ON DUNKL SCHRÖDINGER SEMIGROUPS WITH GREEN BOUNDED POTENTIALS

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ABSTRACT. On $\mathbb{R}^N$ equipped with a normalized root system R, a multiplicity function $k(\alpha) > 0$, and the associated measure

$$dw(\mathbf{x}) = \prod_{\alpha\in R} |\langle \mathbf{x}, \alpha\rangle|^{k(\alpha)}\, d\mathbf{x},$$

we consider a Dunkl Schrödinger operator $L = -\Delta_k + V$, where Δ_k is the Dunkl Laplace operator and $V \in L^1_{\rm loc}(dw)$ is a non-negative potential. Let $h_t(\mathbf{x},\mathbf{y})$ and $k_t^{\{V\}}(\mathbf{x},\mathbf{y})$ denote the Dunkl heat kernel and the integral kernel of the semigroup generated by $-L$ respectively. We prove that $k_t^{\{V\}}(\mathbf{x},\mathbf{y})$ satisfies the following heat kernel lower bounds: there are constants $C, c > 0$ such that

$$h_{ct}(\mathbf{x},\mathbf{y}) \leq C k_t^{\{V\}}(\mathbf{x},\mathbf{y})$$

if and only if

$$\sup_{\mathbf{x}\in\mathbb{R}^N} \int_0^\infty \int_{\mathbb{R}^N} V(\mathbf{y}) w(B(\mathbf{x},\sqrt{t}))^{-1} e^{-\|\mathbf{x}-\mathbf{y}\|^2/t}\, dw(\mathbf{y})\, dt < \infty,$$

where $B(\mathbf{x},\sqrt{t})$ stands for the Euclidean ball centered at $\mathbf{x} \in \mathbb{R}^N$ and radius $\sqrt{t}$.

1. Introduction and statement of the results

Let $A = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^N$, $N \geq 3$. It is well-known (see [19]) that if $V \geq 0$, $V \in L^1_{\rm loc}(\mathbb{R}^N(dx))$, then the kernel $k_t(x,y)$ of the semigroup $\{e^{-tA}\}_{t\geq 0}$ satisfies the Gaussian (heat kernel) lower bounds

$$t^{-N/2} e^{-\|x-y\|^2/4ct} \leq C k_t(x,y)$$

with certain constants $C, c > 0$ if and only if the potential V is Green bounded, that is,

$$\sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{V(y)}{\|x-y\|^{N-2}}\, dy < \infty.$$

The aim of this paper is to prove similar results in the Dunkl setting.

On the Euclidean space $\mathbb{R}^N$ equipped with a normalized root system R and a multiplicity function $k : R \longmapsto (0,\infty)$, let Δ_k denote the Dunkl Laplace operator (see Section 2). Let $dw(\mathbf{x}) = w(\mathbf{x})\, d\mathbf{x}$ be the associated measure, where

$$w(\mathbf{x}) = \prod_{\alpha\in R} |\langle \mathbf{x}, \alpha\rangle|^{k(\alpha)} \tag{1.1}$$

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is its density with respect to the Lebesgue measure $d\mathbf{x}$. For a Lebesgue measurable set $F \subseteq \mathbb{R}^N$, we denote

$$w(F) := \int_F dw(\mathbf{x}). \tag{1.2}$$

It is well-known that Δ_k generates a semigroup $\{H_t\}_{t\geq 0} = \{e^{t\Delta_k}\}_{t\geq 0}$ of linear operators on $L^2(dw)$ which has the form

$$H_t f(\mathbf{x}) = \int_{\mathbb{R}^N} h_t(\mathbf{x},\mathbf{y}) f(\mathbf{y})\, dw(\mathbf{y}),$$

where $0 < h_t(\mathbf{x},\mathbf{y})$ is a smooth function called the Dunkl heat kernel (see Section 2.3 for more details).

Let $V \in L^1_{\rm loc}(dw)$ be a non-negative potential. Consider the Dunkl Schrödinger operator

$$L = -\Delta_k + V.$$

Then $-L$ generates a semigroup $\{e^{-tL}\}_{t\geq 0}$ of self-adjoint linear contractions on $L^2(dw)$. The semigroup $\{e^{-tL}\}_{t\geq 0}$ has the form

$$e^{-tL} f(\mathbf{x}) = \int_{\mathbb{R}^N} k_t^{\{V\}}(\mathbf{x},\mathbf{y}) f(\mathbf{y})\, dw(\mathbf{y}),$$

where the integral kernel $k_t^{\{V\}}(\mathbf{x},\mathbf{y})$ satisfies upper heat kernel bounds

$$0 \leq k_t^{\{V\}}(\mathbf{x},\mathbf{y}) \leq h_t(\mathbf{x},\mathbf{y}). \tag{1.3}$$

The main goal of this paper is to characterize non-negative potentials $V \in L^1_{\rm loc}(dw)$ for which $k_t^{\{V\}}(\mathbf{x},\mathbf{y})$ satisfies the following heat kernel lower bound

$$h_{Ct}(\mathbf{x},\mathbf{y}) \leq C k_t^{\{V\}}(\mathbf{x},\mathbf{y}).$$

In order to state the result we need to introduce some notation. For $\alpha \in R$, let

$$\sigma_\alpha(\mathbf{x}) := \mathbf{x} - 2\frac{\langle \mathbf{x},\alpha\rangle}{\|\alpha\|^2}\alpha \tag{1.4}$$

be the reflection with respect to the subspace perpendicular to α. Let G denote the reflection group generated by the reflections σ_α, $\alpha \in R$. We define the distance of the orbit of $\mathbf{x}$ to the orbit of $\mathbf{y}$ by

$$d(\mathbf{x},\mathbf{y}) = \min\{\|\mathbf{x} - \sigma(\mathbf{y})\| : \sigma \in G\}. \tag{1.5}$$

Obviously,

$$d(\mathbf{x},\mathbf{y}) = d(\mathbf{x},\sigma(\mathbf{y})) \quad \text{for all } \mathbf{x},\mathbf{y} \in \mathbb{R}^N \text{ and } \sigma \in G.$$

Let

$$B(\mathbf{x}, r) = \{\mathbf{x}' \in \mathbb{R}^N : \|\mathbf{x} - \mathbf{x}'\| \leq r\}$$

stands for the (closed) Euclidean ball centered at $\mathbf{x} \in \mathbb{R}^N$ and radius $r > 0$.

Let $\mathbf{N} = N + \sum_{\alpha\in R} k(\alpha)$ be the homogeneous dimension of the system (R,k). Throughout this paper we shall assume that $\mathbf{N} > 2$

Our goal is to prove the following theorem.

Theorem 1.1. *Assume that* $\mathbf{N} > 2$ *and* $V\colon \mathbb{R}^N \longmapsto [0,\infty)$, $V \in L^1_{\text{loc}}(dw)$. *Then the following are equivalent.*

(a) The kernel $k_t^{\{V\}}(\mathbf{x},\mathbf{y})$ *satisfies the following Dunkl heat kernel lower bound: there are constants* $C, c > 0$ *such that for all* $\mathbf{x},\mathbf{y} \in \mathbb{R}^N$ *and* $t > 0$ *we have*

$$h_{ct}(\mathbf{x},\mathbf{y}) \leq C k_t^{\{V\}}(\mathbf{x},\mathbf{y}).$$

(b) There is a constant $\delta > 0$ *such that for all* $\mathbf{x} \in \mathbb{R}^N$ *and* $t > 0$ *we have*

$$\int_{\mathbb{R}^N} k_t^{\{V\}}(\mathbf{x},\mathbf{y})\, dw(\mathbf{y}) \geq \delta.$$

(c) The potential V *is Green bounded, that is,*

$$\sup_{\mathbf{x}\in\mathbb{R}^N} \int_0^\infty \int_{\mathbb{R}^N} V(\mathbf{y}) h_s(\mathbf{x},\mathbf{y})\, dw(\mathbf{y})\, ds < \infty.$$

Remark 1.2. Condition (c) of Theorem 1.1 is equivalent to any of the following ones:

(c')

$$\sup_{\mathbf{x}\in\mathbb{R}^N} \int_0^\infty \int_{\mathbb{R}^N} V(\mathbf{y}) w(B(\mathbf{x},\sqrt{s}))^{-1} e^{-d(\mathbf{x},\mathbf{y})^2/s}\, dw(\mathbf{y})\, ds < \infty,$$

(c")

$$\sup_{\mathbf{x}\in\mathbb{R}^N} \int_0^\infty \int_{\mathbb{R}^N} V(\mathbf{y}) w(B(\mathbf{x},\sqrt{s}))^{-1} e^{-\|\mathbf{x}-\mathbf{y}\|^2/s}\, dw(\mathbf{y})\, ds < \infty.$$

The equivalences are proved in Proposition 7.1.

The proof of Theorem 1.1 depends very much on the upper and lower bounds for $h_t(\mathbf{x},\mathbf{y})$ derived in [9]. We present them in Subsection 2.4.

2. Preliminaries

2.1. **Basic definitions of the Dunkl theory.** In this section we present basic facts concerning the theory of the Dunkl operators. For more details we refer the reader to [5], [14], [16], and [18].

We consider the Euclidean space $\mathbb{R}^N$ with the scalar product $\langle \mathbf{x},\mathbf{y}\rangle = \sum_{j=1}^N x_j y_j$, where $\mathbf{x} = (x_1,\ldots,x_N)$, $\mathbf{y} = (y_1,\ldots,y_N)$, and the norm $\|\mathbf{x}\|^2 = \langle \mathbf{x},\mathbf{x}\rangle$.

A *normalized root system* in $\mathbb{R}^N$ is a finite set $R \subset \mathbb{R}^N \setminus \{0\}$ such that $R \cap \alpha\mathbb{R} = \{\pm\alpha\}$, $\sigma_\alpha(R) = R$, and $\|\alpha\| = \sqrt{2}$ for all $\alpha \in R$, where σ_α is defined by (1.4).

The finite group G generated by the reflections σ_α, $\alpha \in R$, is called the *reflection group* of the root system. Clearly, $|G| \geq |R|$.

A *multiplicity function* is a G-invariant function $k : R \to \mathbb{C}$ which will be fixed and > 0 throughout this paper.

Recall that $\mathbf{N} = N + \sum_{\alpha\in R} k(\alpha)$. Then,

$$w(B(t\mathbf{x},tr)) = t^{\mathbf{N}} w(B(\mathbf{x},r)) \ \text{ for all } \mathbf{x} \in \mathbb{R}^N,\ t,r > 0,$$

where w is the associated measure defined in (1.2). Observe that there is a constant $C > 0$ such that for all $\mathbf{x} \in \mathbb{R}^N$ and $r > 0$ we have

$$C^{-1} w(B(\mathbf{x},r)) \leq r^N \prod_{\alpha\in R} (|\langle \mathbf{x},\alpha\rangle| + r)^{k(\alpha)} \leq C w(B(\mathbf{x},r)), \tag{2.1}$$

so $dw(\mathbf{x})$ is doubling, that is, there is a constant $C>0$ such that

$$w(B(\mathbf{x},2r))\leq Cw(B(\mathbf{x},r)) \quad \text{for all } \mathbf{x}\in\mathbb{R}^N,\ r>0. \tag{2.2}$$

Let us also remark the the sets of measure zero with respect to the measure $dw(\mathbf{x})$ and the Lebesgue measure $d\mathbf{x}$ coincide.

For $\xi\in\mathbb{R}^N$, the *Dunkl operators* T_ξ are the following k-deformations of the directional derivatives ∂_ξ by difference operators:

$$T_\xi f(\mathbf{x})=\partial_\xi f(\mathbf{x})+\sum_{\alpha\in R}\frac{k(\alpha)}{2}\langle\alpha,\xi\rangle\frac{f(\mathbf{x})-f(\sigma_\alpha(\mathbf{x}))}{\langle\alpha,\mathbf{x}\rangle}.$$

The Dunkl operators T_ξ, which were introduced in [5], commute and are skew-symmetric with respect to the G-invariant measure dw. Let us denote $T_j=T_{e_j}$, where $\{e_j\}_{1\leq j\leq N}$ is a canonical orthonormal basis of $\mathbb{R}^N$.

2.2. **Dunkl kernel and Dunkl transform.** For fixed $\mathbf{y}\in\mathbb{R}^N$ the *Dunkl kernel* $E(\mathbf{x},\mathbf{y})$ is a unique analytic solution to the system

$$T_\xi f=\langle\xi,\mathbf{y}\rangle f,\quad f(0)=1.$$

The function $E(\mathbf{x},\mathbf{y})$, which generalizes the exponential function $e^{\langle\mathbf{x},\mathbf{y}\rangle}$, has a unique extension to a holomorphic function on $\mathbb{C}^N\times\mathbb{C}^N$.

