LOCALITY OF VORTEX STRETCHING FOR THE 3D EULER EQUATIONS

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ABSTRACT. We consider the 3D incompressible Euler equations under the following situation: small-scale vortex blob being stretched by a prescribed large-scale stationary flow. More precisely, we clarify what kind of large-scale stationary flows really stretch small-scale vortex blobs in alignment with the straining direction. The key idea is constructing a Lagrangian coordinate so that the Lie bracket is identically zero (c.f. the Frobenius theorem), and investigate the locality of the pressure term by using it.

1. INTRODUCTION

The most important features of the Navier-Stokes turbulence is that turbulence is not random but composed of vortex stretching behavior. More precisely, recent DNS [2, 3, 12, 13] of the Navier-Stokes turbulence at sufficiently high Reynolds numbers have reported that there exists a hierarchy of vortex stretching motions. In particular, Goto-Saito-Kawahara [3] clearly observed that turbulence at sufficiently high Reynolds numbers in a periodic cube is composed of a self-similar hierarchy of antiparallel pairs of vortex tubes, and it is sustained by creation of smaller-scale vortices due to stretching in larger-scale strain fields. This observation is further investigated by Y-Goto-Tsuruhashi [15] (see also [14]). Thus we could conclude physically that local-scale energy transfer is mainly induced by vortex stretching (see also [5, 6, 7] for the related mathematical results). Therefore as the sequence of these studies, our next study will be clarifying the vortex stretching dynamics precisely.

In this paper we mathematically consider the locality of small-scale vortex dynamics in the 3D incompressible Euler equations. More precisely, we consider the inviscid flow under the following situation: small-scale vortex blob being stretched by a prescribed large-scale stationary flow, and we clarify what kind of large-scale stationary flows really stretch smaller-scale vortex blobs in alignment with the straining direction. Now let us describe the incompressible Euler equations (inviscid flow) as follows:

$$(1)\quad \partial_t u + (u\cdot\nabla)u = (\partial_t(u\circ\Phi))\circ\Phi^{-1} = \partial_t^2\Phi\circ\Phi^{-1} = -\nabla p, \quad \nabla\cdot u = 0 \quad \text{in} \quad \mathbb{R}^3,$$
$$u|_{t=0} = u_0,$$

where Φ is the associated Lagrangian flow given by

$$\partial_t\Phi(t,x) = u(t,\Phi(t,x)) =: u\circ\Phi \quad \text{with} \quad \Phi(0,x) = x\in\mathbb{R}^3.$$

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Let $u^S : (-\epsilon,\epsilon)\times\mathbb{R}^3 \to \mathbb{R}^3$ be the flow of a small-scale vortex blob and $u^L : \mathbb{R}^3 \to \mathbb{R}^3$ be a prescribed large-scale stationary flow. Then the associated Lagrangian flows η^L and η^S satisfying $\Phi(t,x) = \eta^L(t,\eta^S(t,x)) = \eta^L \circ \eta^S$ are given by

$$\partial_t\eta^L(t,x) = u^L(\eta^L(t,x)) =: u^L \circ \eta^L \quad \text{with} \quad \eta^L(0,x) = x \in \mathbb{R}^3,$$
$$\partial_t\eta^S(t,x) = u^S(t,\eta^S(t,x)) =: u^S \circ \eta^S \quad \text{with} \quad \eta^S(0,x) = x \in \mathbb{R}^3.$$

Let us assume $\eta^S(t,x_0) = \eta^S(0,x_0) = x_0 \in \ell$, where

$$\ell := \bigcup_{t\in(-\epsilon,\epsilon)} \eta^L(t,x_*) \quad \text{for some} \quad x_* \in \mathbb{R}^3. \tag{2}$$

ℓ represents the rotating axis of the small-scale vortex blob which aligns with the large-scale straining flow. Based on DNS of homogeneous isotropic turbulence, Hamlington-Schumacher-Dahm [4] showed vorticity tends to align with the stretching direction of the background strain. Note that, in their study, the background strain means the strain rate induced by the vorticity beyond radius $\sim 12\eta$ (η is the Kolmogorov scale). See also [3]. With the aid of this physical observation, we constructed the mathematical model (2).

