

# Retrieval from Mixed Sampling Frequency: Generic Identifiability in the Unit Root VAR

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July 7, 2023

## Abstract

The “*RE*trieval from *MI*xed Frequency Sampling” (REMIS) approach based on blocking developed in [Anderson et al. \(2016a\)](#) is concerned with retrieving an underlying high frequency model from mixed frequency observations.

In this paper we investigate parameter-identifiability in the [Johansen \(1995\)](#) vector error correction model for mixed frequency data. We prove that from the second moments of the blocked process after taking differences at lag  $N$  ( $N$  is the slow sampling rate), the parameters of the high frequency system are generically identified. We treat the stock and the flow case as well as deterministic terms.

*Keywords:* *Keywords:* Mixed Frequency, REMIS, VAR, Cointegration, Vector Error Correction Model, Identifiability

*MSC:* 62M10, 62P20

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# 1 Introduction

Econometric analysis is often encountered with multivariate time series data sampled at mixed frequencies. Examples for treating this are [Zadrozny \(1988\)](#), [Ghysels et al. \(2007\)](#)[MIDAS-regression], [Anderson et al. \(2012\)](#), [Schorfheide and Song \(2015\)](#), [Ghysels \(2016\)](#), [Anderson et al. \(2016a\)](#) and [Chambers \(2020\)](#). Identifiability is a prerequisite for consistent estimation (see, e.g., [Deistler and Seifert, 1978](#); [Pötscher and Prucha, 1997](#)) and often is needed for economic interpretation of effects related to particular model parameters. This article investigates *identifiability* of the model parameters in a [Johansen \(1995\)](#) vector error correction model.

The general question is whether the internal characteristics, i.e. the model parameters  $\theta$ , can be retrieved from the external characteristics – in our case observable second moments. Identifiability means that the mapping from the parameters to these second moments is injective. Often injectivity of this mapping can only be achieved for a certain subset of the parameterspace. Here, we prove that identifiability can be obtained for a generic subset of the parameterspace (see [Anderson et al., 2016a](#)).

As opposed to MIDAS-regression, where the observations at high frequency are considered as additional information, we consider mixed frequency as either a “missing-values” or a “dis-aggregation”-problem, by which we mean the following: We commence from an underlying *high frequency system* (e.g., a VECM) parameterised by  $\theta$  for a multivariate process

$$(y_t)_{t \in \mathbb{Z}} = \left( \left( \begin{array}{c} y_t^f \\ y_t^s \end{array} \right) \right)_{t \in \mathbb{Z}},$$

with dimensions  $n$ ,  $n_f$  and  $n_s$  for  $y_t$ ,  $y_t^f$  and  $y_t^s$  respectively. Our aim is to identify and estimate the high frequency system from the observed (mixed frequency) data. The *observational scheme* is as follows: While the fast variables  $y_t^f$  are observed at  $t \in \mathbb{Z}$ , for the slow variables  $y_t^s$  we consider:

1. *Stock-Case*:  $y_t^s$  is observed only at  $t \in N\mathbb{Z}$  for some sampling rate  $N \geq 2$ , hence we have a *missing-value problem*.
2. *Affine aggregation*: we observe an affine transformation

$$w_t := c_w + c_0 y_t^s + \dots + c_{p_c} y_{t-p_c}^s, \tag{1}$$

where  $c_i$  are known constant matrices for  $i \geq 0$ ,  $c_w$  is a known vector and  $w_t$  is observed at  $t \in N\mathbb{Z}$ . A special case of affine aggregation are flow variables:

For example suppose  $y_t^s = GDP_t$ , the monthly gross domestic product of a country. The quarterly GDP,  $w_t$ , is the sum of three monthly GDPs. We call  $y_t^s$  *latent* whenever it is not directly observed. Hence, our aim is to retrieve the underlying high frequency parameters  $\theta$  from data observed according to the observational schemes described above.<sup>1</sup>

With the procedure described above, we are able to model all kinds of linear dynamic relationships between latent and observed variables, whereas the MIDAS (see, e.g., Ghysels, 2016) approach only covers relationships between observed variables. After identifying the parameters one may interpolate missing values or dis-aggregate observations in a model based way by using the retrieved parameters of the underlying high frequency system.

Estimation of continuous time models from mixed frequency data are investigated in Chambers (2003, 2016, 2020). In particular, Chambers (2003, 2020) consider co-integrating regressions and show that the scaled estimators proposed, converge in distribution to functionals of Brownian motion and to stochastic integrals. Hence, the estimators are (weakly) consistent. Then, by Gabrielsen (1978) – and for the case of strong consistency by Deistler and Seifert (1978) – the model parameters are identified.

For the stable vector auto-regressive model Anderson et al. (2012) and Anderson et al. (2016a) either used the *blocking approach* (see also Filler, 2010; Ghysels, 2016) or the *extended Yule-Walker equations* (see Chen and Zdrozny, 1998; Anderson et al., 2016a) to show g-identifiability. For the same model class Gersing and Deistler (2021) present an alternative proof for identifiability using the so-called canonical projection form. This idea is also applied in this paper. On the other hand, Deistler et al. (2017) show that the parameters need not be identified in the auto-regressive-moving average (VARMA) case, if the order of the MA polynomial exceeds the order of the AR polynomial. This article is organised as follows: Section 2 starts with the vector error correction model developed in Johansen (1995) as the underlying high frequency model. In Section 2.2 we describe the observational schemes considered in detail. In particular, we introduce a stationary blocked process containing all observed variables. Section 2.3 introduces conditions, which are later shown to be sufficient for identifiability. We prove that these conditions hold generically in the underlying high frequency parameterspace. Section 3 extends the

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<sup>1</sup>In this example we assume that the variable considered,  $y_t$ , is integrated of order one. If by contrast  $(\log y_t)$  is integrated of order one, the affine approximation of Aadland (2000) in combination with the methodology developed in this article can be applied.

REMIS approach to the non-stationary case: Here, we use the result from [Chambers \(2020\)](#) that the cointegrating vectors can be identified from mixed frequency data. First, we derive a state-space representation of the blocked process that we call Canonical Projection Form (CPF). In this representation, the system matrices are simple transformations of the parameters of the underlying high-frequency model. After that we start from the unique factor of the spectrum of the blocked process (see, e.g., [Deistler and Scherrer, 2022](#), chapter 6.2 and 7.3) to get an arbitrary minimal realisation for this factor and relate this to the CPF. From there we can retrieve the parameters of the underlying high frequency system using the structural properties of the CPF. Section 4 adds deterministic terms. Finally, Section 5 concludes.