The *Dunkl transform* is defined by

$$\mathcal{F}f(\xi)=\boldsymbol{c}_k^{-1}\int_{\mathbb{R}^N}E(-i\xi,\mathbf{x})f(\mathbf{x})\,dw(\mathbf{x}), \tag{2.3}$$

where

$$\boldsymbol{c}_k=\int_{\mathbb{R}^N}e^{-\frac{\|\mathbf{x}\|^2}{2}}\,dw(\mathbf{x})>0,$$

for $f\in L^1(dw)$ and $\xi\in\mathbb{R}^N$. It was introduced in [6] for $k\geq 0$ and further studied in [4]. It was proved in [6, Corollary 2.7] (see also [4, Theorem 4.26]) that it is an isometry on $L^2(dw)$, i.e.,

$$\|f\|_{L^2(dw)}=\|\mathcal{F}f\|_{L^2(dw)} \text{ for all } f\in L^2(dw). \tag{2.4}$$

2.3. **Dunkl Laplacian and Dunkl heat semigroup.** The *Dunkl Laplacian* associated with R and k is the differential-difference operator $\Delta_k=\sum_{j=1}^N T_j^2$, which acts on $C^2(\mathbb{R}^N)$-functions by

$$\Delta_k f(\mathbf{x})=\Delta_{\rm eucl}f(\mathbf{x})+\sum_{\alpha\in R}k(\alpha)\delta_\alpha f(\mathbf{x}),\quad \delta_\alpha f(\mathbf{x})=\frac{\partial_\alpha f(\mathbf{x})}{\langle\alpha,\mathbf{x}\rangle}-\frac{\|\alpha\|^2}{2}\frac{f(\mathbf{x})-f(\sigma_\alpha(\mathbf{x}))}{\langle\alpha,\mathbf{x}\rangle^2}.$$

The operator Δ_k is essentially self-adjoint on $L^2(dw)$ (see for instance [1, Theorem 3.1]) and generates a semigroup $\{H_t\}_{t>0}$ of linear self-adjoint contractions on $L^2(dw)$. The semigroup has the form

$$H_tf(\mathbf{x})=\int_{\mathbb{R}^N}h_t(\mathbf{x},\mathbf{y})f(\mathbf{y})\,dw(\mathbf{y}), \tag{2.5}$$

where the heat kernel

$$h_t(\mathbf{x},\mathbf{y})=\boldsymbol{c}_k^{-1}(2t)^{-\mathbf{N}/2}E\Big(\frac{\mathbf{x}}{\sqrt{2t}},\frac{\mathbf{y}}{\sqrt{2t}}\Big)e^{-(\|\mathbf{x}\|^2+\|\mathbf{y}\|^2)/(4t)} \tag{2.6}$$

is a C^∞-function of the all variables $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$, $t>0$, and satisfies

$$0<h_t(\mathbf{x},\mathbf{y})=h_t(\mathbf{y},\mathbf{x}), \tag{2.7}$$

$$\int_{\mathbb{R}^N} h_t(\mathbf{x},\mathbf{y})\,dw(\mathbf{y})=1. \tag{2.8}$$

The following specific formula for the Dunkl heat kernel was obtained by Rösler [15]:

$$h_t(\mathbf{x},\mathbf{y})=\boldsymbol{c}_k^{-1}2^{-\mathbf{N}/2}t^{-\mathbf{N}/2}\int_{\mathbb{R}^N}\exp(-A(\mathbf{x},\mathbf{y},\eta)^2/4t)\,d\mu_{\mathbf{x}}(\eta) \text{ for all } \mathbf{x},\mathbf{y}\in\mathbb{R}^N, t>0. \tag{2.9}$$

Here

$$A(\mathbf{x},\mathbf{y},\eta)=\sqrt{\|\mathbf{x}\|^2+\|\mathbf{y}\|^2-2\langle\mathbf{y},\eta\rangle}=\sqrt{\|\mathbf{x}\|^2-\|\eta\|^2+\|\mathbf{y}-\eta\|^2} \tag{2.10}$$

and $\mu_{\mathbf{x}}$ is a probability measure, which is supported in the convex hull $\operatorname{conv}\mathcal{O}(\mathbf{x})$ of the orbit $\mathcal{O}(\mathbf{x})=\{\sigma(\mathbf{x}):\sigma\in G\}$.

2.4. **Upper and lower heat kernel bounds.** The closures of connected components of

$$\{\mathbf{x}\in\mathbb{R}^N\ :\ \langle\mathbf{x},\alpha\rangle\neq 0 \text{ for all } \alpha\in R\}$$

are called (closed) *Weyl chambers.*

Lemma 2.1. *Fix $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $\sigma\in G$. Then $d(\mathbf{x},\mathbf{y})=\|\mathbf{x}-\sigma(\mathbf{y})\|$ if and only if $\sigma(\mathbf{y})$ and $\mathbf{x}$ belong to the same Weyl chamber.*

Proof. See [11, Chapter VII, proof of Theorem 2.12]. □

For a finite sequence $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_m)$ of elements of R, $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $t>0$, let

$$\ell(\boldsymbol{\alpha}):=m \tag{2.11}$$

be the length of $\boldsymbol{\alpha}$,

$$\sigma_{\boldsymbol{\alpha}}:=\sigma_{\alpha_m}\circ\sigma_{\alpha_{m-1}}\circ\ldots\circ\sigma_{\alpha_1}, \tag{2.12}$$

and

$$\begin{aligned}&\rho_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y},t)\\ &:=\left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2}\left(1+\frac{\|\mathbf{x}-\sigma_{\alpha_1}(\mathbf{y})\|}{\sqrt{t}}\right)^{-2}\left(1+\frac{\|\mathbf{x}-\sigma_{\alpha_2}\circ\sigma_{\alpha_1}(\mathbf{y})\|}{\sqrt{t}}\right)^{-2}\times\ldots\\ &\times\left(1+\frac{\|\mathbf{x}-\sigma_{\alpha_{m-1}}\circ\ldots\circ\sigma_{\alpha_1}(\mathbf{y})\|}{\sqrt{t}}\right)^{-2}.\end{aligned} \tag{2.13}$$

For $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$, let $n(\mathbf{x},\mathbf{y})=0$ if $d(\mathbf{x},\mathbf{y})=\|\mathbf{x}-\mathbf{y}\|$ and

$$n(\mathbf{x},\mathbf{y})=\min\{m\in\mathbb{Z}:d(\mathbf{x},\mathbf{y})=\|\mathbf{x}-\sigma_{\alpha_m}\circ\ldots\circ\sigma_{\alpha_2}\circ\sigma_{\alpha_1}(\mathbf{y})\|,\quad \alpha_j\in R\} \tag{2.14}$$

otherwise. In other words, $n(\mathbf{x},\mathbf{y})$ is the smallest number of reflections σ_α which are needed to move $\mathbf{y}$ to the (closed) Weyl chamber of $\mathbf{x}$. We also allow $\boldsymbol{\alpha}$ to be the empty sequence, denoted by $\boldsymbol{\alpha}=\emptyset$. Then for $\boldsymbol{\alpha}=\emptyset$, we set: $\sigma_{\boldsymbol{\alpha}}=I$ (the identity operator), $\ell(\boldsymbol{\alpha})=0$, and $\rho_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y},t)=1$ for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $t>0$.

We say that a finite sequence $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_m)$ of roots is *admissible for the pair* $(\mathbf{x},\mathbf{y})\in\mathbb{R}^N\times\mathbb{R}^N$ if $n(\mathbf{x},\sigma_{\boldsymbol{\alpha}}(\mathbf{y}))=0$. In other words, the composition $\sigma_{\alpha_m}\circ\sigma_{\alpha_{m-1}}\circ\ldots\circ\sigma_{\alpha_1}$ of the reflections σ_{α_j} maps $\mathbf{y}$ to the Weyl chamber of $\mathbf{x}$. The set of the all admissible sequences $\boldsymbol{\alpha}$ for the pair $(\mathbf{x},\mathbf{y})$ will be denoted by $\mathcal{A}(\mathbf{x},\mathbf{y})$. Note that if $n(\mathbf{x},\mathbf{y})=0$, then $\boldsymbol{\alpha}=\emptyset\in\mathcal{A}(\mathbf{x},\mathbf{y})$.

Let us define

$$\Lambda(\mathbf{x},\mathbf{y},t):=\sum_{\boldsymbol{\alpha}\in\mathcal{A}(\mathbf{x},\mathbf{y}),\ \ell(\boldsymbol{\alpha})\leq 2|G|}\rho_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y},t). \tag{2.15}$$

Note that for any $c>1$ and for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $t>0$ we have

$$c^{-2|G|}\Lambda(\mathbf{x},\mathbf{y},ct)\leq\Lambda(\mathbf{x},\mathbf{y},t)\leq\Lambda(\mathbf{x},\mathbf{y},ct). \tag{2.16}$$

The following upper and lower bounds for $h_t(\mathbf{x},\mathbf{y})$ were proved in [9].

Theorem 2.2. *Assume that $0<c_u<1/4$ and $c_l>1/4$. There are constants $C_u,C_l>0$ such that for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $t>0$ we have*

$$C_l w(B(\mathbf{x},\sqrt{t}))^{-1}e^{-c_l\frac{d(\mathbf{x},\mathbf{y})^2}{t}}\Lambda(\mathbf{x},\mathbf{y},t)\leq h_t(\mathbf{x},\mathbf{y}), \tag{2.17}$$

$$h_t(\mathbf{x},\mathbf{y})\leq C_u w(B(\mathbf{x},\sqrt{t}))^{-1}e^{-c_u\frac{d(\mathbf{x},\mathbf{y})^2}{t}}\Lambda(\mathbf{x},\mathbf{y},t). \tag{2.18}$$

Remark 2.3. In Theorem 2.2, we can replace $\Lambda(\mathbf{x},\mathbf{y},t)$ by the function

$$\tilde{\Lambda}(\mathbf{x},\mathbf{y},t):=\sum_{\boldsymbol{\alpha}\in\mathcal{A}(\mathbf{x},\mathbf{y}),\ \ell(\boldsymbol{\alpha})\leq |G|}\rho_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y},t). \tag{2.19}$$

Indeed, $\tilde{\Lambda}(\mathbf{x},\mathbf{y},t)\leq\Lambda(\mathbf{x},\mathbf{y},t)$ for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $t>0$. We turn to prove

$$\Lambda(\mathbf{x},\mathbf{y},t)\leq|R|^{2|G|}\tilde{\Lambda}(\mathbf{x},\mathbf{y},t). \tag{2.20}$$

To this end, fix $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$, $t>0$, and take $\boldsymbol{\beta}\in\mathcal{A}(\mathbf{x},\mathbf{y})$ of the minimal length $\ell(\boldsymbol{\beta})$ which satisfies $\ell(\boldsymbol{\beta})\leq 2|G|$, and

$$\rho_{\boldsymbol{\beta}}(\mathbf{x},\mathbf{y},t)=\max_{\boldsymbol{\alpha}\in\mathcal{A}(\mathbf{x},\mathbf{y}),\ \ell(\boldsymbol{\alpha})\leq 2|G|}\rho_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y},t). \tag{2.21}$$

Obviously, $\Lambda(\mathbf{x},\mathbf{y},t)\leq|R|^{2|G|}\rho_{\boldsymbol{\beta}}(\mathbf{x},\mathbf{y},t)$. If $|\boldsymbol{\beta}|\leq|G|$, then (2.20) is proved. If $m=|\boldsymbol{\beta}|>|G|$, then let us consider the sequence

$$I,\sigma_{\beta_1},\sigma_{\beta_2}\circ\sigma_{\beta_1},\ldots,\sigma_{\beta_{m-1}}\circ\cdots\circ\sigma_{\beta_1}. \tag{2.22}$$

Since there are $|\boldsymbol{\beta}|>|G|$ elements in the sequence (2.22), at least two of them coincide.

Assume first that for some $j,s\in\{1,2,\ldots,m-1\}$, $j<s$, we have

$$\sigma_{\beta_j}\circ\sigma_{\beta_{j-1}}\circ\cdots\circ\sigma_{\beta_1}=\sigma_{\beta_s}\circ\cdots\circ\sigma_{\beta_j}\circ\sigma_{\beta_{j-1}}\circ\ldots\circ\sigma_{\beta_1}\neq I.$$

Set $\widetilde{\boldsymbol{\beta}}=(\beta_1,\beta_2,\ldots,\beta_j,\beta_{s+1},\ldots,\beta_m)$. Then $\widetilde{\boldsymbol{\beta}}\in\mathcal{A}(\mathbf{x},\mathbf{y})$, $\ell(\widetilde{\boldsymbol{\beta}})<\ell(\boldsymbol{\beta})$, and $\rho_{\boldsymbol{\beta}}(\mathbf{x},\mathbf{y},t)\leq\rho_{\widetilde{\boldsymbol{\beta}}}(\mathbf{x},\mathbf{y},t)$.

If there is $s\in\{1,2,\ldots,m-1\}$ such that $\sigma_{\beta_s}\circ\cdots\circ\sigma_{\beta_j}\circ\sigma_{\beta_{j-1}}\circ\ldots\circ\sigma_{\beta_1}=I$, then we set $\widetilde{\boldsymbol{\beta}}=(\beta_{s+1},\ldots,\beta_m)$ and argue as above. Thus (2.19) is established.

In order to obtain our regularity results, we will need the following theorem proved in [9].

Theorem 2.4 ([9], Theorem 6.1)**.** *There are constants $C_4,c_4>0$ such that for all $\mathbf{x},\mathbf{y},\mathbf{y}'\in\mathbb{R}^N$ and $t>0$ satisfying $\|\mathbf{y}-\mathbf{y}'\|<\frac{\sqrt{t}}{2}$ we have*

$$|h_t(\mathbf{x},\mathbf{y})-h_t(\mathbf{x},\mathbf{y}')|\leq C_4\frac{\|\mathbf{y}-\mathbf{y}'\|}{\sqrt{t}}h_{c_4t}(\mathbf{x},\mathbf{y}). \tag{2.23}$$

We will also need some auxiliary estimates of the generalized heat kernel $h_t(\mathbf{x},\mathbf{y})$.

