)

By item 1 of Proposition 2.4, we have $\mathcal{N}_{\mathfrak{p}} = p \equiv 1 \pmod{n}$. Since $n \geq n_0$ and $n_0 = \exp \exp(\max\{10^{10}, 3|D_K|\})$ is bigger than p_0 in Theorem 2.1, the underlying prime p is bigger than p_0 . So Theorem 2.1 applies,

$$\nu_{\mathfrak{p}}(\Phi_n(\gamma)) = \nu_{\mathfrak{p}}(\gamma^n - 1) \leq p \exp\left(-0.001 \frac{\log p}{\log \log p}\right) h(\gamma) \log n.$$

We obtain

$$\begin{aligned} \Sigma_{\mathfrak{p}1} &\leq \sum_{\substack{p \equiv 1 \pmod{n} \\ p \leq P}} \max\{0, \nu_{\mathfrak{p}}(\Phi_n(\gamma))\} \log p \\ &\leq \pi(P; n, 1) P \exp\left(-0.001 \frac{\log n}{\log \log n}\right) h(\gamma) \log n \log P, \end{aligned} \quad (3.11)$$

where $\pi(P; m, a)$ counts primes $p \leq x$ satisfying $p \equiv a \pmod{m}$. We estimate trivially $\pi(P; n, 1) \leq P/n$. Thus

$$\Sigma_{\mathfrak{p}1} \leq 2P^2 \exp\left(-0.001 \frac{\log n}{\log \log n}\right) h(\gamma) \frac{(\log n)^2}{n}. \quad (3.12)$$

Combining this with (3.9) and use (2.5), we have

$$P^2 \geq 0.1 \frac{n^2}{(\log n)^2 \log \log n} \exp(0.001 \frac{\log n}{\log \log n}).$$

This implies (3.1) for $n \geq n_0$.

3.2 Case (3.10)

In this case, since $f_{\mathfrak{p}} = 2$, we write p instead of \mathfrak{p} . Suppose σ is the generator of $\text{Gal}(K/\mathbb{Q})$. For such p we have $\nu_p(\gamma^n - 1) = \nu_p((\gamma^\sigma)^n - 1)$. Let $\beta = \gamma^\sigma/\gamma$, we obtain the following inequalities:

$$\nu_p(\beta^n - 1) \geq \nu_p((\gamma^\sigma)^n - 1 - (\gamma^n - 1)) \geq \nu_p(\gamma^n - 1) \geq \nu_p(\Phi_n(\gamma)).$$

Hence (3.10) implies

$$\sum_{p \in \mathcal{P}} \nu_p(\beta^n - 1) \log p \geq 0.2 \varphi(n) h(\gamma). \quad (3.13)$$

where

$$\mathcal{P} = \{p : p \text{ is inert in } K \text{ and } \nu_p(\gamma^n - 1) > 0\}.$$

Denote $v_n = \beta^n - 1$. If $\nu_p(v_n) > 0$ then there exists a unique divisor d of n such that p is primitive for v_n/d . We denote it by d_p . Since $\beta^n - 1 = \prod_{d|n} \Phi_d(\beta)$,

$$\nu_p(v_n) \leq \nu_p(v_n/d_p) + \sum_{\substack{m|n \\ m \neq n/d_p}} \nu_p(\Phi_m(\beta)).$$

By item 2 of Proposition 2.4,

$$\sum_{\substack{m|n \\ m \neq n/d_p}} \nu_p(\Phi_m(\beta)) \leq \sum_{m|n} \nu_p(m) + \sum_{m=1}^7 \nu_p(\Phi_m(\beta)).$$

It follows that

$$\sum_{p \in \mathcal{P}} \nu_p(\beta^n - 1) \log p \leq \nu_p(v_{n/d_p}) \log p + \sum_{m|n} \log m + \sum_{m=1}^7 \sum_{p \in \mathcal{P}} \nu_p(\Phi_m(\beta)) \log p. \quad (3.14)$$