## 2 Notation and Model Class

### 2.1 Representations and Parameterspace of the Underlying High Frequency System

In the first step, we introduce the class of underlying high frequency systems: We commence from a process which is integrated of order one and allows for cointegration. Suppose  $(y_t)_{t \in \mathbb{Z}}$  is  $n \times 1$  and a solution on  $\mathbb{Z}$  of the vector error correction system:

$$\Delta y_t = \Pi y_{t-1} + \sum_{j=1}^{p-1} \Phi_j \Delta y_{t-j} + \nu_t, \quad \nu_t \sim WN(\Sigma_\nu), \quad (2)$$

where  $(\nu_t)_{t \in \mathbb{Z}}$  is white noise and  $\Pi$  is of rank  $r > 0$  in the case of cointegrating relationships, but we also allow the case  $r = 0$ . Such solutions always exist and can be constructed as described in detail in [Bauer and Wagner \(2012\)](#). We obtain a unique factorisation of  $\Pi = \alpha\beta'$  with  $\alpha, \beta \in \mathbb{R}^{n \times r}$  applying the singular value decomposition to  $\Pi$  in the following way:

$$\begin{aligned} \Pi &= \underbrace{U}_{n \times n} \underbrace{\text{diag}(d_1, \dots, d_r, 0, \dots, 0)}_D \underbrace{V'}_{n \times n} = \underbrace{U_1}_{n \times r} \underbrace{\text{diag}(d_1, \dots, d_r)}_{\tilde{D}} \underbrace{V'_1}_{r \times n} \\ &= U_1 \tilde{D} V'_1 = \underbrace{U_1 Q^{-1}}_\alpha \underbrace{Q \tilde{D} V'_1}_{\beta'}, \end{aligned}$$

where  $Q$  is a non-singular matrix of elementary row operations that transforms  $\tilde{D} V'_1$  into its reduced echelon form, such that  $Q \tilde{D} V'_1 = (I_r \quad \beta'_{n-r})$ .

We stack the parameters  $\alpha, \beta, \Phi_1, \dots, \Phi_{p-1}$  to a vector  $\theta_{VECM} \in \mathbb{R}^d$ , where  $d = nr + (n-r)r + (p-1)n^2$ .

We also have a VAR( $p$ ) representation for  $(y_t)$  of the form,

$$y_t = \mathcal{A}_1 y_{t-1} + \dots + \mathcal{A}_p y_{t-p} + \nu_t. \quad (3)$$

Throughout this article, we assume that  $r$  and  $p$  are known a priori. We obtain the representation in (3) by the mapping  $\psi$ :

$\psi : \theta_{VECM} \mapsto \theta_{AR}$ , defined as

$$\mathcal{A}_1 = I_n + \Pi + \Phi_1, \quad \mathcal{A}_j = \Phi_j - \Phi_{j-1} \text{ for } 1 < j < p, \quad \mathcal{A}_p = -\Phi_{p-1},$$

with  $\theta_{AR} = \text{vec}(\mathcal{A}_1 \ \dots \ \mathcal{A}_p)$ . On the other hand for a  $\theta_{AR}$  which has a corresponding VECM representation, we compute  $\theta_{VECM}$  as follows:

$\psi^{-1} : \theta_{AR} \mapsto \theta_{VECM}$

$$\Pi = -I_n + \sum_{j=1}^p \mathcal{A}_j, \quad \Phi_1 = -I_n + \mathcal{A}_1 + \Pi, \quad \Phi_2 = \Phi_1 + \mathcal{A}_2, \quad \dots, \quad \Phi_{p-1} = -\mathcal{A}_p.$$

Now, define the polynomial matrix  $a(z) = I_n - \mathcal{A}_1 z - \dots - \mathcal{A}_p z^p$  where  $z$  is a complex variable or the lag operator on  $\mathbb{Z}$  depending on the context. For  $\check{c} = \begin{pmatrix} I_r \\ 0 \end{pmatrix} \in \mathbb{R}^{n \times r}$  and  $\check{c}_\perp = \begin{pmatrix} 0 \\ I_{n-r} \end{pmatrix} \in \mathbb{R}^{n \times (n-r)}$ ,  $\beta_\perp := (I_n - \check{c}(\beta' \check{c})^{-1} \beta') \check{c}_\perp$ , and  $\alpha_\perp$  defined analogously to  $\beta_\perp$ . We impose the following assumptions (Johansen, 1995, chapter 4):

**Assumption 1** (Cointegrated VAR-System)

(C1)  $\text{rk } \alpha \beta' = r < n$ .

(C2)  $\det(\alpha'_\perp (I_n - \sum_{j=1}^{p-1} \Phi_j) \beta_\perp) \neq 0$ .

(C3)  $\det a(z) = 0 \Rightarrow z = 1$  or  $|z| > 1$ .

(C4)  $\Sigma_\nu = \mathbb{E} \nu_t \nu_t' > 0$ .

We define the parameterspace as follows:<sup>2</sup>

$$\Theta_{VECM,1} := \psi^{-1} \left( \psi \left( \mathbb{R}^d \Big|_{C1,C2} \right) \Big|_{C3} \right), \quad \Theta_1 := \psi(\Theta_{VECM,1})$$

with  $\Theta_1 \xleftrightarrow{\psi} \Theta_{VECM,1}$

Note that under these assumptions  $\psi$  is a homeomorphism. The set of vech  $\Sigma_\nu$  with  $\Sigma_\nu \in \mathbb{R}^{n \times n}$ ,  $\Sigma_\nu = \Sigma'_\nu$  and  $\Sigma_\nu > 0$  (condition (C4) in Assumption 1) is denoted by  $\Theta_2$ . The overall parameterspace for the VAR( $p$ ) representation is

$$\Theta = \Theta_1 \times \Theta_2.$$

We will also need the state-space representation of  $(y_t)_{t \in \mathbb{Z}}$ , which follows from (3):

$$\underbrace{\begin{pmatrix} y_t \\ \vdots \\ y_{t-p+1} \end{pmatrix}}_{X_{t+1}} = \underbrace{\begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 & \cdots & \mathcal{A}_p \\ I_n & & & 0 \\ & \ddots & & \vdots \\ & & I_n & 0 \end{pmatrix}}_{\mathcal{A}} \underbrace{\begin{pmatrix} y_{t-1} \\ \vdots \\ y_{t-p} \end{pmatrix}}_{X_t} + \underbrace{\begin{pmatrix} I_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_B \nu_t \quad (4)$$

$$y_t = \underbrace{(\mathcal{A}_1 \cdots \mathcal{A}_p)}_c X_t + \nu_t. \quad (5)$$

Note that (4), (5) is always controllable as  $\Sigma_\nu$  and therefore  $\Gamma(t) := \mathbb{E}(X_{t+1}X'_{t+1})$  are of full rank. The system (4), (5) is also observable whenever  $\mathcal{A}_p$  is of full rank. This follows since  $\mathcal{A}_p$  is nonsingular (and therefore  $\mathcal{A}$  is non-singular) from the BPH-test (see [Kailath \(1980\)](#) 2.4.3). Hence under Assumption 1 and if  $\mathcal{A}_p$  is nonsingular the system (4), (5) is minimal. For details on controllability and observability see e.g. [Deistler and Scherrer \(2022\)](#), chapter 7 or [Hannan and Deistler \(2012\)](#), chapter 2.