Lemma 2.5. *Assume that $c_0>1$. There is a constant $C_0>0$ such that for all $\mathbf{x},\mathbf{y},\mathbf{y}'\in\mathbb{R}^N$ and $t>0$ satisfying $\|\mathbf{y}-\mathbf{y}'\|<\sqrt{t}$ we have*

$$h_t(\mathbf{x},\mathbf{y})\leq C_0 h_{c_0t}(\mathbf{x},\mathbf{y}'). \tag{2.24}$$

Proof. This is Lemma 6.2 of [9]. For the convenience of the reader, we present an alternative proof here. Let $\mathbf{y},\mathbf{y}'\in\mathbb{R}^N$ be such that $\|\mathbf{y}-\mathbf{y}'\|\leq\sqrt{t}$. Recall that $\|\mathbf{x}\|-\|\eta\|\geq 0$ for all $\eta\in\operatorname{conv}\mathcal{O}(\mathbf{x})$. Put $\varepsilon=c_0-1$. We turn to estimate $A(\mathbf{x},\mathbf{y}',\eta)$ defined in (2.10):

$$\begin{aligned} A(\mathbf{x},\mathbf{y}',\eta)^2&=\|\mathbf{x}\|^2-\|\eta\|^2+\|\mathbf{y}'-\eta\|^2\\ &\leq\|\mathbf{x}\|^2-\|\eta\|^2+(\|\mathbf{y}'-\mathbf{y}\|+\|\mathbf{y}-\eta\|)^2\\ &\leq\|\mathbf{x}\|^2-\|\eta\|^2+\|\mathbf{y}'-\mathbf{y}\|^2+\|\mathbf{y}-\eta\|^2+\varepsilon^{-1}\|\mathbf{y}'-\mathbf{y}\|^2+\varepsilon\|\mathbf{y}-\eta\|^2\\ &\leq(1+\varepsilon)(\|\mathbf{x}\|^2-\|\eta\|^2+\|\mathbf{y}-\eta\|^2)+(1+\varepsilon^{-1})t\\ &=(1+\varepsilon)A(\mathbf{x},\mathbf{y},\eta)^2+(1+\varepsilon^{-1})t, \end{aligned} \tag{2.25}$$

where in the second inequality of (2.25) we have used the inequality $2ab\leq\varepsilon a^2+\varepsilon^{-1}b^2$. Using (2.25) and the Rösler formula (2.9), we get

$$\begin{aligned} h_t(\mathbf{x},\mathbf{y})&=\boldsymbol{c}_k^{-1}2^{-\mathbf{N}/2}t^{-\mathbf{N}/2}\int_{\mathbb{R}^N}e^{-A(\mathbf{x},\mathbf{y},\eta)^2/4t}\,d\mu_{\mathbf{x}}(\eta)\\ &\leq\boldsymbol{c}_k^{-1}2^{-\mathbf{N}/2}t^{-\mathbf{N}/2}\int_{\mathbb{R}^N}e^{-A(\mathbf{x},\mathbf{y}',\eta)^2/4(1+\varepsilon)t}e^{(1+\varepsilon^{-1})/(4(1+\varepsilon))}\,d\mu_{\mathbf{x}}(\eta)\\ &=C_0h_{c_0t}(\mathbf{x},\mathbf{y}'). \end{aligned} \tag{2.26}$$

□

As a consequence of Lemma 2.5, we obtain the next lemma, which will be used in the proof of the main theorem.

Lemma 2.6. *There are constants $C,c>1$ such that for all $s,t>0$ and $\mathbf{x},\mathbf{x}',\mathbf{y}\in\mathbb{R}^N$ we have*

$$\int_{\mathbb{R}^N}\Big|h_s(\mathbf{x},\mathbf{z})-h_s(\mathbf{x}',\mathbf{z})\Big|h_t(\mathbf{z},\mathbf{y})\,dw(\mathbf{z})\leq C\frac{\|\mathbf{x}-\mathbf{x}'\|}{\sqrt{s}}h_{cs+t}(\mathbf{x},\mathbf{y})+C\frac{\|\mathbf{x}-\mathbf{x}'\|}{\sqrt{s}}h_{cs+t}(\mathbf{x}',\mathbf{y}). \tag{2.27}$$

Proof. If $\|\mathbf{x}-\mathbf{x}'\|\geq\frac{\sqrt{s}}{2}$, then (2.27) follows by the semigroup property of $h_t(\mathbf{x},\mathbf{y})$. Assume that $\|\mathbf{x}-\mathbf{x}'\|<\frac{\sqrt{s}}{2}$. Theorem 2.4 asserts that there are constants $C,c>1$ such that for all $s_1>0$ and $\mathbf{x}_1,\mathbf{x}_1',\mathbf{z}_1\in\mathbb{R}^N$ satisfying $\|\mathbf{x}_1-\mathbf{x}_1'\|<\frac{\sqrt{s_1}}{2}$ we have

$$|h_{s_1}(\mathbf{x}_1,\mathbf{z}_1)-\ h_{s_1}(\mathbf{x}_1',\mathbf{z}_1)|\leq C\frac{\|\mathbf{x}_1-\mathbf{x}_1'\|}{\sqrt{s_1}}h_{cs_1}(\mathbf{x}_1,\mathbf{z}_1). \tag{2.28}$$

Hence, by the semigroup property of the generalized heat semigroup, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N}|h_s(\mathbf{x},\mathbf{z})-h_s(\mathbf{x}',\mathbf{z})|\,h_t(\mathbf{z},\mathbf{y})\,dw(\mathbf{z})&\leq C\frac{\|\mathbf{x}-\mathbf{x}'\|}{\sqrt{s}}\int_{\mathbb{R}^N}h_{cs}(\mathbf{x},\mathbf{z})h_t(\mathbf{z},\mathbf{y})\,dw(\mathbf{z})\\ &=C\frac{\|\mathbf{x}-\mathbf{x}'\|}{\sqrt{s}}h_{cs+t}(\mathbf{x},\mathbf{y}). \end{aligned}$$

□

3. Dunkl Schrödinger operators with non-negative potentials - introductory results

For a nonnegative potential $V\colon \mathbb{R}^N \longmapsto [0,\infty)$, $V\in L^1_{\rm loc}(dw)$, let $V_n(x) := \min(V(\mathbf{x}),n)$, $n=1,2,\dots$. We consider the quadratic forms

$$Q_\infty(f,g) := \int_{\mathbb{R}^N} \Big(\sum_{j=1}^N T_jf(\mathbf{x})\overline{T_jg(\mathbf{x})} + V(\mathbf{x})f(\mathbf{x})\overline{g(\mathbf{x})}\Big)\, dw(\mathbf{x}), \tag{3.1}$$

$$Q_n(f,g) := \int_{\mathbb{R}^N} \Big(\sum_{j=1}^N T_jf(\mathbf{x})\overline{T_jg(\mathbf{x})} + V_n(\mathbf{x})f(\mathbf{x})\overline{g(\mathbf{x})}\Big)\, dw(\mathbf{x}), \tag{3.2}$$

with the domains

$$\mathcal{D}(Q_\infty) = \{ f\in L^2(dw) : \|\mathbf{x}\|\mathcal{F}f(\mathbf{x})\in L^2(dw(\mathbf{x})) \ \text{ and } \ \sqrt{V(\mathbf{x})}f(\mathbf{x})\in L^2(dw(\mathbf{x}))\},$$

$$\mathcal{D}(Q_n) = \{ f\in L^2(dw) : \|\mathbf{x}\|\mathcal{F}f(\mathbf{x})\in L^2(dw(\mathbf{x}))\ \},\quad n=1,2,\dots.$$

Observe that $C_c^\infty(\mathbb{R}^N)$ is a dense subspace of $L^2(dw)$ such that $C_c^\infty(\mathbb{R}^N)\subseteq \mathcal{D}(Q_\infty)\subseteq \mathcal{D}(Q_n)$. The forms Q_∞ and Q_n are non-negative and closed. So they define self-adjoint non-negative operators L_∞, L_n respectively:

$$\mathcal{D}(L_n) = \{ f\in \mathcal{D}(Q_n) : |Q_n(f,g)|\leq C_f\|g\|_{L^2(dw)} \quad \text{for all } g\in\mathcal{D}(Q_n),\},\quad n=\infty,1,2,\dots$$

and, for $f\in\mathcal{D}(L_n)$, the operator L_n is defined by the equation

$$\int_{\mathbb{R}^N} (L_nf)(\mathbf{x})\bar{g}(\mathbf{x})\,dw(\mathbf{x}) = Q_n(f,g) \quad \text{for all } g\in\mathcal{D}(Q_n),$$

see e.g. [3, Theorem 4.12]. Moreover, $f\in\mathcal{D}(Q_\infty)$ if and only if $\lim_{n\to\infty}Q_n(f,f)<\infty$. Further, $Q_\infty(f,f)=\lim_{n\to\infty}Q_n(f,f)$ and, by the definition of V_n, the convergence is monotone. Set $L:=L_\infty$. The operator $-L_n$ is the generator of a semigroup of linear contractions on $L^2(dw)$, denoted by $\{e^{-tL_n}\}_{t\geq 0}$ for $n=1,2,\dots$ and $\{e^{-tL}\}_{t\geq 0}$ for $L=L_\infty$. Let $a>0$. Theorem 4.32 of [3] asserts that

$$\lim_{n\to\infty}\Big\{\sup_{0\leq t\leq a}\|e^{-tL_n}f-e^{-tL}f\|_{L^2(dw)}\Big\}=0 \quad \text{for all } f\in L^2(dw). \tag{3.3}$$

In the forthcoming sections we provide rigorous proofs of existence, regularity and bounds for the kernels of the semigroups $\{e^{-tL_n}\}_{t\geq 0}$, $n=\infty,1,2,\dots$. The main tools are the following product formula and Duhamel formula for semigroups generated by perturbations of generators by bounded operators on Banach spaces which we state as the theorem.

Theorem 3.1. *Let A be a generator of a semigroup $\{e^{tA}\}_{t\geq 0}$ of linear operators on a Banach space X, and let B be a bounded operator on X. Then $A+B$ is a generator of a semigroup of linear operators on X, denoted by $\{e^{t(A+B)}\}_{t\geq 0}$, and for every $x\in X$ one has*

$$e^{t(A+B)}x = \lim_{n\to\infty}\Big(e^{tA/n}e^{tB/n}\Big)^n x, \tag{3.4}$$

$$e^{tA}x = e^{t(A+B)}x - \int_0^t e^{(t-s)A}Be^{s(A+B)}x\,ds. \tag{3.5}$$

Moreover, if the semigroup $\{e^{tA}\}_{t\geq 0}$ is holomorphic, so is $\{e^{t(A+B)}\}_{t\geq 0}$.

Let us remark that under a stronger assumption, namely $V \in L^2_{\rm loc}(dw)$, it was proved in Amri and Hammi [1] that L is essentially self-adjoint non-negative operator, that is, L is the closure of the operator

$$\mathcal{L}_\infty f = -\Delta_k f + Vf,$$

initially defined on $C_c^\infty(\mathbb{R}^N)$. We will not use this assumption in our forthcoming considerations and keep the weaker assumption $V \in L^1_{\rm loc}(dw)$.

4. Schrödinger semigroups with bounded potentials

In this section we utilize the product formula (3.4) to get existence and regularity of the kernel $k_t^{\{V\}}(\mathbf{x},\mathbf{y})$ from properties of approximation kernels. In this section, we assume that $V \geq 0$ is a bounded potential.

Theorem 4.1. *Assume that $V : \mathbb{R}^N \longmapsto [0,\infty)$ is a bounded measurable function. Then the semigroup $\{e^{t(\Delta_k - V)}\}_{t\geq 0}$ of linear operators generated by $\Delta_k - V$ has the form*

$$e^{t(\Delta_k - V)} f(\mathbf{x}) = \int_{\mathbb{R}^N} k_t^{\{V\}}(\mathbf{x},\mathbf{y}) f(\mathbf{y})\, dw(\mathbf{y}), \quad f \in L^2(dw), \tag{4.1}$$

where $\mathbb{R}^N \times \mathbb{R}^N \times (0,\infty) \ni (\mathbf{x},\mathbf{y},t) \longmapsto k_t^{\{V\}}(\mathbf{x},\mathbf{y})$ is a continuous function such that there are constants $C, c > 0$ such that

$$0 \leq k_t^{\{V\}}(\mathbf{x},\mathbf{y}) \leq h_t(\mathbf{x},\mathbf{y}),$$

$$|k_t^{\{V\}}(\mathbf{x},\mathbf{y}) - k_t^{\{V\}}(\mathbf{x}',\mathbf{y}')| \leq C(1+\sqrt{t\|V\|_\infty})\frac{\|\mathbf{x}-\mathbf{x}'\| + \|\mathbf{y}-\mathbf{y}'\|}{\sqrt{t}} h(ct,\mathbf{x},\mathbf{x}',\mathbf{y},\mathbf{y}'), \tag{4.2}$$

for all $\mathbf{x},\mathbf{x}',\mathbf{y},\mathbf{y}' \in \mathbb{R}^N$ and $t > 0$, where

$$h(ct,\mathbf{x}_1,\mathbf{x}_2,\mathbf{y}_1,\mathbf{y}_2) := \sum_{i=1}^2 \sum_{j=1}^2 h_{ct}(\mathbf{x}_i,\mathbf{y}_j).$$

Moreover, for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^N$, the function $(0,\infty) \ni t \to k_t^{\{V\}}(\mathbf{x},\mathbf{y})$ is differentiable and for any $m \in \mathbb{N}$ there is a constant $C_m > 0$ such that for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^N$ and $t > 0$ we have

$$\Big|\frac{d^m}{dt^m} k_t^{\{V\}}(\mathbf{x},\mathbf{y})\Big| \leq C_m t^{-m} w(B(\mathbf{x},\sqrt{t}))^{-1/2} w(B(\mathbf{y},\sqrt{t}))^{-1/2}. \tag{4.3}$$

The constants C, c, C_m are independent of V.

Proof. We assume that $V \not\equiv 0$. It suffices to prove (4.2) for $0 < t \leq \|V\|_\infty^{-1}$ and then use the semigroup property.