In general, η^S is strongly affected by the large-scale straining flow η^L when $\Phi = \eta^L \circ \eta^S$ comes from an Euler flow, thus, in this study, we need to clarify the nonlinear interaction even partially.

Remark 1. For readers' convenience, in this remark, we state a typical vortex stretching as an example. Let $x_0 \in \{x_1 = x_2 = 0\} = \ell$, and let (r,θ,x_3) be the cylindrical coordinate such that $(x_1,x_2) = (r\cos\theta, r\sin\theta)$ with

$$e_r := \frac{(x_1,x_2,0)}{\sqrt{x_1^2+x_2^2}}, \quad e_\theta := \frac{(-x_2,x_1,0)}{\sqrt{x_1^2+x_2^2}} \quad \text{and} \quad e_{x_3} := (0,0,1).$$

First we give the typical straining flow $(u_r^L, u_\theta^L, u_{x_3}^L)$ as follows:

$$u_r^L := u^L\cdot e_r = -r, \quad u_{x_3}^L := u^L\cdot e_{x_3} = 2x_3 \quad \text{and} \quad u_\theta^L := u^L\cdot e_\theta = 0.$$

In this case the corresponding Lagrangian flow is

$$(\eta_r^L, \eta_\theta^L, \eta_{x_3}^L) = (e^{-t}r_0, \theta_0, e^{2t}x_{3,0}),$$

where $(r_0,\theta_0,x_{3,0})$ represents the initial position. For $\omega := \nabla\times u$, let $\omega^S := \nabla\times u^S$ be axisymmetric vorticity depending only on r variable, such that

$$\omega_\theta^S(t,r) := \omega^S\cdot e_\theta, \quad \omega_{x_3}^S := \omega^S\cdot e_{x_3} = 0 \quad \text{and} \quad \omega_r^S := \omega^S\cdot e_r = 0.$$

Then we have the following explicit solution (vorticity) to the Euler equations:

$$\omega_\theta^S(t,\eta_r^L(t)) = e^t\omega_{0,\theta}^S(e^{-t}r_0) \quad \text{and} \quad \omega_{0,\theta}^S = \omega_\theta^S(t)|_{t=0}. \tag{3}$$

With the aid of the typical stretching motion in Remark 1, we rigorously define the meaning of "stably stretching" as follows.

Definition 1. (Stably stretching.) η^L is said to be *stably stretching* along the rotating axis $\ell\subset\mathbb{R}^3$, if the following two conditions hold.

- The rotation axis ℓ is stretched along $\partial_t\eta^L$ direction, that is

$$\partial_t|\partial_t\eta^L(t,x_0)| > 0 \quad \text{for each} \quad x_0 \in \ell. \tag{4}$$

- Let

$$x_0^\perp := \{x \in \mathbb{R}^3 : (x_0 - x)\cdot \partial_t \eta^L(0, x_0) = 0,\ |x_0 - x| < \delta\} \quad \text{for each} \quad x_0 \in \ell.$$

  The time evolution of the surface $x_0^\perp$ (accompanied by the fluid particles) is always perpendicular to the stretching direction at $\eta^L(t, x_0)$ $(t > 0)$, that is,

$$\partial_t \eta^L(t, x_0) \perp \Phi(t, x_0^\perp) \quad \text{for each} \quad x_0 \in \ell. \tag{5}$$

Remark 2. (3) satisfies (4) and (5).