Trivially

$$\sum_{m|n} \log m \leq \tau(n) \log n. \quad (3.15)$$

Notice that

$$\nu_p(\Phi_m(\beta)) \leq \nu_p(v_m) \leq \frac{1}{2} \nu_p((\gamma^m - (\gamma^\sigma)^m)^2)$$

and $(\gamma^m - (\gamma^\sigma)^m)^2$ is a rational integer of absolute value not exceeding $4(\max\{|\gamma|, |\gamma^\sigma|\})^{2m}$, we have

$$\sum_p \nu_p(v_m) \log p \leq \log 2 + m \log^+(\max\{|\gamma|, |\gamma^\sigma|\}) \leq \log 2 + 2mh(\gamma). \quad (3.16)$$

Hence

$$\sum_{m=1}^7 \sum_{p \in \mathcal{P}} \nu_p(\Phi_m(\beta)) \log p \leq 7 \log 2 + 56h(\gamma). \quad (3.17)$$

Combining (3.13)(3.14)(3.15)(3.17), we obtain

$$\sum_{p \in \mathcal{P}} \nu_p(v_{n/d_p}) \log p \geq 0.2\varphi(n)h(\gamma) - \tau(n) \log n - 56h(\gamma) - 7 \log 2.$$

3.2.1 Big d_p

Using (3.16),

$$\sum_{d_p \geq \tau(n) \log n} \sum_{p \in \mathcal{P}} \nu_p(v_{n/d_p}) \log p \leq 2nh(\gamma) \sum_{\substack{d|n \\ d \geq \tau(n) \log n}} \frac{1}{d} + \tau(n) \log 2.$$

The sum on the right is trivially bounded by

$$\frac{\tau(n)}{\tau(n) \log n} = \frac{1}{\log n}.$$

Hence

$$\sum_{d_p \geq \tau(n) \log n} \nu_p(v_{n/d_p}) \log p \leq \frac{2nh(\gamma)}{\log n} + \tau(n) \log 2.$$

Denote by \mathcal{P}' the subset of \mathcal{P} consisting of p with $d_p < \tau(n) \log n$:

$$\mathcal{P}' = \{p \in \mathcal{P} : d_p < \tau(n) \log n\}.$$

So we have

$$\begin{aligned} \sum_{p \in \mathcal{P}'} \nu_p(v_{n/d_p}) \log p &\geq 0.2\varphi(n)h(\gamma) - \tau(n) \log n - 56h(\gamma) - 7 \log 2 \\ &\quad - \frac{2nh(\gamma)}{\log n} - \tau(n) \log 2. \end{aligned}$$

Using (2.5)(2.7), since $n \geq \exp \exp(10^{10})$, we obtain

$$\sum_{p \in \mathcal{P}'} \nu_p(v_{n/d_p}) \log p \geq 0.1\varphi(n)h(\gamma). \quad (3.18)$$

3.2.2 Small $d_p : d_p < \tau(n) \log n$

In subsections 3.2.2 and 3.2.3 of [2], the authors use estimates of counting function for S -units to bound $\#\{d \mid n : d < \tau(n) \log n\}$ and then give a upper bound for $\#\mathcal{P}'$. One can verify that these bounds are still effective in our case, as following:

$$\#\{d \mid n : d < \tau(n) \log n\} \leq \exp\left(70 \frac{\log n \log \log \log n}{(\log \log n)^2}\right). \quad (3.19)$$

$$\#\mathcal{P}' \leq \left(\frac{P}{n} + 1\right) \exp\left(80 \frac{\log n \log \log \log n}{(\log \log n)^2}\right). \quad (3.20)$$