Let us consider the integral kernels $Q_{n,t}(\mathbf{x},\mathbf{y})$ of the operators $(H_{t/n} e^{-tV/n})^n$. We write

$$Q_{n,t}(\mathbf{x},\mathbf{y}) := q_{n,t}(\mathbf{x},\mathbf{y}) e^{-tV(\mathbf{y})/n}, \tag{4.4}$$

where

$$\begin{aligned} &0 \leq q_{n,t}(\mathbf{x},\mathbf{y}) \\ &:= \int_{\mathbb{R}^N} \dots \int_{\mathbb{R}^N} h_{t/n}(\mathbf{x},\mathbf{z}_1) e^{-tV(\mathbf{z}_1)/n} h_{t/n}(\mathbf{z}_1,\mathbf{z}_2) e^{-tV(\mathbf{z}_2)/n} \dots h_{t/n}(\mathbf{z}_{n-1},\mathbf{y})\, dw(\mathbf{z}_{n-1}) \dots dw(\mathbf{z}_1) \\ &\leq h_t(\mathbf{x},\mathbf{y}). \end{aligned} \tag{4.5}$$

We prove that the functions $\mathbb{R}^N \times \mathbb{R}^N \ni (\mathbf{x},\mathbf{y}) \mapsto q_{n,t}(\mathbf{x},\mathbf{y})$, which are clearly continuous, are Lipschitz functions of $(\mathbf{x},\mathbf{y})$. A uniform bound independent of $n \in \mathbb{N}$ will be given.

For $\mathbf{z}_1 \in \mathbb{R}^N$ we write $\exp(-tV(\mathbf{z}_1)/n) = 1 - tW(\mathbf{z}_1)/n$ where $|W(\mathbf{z}_1)| \leq \|V\|_\infty$. Thus, thanks to the fact $\int_{\mathbb{R}^N} h_{t/n}(\mathbf{x},\mathbf{z}_1)h_{t/c}(\mathbf{z}_1,\mathbf{z}_2)\,dw(\mathbf{z}_1) = h_{2t/n}(\mathbf{x},\mathbf{z}_2)$, we get

$$\begin{aligned}
&q_{n,t}(\mathbf{x},\mathbf{y})\\
&= \int_{\mathbb{R}^N}\dots\int_{\mathbb{R}^N} h_{2t/n}(\mathbf{x},\mathbf{z}_2)e^{-tV(\mathbf{z}_2)/n}h_{t/n}(\mathbf{z}_2,\mathbf{z}_3)e^{-tV(\mathbf{z}_3)/n}\dots h_{t/n}(\mathbf{z}_{n-1},\mathbf{y})\,dw(\mathbf{z}_{n-1})\dots dw(\mathbf{z}_2)\\
&- \frac{t}{n}\int_{\mathbb{R}^N}\dots\int_{\mathbb{R}^N} h_{t/n}(\mathbf{x},\mathbf{z}_1)W(\mathbf{z}_1)h_{t/n}(\mathbf{z}_1,\mathbf{z}_2)e^{-tV(\mathbf{z}_2)/n}\dots h_{t/n}(\mathbf{z}_{n-1},\mathbf{y})\,dw(\mathbf{z}_{n-1})\dots dw(\mathbf{z}_1)\\
&=: J_1^{[1]}(\mathbf{x},\mathbf{y}) - J_2^{[1]}(\mathbf{x},\mathbf{y}).
\end{aligned}$$

Observe that by Lemma 2.6, we have

$$\begin{aligned}
|J_2^{[1]}(\mathbf{x},\mathbf{y}) - J_2^{[1]}(\mathbf{x}',\mathbf{y})| &\leq \frac{t\|V\|_\infty}{n}\int_{\mathbb{R}^N} |h_{t/n}(\mathbf{x},\mathbf{z}_1) - h_{t/n}(\mathbf{x}',\mathbf{z}_1)|h_{(n-1)t/n}(\mathbf{z}_1,\mathbf{y})\;dw(\mathbf{z}_1)\\
&\leq C\frac{t\|V\|_\infty}{n}\frac{\|\mathbf{x}-\mathbf{x}'\|}{\sqrt{t/n}}\Big(h_{ct/n+(n-1)t/n}(\mathbf{x},\mathbf{y}) + h_{ct/n+(n-1)t/n}(\mathbf{x}',\mathbf{y})\Big)\\
&\leq C\frac{t\|V\|_\infty}{\sqrt{n}}\frac{\|\mathbf{x}-\mathbf{x}'\|}{\sqrt{t}}\Big(h_{ct/n+(n-1)t/n}(\mathbf{x},\mathbf{y}) + h_{ct/n+(n-1)t/n}(\mathbf{x}',\mathbf{y})\Big).
\end{aligned}$$

It follows from (2.9) that $h_{ct/n+(n-1)t}(\mathbf{x},\mathbf{y}) \leq C'h_{ct}(\mathbf{x},\mathbf{y})$. Hence, in the first step we have got

$$\begin{aligned}
|q_{n,t}(\mathbf{x},\mathbf{y}) - q_{n,t}(\mathbf{x}',\mathbf{y})| &\leq |J_1^{[1]}(\mathbf{x},\mathbf{y}) - J_1^{[1]}(\mathbf{x}',\mathbf{y})|\\
&\quad + C\frac{t\|V\|_\infty}{\sqrt{n}}\frac{\|\mathbf{x}-\mathbf{x}'\|}{\sqrt{t}}\Big(h_{ct}(\mathbf{x},\mathbf{y}) + h_{ct}(\mathbf{x}',\mathbf{y})\Big).
\end{aligned} \tag{4.6}$$

In the second step, we deal with $J_1^{[1]}(\cdot,\cdot)$. For $\mathbf{z}_2 \in \mathbb{R}^N$ we write $\exp(-tV(\mathbf{z}_2)/n) = 1 - tW(\mathbf{z}_2)/n$, where $|W(\mathbf{z}_2)| \leq \|V\|_\infty$ and plug to the formula for $J_1^{[1]}(\cdot,\cdot)$. Thus

$$\begin{aligned}
&J_1^{[1]}(\mathbf{x},\mathbf{y})\\
&= \int_{\mathbb{R}^N}\dots\int_{\mathbb{R}^N} h_{2t/n}(\mathbf{x},\mathbf{z}_2)(1-\frac{t}{n}W(\mathbf{z}_2))h_{t/n}(\mathbf{z}_2,\mathbf{z}_3)e^{-tV(\mathbf{z}_3)/n}\dots h_{t/n}(\mathbf{z}_{n-1},\mathbf{y})\,dw(\mathbf{z}_{n-1})\dots\;dw(\mathbf{z}_2)\\
&= \int_{\mathbb{R}^N}\dots\int_{\mathbb{R}^N} h_{3t/n}(\mathbf{x},\mathbf{z}_3)e^{-tV(\mathbf{z}_3)/n}h_{t/n}(\mathbf{z}_3,\mathbf{z}_4)e^{-tV(\mathbf{z}_4)/n}\dots h_{t/n}(\mathbf{z}_{n-1},\mathbf{y})\,dw(\mathbf{z}_{n-1})\dots\;dw(\mathbf{z}_3)\\
&- \frac{t}{n}\int_{\mathbb{R}^N}\dots\int_{\mathbb{R}^N} h_{2t/n}(\mathbf{x},\mathbf{z}_2)W(\mathbf{z}_2)h_{t/n}(\mathbf{z}_2,\mathbf{z}_3)e^{-tV(\mathbf{z}_3)/n}\dots h_{t/n}(\mathbf{z}_{n-1},\mathbf{y})\,dw(\mathbf{z}_{n-1})\dots\;dw(\mathbf{z}_2)\\
&=: J_1^{[2]}(\mathbf{x},\mathbf{y}) - J_2^{[2]}(\mathbf{x},\mathbf{y}).
\end{aligned}$$

Further, by Lemma 2.6,

$$\begin{aligned}
|J_2^{[2]}(\mathbf{x},\mathbf{y}) - J_2^{[2]}(\mathbf{x}',\mathbf{y})| &\leq \frac{Ct\|V\|_\infty}{n}\int_{\mathbb{R}^N} |h_{2t/n}(\mathbf{x},\mathbf{z}_2) - h_{2t/n}(\mathbf{x}',\mathbf{z}_2)|h_{(n-2)t/n}(\mathbf{z}_2,\mathbf{y})\;dw(\mathbf{z}_2)\\
&\leq \frac{Ct\|V\|_\infty}{n}\frac{\|\mathbf{x}-\mathbf{x}'\|}{\sqrt{2t/n}}C\Big(h_{2ct/n+(n-2)t/n}(\mathbf{x},\mathbf{y}) + h_{2ct/n+(n-2)t/n}(\mathbf{x},\mathbf{y})\Big)\\
&\leq \frac{Ct\|V\|_\infty}{\sqrt{2n}}\frac{\|\mathbf{x}-\mathbf{x}'\|}{\sqrt{t}}\Big(h_{ct}(\mathbf{x},\mathbf{y}) + h_{ct}(\mathbf{x},\mathbf{y})\Big).
\end{aligned}$$

Thus, at the end of the second step, we have

$$\begin{aligned} |q_{n,t}(\mathbf{x},\mathbf{y})-q_{n,t}(\mathbf{x}',\mathbf{y})| &\leq |J_1^{[2]}(\mathbf{x},\mathbf{y})-J_1^{[2]}(\mathbf{x},\mathbf{y})| \\ &\quad + Ct\|V\|_\infty \frac{\|\mathbf{x}-\mathbf{x}'\|}{\sqrt{t}}\Big(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{2n}}\Big)\Big(h_{ct}(\mathbf{x},\mathbf{y})+h_{ct}(\mathbf{x},\mathbf{y})\Big). \end{aligned} \tag{4.7}$$

We continue this procedure, obtaining at the of the m-th step, $1\leq m\leq n-1$, the bound

$$\begin{aligned} |q_{n,t}(\mathbf{x},\mathbf{y})-q_{n,t}(\mathbf{x}',\mathbf{y})| &\leq |J_1^{[m]}(\mathbf{x},\mathbf{y})-J_1^{[m]}(\mathbf{x}',\mathbf{y})| \\ &\quad + Ct\|V\|_\infty \frac{\|\mathbf{x}-\mathbf{x}'\|}{\sqrt{t}}\Big(\sum_{\ell=1}^{m}\frac{1}{\sqrt{\ell n}}\Big)\Big(h_{ct}(\mathbf{x},\mathbf{y})+h_{ct}(\mathbf{x},\mathbf{y})\Big), \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} J_1^{[m]}(\mathbf{x},\mathbf{y}) := \int_{\mathbb{R}^N}\dots\int_{\mathbb{R}^N} & h_{(m+1)t/n}(\mathbf{x},\mathbf{z}_{m+1})e^{-tV(\mathbf{z}_{m+1})}h_{t/n}(\mathbf{z}_{m+1},\mathbf{z}_{m+2})e^{-tV(\mathbf{z}_{m+2})/n}\times\dots \\ &\times h_{t/n}(\mathbf{z}_{n-1},\mathbf{y})\,dw(\mathbf{z}_{n-1})\dots\,dw(\mathbf{z}_{m+1}). \end{aligned}$$

Finally, we end up with the bound

$$\begin{aligned} |q_{n,t}(\mathbf{x},\mathbf{y})-q_{n,t}(\mathbf{x}',\mathbf{y})| &\leq |h_t(\mathbf{x},\mathbf{y})-h_t(\mathbf{x}',\mathbf{y})| \\ &\quad + Ct\|V\|_\infty \frac{\|\mathbf{x}-\mathbf{x}'\|}{\sqrt{t}}\Big(\sum_{\ell=1}^{n-1}\frac{1}{\sqrt{\ell n}}\Big)\Big(h_{ct}(\mathbf{x},\mathbf{y})+h_{ct}(\mathbf{x}',\mathbf{y})\Big) \\ &\leq |h_t(\mathbf{x},\mathbf{y})-h_t(\mathbf{x}',\mathbf{y})| + Ct\|V\|_\infty \frac{\|\mathbf{x}-\mathbf{x}'\|}{\sqrt{t}}\Big(h_{ct}(\mathbf{x},\mathbf{y})+h_{ct}(\mathbf{x}',\mathbf{y})\Big) \\ &\leq C(1+t\|V\|_\infty)\frac{\|\mathbf{x}-\mathbf{x}'\|}{\sqrt{t}}\Big(h_{ct}(\mathbf{x},\mathbf{y})+h_{ct}(\mathbf{x}',\mathbf{y})\Big). \end{aligned}$$

By the same argument,

$$|q_{n,t}(\mathbf{x},\mathbf{y})-q_{n,t}(\mathbf{x},\mathbf{y}')| \leq C(1+t\|V\|_\infty)\frac{\|\mathbf{y}-\mathbf{y}'\|}{\sqrt{t}}\Big(h_{ct}(\mathbf{x},\mathbf{y})+h_{ct}(\mathbf{x},\mathbf{y}')\Big). \tag{4.9}$$

Recall that $0<t\leq\|V\|_\infty^{-1}$. By the Arzelá-Ascoli theorem, there is a subsequence $\{n_j\}_{j\in\mathbb{N}}$ such that $\{q_{n_j,t}(\mathbf{x},\mathbf{y})\}_{j\in\mathbb{N}}$ converges uniformly on all compact sets of $\mathbb{R}^N\times\mathbb{R}^N$ to a continuous function $(\mathbf{x},\mathbf{y})\longmapsto k_t^{\{V\}}(\mathbf{x},\mathbf{y})$, which satisfies:

$$\begin{aligned} &0\leq k_t^{\{V\}}(\mathbf{x},\mathbf{y})\leq h_t(\mathbf{x},\mathbf{y}), \\ &|k_t^{\{V\}}(\mathbf{x},\mathbf{y})-k_t^{\{V\}}(\mathbf{x}',\mathbf{y}')|\leq C\frac{\|\mathbf{x}-\mathbf{x}'\|+\|\mathbf{y}-\mathbf{y}'\|}{\sqrt{t}}h(ct,\mathbf{x},\mathbf{y},\mathbf{x}',\mathbf{y}'). \end{aligned} \tag{4.10}$$

Observe that the sequence $\{Q_{n_j,t}(\mathbf{x},\mathbf{y})\}_{j\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{R}^N\times\mathbb{R}^N$ to $k_t^{\{V\}}(\mathbf{x},\mathbf{y})$ as well. By the product formula (3.4), for all $f\in L^2(dw)$, we have

$$e^{t(\Delta_k-V)}f(\mathbf{x})=\lim_{n\to\infty}\int_{\mathbb{R}^N}Q_{t,n}(\mathbf{x},\mathbf{y})f(\mathbf{y})\,dw(\mathbf{y}) \tag{4.11}$$

with the convergence in the $L^2(dw(\mathbf{x}))$-norm. Recall that $Q_{t,n}(\mathbf{x},\mathbf{y})\leq h_t(\mathbf{x},\mathbf{y})$ (see (4.4) and (4.5)). Thus, by the Lebesgue dominated convergence theorem, for all $\mathbf{x}\in\mathbb{R}^N$, one has

$$\lim_{j\to\infty}\int_{\mathbb{R}^N} Q_{t,n_j}(\mathbf{x},\mathbf{y})f(\mathbf{y})\,dw(\mathbf{y})=\int_{\mathbb{R}^N} k_t^{\{V\}}(\mathbf{x},\mathbf{y})f(\mathbf{y})\,dw(\mathbf{y}).$$

Thus (4.1) is established.