To state our theorem, we need to prepare "curvature". First let us choose a point $x_0 \in \ell$ (reference point) and fix it. Identifying $\frac{d}{dz}$ with ∂_z, we define $t(z)$ as

$$\partial_z t > 0, \quad |\partial_z \eta^L| := |\partial_z \eta^L(t(z), x_0)| = 1 \quad \text{and} \quad t(0) = 0. \tag{6}$$

In this case we immediately have $\partial_z t = |\partial_t \eta^L|^{-1}$ and its inverse is $\partial_t z = |\partial_t \eta^L|$ (with the variables omitted). Then define the unit tangent vector τ as

$$\tau(z) = \partial_z \eta^L(t(z), x_0),$$

the unit curvature vector n as $\kappa n = \partial_z \tau$ with a curvature function $\kappa(z) > 0$, the unit torsion vector b as : $b(z) := \tau(z) \times n(z)$, where $\times$ is the exterior product. Without loss of generality, we can assume the torsion function $T(z)$ is positive by choosing the orientation of the torsion vector $b(z)$. Then we now state our main theorem.

Theorem 1. *Let $(\partial_t \Phi)\circ \Phi^{-1} = \partial_t(\eta^L \circ \eta^S)\circ(\eta^L\circ\eta^S)^{-1}$ be a solution to the Euler equations. If η^L is stably stretching along $\ell \subset \mathbb{R}^3$, then we have*

$$\partial_z(\kappa |\partial_t \eta^L|^2) = 0 \quad \textit{for each} \quad x_0 \in \ell, \tag{7}$$

in particular, we do not need any information of η^S,

Remark 3. Since $\partial_t |\partial_t \eta^L| > 0$, we obtain a necessary condition $\partial_z \kappa(z) \le 0$.

Finally, we examine a typical straining flow $(-x_1, x_2, 0)$ whether or not it satisfies the formula (7). Note that we could also examine the straining flow $(-x_1, -x_2, 2x_3)$ which is already appeared in Remark 1, but in this case the formula becomes much more complicated, thus we leave it as a reader's exercise.

Corollary 2. *If u^L is a straining flow such that $u^L = (-x_1, x_2, 0)$, then we have*

$$\partial_z(\kappa|\partial_t \eta^L|^2) = -\frac{\tanh 2t}{\cosh 2t} \quad \textit{with} \quad t = \frac{1}{2}\log\left(\frac{x_1}{x_2}\right).$$

This means that, if $\ell \not\subset \{x_1 = 0\}$ and is in the stretching region: $\ell \subset \{x : |x_2| > |x_1|\}$, then the pair of η^L and $\Phi = \eta^L \circ \eta^S$ does not satisfy (5)*, which implies that η^L is "unstably" stretching along ℓ.*

2. Proof of Theorem 1

First we rephrase the initial flat plane $x_0^\perp$ such that

$$x_0^\perp = \{x_0 + r_1 n(0) + r_2 b(0) : \sqrt{r_1^2 + r_2^2} < \delta\}. \tag{8}$$

For any initial particle on the plane $x = x_0 + r_1 n(0) + r_2 b(0) \in x_0^\perp$, $\Phi(t,x)$ is uniquely expressed as (we omit the change of variables)

$$\begin{aligned}\Phi(t,x) &=: \Phi(z,r_1,r_2)\\ &= \eta^L(t(z),x_0) + Z(z,r_1,r_2)\tau(z) + R_1(z,r_1,r_2)n(z) + R_2(z,r_1,r_2)b(z),\end{aligned} \tag{9}$$

for sufficiently small r_1 and r_2, with $Z(z,0,0)=0$, $Z(0,r_1,r_2)=0$, $R_1(0,r_1,r_2)=r_1$ and $R_2(0,r_1,r_2)=r_2$.