## 2.2 Mixed Frequency Data: Stock and Flow Variables

A main challenge of the identifiability proof in the integrated case – as opposed to the stationary case ([Anderson et al., 2016a](#)) – is that the second

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<sup>2</sup>We write  $\mathbb{R}^d \Big|_{C1,C2}$  to denote the set of real vectors in  $\mathbb{R}^d$  for which C1 and C2 hold.

moments of an integrated process (that is,  $\mathbb{E}y_s y_t$ ,  $s, t \in \mathbb{Z}$ ) are time dependent and cannot be estimated directly. Instead, for the sake of practical relevance of identifiability considerations, we identify from observable second moments of stationary transformations of the level process (that is,  $(y_t)_{t \in \mathbb{Z}}$ ). Suppose for the moment, that the matrix of cointegration vectors  $\beta$  is known. Our proof commences from what we call the “blocked process”, where we distinguish between the Stock- and the Flow-case:

**1. Stock Variables:** In this case for  $t \in N\mathbb{Z}$ , we get the co-stationary vector  $\tilde{y}_t$  of “observed” random variables. We will use  $\tilde{n} := r + n + (N - 1)n_f$  for the dimension of  $\tilde{y}_t$  henceforth. Let  $u_t^S := \beta' y_t$ ,  $\Delta_N y_t := y_t - y_{t-N} = \sum_{j=0}^{N-1} \Delta y_{t-j}$ , and

$$\tilde{y}_t = \begin{pmatrix} \beta' y_t \\ y_t - y_{t-N} \\ y_{t-1}^f - y_{t-N-1}^f \\ \vdots \\ y_{t-N+1}^f - y_{t-2N+1}^f \end{pmatrix} = \begin{pmatrix} u_t^S \\ \Delta_N y_t \\ \Delta_N y_{t-1}^f \\ \vdots \\ \Delta_N y_{t-N+1}^f \end{pmatrix}. \quad (6)$$

The blocked process  $(\tilde{y}_t)$  is similar to the blocked process in [Anderson et al. \(2016a\)](#) with the distinction that we added the variable  $\beta' y_t = u_t^S$  and take differences at lag  $N$ . Admittedly, the true  $\beta$  is in fact not observed, however since  $\beta$  can be estimated consistently, for the purpose of the analysis of identifiability we can assume  $\beta' y_t$  to be observed.

**2. Flow Variables:** In a similar way, we may consider the case where all slow variables are flow variables, in which case we are able to observe the temporal aggregate  $w_t := \sum_{j=0}^{N-1} y_{t-j}^s$  at  $t \in N\mathbb{Z}$ . So

$$\Delta_N^\Sigma y_t := \sum_{j=0}^{N-1} y_{t-j} - \sum_{j=0}^{N-1} y_{t-N-j} = \Delta_N \sum_{j=0}^{N-1} \begin{pmatrix} y_t^f \\ y_t^s \end{pmatrix}.$$

If all slow variables are flow variables, we can observe  $\sum_{j=0}^{N-1} y_{t-j} = \left( w_t', \sum_{j=0}^{N-1} y_{t-j}^{f'} \right)'$ ,  $t \in N\mathbb{Z}$ . Since  $\beta' y_t$  is stationary, we have that  $(\beta' y_t)_{t \in N\mathbb{Z}}$  and  $u_t^F := \beta' \sum_{j=0}^{N-1} y_{t-j} \in \mathbb{R}^r$  are integrated of order zero. For the flow case we de-

fine the co-stationary vector process

$$\tilde{y}_t = \begin{pmatrix} u_t^{\mathcal{F}} \\ \Delta_N^{\Sigma} y_t \\ \Delta_N y_{t-1}^f \\ \vdots \\ \Delta_N y_{t-N+1}^f \end{pmatrix}. \quad (7)$$

We call the autocovariance function of the (stationary) blocked process

$$\tilde{\gamma} : h \mapsto \mathbb{E} \tilde{y}_{t+h} \tilde{y}_t', \quad \text{where } h \in N\mathbb{Z}, \quad (8)$$

*observed second moments*, which can be consistently estimated from the data (if  $\beta$  is known) under standard assumptions.

The motivation to consider this blocked process for identifiability is the following:

1. We take differences at lag  $N$  (as opposed to lag one) because these differences can be directly computed from the mixed frequency data and are stationary.
2. Note that the set of observable autocovariances given mixed frequency data is

$$\begin{aligned} \gamma_{\Delta_N y}^{ff}(h) &:= \mathbb{E} \Delta_N y_{t+h}^f \Delta_N y_t^{f'} & h \in \mathbb{Z} \\ \gamma_{\Delta_N y}^{fs}(h) &:= \mathbb{E} \Delta_N y_{t+h}^f \Delta_N y_t^{s'} & h \in \mathbb{Z} \\ \gamma_{\Delta_N y}^{ss}(h) &:= \mathbb{E} \Delta_N y_{t+h}^s \Delta_N y_t^{s'} & h \in N\mathbb{Z} \\ \gamma_{\beta}(h) &:= \mathbb{E} u_{t+h} u_t' & h \in N\mathbb{Z}, \end{aligned}$$

where the superscript “.” is shorthand for  $\mathcal{S}$  or  $\mathcal{F}$ . Note that these are exactly the second moments of the autocovariance function  $\tilde{\gamma}$  of the blocked process defined in equations (6) for the stock case. In an obvious way this is treated accordingly in the flow case (7). So the blocked process “contains the whole second moment information available” from which we can identify. The same idea is also applied for the stationary case in [Anderson et al. \(2016a\)](#).

3. Our interest in the particular blocked process (6), (7) having  $u_t$  in the first coordinates, originates in the fact that we can obtain a minimal representation for this process (see Section 3), where the parameters are fairly simple functions of the parameters of the underlying high frequency system. This will finally help us to retrieve the high frequency model parameters.

Next, we define the concept of generic identifiability. Here, identifiability is concerned with the problem whether the parameters of the underlying high frequency system (4), (5) or (2) are uniquely determined from the observable second moments (defined below in this section). To be more precise, a subset  $\Theta_I \subset \Theta$  is called identifiable, if the mapping attaching the observable second moments to the parameters  $\theta \in \Theta_I$  is injective. In our setting identifiability for the whole set  $\Theta$  cannot be obtained. To see this, we consider a simple example where  $p = 1$ ,  $r = 1$ , and  $n = 2$ , the first coordinate of  $y_t$  is a fast variable, denoted  $y_t^f$ , while the second coordinate,  $y_t^s$ , is a slow stock variable. We assume that the cointegrating vector  $\beta = (1, \beta_s)$  is known. Recall that the observed second moments are as described in equations (6) and (8). Let  $\sigma_{ff}$ ,  $\sigma_{fs} = \sigma_{sf}$ , and  $\sigma_{ss}$  denote the elements of the covariance matrix  $\Sigma_\nu$ . Appendix B shows that there exist two parameter vectors  $\theta^I := (\alpha_f^I, \alpha_s^I, 1, \beta_s, \sigma_{ff}^I, \sigma_{fs}^I, \sigma_{ss}^I)' \neq \theta^{II} := (\alpha_f^{II}, \alpha_s^{II}, 1, \beta_s, \sigma_{ff}^{II}, \sigma_{fs}^{II}, \sigma_{ss}^{II})'$  such that all observable second moments are the same; hence in this case the mapping from the model parameters to observable second moments cannot be injective and the model parameters are not identified from observed second moments. In this example  $\alpha_f^I = \alpha_f^{II} = 0$ . This implies that the fast coordinate follows a random walk and does not provide any information on the parameter  $\alpha_s$ , that is on how  $\beta' y_t$  affects  $\Delta y_t^s$ ,  $t \in 2\mathbb{Z}$ . However, in this paper we prove that identifiability holds for a so called generic subset of  $\Theta$ . Note that a set  $\Theta_I \subset \Theta$  is called *generic* in  $\Theta$ , if it contains a subset that is open and dense in  $\Theta$ .

Let  $\Theta_I := (G \cap \Theta_1) \times \Theta_2$ , where  $G \subset \mathbb{R}^{n^2 p}$  is defined in Assumption 2 below. In this paper we show firstly that  $\Theta_I$  is generic in  $\Theta$  (see Section 2.3) and secondly that the set of high frequency systems corresponding to  $\Theta_I$  is identifiable from the observable second moments (see Section 3). Or formally, we show that

$$\pi : \theta \mapsto \tilde{\gamma} \tag{9}$$

is injective on  $\Theta_I \subset \Theta$ .