We now turn to prove (4.3). The operator $-\Delta_k+V$ is non-negative and self-adjoint on $L^2(dw)$. Thus, by the spectral theorem, the mapping $(0,\infty)\ni t\mapsto e^{t(\Delta_k-V)}\in\mathcal{L}(L^2(dw))$ is a smooth function, and for any $m\in\mathbb{N}$ there is $C_m>0$ such that for all measurable and bounded $V\geq 0$ and $t>0$ we have

$$\left\|\frac{d^m}{dt^m}e^{t(\Delta_k-V)}\right\|_{\mathcal{L}(L^2(dw))}\leq C_m t^{-m}.$$

Here $\mathcal{L}(L^2(dw))$ denotes the Banach space of bounded linear operators on $L^2(dw)$. Thus,

$$\left|\frac{d^m}{dt^m}\langle e^{t(\Delta_k-V)}f,g\rangle_{L^2(dw)}\right|\leq C_m t^{-m}\|f\|_{L^2(dw)}\|g\|_{L^2(dw)}.$$

For $t>0$, set $t_0=t/4$. Then for fixed $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$, we have

$$k_t^{\{V\}}(\mathbf{x},\mathbf{y})=\left\langle e^{(t-2t_0)(\Delta_k-V)}k_{t_0}(\cdot,\mathbf{y}),k_{t_0}(\mathbf{x},\cdot)\right\rangle_{L^2(dw)}.$$

Hence (4.3) follows, since $\|k_{t_0}(\cdot,\mathbf{y})\|_{L^2(dw)}\leq Cw(B(\mathbf{y},\sqrt{t}))^{-1/2}$, by Theorem 2.2. □

Corollary 4.2. *Assume that $V_1,V_2\colon\mathbb{R}^N\longmapsto[0,\infty)$, V_1,V_2 are bounded, and $V_1(\mathbf{y})\leq V_2(\mathbf{y})$ for all $\mathbf{y}\in\mathbb{R}^N$. Then for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $t>0$ we have*

$$k_t^{\{V_2\}}(\mathbf{x},\mathbf{y})\leq k_t^{\{V_1\}}(\mathbf{x},\mathbf{y}).$$

Proof. It is enough to note that the assumption $V_1(\mathbf{y})\leq V_2(\mathbf{y})$ implies

$$\begin{aligned}&k_t^{\{V_2\}}(\mathbf{x},\mathbf{y})\\&=\lim_{n\to\infty}\int_{\mathbb{R}^N}\dots\int_{\mathbb{R}^N}h_{t/n}(\mathbf{x},\mathbf{z}_1)e^{-tV_2(\mathbf{z}_1)/n}h_{t/n}(\mathbf{z}_1,\mathbf{z}_2)e^{-tV_2(\mathbf{z}_2)/n}\dots h_{t/n}(\mathbf{z}_{n-1},\mathbf{y})\,dw(\mathbf{z}_{n-1})\dots dw(\mathbf{z}_1)\\&\leq\lim_{n\to\infty}\int_{\mathbb{R}^N}\dots\int_{\mathbb{R}^N}h_{t/n}(\mathbf{x},\mathbf{z}_1)e^{-tV_1(\mathbf{z}_1)/n}h_{t/n}(\mathbf{z}_1,\mathbf{z}_2)e^{-tV_1(\mathbf{z}_2)/n}\dots h_{t/n}(\mathbf{z}_{n-1},\mathbf{y})\,dw(\mathbf{z}_{n-1})\dots dw(\mathbf{z}_1)\\&=k_t^{\{V_1\}}(\mathbf{x},\mathbf{y}).\end{aligned}$$

□

5. Upper bounds for Schrödinger semigroups with non-negative potentials

Theorem 5.1. *Assume that $V\colon\mathbb{R}^N\longmapsto[0,\infty)$, $V\in L^1_{\rm loc}(dw)$. Let $V_n=\min(V,n)$ and $L_n=-\Delta_k+V_n$, $n\in\mathbb{N}$. Then, for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $t>0$ the sequence $\{k_t^{\{V_n\}}(\mathbf{x},\mathbf{y})\}_{n\in\mathbb{N}}$ converges monotonically to the kernel of the semigroup $\{e^{-tL}\}_{t\geq 0}$, that is, for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $t>0$ we have*

$$\lim_{n\to\infty}k_t^{\{V_n\}}(\mathbf{x},\mathbf{y})=k_t^{\{V\}}(\mathbf{x},\mathbf{y})$$

and

$$e^{-tL}f(\mathbf{x})=\int_{\mathbb{R}^N}k_t^{\{V\}}(\mathbf{x},\mathbf{y})f(\mathbf{y})\,dw(\mathbf{y}).\tag{5.1}$$

Moreover, for any $m \in \mathbb{N}$ there is a constant $C_m > 0$ such that for all $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^N \times \mathbb{R}^N$ the function $(0, \infty) \ni t \mapsto k_t^{\{V\}}(\mathbf{x}, \mathbf{y})$ is smooth and

$$\left|\frac{d^m}{dt^m} k_t^{\{V\}}(\mathbf{x}, \mathbf{y})\right| \le C_m t^{-m} w(B(\mathbf{x}, \sqrt{t}))^{-1/2} w(B(\mathbf{y}, \sqrt{t}))^{-1/2}. \tag{5.2}$$

Proof. By the results of the previous section (see Theorem 4.1 and Corollary 4.2) the kernels $\{k_t^{\{V_n\}}(\mathbf{x}, \mathbf{y})\}_{n \in \mathbb{N}}$ of the semigroups $\{e^{-tL_n}\}_{t \ge 0}$, form a monotonic family of continuous functions of $(t, \mathbf{x}, \mathbf{y})$, that is,

$$0 \le k_t^{\{V_{n+1}\}}(\mathbf{x}, \mathbf{y}) \le k_t^{\{V_n\}}(\mathbf{x}, \mathbf{y}) \le h_t(\mathbf{x}, \mathbf{y}).$$

Hence, for all $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^N \times \mathbb{R}^N$ and $t > 0$ the limit $\lim_{n \to \infty} k_t^{\{V_n\}}(\mathbf{x}, \mathbf{y})$ exists and defines a kernel $k_t^{\{V\}}(\mathbf{x}, \mathbf{y}) \le h_t(\mathbf{x}, \mathbf{y})$. Moreover, applying the Arzelà–Ascoli theorem, we deduce (5.2) from the inequalities

$$\left|\frac{d^m}{dt^m} k_t^{\{V_n\}}(\mathbf{x}, \mathbf{y})\right| \le C_m t^{-m} w(B(\mathbf{x}, \sqrt{t}))^{-1/2} w(B(\mathbf{y}, \sqrt{t}))^{-1/2},$$

which hold for $k_t^{\{V_n\}}(\mathbf{x}, \mathbf{y})$ thanks to Theorem 4.1 (see (4.3)). Further, by the Lebesgue dominated convergence theorem, for each $\mathbf{x} \in \mathbb{R}^N$ and all $f \in L^2(dw)$, the limit

$$\lim_{n \to \infty} e^{-tL_n} f(\mathbf{x}) = \lim_{n \to \infty} \int_{\mathbb{R}^N} k_t^{\{V_n\}}(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, dw(\mathbf{y})$$

exists and defines a bounded functional such that

$$\lim_{n \to \infty} e^{-tL_n} f(\mathbf{x}) = \int_{\mathbb{R}^N} k_t^{\{V\}}(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, dw(\mathbf{y}).$$

On the other hand, by (3.3) for each $f \in L^2(dw)$, the sequence $\{e^{-tL_n} f\}_{n \in \mathbb{N}}$ converges in the $L^2(dw)$-norm to $e^{-tL} f$, hence (5.1) follows. □

Corollary 5.2. *Assume that $V_1, V_2 \colon \mathbb{R}^N \longmapsto [0, \infty)$, $V_1, V_2 \in L^1_{\mathrm{loc}}(dw)$ and $V_1(\mathbf{y}) \le V_2(\mathbf{y})$ for all $\mathbf{y} \in \mathbb{R}^N$. Then for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ and $t > 0$ we have*

$$k_t^{\{V_2\}}(\mathbf{x}, \mathbf{y}) \le k_t^{\{V_1\}}(\mathbf{x}, \mathbf{y}).$$

Proof. It is a consequence of Corollary 4.2 and Theorem 5.1. □

6. The Feynman-Kac formula

In this section we elaborate the Feynman-Kac formula for Dunkl Schrödinger operators with continuous bounded potentials. Our approach is standard and uses the product formula (3.4) (see also (4.11)). For the reader convenience, we provide some details. Then the Feynman–Kac formula will be used in proving the implication (c) $\Longrightarrow$ (a) of Theorem 1.1.

Let $I \subset \mathbb{R}$ be an interval. Recall that a function $I \ni t \longmapsto X_t \in \mathbb{R}^N$ is said to be *càdlàg* if it is right continuous, and it has left limits.

Proposition 6.1. *Assume that $f \colon [a, b] \to \mathbb{R}$ is a bounded càdlàg function. Then*

$$\lim_{n \to \infty} \frac{b-a}{n} \sum_{k=0}^{n-1} f\left(a + \frac{k(b-a)}{n}\right) = \int_a^b f(t)\, dt.$$

The right-hand side of the formula above is understood as the Lebesgue integral.

The proposition can be proved by standard arguments. For the completeness we elaborate it in Appendix A.

Let $(X_t, \Omega, \mathbb{P}^{\mathbf{x}})$, $X_t : \Omega \to \mathbb{R}^N$, be a Dunkl process associated with the transition probabilities

$$P_t(\mathbf{x}, E) = \int_E h_t(\mathbf{x}, \mathbf{y})\, dw(\mathbf{y}),$$

that is, a Markov process with càdlàg realizations $[0, \infty) \ni t \mapsto X_t(\omega)$ satisfying

(6.1)
$$\mathbb{P}^{\mathbf{x}}\{\omega \in \Omega : X_{t_1} \in E_1, X_{t_2} \in E_2, \dots, X_{t_n} \in E_n\}$$
$$= \int_{E_1}\int_{E_2}\dots\int_{E_n} h_{t_1}(\mathbf{x}, \mathbf{x}_1)h_{t_2-t_1}(\mathbf{x}_1, \mathbf{x}_2)\dots h_{t_n-t_{n-1}}(\mathbf{x}_{n-1}, \mathbf{x}_n)\, dw(\mathbf{x}_n)\, dw(\mathbf{x}_{n-1})\dots dw(\mathbf{x}_1).$$

for any finite sequence $0 < t_1 < t_t < \dots < t_n$ and any measurable sets $E_1, E_2, \dots, E_n \subseteq \mathbb{R}^N$ (see Rösler-Voit [17]). The formula implies that for a reasonable measurable function F defined on $(\mathbb{R}^N)^n$ one has

(6.2)
$$\mathbb{E}^{\mathbf{x}}(F(X_{t_1}, X_{t_2}, \dots, X_{t_n}))$$
$$= \int_{(\mathbb{R}^N)^n} F(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)h_{t_1}(\mathbf{x}, \mathbf{x}_1)h_{t_2-t_1}(\mathbf{x}_1, \mathbf{x}_2)\dots h_{t_n-t_{n-1}}(\mathbf{x}_{n-1}, \mathbf{x}_n)\, dw(\mathbf{x}_n)\, dw(\mathbf{x}_{n-1})\dots dw(\mathbf{x}_1).$$

Assume that $V \geq 0$ is a bounded continuous function. Let $Q_{n,t}(\mathbf{x}, \mathbf{y})$ be as in the proof of Theorem 4.1 (see (4.4) and (4.5)). Let $f \in L^2(dw)$ and $t > 0$. Putting $t_k := \frac{k}{n}t$, $1 \leq k \leq n$, and using (6.2), we have

(6.3)
$$\begin{aligned}
(H_{t/n}e^{-\frac{t}{n}V})^n f(\mathbf{x}) &= \int_{\mathbb{R}^N} Q_{n,t}(\mathbf{x}, \mathbf{y})f(\mathbf{y})\, dw(\mathbf{y})\\
&= \int_{(\mathbb{R}^N)^n} h_{t/n}(\mathbf{x}, \mathbf{x}_1)h_{t/n}(\mathbf{x}_1, \mathbf{x}_2)\dots h_{t/n}(\mathbf{x}_{n-1}, \mathbf{x}_n)\\
&\qquad\qquad \times \underbrace{e^{-\frac{t}{n}(V(\mathbf{x}_1)+V(\mathbf{x}_2)+\dots+V(\mathbf{x}_n))}f(\mathbf{x}_n)}_{F(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)}\, dw(\mathbf{x}_n)\, dw(\mathbf{x}_{n-1})\ \dots\ dw(\mathbf{x}_1)\\
&= E^{\mathbf{x}}(F(X_{t_1}, X_{t_2}, \dots, X_{t_n}))\\
&= E^{\mathbf{x}}\Big[\exp\Big(-\frac{t}{n}\sum_{k=1}^n V(X_{t_k})\Big) f(X_t)\Big].
\end{aligned}$$

Recall that it was established in the proof of Theorem 4.1 that there is a subsequence $\{n_j\}_{j\in\mathbb{N}}$ such that the continuous functions $Q_{n_j,t}(\mathbf{x}, \mathbf{y}) \leq h_t(\mathbf{x}, \mathbf{y})$ converge uniformly on compact subsets to $k_t^{\{V\}}(\mathbf{x}, \mathbf{y})$. Hence taking into account integration of càdlàg functions (see Proposition 6.1), we obtain the following corollary.