Remark 4. We can rephrase (5) as follows:

$$\partial_{r_1} Z|_{r_1=r_2=0} = \partial_{r_2} Z|_{r_1=r_2=0} = 0 \quad \text{for} \quad z>0. \tag{10}$$

This is due to the fact that

$$\Phi(t,x_0^\perp) = \bigcup_{\sqrt{r_1^2+r_2^2}<\delta} \Phi(z(t),r_1,r_2)$$

by (8) and (9), and

$$\partial_t \eta^L(t,x_0)\cdot\partial_{r_1}\Phi(z(t),r_1,r_2)|_{r_1=r_2} = \partial_t\eta^L(t,x_0)\cdot\partial_{r_2}\Phi(z(t),r_1,r_2)|_{r_1=r_2}=0.$$

Since the corresponding Jacobian $\frac{\partial(R_1,R_2)}{\partial(r_1,r_2)}$ is clearly nonzero for sufficiently small $z>0$, so, by the inverse function theorem (for each $z>0$), we can rewrite the equality (9) as follows: Let R_1 and R_2 be variables, and r_1 and r_2 be the corresponding inverse functions. For any particle $x = x_0 + r_1(z,R_1,R_2)n(0) + r_2(z,R_1,R_2)b(0) \in x_0^\perp$,

$$\begin{aligned}\Phi(t,x) &=: \Phi(z,R_1,R_2)\\ &= \eta^L(t(z),x_0) + Z(z,r_1(z,R_1,R_2),r_2(z,R_1,R_2))\tau(z) + R_1 n(z) + R_2 b(z)\end{aligned} \tag{11}$$

for sufficiently small $|R_1|,|R_2|, z>0$. Let us recall the Frenet-Serret formulas:

$$\frac{d}{dz}\begin{pmatrix}\tau\\ n\\ b\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & T\\ 0 & -T & 0\end{pmatrix}\begin{pmatrix}\tau\\ n\\ b\end{pmatrix}.$$

By combining the Frenet-Serret formulas, (10) and (11), we have

$$\begin{cases}\partial_z\Phi &= \tau + R_1(Tb-\kappa\tau) - R_2Tn + \mathcal{O}(R_1^2,R_2^2),\\ \partial_{R_1}\Phi &= n + \mathcal{O}(R_1,R_2),\\ \partial_{R_2}\Phi &= b+\mathcal{O}(R_1,R_2),\end{cases} \tag{12}$$

where $\mathcal{O}$ is Landau's notation. A direct calculation yields

$$\begin{aligned}\partial_z^2\Phi =& \kappa n - R_1(T^2+\kappa^2)n + R_1((\partial_z T)b - (\partial_z\kappa)\tau)\\ &- R_2T(-\kappa\tau+Tb) - R_2(\partial_z T)n + \mathcal{O}(R_1^2,R_2^2)\end{aligned}$$

and then

$$\begin{aligned}\partial_{R_1}\partial_z\Phi|_{R_1=R_2=0} &= Tb-\kappa\tau,\\ \partial_{R_1}\partial_z^2\Phi|_{R_1=R_2=0} &= -(T^2+\kappa^2)n + ((\partial_z T)b - (\partial_z\kappa)\tau),\\ \partial_{R_2}\partial_z\Phi|_{R_1=R_2=0} &= -Tn,\\ \partial_{R_2}\partial_z^2\Phi|_{R_1=R_2=0} &= -T(-\kappa\tau+Tb) - (\partial_z T)n.\end{aligned}$$

Thus

$$(\partial_{R_1}\partial_z^2\Phi)\cdot\tau|_{R_1,R_2=0} = -\partial_z\kappa \quad \text{and} \quad (\partial_{R_1}\partial_z\Phi)\cdot\tau|_{R_1,R_2=0} = -\kappa.$$

On the other hand, by the Leibniz rule, we see

$$\partial_t^2\Phi = \partial_z^2\Phi(\partial_t z)^2 + \partial_z\Phi\partial_t^2 z \quad \text{with the variables omitted.}$$