Finally, in terms of identifiability, we may suppose without loss of generality that  $\beta$  is known. For instance Miller (2016) or Chambers (2020) propose estimators, accounting for stock and flow variables, respectively. The estimators of  $\beta$  scaled by  $T$  weakly converge to a random variable bounded in probability. Hence, e.g. by White (2001), the estimator is weakly consistent. By Gabrielsen (1978) the matrix of cointegrating vectors  $\beta \in \mathbb{R}^{n \times r}$  is identified from mixed frequency observations given the assumptions imposed

in [Chambers \(2020\)](#) or [Miller \(2016\)](#). These assumptions are only posed on the stochastic properties of the high frequency innovations  $(\nu_t)_{t \in \mathbb{Z}}$  and therefore do not restrict our results on the genericity of the identifiability conditions from [Section 2.3](#). If strong consistency could be established for some estimator of  $\beta$ , the results of [Deistler and Seifert \(1978\)](#) apply and  $\beta$  is identified.

## 2.3 Generic Identifiability and Topological Properties of the Parameterspace

In this section we define the conditions that we need for identifiability and prove that these conditions result in a generic subset of the parameterspace. Define a set  $G \subset \mathbb{R}^{n^2 p}$  by the following assumptions:

**Assumption 2** (g-Identifiability Assumptions)

- (I1)**  $\text{rk } \mathcal{A}_p = n$ .
- (I2)**  $\text{rk } \Gamma(t) = np$  where  $\Gamma(t) = \mathbb{E}(X_{t+1} X'_{t+1})$ .
- (I3)** The eigenvalues of  $\mathcal{A}$  are of the form:  $(1, \dots, 1, \lambda_{n-r+1}, \dots, \lambda_{np})$  where  $|\lambda_j| < 1$  and  $\lambda_i \neq \lambda_j$  for  $i \neq j$  with  $i, j = n - r + 1, \dots, np$ .
- (I4)** For non-unit eigenvalues  $\lambda_i \neq \lambda_j$  it follows that  $\lambda_i^N \neq \lambda_j^N$ .
- (I5)** For all eigenvalues  $\lambda$  of  $\mathcal{A}$  smaller than one, it holds that  $1 + \lambda + \dots + \lambda^N \neq 0$  or  $v_1$  consisting of the first  $n$  elements of the eigenvector  $v$  of  $\mathcal{A}$  corresponding to  $\lambda$ , it holds that  $\beta' v_1 \neq 0$ .
- (I6)** The pair  $(S_{n_f}^{(1)}, A)$  is observable, where  $S_{n_f}^{(1)}$  is defined in [equations \(14\), \(15\)](#) and  $A$  is defined in [equation \(10\)](#).

Assumption (I2) already follows from  $\Sigma > 0$ . Recall that  $\Theta_I = (G \cap \Theta_1) \times \Theta_2$ . These assumptions are similar to the stationary case considered in [Felsenstein \(2014\)](#); [Anderson et al. \(2016a,b\)](#). There, the stability condition defines an open set  $\Theta' \subset \mathbb{R}^{n^2 p}$ . We also have a corresponding set  $G'$  defining the identifiability conditions for the stationary case, which is generic in  $\mathbb{R}^{n^2 p}$ . Then, the intersection  $\Theta' \cap G'$  is generic in  $\Theta'$ . However, in the integrated case, where unit roots occur, the situation is more intricate since neither  $\Theta_1$  nor  $G$  is open in  $\mathbb{R}^{n^2 p}$ . This follows from the fact that for a process with  $n - r$

common trends, the  $n - r$  eigenvalues of  $\mathcal{A}$  in (4) are equal to one [note that the eigenvalues of  $\mathcal{A}$  are the reciprocals of the zeros of  $a(z)$ ]. The following Theorem 1 implies that the identifiability conditions are generically fulfilled in  $\Theta$ :

**Theorem 1**

Let  $\Theta_1$  be endowed with the Euclidean norm  $d$ . The set  $\Theta_1 \cap G$  is open and dense in  $\Theta_1$ .

Since genericity is a topological property, it also holds for the homeomorphic parameterspace corresponding the vector error correction representation in (2) defined by Assumption 1.

### 3 Generic Identifiability

In this section, we first define a canonical state-space representation for the blocked process running on  $t \in N\mathbb{Z}$ . We prove that this representation is minimal under our identifiability conditions. Then under an additional assumption on the lag order  $p$ , we show that the high frequency parameters are generically identifiable. The proofs of minimality and identifiability make use of the canonical representation.

We follow Hansen and Johansen (1999) and obtain from (2) the following state-space system for  $\beta'y_t$  and first differences of  $y_t$ , that is  $\Delta y_t = y_t - y_{t-1}$ . Then,

$$\underbrace{\begin{pmatrix} \beta'y_t \\ \Delta y_t \\ \vdots \\ \Delta y_{t-p+2} \end{pmatrix}}_{\underline{x}_{t+1} \in \mathbb{R}^{r+n(p-1)}} = \underbrace{\begin{pmatrix} \beta'\alpha + I_r & \beta'\Phi_1 & \cdots & \cdots & \beta'\Phi_{p-1} \\ \alpha & \Phi_1 & \cdots & \cdots & \Phi_{p-1} \\ 0_{n \times r} & I_n & & & 0_{n \times n} \\ \vdots & & \ddots & & \vdots \\ & & & I_n & 0 \end{pmatrix}}_{A \in \mathbb{R}^{r+n(p-1) \times r+n(p-1)}} \underbrace{\begin{pmatrix} \beta'y_{t-1} \\ \Delta y_{t-1} \\ \vdots \\ \Delta y_{t-p+1} \end{pmatrix}}_{\underline{x}_t} + \underbrace{\begin{pmatrix} \beta' \\ I_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_B \nu_t \quad (10)$$

$$\underbrace{\begin{pmatrix} \beta'y_t \\ \Delta y_t \end{pmatrix}}_{\in \mathbb{R}^{r+n}} = \underbrace{\begin{pmatrix} \beta'\alpha + I_r & \beta'\Phi_1 & \cdots & \cdots & \beta'\Phi_{p-1} \\ \alpha & \Phi_1 & \cdots & \cdots & \Phi_{p-1} \end{pmatrix}}_{C \in \mathbb{R}^{r+n \times r+n(p-1)}} \begin{pmatrix} \beta'y_{t-1} \\ \Delta y_{t-1} \\ \vdots \\ \Delta y_{t-p+1} \end{pmatrix} + \underbrace{\begin{pmatrix} \beta' \\ I_n \end{pmatrix}}_{D \in \mathbb{R}^{r+n \times n}} \nu_t \quad (11)$$



dimension exceeds ( $Nn$ ) the output dimension ( $\tilde{n}$ ), noting that  $\Sigma_\nu > 0$ ):

$$x_{t+1} = \underbrace{c A^N c^{-1}}_{:=A_b} x_{t-N+1} + \underbrace{c B_b}_{B_{b,c}} \nu_t^b \quad (13)$$

$$\tilde{y}_t = \underbrace{S_\zeta A^N c^{-1}}_{:=C_b} x_{t-N+1} + D_b \nu_t^b, \quad (14)$$