Corollary 6.2 (Feynman-Kac formula)**.** *Let $V \geq 0$ be a bounded continuous function. Then for $t > 0$, $\mathbf{x} \in \mathbb{R}^N$ and all $f \in L^2(dw)$ we have*

$$e^{t(\Delta_k - V)}f(\mathbf{x}) = E^{\mathbf{x}}\Big[\exp\Big(-\int_0^t V(X_s)\, ds\Big) f(X_t)\Big]. \tag{6.4}$$

7. Schrödinger semigroups with Green bounded potentials

7.1. **Green bounded potentials.** In the proposition below we elaborate the equivalences stated in Remark 1.2. For a measurable function $V:\mathbb{R}^N \longmapsto [0,\infty)$ and $\mathbf{x}\in\mathbb{R}^N$, let

$$\mathbf{G}(V)(\mathbf{x}) := \int_0^\infty \int_{\mathbb{R}^N} w(B(\mathbf{x},\sqrt{s}))^{-1} e^{-\|\mathbf{x}-\mathbf{y}\|^2/s} V(\mathbf{y})\,dw(\mathbf{y})\,ds, \tag{7.1}$$

$$\mathbf{G}_1(V)(\mathbf{x}) := \int_0^\infty \int_{\mathbb{R}^N} h_s(\mathbf{x},\mathbf{y}) V(\mathbf{y})\,dw(\mathbf{y})\,ds, \tag{7.2}$$

$$\mathcal{G}(V)(\mathbf{x}) := \int_0^\infty \int_{\mathbb{R}^N} w(B(\mathbf{x},\sqrt{s}))^{-1} e^{-d(\mathbf{x},\mathbf{y})^2/s} V(\mathbf{y})\,dw(\mathbf{y})\,ds. \tag{7.3}$$

Proposition 7.1. *There are constants $C_1, C_2, C_3 > 0$ such that for all measurable non-negative functions $V:\mathbb{R}^N \longmapsto [0,\infty)$ one has*

$$\|\mathbf{G}(V)\|_{L^\infty} \leq C_1 \|\mathbf{G}_1(V)\|_{L^\infty} \leq C_2 \|\mathcal{G}(V)\|_{L^\infty} \leq C_3 \|\mathbf{G}(V)\|_{L^\infty}. \tag{7.4}$$

Proof. It follows from Theorem 2.2 that there are constants $C, c > 0$ such that for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $s>0$ we have

$$C^{-1} w(B(\mathbf{x},\sqrt{s}))^{-1} e^{-c^{-1}\|\mathbf{x}-\mathbf{y}\|^2/s} \leq h_s(\mathbf{x},\mathbf{y}) \leq C w(B(\mathbf{x},\sqrt{s}))^{-1} e^{-cd(\mathbf{x},\mathbf{y})^2/s} \tag{7.5}$$

(see also [2, Theorems 4.1 and 4.4]). Further, by the definition of $d(\mathbf{x},\mathbf{y})$ (see (1.5)),

$$w(B(\mathbf{x},\sqrt{s}))^{-1} e^{-cd(\mathbf{x},\mathbf{y})^2/s} \leq \sum_{\sigma\in G} w(B(\mathbf{x},\sqrt{s}))^{-1} e^{-c\|\sigma(\mathbf{x})-\mathbf{y}\|^2/s}. \tag{7.6}$$

The proposition is a direct consequence of the inequalities (7.5), (7.6), and the doubling property of the measure dw (see (2.2)). ☐

In order to establish Theorem 1.1, we prove the implications: (a) $\Longrightarrow$ (b) (in Lemma 7.2), then (b) $\Longrightarrow$ (c) (in Lemma 7.4), and finally, (c) $\Longrightarrow$ (a) (in Subsection 7.3). We prove (c) $\Longrightarrow$ (a) in the separate subsection, because it is relatively more involving and it uses the heat kernel estimates (2.17), (2.18), and the Feynman–Kac formula (see Section 6).

7.2. **Proofs of the implications (a) $\Longrightarrow$ (b) and (b) $\Longrightarrow$ (c).**

Lemma 7.2. *Assume that $V:\mathbb{R}^N \longmapsto [0,\infty)$, $V\in L^1_{\text{loc}}(dw)$. Assume that there are constants $C, c>0$ such that for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $t>0$ we have*

$$h_{ct}(\mathbf{x},\mathbf{y}) \leq C k_t^{\{V\}}(\mathbf{x},\mathbf{y}). \tag{7.7}$$

Then there is a constant $\delta>0$ such for all $\mathbf{x}\in\mathbb{R}^N$ and $t>0$ we have

$$\int_{\mathbb{R}^N} k_t^{\{V\}}(\mathbf{x},\mathbf{y})\,dw(\mathbf{y}) > \delta.$$

Proof. It is enough to integrate (7.7) with respect to $dw(\mathbf{y})$ and apply (2.8). ☐

Lemma 7.3. *Assume that $V:\mathbb{R}^N \longmapsto [0,\infty)$ is measurable and bounded. Then for all $t>0$ and $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ we have*

$$h_t(\mathbf{x},\mathbf{y}) = k_t^{\{V\}}(\mathbf{x},\mathbf{y}) + \int_0^t \int_{\mathbb{R}^N} h_s(\mathbf{x},\mathbf{z}) V(\mathbf{z}) k_{t-s}^{\{V\}}(\mathbf{z},\mathbf{y})\,dw(\mathbf{z})\,ds. \tag{7.8}$$

Proof. See Theorem 3.1. ☐

Lemma 7.4. *Assume that $V : \mathbb{R}^N \longmapsto [0,\infty)$, $V \in L^1_{\rm loc}(dw)$. Assume that there is $\delta > 0$ such for all $\mathbf{x} \in \mathbb{R}^N$ and $t > 0$ we have*

$$\int_{\mathbb{R}^N} k_t^{\{V\}}(\mathbf{x},\mathbf{y})\,dw(\mathbf{y}) > \delta. \tag{7.9}$$

Then V is Green bounded.

Proof. The proof is standard. Let $V_n = \min(V,n)$, $n \in \mathbb{N}$. Recall that $k_t^{\{V_n\}}(\mathbf{x},\mathbf{y}) \geq k_t^{\{V\}}(\mathbf{x},\mathbf{y})$ (see Corollaries 4.2 and 5.2). By the perturbation formula (7.8) applied to $k_t^{\{V_n\}}(\mathbf{x},\mathbf{y})$, (2.8), and the assumption (7.9) we have

$$1 \geq \int_{\mathbb{R}^N}\int_0^t\int_{\mathbb{R}^N} h_s(\mathbf{x},\mathbf{z})V_n(\mathbf{z})k_{t-s}^{\{V_n\}}(\mathbf{z},\mathbf{y})\,dw(\mathbf{z})\,ds\,dw(\mathbf{y}) \geq \delta\int_0^t\int_{\mathbb{R}^N} h_s(\mathbf{x},\mathbf{z})V_n(\mathbf{z})\,dw(\mathbf{z})\,ds$$

with δ independent of $n \in \mathbb{N}$. Letting $t \to \infty$, we obtain the bound $\|\mathbf{G}_1(V_n)\|_{L^\infty} \leq \delta^{-1}$. Now, letting $n \to \infty$, we get the lemma by applying the Lebesgue monotone convergence theorem and Proposition 7.1. □

7.3. **Implication** (c) $\Longrightarrow$ (a)**.** In order to prove the implication we adapt to the Dunkl setting general patterns of proofs of lower bounds for the classical Schrödinger operators or Bessel-Schrödinger operators (see [19], [7]). Thus, first we prove the lower bounds in the case of continuous and bounded V with the property $\|\mathcal{G}(V)\|$ being small enough. Then we extend the lower estimates to all non-negative Green bounded potentials V. The main difficulties we face concern the fact that the upper and lower estimates of the Dunkl heat kernel $h_t(\mathbf{x},\mathbf{y})$ have rather complex forms which involve both - the orbit distance $d(\mathbf{x},\mathbf{y})$ and the Euclidean distances $\|\mathbf{x}-\sigma_{(\alpha_1,\alpha_2,\dots,\alpha_j)}(\mathbf{y})\|$ contained in the definition of the function Λ (see (2.15)). To this end we need a preparation.

For $\mathbf{x},\mathbf{y} \in \mathbb{R}^N$ and $t > 0$ we set

$$\mathcal{G}_t(\mathbf{x},\mathbf{y}) := w(B(\mathbf{y},\sqrt{t}))^{-1}e^{-d(\mathbf{x},\mathbf{y})^2/t}. \tag{7.10}$$

Let us begin with a proposition concerning the properties of the generalized heat kernel. In its proof, the specific generalized heat kernel bounds from Theorem 2.2 are utilized.

Proposition 7.5. *There are constants $C_1 > 0$, $c_1 > 1$, such that for all $0 < s \leq t/2$ and all $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^N$ one has*

$$h_{t-s}(\mathbf{x},\mathbf{z})h_s(\mathbf{z},\mathbf{y}) \leq C_1 h_{c_1t}(\mathbf{x},\mathbf{y})\mathcal{G}_{c_1s}(\mathbf{z},\mathbf{y}). \tag{7.11}$$

Proof. Let $c_0 > 1$. By Lemma 2.5, for all $\mathbf{z}' \in \mathbb{R}^N$ such that $\|\mathbf{z}-\mathbf{z}'\| \leq \sqrt{s} \leq \sqrt{t-s}$ we have

$$h_{t-s}(\mathbf{x},\mathbf{z})h_s(\mathbf{z},\mathbf{y}) \leq C_0^2 h_{c_0(t-s)}(\mathbf{x},\mathbf{z}')h_{c_0s}(\mathbf{z}',\mathbf{y}). \tag{7.12}$$

Note that

$$e^{-c_u\frac{d(\mathbf{z}',\mathbf{y})^2}{c_0s}} \leq Ce^{-c_u\frac{d(\mathbf{z}',\mathbf{y})^2}{2c_0s}}e^{-c_u\frac{d(\mathbf{z},\mathbf{y})^2}{4c_0s}} \quad \text{for } \|\mathbf{z}-\mathbf{z}'\| \leq \sqrt{s}.$$

Hence, applying (2.18), (2.16), the doubling property of dw (see (2.2)), and Theorem 2.2, we get

(7.13)
$$\begin{aligned}
&h_{c_0(t-s)}(\mathbf{x},\mathbf{z}')h_{c_0s}(\mathbf{z}',\mathbf{y})\\
&\leq Cw(B(\mathbf{z}',\sqrt{c_0(t-s)}))^{-1}w(B(\mathbf{z}',\sqrt{c_0s}))^{-1}\Lambda(\mathbf{x},\mathbf{z}',c_0(t-s))\Lambda(\mathbf{z}',\mathbf{y},c_0s)e^{-c_u\frac{d(\mathbf{x},\mathbf{z}')^2}{c_0(t-s)}-c_u\frac{d(\mathbf{z}',\mathbf{y})^2}{c_0s}}\\
&\leq C\left(w(B(\mathbf{z}',\sqrt{t-s}))^{-1}\Lambda(\mathbf{x},\mathbf{z}',t-s)\Lambda(\mathbf{z}',\mathbf{y},s)e^{-c_u\frac{d(\mathbf{x},\mathbf{z}')^2}{2c_0(t-s)}-c_u\frac{d(\mathbf{z}',\mathbf{y})^2}{2c_0s}}\right)\left(w(B(\mathbf{z},\sqrt{s}))^{-1}e^{-c_u\frac{d(\mathbf{y},\mathbf{z})^2}{4c_0s}}\right)\\
&\leq Cw(B(\mathbf{z},\sqrt{s}))\mathcal{G}_{c_1s}(\mathbf{y},\mathbf{z})h_{c_1(t-s)}(\mathbf{x},\mathbf{z}')h_{c_1s}(\mathbf{z}',\mathbf{y}).
\end{aligned}$$

Since the estimates (7.12) and (7.13) are given uniformly on $\mathbf{z}'\in B(\mathbf{z},\sqrt{s})$, taking their mean over the ball $B(\mathbf{z},\sqrt{s})$ we get

$$\begin{aligned}
h_{t-s}(\mathbf{x},\mathbf{z})h_s(\mathbf{z},\mathbf{y}) &\leq C\mathcal{G}_{c_1s}(\mathbf{y},\mathbf{z})\int_{B(\mathbf{z},\sqrt{s})}h_{c_1(t-s)}(\mathbf{x},\mathbf{z}')h_{c_1s}(\mathbf{z}',\mathbf{y})\,dw(\mathbf{z}')\\
&\leq C\mathcal{G}_{c_1s}(\mathbf{y},\mathbf{z})\int_{\mathbb{R}^N}h_{c_1(t-s)}(\mathbf{x},\mathbf{z}')h_{c_1s}(\mathbf{z}',\mathbf{y})\,dw(\mathbf{z}')\\
&= C\mathcal{G}_{c_1s}(\mathbf{y},\mathbf{z})h_{c_1t}(\mathbf{x},\mathbf{y}),
\end{aligned}$$

where is the last step we have used the semigroup property of $h_t(\cdot,\cdot)$. □

The following corollary is an consequence of Proposition 7.5.