Combining the facts $\partial_t^2 z = \partial_t|\partial_t\eta^L|$ and $\partial_t z = |\partial_t\eta^L|$, we have

$$-\partial_{R_1}(\nabla p\cdot\tau) = \partial_{R_1}(\partial_t^2\Phi\cdot\tau) = (\partial_{R_1}\partial_t^2\Phi)\cdot\tau = -\kappa\partial_t|\partial_t\eta^L| - \partial_z\kappa|\partial_t\eta^L|^2, \tag{13}$$

$$-\partial_{R_2}(\nabla p\cdot\tau) = \partial_{R_2}(\partial_t^2\Phi\cdot\tau) = (\partial_{R_2}\partial_t^2\Phi)\cdot\tau = +T\kappa|\partial_t\eta^L|^2 \tag{14}$$

for $R_1 = R_2 = 0$. Next we derive other formulae by using the Euler equations. By the Leibniz rule, we see

$$\kappa n = \partial_z^2\eta^L(t(z), x_0) = \partial_z(\partial_t\eta^L\partial_z t) = \partial_t^2\eta^L(\partial_z t)^2 + \partial_t\eta^L\partial_z^2 t.$$

Combining $\partial_z^2 t = \partial_z|\partial_t\eta^L|^{-1} = -|\partial_t\eta^L|^{-2}\partial_z|\partial_t\eta^L| = -|\partial_t\eta^L|^{-3}\partial_t|\partial_t\eta^L|$, we have

$$\partial_t^2\eta^L = |\partial_t\eta^L|^2\kappa n + \partial_t|\partial_t\eta^L|\tau.$$

By using the Euler equations, we have

$$\begin{aligned}
-\nabla p\cdot\tau &= \partial_t^2\Phi\cdot\tau = \partial_t|\partial_t\eta^L|,\\
-\nabla p\cdot n &= \partial_t^2\Phi\cdot n = \kappa|\partial_t\eta^L|^2,\\
-\partial_z(\nabla p\cdot n) &= \partial_z\kappa|\partial_t\eta^L|^2 + 2\kappa\partial_t|\partial_t\eta^L|,\\
-\nabla p\cdot b &= 0
\end{aligned}$$

for $R_1 = R_2 = 0$ with the change of variables $\circ\eta^L$ omitted again. On the other hand, from (12), we have the following inverse matrix:

$$\begin{pmatrix}\tau\\ n\\ b\end{pmatrix} = \begin{pmatrix}(1-\kappa R_1)^{-1} & R_2T(1-\kappa R_1)^{-1} & -R_1T(1-\kappa R_1)^{-1}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\partial_z\Phi\\ \partial_{R_1}\Phi\\ \partial_{R_2}\Phi\end{pmatrix}$$

with the higher order terms omitted since we finally take $R_1, R_2 \to 0$. Then we see

$$\begin{aligned}
\nabla p\cdot\tau =&(1-\kappa R_1)^{-1}(\nabla p\cdot\partial_z\Phi)\\
&+ R_2T(1-\kappa R_1)^{-1}(\nabla p\cdot\partial_{R_1}\Phi) - R_1T(1-\kappa R_1)^{-1}(\nabla p\cdot\partial_{R_2}\Phi)\\
=&(1-\kappa R_1)^{-1}\partial_z(p\circ\Phi)\\
&+ R_2T(1-\kappa R_1)^{-1}\partial_{R_1}(p\circ\Phi) - R_1T(1-\kappa R_1)^{-1}\partial_{R_2}(p\circ\Phi).
\end{aligned}$$

and then (omit the variable $\circ\Phi$)

$$\begin{aligned}
-\partial_{R_1}(\nabla p\cdot\tau)|_{R_1=R_2=0} &= (-\kappa\partial_z p - \partial_{R_1}\partial_z p - T\partial_{R_2}p)\,|_{R_1=R_2=0}\\
(\text{commute } \partial_{R_1} \text{ and } \partial_z) &= (-\kappa(\nabla p\cdot\tau) - \partial_z(\nabla p\cdot n) - T(\nabla p\cdot b))\,|_{R_1=R_2=0}\\
&= 3\kappa\partial_t|\partial_t\eta^L| + \partial_z\kappa|\partial_t\eta^L|^2.
\end{aligned}$$

Combining (13), we have the desired formula.