where

$$S_\zeta := \begin{matrix} (r \times m)\{ \\ S_{n_f}^{(1)} (n_f \times m)\{ \\ S_{n_s}^{(1)} (n_s \times m)\{ \\ S_{n_f}^{(2)} (n_f \times m)\{ \\ \vdots \\ S_{n_f}^{(N)} (n_f \times m)\{ \end{matrix} \begin{pmatrix} I_r & 0 & \cdots & & & & 0 \\ 0 & (I_{n_f}, 0) & \cdots & (I_{n_f}, 0) & 0 & \cdots & 0 \\ 0 & (0, I_{n_s}) & \cdots & (0, I_{n_s}) & 0 & \cdots & 0 \\ 0 & 0 & (I_{n_f}, 0) & \cdots & (I_{n_f}, 0) & & \\ & & & \ddots & & \ddots & \vdots \\ & & & (I_{n_f}, 0) & \cdots & (I_{n_f}, 0) & 0 \end{pmatrix},$$

$$C_b = \begin{pmatrix} (I_r \ 0 \ \cdots \ 0) A^N \\ S_{n_f}^{(1)} A^N \\ S_{n_s}^{(1)} A^N \\ S_{n_f}^{(2)} A^N \\ S_{n_f}^{(3)} A^N \\ \vdots \\ S_{n_f}^{(N)} A^N \end{pmatrix} = \begin{pmatrix} (I_r \ 0 \ \cdots \ 0) A^N \\ S_{n_f}^{(1)} A^N \\ S_{n_s}^{(1)} A^N \\ S_{n_f}^{(1)} A^{N-1} \\ S_{n_f}^{(1)} A^{N-2} \\ \vdots \\ S_{n_f}^{(1)} A \end{pmatrix}, \quad \nu_t^b := \begin{pmatrix} \nu_t \\ \vdots \\ \nu_{t-N+1} \end{pmatrix} \in \mathbb{R}^{Nn}. \quad (15)$$

The matrices  $B_{b,c} \in \mathbb{R}^{r+n(p-1) \times Nn}$  and  $D_b \in \mathbb{R}^{r+n \times Nn}$  are obtained from  $B$  and  $A$ .

**2. Case: Flow Variables:** Next, we obtain the state vector  $x_{t+1}$  for the flow case. Note that  $y_{t-j} = y_t - \sum_{\ell=1}^j \Delta y_{t-\ell}$ , such that  $\sum_{j=0}^{N-1} y_{t-j} = \sum_{j=0}^{N-1} \left( y_t - \sum_{\ell=1}^j \Delta y_{t-\ell} \right) = N y_t - (N-1) \Delta y_{t-1} - \cdots - \Delta y_{t-N+1}$ . Analogously

to equation (12), this yields for  $p \geq 2N + 1$  that

$$\underbrace{\begin{pmatrix} \beta' \sum_{j=0}^{N-1} y_{t-j} \\ \Delta \sum_N y_t \\ \Delta^N y_{t-1} \\ \vdots \\ \Delta^N y_{t-N+1} \\ \Delta y_{t-N} \\ \vdots \\ \Delta y_{t-p+2} \end{pmatrix}}_{x_{t+1} \in \mathbb{R}^{r+n(p-1)}} = \underbrace{\begin{pmatrix} NI_r & -(N-1)\beta' & -(N-2)\beta' & \cdots & -\beta' & 0 & \cdots \\ 0 & I_n & \cdots & I_n & -I_n & \cdots & -I_n & 0 \\ 0 & 0 & I_n & \cdots & I_n & 0 & \cdots \\ \vdots & & & \ddots & & & & \\ \vdots & & & & & & & \\ \vdots & & & & I_n & \cdots & I_n & 0 \\ \vdots & & & \cdots & 0 & I_n & 0 & \cdots \\ \vdots & & & & & & & \ddots \end{pmatrix}}_{c \in \mathbb{R}^{r+n(p-1) \times r+n(p-1)}} \begin{pmatrix} u_t^S \\ \Delta y_t \\ \vdots \\ \Delta y_{t-p+2} \end{pmatrix}. \quad (16)$$

We use the same notation for  $\tilde{y}_t$ ,  $x_t$ ,  $c$  for both cases. With this notation, we obtain the following state-space representation for blocked process in the flow case:

$$x_{t+1} = \underbrace{cA_b c^{-1}}_{A_{b,c}} x_{t-N+1} + \underbrace{cB_b}_{B_{b,c}} \nu_t^b \quad (17)$$

$$\tilde{y}_t = \underbrace{S_\zeta A^N}_{C_b \in \mathbb{R}^{\tilde{n} \times m}} c^{-1} x_{t-N+1} + D_{b,c} \nu_t^b, \quad (18)$$

$$\underbrace{C_b, c}_{C_{b,c} \in \mathbb{R}^{\tilde{n} \times m}}$$

where

$$S_\zeta = \begin{pmatrix} NI_r & -(N-1)\beta' & -(N-2)\beta' & \cdots & -\beta' & 0 & \cdots \\ 0 & I_n & \cdots & I_n & -I_n & \cdots & -I_n & 0 & \cdots \\ 0 & 0 & (I_{n_f}, 0) & \cdots & (I_{n_f}, 0) & 0 & \cdots \\ & & & \ddots & & & & & \\ 0 & & 0 & (I_{n_f}, 0) & \cdots & (I_{n_f}, 0) & 0 & \cdots \end{pmatrix}.$$

The matrix  $D_{b,c} \in \mathbb{R}^{\tilde{n} \times Nn}$  follows from  $D_b$ , the matrix  $c$  and the selection of the corresponding rows resulting in  $\tilde{y}_t$ .

**3. Case: Mixed Case:** Consider the case where we have slow stock as well as slow flow variables: For example, if  $(y_t)$  is a three-dimensional process, where  $n_f = 1$ ,  $n_s = 2$ ,  $N = 2$ ,  $c_1 = I_2$ , and  $c_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$  in equation (1).

Then  $\beta' (y_t^f, w_t^f)'$  is (in general) not stationary. However, in special cases, such as separate cointegrating relationships among the slow flow variables only, or among the slow stock and fast variables only, etc. we can proceed similarly to the flow case. In the following we only consider the stock or the flow case.

The problem with the systems considered above is that the inputs  $\nu_t^b$  are not the innovations of  $\tilde{y}_t$ . However, from the stable miniphase spectral factorisation, we only obtain transfer functions corresponding to systems in innovation form (see, e.g., [Deistler and Scherrer, 2022](#), Chapter 7). The following [Theorem 2](#) is the first step for obtaining a canonical state-space representation for the blocked process. A minimal state-space representation is called “canonical” if its parameters are uniquely determined from the transfer function. We introduce the following notation for specific subspaces of  $L^2(\Omega, \mathcal{A}, P)$ , the space of square integrable random variables on the underlying probability space  $(\Omega, \mathcal{A}, P)$ :

$$\begin{aligned}\mathbb{H}(y) &:= \overline{\text{sp}}(y_{it} \mid t \in \mathbb{Z}, i = 1, \dots, n) \\ \mathbb{H}_t(y) &:= \overline{\text{sp}}(y_{is} \mid s \leq t, i = 1, \dots, n) \\ {}_N\mathbb{H}(y) &:= \overline{\text{sp}}(y_{it} \mid t \in N\mathbb{Z}, i = 1, \dots, n) \\ {}_N\mathbb{H}_t(y) &:= \overline{\text{sp}}(y_{is} \mid s \leq t \text{ and } s \in N\mathbb{Z}, i = 1, \dots, n) ,\end{aligned}$$

where  $\overline{\text{sp}}(\cdot)$  denotes the closed span and  $\text{proj}(v \mid U)$  the projection of  $v$  on a closed subspace  $U$  of  $L^2$ .