Corollary 7.6. *Assume that $V:\mathbb{R}^N\longmapsto[0,\infty)$ is continuous, bounded, and Green bounded. Then there are constants $C_2,c_2>0$ such that for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ we have*

$$\int_0^t\int_{\mathbb{R}^N}h_{t-s}(\mathbf{x},\mathbf{z})V(\mathbf{z})k_s^{\{V\}}(\mathbf{z},\mathbf{y})\,dw(\mathbf{z})\,ds\leq C_2\|\mathcal{G}(V)\|_{L^\infty}h_{c_2t}(\mathbf{x},\mathbf{y}).$$

Proof. By (1.3) we have

$$\begin{aligned}
\int_0^t\int_{\mathbb{R}^N}h_{t-s}(\mathbf{x},\mathbf{z})V(\mathbf{z})k_s^{\{V\}}(\mathbf{z},\mathbf{y})\,dw(\mathbf{z})\,ds &\leq \int_0^t\int_{\mathbb{R}^N}h_{t-s}(\mathbf{x},\mathbf{z})V(\mathbf{z})h_s(\mathbf{z},\mathbf{y})\,dw(\mathbf{z})\,ds\\
&=\int_0^{t/2}\int_{\mathbb{R}^N}\dots+\int_{t/2}^t\int_{\mathbb{R}^N}\dots=:I_1+I_2.
\end{aligned}$$

We will estimate I_1; the case of I_2 can be reduced to the case of I_1 by the change of variables. By Proposition 7.5 we get

$$I_1\leq C_1h_{c_1t}(\mathbf{x},\mathbf{y})\int_0^t\int_{\mathbb{R}^N}\mathcal{G}_{c_1s}(\mathbf{z},\mathbf{y})V(\mathbf{z})\,dw(\mathbf{z})\,ds.$$

Finally, by the change of variables $s_1:=c_1s$ we get

$$\int_0^t\int_{\mathbb{R}^N}\mathcal{G}_{c_1s}(\mathbf{z},\mathbf{y})V(\mathbf{z})\,dw(\mathbf{z})\,ds\leq C\|\mathcal{G}(V)\|_{L^\infty},$$

which finishes the proof. □

Corollary 7.7. *Assume that $V:\mathbb{R}^N\longmapsto[0,\infty)$ is bounded, continuous, and Green bounded. Let $c_3>0$. There are $\widetilde{c}_3,\varepsilon>0$ such that if $\|\mathcal{G}(V)\|_{L^\infty}<\varepsilon$, then for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $t>0$ such that $d(\mathbf{x},\mathbf{y})<c_3\sqrt{t}$ one has*

$$k_t^{\{V\}}(\mathbf{x},\mathbf{y})\geq\widetilde{c}_3h_t(\mathbf{x},\mathbf{y}).\tag{7.14}$$

Proof. From the perturbation formula (7.8) we get

$$h_t(\mathbf{x},\mathbf{y})-k_t^{\{V\}}(\mathbf{x},\mathbf{y})=\int_0^t\int_{\mathbb{R}^N}h_{t-s}(\mathbf{x},\mathbf{z})V(\mathbf{z})k_s^{\{V\}}(\mathbf{z},\mathbf{y})\,dw(\mathbf{z})\,ds.$$

Hence from Corollary 7.6, (2.18), and (2.17), we deduce that

$$\begin{aligned}k_t^{\{V\}}(\mathbf{x},\mathbf{y})&\geq h_t(\mathbf{x},\mathbf{y})-C_2\|\mathcal{G}(V)\|_{L^\infty}h_{c_2t}(\mathbf{x},\mathbf{y})\\&\geq C_{\rm l}w(B(\mathbf{x},\sqrt{t}))^{-1}e^{-c_{\rm l}\frac{d(\mathbf{x},\mathbf{y})^2}{t}}\Lambda(\mathbf{x},\mathbf{y},t)\\&\quad-C_{\rm u}C_2\|\mathcal{G}(V)\|_{L^\infty}w(B(\mathbf{x},\sqrt{c_2t}))^{-1}\Lambda(\mathbf{x},\mathbf{y},c_2t).\end{aligned}\tag{7.15}$$

Note that by the fact $d(\mathbf{x},\mathbf{y})\leq c_3\sqrt{t}$, we have $e^{-c_{\rm l}\frac{d(\mathbf{x},\mathbf{y})^2}{t}}\geq C>0$. Further, by the doubling property of dw and (2.16), if $\varepsilon>0$ is small enough, (7.15) implies

$$k_t^{\{V\}}(\mathbf{x},\mathbf{y})\geq cw(B(\mathbf{x},\sqrt{t}))^{-1}\Lambda(\mathbf{x},\mathbf{y},t)$$

for some constant $c>0$. Finally, (7.14) is a consequence of (2.18). □

Proposition 7.8. *Assume that $V:\mathbb{R}^N\longmapsto[0,\infty)$ is continuous, bounded, and Green bounded. There are $\varepsilon,c_4,C_4>0$ such that if $\|\mathcal{G}(V)\|_{L^\infty}<\varepsilon$, then for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $t>0$ one has*

$$k_t^{\{V\}}(\mathbf{x},\mathbf{y})\geq C_4h_{c_4t}(\mathbf{x},\mathbf{y}).\tag{7.16}$$

Proof. Let $c_3=1$, $\tilde{c}_3$ and $\varepsilon>0$ be as in Corollary 7.7. Without loss of generality we assume that $t=1$. Further, according to Corollary 7.7, it suffices to consider $d(\mathbf{x},\mathbf{y})\geq 1$. Let $\sigma\in G$ be such that $\|\sigma(\mathbf{y})-\mathbf{x}\|=d(\mathbf{y},\mathbf{x})$. Set $\mathbf{y}'=\sigma(\mathbf{y})$, $n=64[d(\mathbf{x},\mathbf{y})^2]=64[\|\mathbf{x}-\mathbf{y}'\|^2]$, and

$$\mathbf{y}_j=\mathbf{x}+j\frac{\mathbf{y}'-\mathbf{x}}{n},\quad j=1,2,\dots,n-1.$$

Consider the balls $B_j=B\left(\mathbf{y}_j,(8\sqrt{n})^{-1}\right)$. By the semigroup property of $k_1^{\{V\}}(\mathbf{x},\mathbf{y})$ and the fact that $k_{t_1}^{\{V\}}(\mathbf{x}_1,\mathbf{x}_2)\geq 0$ for all $\mathbf{x}_1,\mathbf{x}_2\in\mathbb{R}^N$ and $t_1>0$ we have

$$\begin{aligned}&k_1^{\{V\}}(\mathbf{x},\mathbf{y})\\&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\dots\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}k_{\frac{1}{n}}^{\{V\}}(\mathbf{x},\mathbf{z}_1)k_{\frac{1}{n}}^{\{V\}}(\mathbf{z}_1,\mathbf{z}_2)\dots k_{\frac{1}{n}}^{\{V\}}(\mathbf{z}_{n-2},\mathbf{z}_{n-1})k_{\frac{1}{n}}^{\{V\}}(\mathbf{z}_{n-1},\mathbf{y})\,dw(\mathbf{z}_1)\dots dw(\mathbf{z}_{n-1})\\&\geq\int_{B_1}\int_{B_2}\dots\int_{B_{n-2}}\int_{B_{n-1}}k_{\frac{1}{n}}^{\{V\}}(\mathbf{x},\mathbf{z}_1)k_{\frac{1}{n}}^{\{V\}}(\mathbf{z}_1,\mathbf{z}_2)\dots k_{\frac{1}{n}}^{\{V\}}(\mathbf{z}_{n-2},\mathbf{z}_{n-1})k_{\frac{1}{n}}^{\{V\}}(\mathbf{z}_{n-1},\mathbf{y})\,dw(\mathbf{z}_{n-1})\dots dw(\mathbf{z}_1).\end{aligned}\tag{7.17}$$

Observe that for $\mathbf{z}_j\in B_j$ and $\mathbf{z}_{j+1}\in B_{j+1}$ we have

$$\|\mathbf{z}_j-\mathbf{z}_{j+1}\|\leq\frac{4}{8\sqrt{n}}.$$

By Corollary 7.7, Lemma 2.5, (2.17), and the doubling property of the measure dw, we get

$$k_{\frac{1}{n}}^{\{V\}}(\mathbf{z}_j,\mathbf{z}_{j+1})\geq c_5w\left(B\left(\mathbf{y}_j,(\sqrt{n})^{-1}\right)\right)^{-1}.\tag{7.18}$$

Moreover, by the fact that $d(\mathbf{z}_{n-1},\mathbf{y})=d(\mathbf{z}_{n-1},\mathbf{y}')\leq\|\mathbf{z}_{n-1}-\mathbf{y}'\|\leq\frac{4}{8\sqrt{n}}$, Corollary 7.7, and Lemma 2.5 (with $c_0=2$), we obtain

$$k_{\frac{1}{n}}^{\{V\}}(\mathbf{z}_{n-1},\mathbf{y})\geq\widetilde{c}_3h_{\frac{1}{n}}(\mathbf{z}_{n-1},\mathbf{y})\geq\widetilde{c}_3C_0^{-1}h_{1/(2n)}(\mathbf{y},\mathbf{y}').\tag{7.19}$$

Recall that by the doubling property of dw and the definition of B_j we have

$$\frac{w(B_j)}{w\left(B\left(\mathbf{y}_j,(\sqrt{n})^{-1}\right)\right)} \geq c.$$

Therefore, by (7.17), (7.18), and (7.19), for a constant $c_6 > 0$ small enough,

$$k_1^{\{V\}}(\mathbf{x},\mathbf{y}) \geq c_6^{n-1} h_{1/(2n)}(\mathbf{y},\mathbf{y}'). \tag{7.20}$$

Then, by (2.17), doubling property of dw, (2.16), (2.1), and the fact that $d(\mathbf{y},\mathbf{y}')=0$, one gets

$$h_{1/(2n)}(\mathbf{y},\mathbf{y}') \geq C_l w(B(\mathbf{y},(\sqrt{2n})^{-1})^{-1}\Lambda(\mathbf{y},\mathbf{y}',(1/(2n))) \geq Cn^{-\mathbf{N}/2}n^{-2|G|}w(B(\mathbf{y},1))^{-1}\Lambda(\mathbf{y},\mathbf{y}',1). \tag{7.21}$$

Recall that $n = 64[d(\mathbf{x},\mathbf{y})^2]$. Hence, by (7.20), (7.21), and (2.18), we obtain

$$k_1^{\{V\}}(\mathbf{x},\mathbf{y}) \geq Cc_6^{n-1}n^{-\mathbf{N}/2}n^{-2|G|}w(B(\mathbf{y},1))^{-1}\Lambda(\mathbf{y},\mathbf{y}',1) \geq ce^{-c_7 d(\mathbf{x},\mathbf{y})^2}h_1(\mathbf{y},\mathbf{y}').$$

Further, by Proposition 7.5, we get

$$e^{-c_7 d(\mathbf{x},\mathbf{y})^2}h_1(\mathbf{y},\mathbf{y}') \geq Ce^{-c_7 d(\mathbf{x},\mathbf{y})^2}w(B(\mathbf{x},1))h_{1/c_1}(\mathbf{x},\mathbf{y})h_{1/c_1}(\mathbf{x},\mathbf{y}').$$

Since $\|\mathbf{x}-\mathbf{y}'\| = d(\mathbf{x},\mathbf{y})$, by Lemma 2.1 and the definition of $\Lambda(\cdot,\cdot,\cdot)$ (see (2.15)), we have $\emptyset \in \mathcal{A}(\mathbf{x},\mathbf{y}')$, so $\Lambda(\mathbf{x},\mathbf{y}',1/c_1) \geq 1$. Hence, by (2.17), one obtains

$$h_{1/c_1}(\mathbf{x},\mathbf{y}') \geq Cw(B(\mathbf{x},1))^{-1}e^{-c_8 d(\mathbf{x},\mathbf{y})^2}.$$

Thus, using Theorem 2.2, we conclude that

$$k_1^{\{V\}}(\mathbf{x},\mathbf{y}) \geq Ce^{-(c_7+c_8)d(\mathbf{x},\mathbf{y})^2}h_{1/c_1}(\mathbf{x},\mathbf{y}) \geq Cw(B(\mathbf{x},\sqrt{1/c_1}))^{-1}e^{-c_9 d(\mathbf{x},\mathbf{y})^2}\Lambda(\mathbf{x},\mathbf{y},1/c_1).$$

Finally the claim follows by applying (2.16) together with Theorem 2.2. □

Let us note that implication (c) $\implies$ (a) is already proved under the assumption that $\|\mathcal{G}(V)\|_{L^\infty}$ is small enough and V is continuous and bounded. In Proposition 7.9, we will make use of the Feynman–Kac formula to relax the assumption $\|\mathcal{G}(V)\|_{L^\infty} < \varepsilon$ for continuous and bounded functions V. Finally, in the further part of this subsection, we will relax the assumption that V is continuous and bounded.