Remark 5. We can rephrase the commutativity of ∂_{R_1} and ∂_z as

$$[\partial_z, \partial_{R_1}] = \partial_z\partial_{R_1} - \partial_{R_1}\partial_z = 0,$$

where $[\cdot,\cdot]$ is the Lie braket (c.f. the Frobenius theorem, see Chapter 19 in [8] for example). For the previous studies using this property, see Chan-Czubak-Y [1, Section 2.5] and Lichtenfelz-Y [9], more originally, see Ma-Wang [10, (3.7)].

Remark 6. Since $\nabla p\cdot b=\partial_{R_2}p\equiv 0$, then

$$\begin{aligned}-\partial_{R_2}(\nabla p\cdot\tau)&=-\partial_{R_2}\partial_z p-T\partial_{R_1}p\\ (\text{commute }\partial_{R_2}\text{ and }\partial_z)&=-T(\nabla p\cdot n)=T\kappa|\partial_t\eta^L|^2\end{aligned}$$

for $R_1=R_2=0$. However this formula is useless, since it coincides with (14).

3. Proof of Corollary 2

For any $x\in\{x:|x_2|>|x_1|\}\cap\{x_1\neq 0\}$, let us set

$$\eta^L(t,x)=\begin{pmatrix}re^{t+t_0}\\ re^{-(t+t_0)}\\ x_3\end{pmatrix},$$

where $r:=\sqrt{x_1x_2}$ and $t_0=\frac{1}{2}\log(x_1/x_2)$. In this case we see

$$\partial_z t:=|\partial_t\eta^L|^{-1}=\frac{1}{r(e^{2t}+e^{-2t})^{1/2}}=\frac{1}{r\sqrt{2}(\cosh 2t)^{1/2}}$$

and

$$\partial_z^2 t=-\frac{(\sinh 2t)\partial_z t}{r\sqrt{2}(\cosh 2t)^{3/2}}=-\frac{\sinh 2t}{2r^2(\cosh 2t)^2}=-\frac{\tanh 2t}{2r^2\cosh 2t}.$$

On the other hand,

$$\begin{aligned}\kappa n&=\partial_t^2\eta^L(\partial_z t)^2+\partial_t\eta^L\partial_z^2 t\\ &=\frac{1}{2r^2\cosh 2t}\begin{pmatrix}re^t\\ re^{-t}\\ 0\end{pmatrix}-\frac{\tanh 2t}{2r^2\cosh 2t}\begin{pmatrix}re^t\\ -re^{-t}\\ 0\end{pmatrix}.\end{aligned}$$

Thus

$$\kappa^2=\frac{1}{2r^2\cosh 2t}-\frac{(\tanh 2t)^2}{r^2\cosh 2t}+\frac{(\tanh 2t)^2}{2r^2\cosh 2t}=\frac{1}{2r^2(\cosh 2t)^3}$$

and then

$$\kappa=\frac{1}{\sqrt{2}r(\cosh 2t)^{3/2}},\quad \partial_z\kappa=-\frac{3\tanh 2t}{2r^2(\cosh 2t)^2},$$

$$|\partial_t\eta^L|^2=2r^2\cosh 2t\quad\text{and}\quad \partial_t|\partial_t\eta^L|=\sqrt{2}r(\tanh 2t)^{1/2}(\sinh 2t)^{1/2}.$$

Combining the above calculations, we have the following desired formula:

$$\partial_z(\kappa|\partial_t\eta^L|^2)=2\kappa\partial_t|\partial_t\eta^L|+\partial_z\kappa|\partial_t\eta^L|^2=-\frac{\tanh 2t}{\cosh 2t}.$$

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