**Theorem 2**

Suppose that [Assumption 1](#) holds. Consider the blocked process  $(\tilde{y}_t)_{t \in N\mathbb{Z}}$  and set

$$\begin{aligned}s_{t-N+1} &:= \text{proj}(x_{t-N+1} \mid {}_N\mathbb{H}_{t-N}(\tilde{y})) \\ \tilde{v}_t &:= \tilde{y}_t - \text{proj}(\tilde{y}_t \mid {}_N\mathbb{H}_{t-N}(\tilde{y})) .\end{aligned}$$

Then there exists  $\tilde{B}_c \in \mathbb{R}^{np \times \tilde{n}}$  such that

$$s_{t+1} = A_{b,c}s_{t-N+1} + \tilde{B}_c\tilde{v}_t \tag{19}$$

$$\tilde{y}_t = C_{b,c}s_{t-N+1} + \tilde{v}_t \tag{20}$$

is a miniphase and stable state-space representation of  $(\tilde{y}_t)_{t \in N\mathbb{Z}}$ , i.e. it is in innovation form.

We call the representation in [\(19\)](#), [\(20\)](#) *canonical projection form* (CPF) of  $\tilde{y}_t$ . Note that the CPF provides an algorithm for computing the transfer function  $\tilde{k}(\tilde{z})$  of  $(\tilde{y}_t)_{t \in N\mathbb{Z}}$  which corresponds to the Wold representation, where  $\tilde{z} := z^N$ .

Next we show that the system [\(19\)](#) and [\(20\)](#) is observable and controllable and therefore minimal (see, e.g., [Hannan and Deistler, 2012](#), Theorem 2.3.3) for all  $\theta \in \Theta_I$ .

**Theorem 3**

For  $\theta \in \Theta_I$ , the system (19) and (20) is minimal.

By Theorem 3, we know that the McMillan degree of the transfer function of the blocked process  $(\tilde{y}_t)_{t \in \mathbb{N}\mathbb{Z}}$  corresponding to an underlying high-frequency VECM is  $m = r + n(p - 1)$ . This will be used in the proof of the subsequent Theorem 4, where we can relate an arbitrary minimal realisation  $(\bar{A}_{b,c}, \bar{B}_{b,c}, \bar{C}_{b,c})$  of the transfer function  $\tilde{k}(\tilde{z}) = (\bar{C}_{b,c} (I_m \tilde{z}^{-1} - \bar{A}_{b,c}) \bar{B}_{b,c} + I_{\tilde{n}})$  (where  $\tilde{z} := z^N$ ) to the CPF  $(A_{b,c}, \tilde{B}_c, C_{b,c})$ . The minimal realisation  $(\bar{A}_{b,c}, \bar{B}_{b,c}, \bar{C}_{b,c})$  can be either obtained by the spectral factorisation and e.g. the echelon realisation from the Hankel matrix of the transfer function (see e.g. Hannan and Deistler, 2012, Theorem 2.6.2) or directly from the Hankel matrix of the observed second moments (see, e.g. Anderson et al., 2016a, Proof of Theorem 8). In the next step we relate the CPF to the underlying VECM/VAR – exploiting the fact that the parameters  $\theta$  of the underlying VECM reappear in the CPF.

Finally, we show that the parameters of the high frequency system are generically identifiable from the observed second moments, i.e. from  $\tilde{\gamma}$ .

**Theorem 4** (Generic-Identifiability: Flow or Stock Case)

Let  $p \geq N + 2$  for stock case or  $p \geq 2N + 1$  for the flow case. Then,

1. The mapping,  $\pi$  in equation (9) which attaches the second moments of  $(\tilde{y}_t)_{t \in \mathbb{N}\mathbb{Z}}$  to the high frequency parameters  $\theta$  is injective on  $\Theta_I$ .
2. Its inverse,  $\pi^{-1}$ , is continuous on  $\pi(\Theta_I)$ .

Since by Theorem 1,  $\Theta_I$  is a generic subset of  $\Theta$ , we say that  $\theta$  is generically identifiable from the observed autocovariance function  $\tilde{\gamma}$ . Theorems 3 and 4 imply that the representation (19), (20) is indeed canonical on  $\pi(\Theta_I)$ . Since the second moments of  $(\tilde{y}_t)$  can be consistently estimated from the data under mild conditions, by the continuity of  $\pi^{-1}$  it follows that we have a consistent estimator for  $\theta$ . The mapping  $\pi^{-1}$  is also called *realisation procedure*, since we realise the system parameters from the external characteristics of the data, i.e. the second moments, the spectrum or the transfer function respectively.

Finally, we consider the question whether  $\pi^{-1}(\pi(\Theta_I)) = \Theta_I$ . This is important to ensure that outside that the identified parameter set  $\Theta_I$  there are no elements, say  $\theta_{-I}$ , which result in the same observable second moments as some  $\theta \in \Theta_I$ :

**Theorem 5**

For all  $\theta_{-I} \in \Theta \setminus \Theta_I$  there exists no  $\theta \in \Theta_I$  such that  $\pi(\theta_{-I}) = \pi(\theta) = \tilde{\gamma}$ .

## 4 Deterministic Terms

This section investigates the VECM

$$\Delta y_t = \mu_0 + \mu_1 t + \Pi y_{t-1} + \sum_{j=1}^{p-1} \Phi_j \Delta y_{t-j} + \nu_t, \quad \nu_t \sim WN(\Sigma_\nu), \quad (21)$$

containing the deterministic terms  $\mu_0$  and  $\mu_1 t$ . The five cases following from (21), namely “ $H_2(r)$ ,  $H_1(r)$ ,  $H_1^*(r)$ ,  $H(r)$ , and  $H^*(r)$ ”, are obtained and defined in Johansen (1995)[page 81 and our Appendix H]. Recall that the cointegrating vectors  $\beta$  can be identified from mixed frequency data (see Chambers, 2020). For high frequency data  $(\Delta y_t)_{t \in \mathbb{Z}}$  and  $(\beta' y_t)_{t \in \mathbb{Z}}$  we can compute the expectations  $\mathbb{E} \Delta y_t$  and  $\mathbb{E} \beta' y_t$ , while for mixed frequency case we get  $\mathbb{E} \beta' y_t$  and  $\mathbb{E} \Delta_N y_t = \mathbb{E} (y_t - y_{t-N})$  for the stock case and  $\mathbb{E} \beta' w_t = \mathbb{E} \sum_{j=0}^{N-1} \beta' y_{t-j}$  and  $\mathbb{E} \Delta_N^\Sigma y_t = \mathbb{E} \sum_{j=0}^{N-1} \Delta_N y_{t-j}$  for the flow case, respectively. To identify the deterministic terms in (21) we can proceed as follows:

1. Remove deterministic trends from  $(\beta' y_t)_{t \in N\mathbb{Z}}$  and  $(\Delta_N y_t)_{t \in N\mathbb{Z}}$  in the stock case or  $(\beta' w_t)_{t \in N\mathbb{Z}}$  and  $(\Delta_N^\Sigma y_t)_{t \in N\mathbb{Z}}$  for the flow case [given  $\beta$ ], such that  $\mathbb{E} \beta' y_t = 0$  and  $\mathbb{E} \Delta_N y_t = 0$ ,  $\mathbb{E} \beta' w_t = 0$ , and  $\mathbb{E} \Delta_N^\Sigma y_t = 0$ .
2. Apply Theorem 4 to obtain the parameters  $\theta$ .
3. Given the parameters  $\theta$ , Appendix H shows that the parameters  $\mu_0$  and  $\mu_1$  can be identified from moments following from data observed.