Proposition 7.9. *Assume that $V : \mathbb{R}^N \longmapsto [0,\infty)$ is continuous, bounded, and Green bounded. Then there are constants $C_{\mathcal{G}}, c_{\mathcal{G}} > 0$, $C_{\mathcal{G}} < 1$, which depend only on the bound of $\|\mathcal{G}(V)\|_{L^\infty}$ such that for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^N$ and $t > 0$ we have*

$$k_t^{\{V\}}(\mathbf{x},\mathbf{y}) \geq C_{\mathcal{G}} h_{c_{\mathcal{G}}t}(\mathbf{x},\mathbf{y}). \tag{7.22}$$

Proof. Let ε be the same as in Proposition 7.8. We may assume that $\|\mathcal{G}(V)\|_{L^\infty} \geq \varepsilon$. Let $1 < p < \infty$ be such that $\|\mathcal{G}(\frac{1}{p}V)\|_{L^\infty} = \varepsilon/2$. Recall that $k_t^{\{V\}}(\mathbf{x},\mathbf{y})$ is a continuous function (see Theorem 4.1). By the Lebesgue differentiation theorem, for all $(\mathbf{x}_0,\mathbf{y}_0) \in \mathbb{R}^N \times \mathbb{R}^N$ and $t > 0$, we have

$$\begin{aligned} k_t^{\{\frac{1}{p}V\}}(\mathbf{x}_0,\mathbf{y}_0) &= \lim_{r\to 0^+}\frac{1}{w(B(\mathbf{y}_0,r))}\int_{B(\mathbf{y}_0,r)} k_t^{\{\frac{1}{p}V\}}(\mathbf{x}_0,\mathbf{y})\,dw(\mathbf{y}) \\ &= \lim_{r\to 0^+}\frac{1}{w(B(\mathbf{y}_0,r))}E^{\mathbf{x}_0}\Big[\exp\Big(-\int_0^t \frac{1}{p}V(X_s)\,ds\Big)\chi_{B(\mathbf{y}_0,r)}(X_t)\Big], \end{aligned} \tag{7.23}$$

where in the last equality we have used the Feynman-Kac formula (6.4). Now, we apply the Hölder's inequality with the exponents $p+p'=pp'$, and then Theorem 2.2 obtaining

(7.24)
$$\begin{aligned}k_t^{\{\frac{1}{p}V\}}(\mathbf{x}_0,\mathbf{y}_0) &\leq \lim_{r\to 0^+}\frac{1}{w(B(\mathbf{y}_0,r))}\Big\{E^{\mathbf{x}_0}\Big(e^{-\int_0^t V(X_s)\,ds}\chi_{B(\mathbf{y}_0,r)}(X_t)\Big)\Big\}^{1/p}\Big\{E^{\mathbf{x}_0}\Big(\chi_{B(\mathbf{y}_0,r)}(X_t)\Big)\Big\}^{1/p'}\\ &= \Big\{k_t^{\{V\}}(\mathbf{x}_0,\mathbf{y}_0)\Big\}^{1/p} h_t(\mathbf{x}_0,\mathbf{y}_0)^{1/p'}\\ &\leq C\Big\{k_t^{\{V\}}(\mathbf{x}_0,\mathbf{y}_0)\Big\}^{1/p}\frac{\Lambda(\mathbf{x}_0,\mathbf{y}_0,t)^{1/p'}}{w(B(\mathbf{x}_0,\sqrt{t}))^{1/p'}}.\end{aligned}$$

By Proposition 7.8, (2.17), the doubling property (2.2), and (2.16), we have

$$k_t^{\{\frac{1}{p}V\}}(\mathbf{x}_0,\mathbf{y}_0)\geq C_4 h_{c_4t}(\mathbf{x}_0,\mathbf{y}_0)\geq c'' w(B(\mathbf{x}_0,\sqrt{t}))^{-1}e^{-c'd(\mathbf{x}_0,\mathbf{y}_0)^2/4t}\Lambda(\mathbf{x}_0,\mathbf{y}_0,t). \tag{7.25}$$

Thus, combining (7.24) together with (7.25), we get

$$c'' w(B(\mathbf{x}_0,\sqrt{t}))^{-1}e^{-c'd(\mathbf{x}_0,\mathbf{y}_0)^2/4t}\Lambda(\mathbf{x}_0,\mathbf{y}_0,t)\leq C\Big\{k_t^{\{V\}}(\mathbf{x}_0,\mathbf{y}_0)\Big\}^{1/p}\frac{\Lambda(\mathbf{x}_0,\mathbf{y}_0,t)^{1/p'}}{w(B(\mathbf{x}_0,\sqrt{t}))^{1/p'}}. \tag{7.26}$$

Finally, by Theorem 2.2,

$$k_t^{\{V\}}(\mathbf{x}_0,\mathbf{y}_0)\geq\Big(\frac{c''}{C}\Big)^p e^{-c'pd(\mathbf{x}_0,\mathbf{y}_0)^2/4t}\frac{\Lambda(\mathbf{x}_0,\mathbf{y}_0,t)}{w(B(\mathbf{x}_0,\sqrt{t}))}\geq c h_{t/c'}(\mathbf{x}_0,\mathbf{y}_0). \tag{7.27}$$

□

Proposition 7.10. *Assume that $V\colon\mathbb{R}^N\longmapsto[0,\infty)$ is measurable, bounded, and Green bounded. Then there are constant $\widetilde{C}_{\mathcal{G}},\widetilde{c}_{\mathcal{G}}>0$, which depend only on the bound of $\|\mathcal{G}(V)\|_{L^\infty}$, such that for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $t>0$ we have*

$$k_t^{\{V\}}(\mathbf{x},\mathbf{y})\geq\widetilde{C}_{\mathcal{G}}h_{\widetilde{c}_{\mathcal{G}}t}(\mathbf{x},\mathbf{y}). \tag{7.28}$$

Proof. For $n\in\mathbb{N}$ we consider

$$\widetilde{V}_n(\mathbf{x})=\int_{\mathbb{R}^N}h_{1/n}(\mathbf{x},\mathbf{y})V(\mathbf{y})\,dw(\mathbf{y}).$$

Then, $\lim_{n\to\infty}\widetilde{V}_n(\mathbf{x})=V(\mathbf{x})$ for almost all $\mathbf{x}\in\mathbb{R}^N$ (see e.g. [2, Remark 5.5]) and, by the regularity of the heat semigroup, $\widetilde{V}_n$ are continuous functions. Moreover, by (2.8), $\|\widetilde{V}_n\|_{L^\infty}\leq\|V\|_{L^\infty}$, and, by Proposition 7.1, there is a constant $C>0$ such that

$$\|\mathcal{G}(\widetilde{V}_n)\|_{L^\infty}\leq C\|\mathcal{G}(V)\|_{L^\infty}.$$

Recall that $\{e^{-t\tilde{L}_n}\}_{t\geq 0}$ and $\{e^{-tL}\}_{t\geq 0}$ are the contraction semigroups on $L^2(dw)$ generated by the operators $-\tilde{L}_n=\Delta_k-\widetilde{V}_n$ and $-L=\Delta_k-V$ respectively. Then, for $f\in\mathcal{D}(L)=\mathcal{D}(\tilde{L}_n)=\mathcal{D}(\Delta_k)$, we have $\lim_{n\to\infty}\tilde{L}_nf=Lf$. Hence Theorem 3.4.5 of [13] asserts that

$$\lim_{n\to\infty}e^{-t\tilde{L}_n}f=e^{-tL}f\quad\text{in } L^2(dw)-\text{norm},\quad\text{for all } f\in L^2(dw). \tag{7.29}$$

Further, Proposition 7.9 and Theorem 4.1 imply that there are constants $C_{\mathcal{G}},c_{\mathcal{G}}>0$, $C_{\mathcal{G}}<1$, such that for all n, we have

$$C_{\mathcal{G}}h_{c_{\mathcal{G}}t}(\mathbf{x},\mathbf{y})\leq k_t^{\{\widetilde{V}_n\}}(\mathbf{x},\mathbf{y})\text{ for all }(t,\mathbf{x},\mathbf{y})\in(0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N. \tag{7.30}$$

Assume towards contradiction that $C_{\mathcal{G}} h_{c_{\mathcal{G}} t}(\mathbf{x}_0, \mathbf{y}_0) > k_t^{\{V\}}(\mathbf{x}_0, \mathbf{y}_0)$ for some $(t_0, \mathbf{x}_0, \mathbf{y}_0) \in (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N$. Then, by the fact that $k_t^{\{V\}}(\cdot,\cdot)$ and $h_t(\cdot,\cdot)$ are continuous, there are $\varepsilon, \delta > 0$ such that $C_{\mathcal{G}} h_{c_{\mathcal{G}} t_0}(\mathbf{x}, \mathbf{y}) > k_{t_0}^{\{V\}}(\mathbf{x}, \mathbf{y}) + \varepsilon$ for all $(\mathbf{x}, \mathbf{y}) \in B(\mathbf{x}_0, \delta) \times B(\mathbf{y}_0, \delta)$. Hence, applying (7.29) to $f = \chi_{B(\mathbf{y}_0, \delta)}$ we obtain a contradiction. □

Proof of the implication (c) $\Longrightarrow$ (a). Assume that $V\colon \mathbb{R}^N \longmapsto [0,\infty)$, $V \in L^1_{\rm loc}(dw)$, is Green bounded. Consider the operators $L_n = -\Delta_k + V_n$, $V_n = \min(V, n)$, $n \in \mathbb{N}$. By Proposition 7.10 there are $\widetilde{C}_{\mathcal{G}}, \widetilde{c}_{\mathcal{G}} > 0$ such that for all $n \in \mathbb{N}$ we have

$$k_t^{\{V_n\}}(\mathbf{x}, \mathbf{y}) \geq \widetilde{C}_{\mathcal{G}} h_{\widetilde{c}_{\mathcal{G}} t}(\mathbf{x}, \mathbf{y})$$

for all $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^N \times \mathbb{R}^N$. Now the required lower bound for $k_t^{\{V\}}(\mathbf{x}, \mathbf{y})$ follows from Theorem 5.1.

□

Appendix A. Proof of Proposition 6.1

Lemma A.1. *Assume that $f : [a,b] \to \mathbb{R}$ is a bounded càdlàg function. Define*

$$J_f(t_0) := \Big| \lim_{t \to t_0^-} f(t) - \lim_{t \to t_0^+} f(t) \Big| = \Big| \lim_{t \to t_0^-} f(t) - f(t_0) \Big|. \tag{A.1}$$

Then, for all $\varepsilon > 0$, one has

$$\#\{t \in [a,b] : J_f(t) \geq \varepsilon\} = C_\varepsilon < \infty.$$

Proof. Aiming for a contradiction, suppose that the set $A = \{t \in [0,1] : J_f(t) \geq \varepsilon\}$ is infinite. Let t_0 be a accumulation point of A. There is $\delta > 0$ such that $|f(t) - \lim_{t \to t_0^-} f(t)| < \varepsilon/4$ for $t_0 - \delta < t < t_0$. Thus $|f(t) - f(t')| \leq \varepsilon/2$ for $t_0 - \delta < t, t' < t_0$. We proceed similarly to obtain $|f(t) - f(t')| \leq \varepsilon/2$ for $t_0 < t, t' < t_0 + \delta'$. So $J_f(t) \leq \varepsilon/2$ for $t \in (t_0 - \delta, t_0 + \delta')$, and we get the contradiction. □

Proof of Proposition 6.1. We may assume that $a = 0$, $b = 1$, and $|f(t)| \leq 1$. Fix $\varepsilon > 0$. Consider the finite set

$$A = \{t \in [0,1] : J_f(t) \geq \varepsilon\} = \{t_1, t_2, \dots, t_{m-1}\}$$

(see Lemma A.1). Let U be an open set such that $\{t_1, t_2, \dots, t_{m-1}\} \subset U$, $|U| < \varepsilon$, and $[0,1] \setminus U$ is a finite union of closed disjoint intervals $I_1, \dots I_m$. Then

$$\int_U |f(t)|\, dt < \varepsilon.$$

Consider $I_j = [a_j, b_j]$. For every $t \in [a_j, b_j]$ there is $\delta_t > 0$ such that $|f(t) - f(t')| < 4\varepsilon$ for $t' \in [a_j, b_j]$, $|t' - t| < \delta_t$, because $J_f f(t) < \varepsilon$. By compactness, there is $\delta_j > 0$ such that $|f(t) - f(t')| \leq 8\varepsilon$ for all $t, t' \in [a_j, b_j]$, $|t - t'| < \delta_j$. Take $\delta = \min\{\delta_1, \dots, \delta_m\}$. If $n \in \mathbb{N}$ is such that $\frac{1}{n} \leq \delta/2$ and $[\frac{k}{n}, \frac{k+1}{n}] \subseteq [a_j, b_j]$, then

$$\Big| \int_{k/n}^{(k+1)/n} f(t)\, dt - \frac{1}{n} f(k/n) \Big| < 8\varepsilon/n.$$

So, we easily conclude that

$$\Big| \frac{1}{n} \sum_{k=0}^{n-1} f(k/n) - \int_0^1 f(t)\, dt \Big| \leq 20\varepsilon$$

for $n \in \mathbb{N}$ large enough. □

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