3.2.3 Using Theorem 2.2

By item 1 of Proposition 2.4, for $p \in \mathcal{P}'$, we have $n \mid p^2 - 1$. Hence $p > n^{1/2} \geq n_0^{1/2}$. Since $n_0^{1/2} \geq p_0 = \exp \exp(\max\{10^8, 2|D_K|\})$, Theorem 2.2 applies:

$$\begin{aligned} \nu_p(\beta^n - 1) &\leq p \exp\left(-0.001 \frac{\log p}{\log \log p}\right) h(\beta) \log n \\ &\leq 2P \exp\left(-0.0005 \frac{\log n}{\log \log n}\right) h(\gamma) \log n. \end{aligned} \quad (3.21)$$

The last inequality holds because

$$p \leq P, \quad \frac{\log p}{\log \log p} \geq \frac{1}{2} \frac{\log n}{\log \log n}, \quad h(\beta) \leq 2h(\gamma).$$

Since $\nu_p(v_{n/d_p}) \leq \nu_p(\beta^n - 1)$, we obtain,

$$\sum_{p \in \mathcal{P}'} \nu_p(v_{n/d_p}) \log p \leq 2P \exp\left(-0.0005 \frac{\log n}{\log \log n}\right) h(\gamma) \log P \log n \#\mathcal{P}'. \quad (3.22)$$

Combining (3.18)(3.22),

$$0.1\varphi(n)h(\gamma) \leq 2P \exp\left(-0.0005\frac{\log n}{\log \log n}\right) h(\gamma) \log P \log n \#P'.$$

Using (3.20) and (2.5), we obtain, for $n \geq \exp \exp(10^{10})$,

$$\begin{aligned} 40P(P+n) \log P &\geq \frac{n^2}{\log n \log \log n} \exp\left(\frac{\log n(\log \log n - 160000 \log \log \log n)}{2000(\log \log n)^2}\right) \\ &\geq n^2 \exp\left(0.0004\frac{\log n}{\log \log n}\right). \end{aligned}$$

Obviously $P \geq n$, so we have

$$80P^2 \log P \geq n^2 \exp\left(0.0004\frac{\log n}{\log \log n}\right),$$

which implies (3.1) and we are done.

Acknowledgments The author thanks Yuri Bilu for checking the proof, polishing the exposition and helpful discussions. The author also acknowledges support of China Scholarship Council grant CSC202008310189.

References

- [1] Yuri Bilu, Haojie Hong, and Sanoli Gun, *Uniform explicit Stewart's theorem on prime factors of linear recurrences*, arXiv:2108.09857 (2021).
- [2] Yuri Bilu, Haojie Hong, and Florian Luca, *Big prime factors in orders of elliptic curves over finite fields*, arXiv:2112.07046 (2021).
- [3] Yuri Bilu and Florian Luca, *Binary polynomial power sums vanishing at roots of unity*, Acta Arith. **198** (2021), no. 2, 195–217. MR 4228301
- [4] P. Erdős, *Some recent advances and current problems in number theory*, Lectures on Modern Mathematics, Vol. III, pp. 196–244. Wiley, New York, 1965.
- [5] J.-L. Nicolas and G. Robin, *Majorations explicites pour le nombre de diviseurs de N* , Canad. Math. Bull. **26** (1983), no. 4, 485–492. MR 716590
- [6] Guy Robin, *Estimation de la fonction de Tchebychef θ sur le k -ième nombre premier et grandes valeurs de la fonction $\omega(n)$ nombre de diviseurs premiers de n* , Acta Arith. **42** (1983), no. 4, 367–389. MR 736719
- [7] J. Barkley Rosser and Lowell Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94. MR 137689
- [8] A. Schinzel, *On primitive prime factors of $a^n - b^n$* , Proc. Cambridge Philos. Soc. **58** (1962), 555–562. MR 0143728
- [9] A. Schinzel, *Primitive divisors of the expression $A^n - B^n$ in algebraic number fields*, J. Reine Angew. Math. **268(269)** (1974), 27–33. MR 344221
- [10] Cameron L. Stewart, *On divisors of Lucas and Lehmer numbers*, Acta Math. **211** (2013), no. 2, 291–314. MR 3143892

Haojie Hong: Institut de Mathématiques de Bordeaux, Université de Bordeaux & CNRS, Talence, France