This results in:

**Theorem 6**

Under the assumptions of Theorem 4 the parameters  $\mu_0$  and  $\mu_1$  can be identified generically from observable first and second moments.

**Proof.** See Appendix H. ■

## 5 Conclusion

In this paper, we generalise the results on identifiability from mixed frequency data in [Anderson et al. \(2016a,b\)](#) obtained for stationary VAR-systems to the case of unit-roots and cointegrating relationships. As is well known these systems have also a *vector error correction representation*. The corresponding parameterspaces are homeomorphic.

We commence from a solution of the (unstable) VAR system on the integers  $\mathbb{Z}$  (see [Bauer and Wagner, 2012](#), for the existence of such a solution). Then we take differences at lag  $N$  (which is the sampling rate of the slow/aggregated process) and stack these to what we call the “blocked process”. In addition, the blocked process also contains the stationary process  $\beta'y_t$ , where  $\beta$  is the matrix of cointegrating relationships. This matrix is identified from mixed frequency data as already shown in [Chambers \(2020\)](#). This blocked process is stationary and contains all relevant differences of the observations.

The contribution of this paper can be seen as an extension of the results in [Chambers \(2020\)](#), by proving that also the remaining parameters of the vector error correction model (i.e. besides  $\beta$ ) are (generically) identified from mixed frequency observations.

The identifiability proof consists of two steps: In the first step, we derive a state-space representation of the blocked process (“the canonical projection form”) which is minimal, in innovation form (both, for the stock and the flow case) and unique. In the second step, we derive an algorithm, that retrieves the parameters of the underlying high frequency system from the parameters of the canonical projection form.

We show that the conditions ([Assumption 2](#)) which are sufficient for identifiability are generic in the parameterspace. This is more intricate than in the stationary case, since the parameterspace is not an open subspace of the Euclidean space, due to the fact that we allow for unit roots. Since the VECM and the VAR parameterspaces are homeomorphic, the genericity result holds for both.

Finally, we show that all common cases of deterministic terms in the VECM can be reduced to the case of non-deterministic terms.

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## Conflict of Interest Statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

## Acknowledgements

The authors would like to thank the participants of the Economics Research Seminar of the University of Regensburg, the CFE 2020 conference, the 3rd Italian Workshop of Econometrics and Empirical Economics (IWEEE 2022), and the 10th Italian Congress of Econometrics and Statistics (ICEEE 2023), as well as Christoph Rust for helpful comments that lead to improvement of the paper. The authors gratefully acknowledge financial support from the Austrian Central Bank under Anniversary Grant No. 18287 and the DOC-Fellowship of the Austrian Academy of Sciences (ÖAW). Manfred Deistler and Leopold Sögner acknowledge support by the Cost Action HiTEc - CA21163.

## A Moments Observed

In Section 2.2, we call the moments  $\mathbb{E} \tilde{y}_{t+h} \tilde{y}'_t$  in equation (8) also *observed second moments* since they can be consistently estimated from the data as we will argue in this section. Those moments in  $\mathbb{E} \tilde{y}_{t+h} \tilde{y}'_t$  which involve only  $N$ -differences from the data do not require any further discussion. We are left with considering moments that involve  $\beta$ .

In the following, if  $N$  is the sampling rate of the slow variables and  $T$  is the number of time observations of high-frequency data, we denote by  $T_s$  the number of time observations available for the slow/ aggregated coordinates which is approximately  $T/N$  and by  $\mathbb{I}_s$  the index set of slow/ aggregated time observations, e.g. for  $N = 3$  we could have  $\mathbb{I}_s = \{3, 6, 9, \dots, T\}$ .

In Miller (2016); Chambers (2020) consistent estimators  $\hat{\beta}_{T_s}$  that converge in probability to  $\beta$  are provided for the mixed frequency case. In both articles, the corresponding estimator  $\hat{\beta}_{T_s}$  converges weakly to a product of the inverse of a functional of Brownian motions and a stochastic integral. By applying (White, 2001, Lemmata 4.5 and 4.6), we observe that

$$(\hat{\beta}_T - \beta) = O_p(T_s^{-1}) .$$

Next we consider the high-frequency moments  $\mathbb{E} \beta' y_t$ ,  $\mathbb{E} \beta' y_t \Delta y'_s$ , and  $\mathbb{E} \beta' y_t y'_s \beta$ ,  $s, t \in \mathbb{Z}$ . The following calculations work in the same way for the mixed frequency counterparts. We can estimate  $\mathbb{E} \beta' y_t$  by  $\frac{1}{T_s} \sum_{t \in \mathbb{I}_s} \hat{\beta}'_{T_s} y_t$ . We impose the standard assumptions that a weak law of large number and a functional central limit theorem can be applied to properly scaled terms. For a ‘‘Low-Frequency Invariance Principle’’ see, e.g., Miller (2016)[Section 2.3]. Let  $B(\tau)$  denote some  $n$ -dimensional Brownian motion. Define  $\hat{u}_t^S := \hat{\beta}'_{T_s} y_t$  and  $\hat{u}_t^F := \hat{\beta}'_{T_s} \sum_{j=0}^{N-1} y_{t-j}$ . This results in

$$\begin{aligned} T_s^{-1} \sum_{t \in \mathbb{I}_s} \hat{u}_t^S &= T_s^{-1} \sum_{t \in \mathbb{I}_s} \hat{\beta}'_{T_s} y_t = T_s^{-1} \sum_{t \in \mathbb{I}_s} \beta' y_t + T_s^{-1} \sum_{t \in \mathbb{I}_s} (\hat{\beta}_{T_s} - \beta)' y_t \\ &= \mathbb{E} \beta' y_t + o_p(1) + T_s^{1/2} (\hat{\beta}_{T_s} - \beta)' T_s^{-3/2} \underbrace{\sum_{t \in \mathbb{I}_s} y_t}_{\Rightarrow \int_0^1 B(\tau) d\tau} \\ &= \mathbb{E} \beta' y_t + o_p(1) + O_p(T^{-1/2}) O_p(1) . \end{aligned} \tag{A.1}$$

Next, we look at the second moments for the stock case appearing in  $\tilde{\gamma}$ , i.e.

second moments of the form  $\mathbb{E} u_t^S \Delta_N y_s$ . Observe that we have

$$\begin{aligned}
T_s^{-1} \sum_{t \in \mathbb{I}_s} \widehat{u}_t^S \Delta y'_s &= T_s^{-1} \sum_{t \in \mathbb{I}_s} \widehat{\beta}_{T_s}' y_t \Delta y'_s \\
&= T_s^{-1} \sum_{t \in \mathbb{I}_s} \beta' y_t \Delta y'_s + \underbrace{\left( \widehat{\beta}_{T_s} - \beta \right)' T_s^{-1} \sum_{t \in \mathbb{I}_s} y_t \Delta y'_s}_{\Rightarrow \int_0^1 B(\tau) dB(\tau)'} \\
&= \mathbb{E} u_t^S \Delta y'_s + o_p(1) + O_p(T_s^{-1}) O_p(1) . \tag{A.2}
\end{aligned}$$

It follows immediately that also

$$T_s^{-1} \sum_{t \in \mathbb{I}_s} \widehat{u}_t^S \Delta_N y_s \xrightarrow{P} \mathbb{E} u_t^S \Delta_N y_s ,$$

where “ $\xrightarrow{P}$ ” denotes convergence in probability and “.” is shorthand for  $s, f$  (“fast” and “slow” coordinates). Finally we look the estimation of  $\mathbb{E} u_t^S u_s^{S'}$  for  $t - s \in N\mathbb{Z}$ :

$$\begin{aligned}
T_s^{-1} \sum_{t \in \mathbb{I}_s} \widehat{u}_t^S (\widehat{u}_s^S)' &= T_s^{-1} \sum_{t \in \mathbb{I}_s} \beta' y_t y_s' \beta + T_s^{1/2} (\widehat{\beta}_{T_s} - \beta)' T_s^{-2} \underbrace{\left( \sum_{t \in \mathbb{I}_s} y_t y_s' \right)}_{\Rightarrow \int_0^1 B(\tau) B(\tau)' d\tau} T_s^{1/2} (\widehat{\beta}_{T_s} - \beta) \\
&= \mathbb{E} u_t^S (u_s^S)' + o_p(1) + O_p(T_s^{-1/2}) O_p(1) O_p(T_s^{-1/2}) . \tag{A.3}
\end{aligned}$$

We obtain analogous results for moments involving  $u_t^F$ , by noting that  $u_t^F = \sum_{j=0}^{N-1} \beta' y_{t-j}$ . Hence, equations (A.1) to (A.3) demonstrate why the above moments can be considered as moments observed although  $\beta$  has to be estimated.

## B Example for Non-Identifiability

We consider a VECM where  $n = 2$ ,  $r = 1$ , and  $p = 1$ . The cointegrating vector is  $\beta = (1, \beta_s)'$ ,  $\alpha = (\alpha_f, \alpha_s)'$ , and  $\Pi = \alpha\beta'$ . That is

$$\Delta y_t = \Pi y_{t-1} + \nu_t, \quad \nu_t \sim WN(\Sigma_\nu), \quad \text{such that } \Sigma_\nu = \begin{pmatrix} \sigma_{ff} & \sigma_{fs} \\ \sigma_{sf} & \sigma_{ss} \end{pmatrix}, \quad \text{and}$$

$$y_t = \underbrace{(I_n + \Pi)}_{\mathcal{A}_1} y_{t-1} + \nu_t = \begin{pmatrix} \underbrace{1 + \alpha_f}_{a_{ff}} & \underbrace{\alpha_f \beta_s}_{a_{fs}} \\ \underbrace{\alpha_s}_{a_{sf}} & \underbrace{1 + \alpha_s \beta_s}_{a_{ss}} \end{pmatrix} y_{t-1} + \nu_t. \quad (\text{A.4})$$

By Theorem 1 of [Anderson et al. \(2016a\)](#) the model parameters are not identified if and only if (i)  $a_{fs} = 0$ , (ii)  $a_{sf} + \frac{\sigma_{sf}}{\sigma_{ff}}(a_{ss} - a_{ff}) = 0$ , and (iii)  $a_{ss} \neq 0$ .  $a_{fs} = 0$  implies  $\alpha_f = 0$  [with  $\beta_s = 0$  we do not have a cointegrating relationship]. From  $a_{sf} + \frac{\sigma_{sf}}{\sigma_{ff}}(a_{ss} - a_{ff}) = \alpha_s \left(1 + \frac{\sigma_{sf}}{\sigma_{ff}}\beta_s\right) = 0$ , we conclude that  $\alpha_s$  must be non-zero, otherwise  $(y_t)$  is a white noise process. Hence, we get  $0 = \left(1 + \frac{\sigma_{fs}}{\sigma_{ff}}\beta_s\right)$ . The third constraint results in  $0 = 1 + \alpha_s\beta_s$ . Hence, by considering the second moments in levels and the arguments provided in the proof Theorem 1 of [Anderson et al. \(2016a\)](#), we get identifiability on a generic set. Unfortunately these moments are not “observed moments” since they cannot be estimated from data observed as in the stationary case.

By contrast  $\mathbb{E}\Delta_N \tilde{y}_t \Delta_N \tilde{y}'_{t-\ell}$ ,  $\ell \in \mathbb{Z}$ , are observed moments, where we consider stock variables only and let  $\Delta_N \tilde{y}_t = y_t - y_{t-N}$ . In addition, when  $\beta$  is known [or can be consistently estimated as shown in [Miller \(2016\)](#); [Chambers \(2020\)](#)] also the moments  $\mathbb{E}\beta \tilde{y}_t \Delta_N \tilde{y}'_{t-\ell}$ ,  $t, t - \ell \in 2\mathbb{Z}$ , can be observed. Applying [\(10\)](#) and [\(11\)](#) to the current example yields

$$\begin{pmatrix} \beta' y_t \\ \Delta y_t \\ \Delta y_{t-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \beta' \alpha + 1 & 0_{1 \times 2} & 0_{1 \times 2} \\ \alpha & 0_{2 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 1} & I_2 & 0_{2 \times 2} \end{pmatrix}}_{A \in \mathbb{R}^{r+2n \times r+2n}} \begin{pmatrix} \beta' y_{t-1} \\ \Delta y_{t-1} \\ \Delta y_{t-2} \end{pmatrix} + \begin{pmatrix} \beta' \\ I_n \\ 0 \end{pmatrix} \nu_t, \quad t \in \mathbb{Z}. \quad (\text{A.5})$$

Next, by direct calculations we get

$$\begin{aligned}
\Delta_2 y_t &= \Delta y_t + \Delta y_{t-1} = \Pi y_{t-1} + \nu_t + \Pi y_{t-2} + \nu_{t-1} \\
&= \Pi (y_{t-2} + \Delta y_{t-1}) + \nu_t + \Pi y_{t-2} + \nu_{t-1} \\
&= 2\alpha\beta' y_{t-2} + \alpha\beta'\alpha\beta' y_{t-2} + \Pi\nu_{t-1} + \nu_{t-1} + \nu_t \\
&= \underbrace{\begin{pmatrix} 2\alpha_f + \alpha_f^2 + \alpha_f\beta_s\alpha_s \\ 2\alpha_s + \alpha_f\alpha_s + \alpha_s^2\beta_s \end{pmatrix}}_{=: M_{\alpha,\beta}} \beta y_{t-2} + \Pi\nu_{t-1} + \nu_{t-1} + \nu_t. \tag{A.6}
\end{aligned}$$

In addition,  $\beta' y_t = \beta' (y_{t-2} + \Delta_2 y_t)$  and  $\Delta y_{t-1}^f := y_{1t-1} = \alpha_f \beta y_{t-2} + (1, 0)' \nu_{t-1}$ . This results in the state-space system

$$\begin{aligned}
\underbrace{\begin{pmatrix} \beta' y_t \\ \Delta_2 y_t \\ \Delta y_{t-1}^f \end{pmatrix}}_{x_t} &= \underbrace{\begin{pmatrix} 1 + \beta' M_{\alpha,\beta} & 0_{1 \times 2} & 0_{1 \times 2} \\ M_{\alpha,\beta} & 0_{2 \times 2} & 0_{2 \times 2} \\ \alpha_f & 0_{1 \times 2} & 0_{1 \times 2} \end{pmatrix}}_{\tilde{A} \in \mathbb{R}^{r+n+1 \times r+n+1}} \begin{pmatrix} \beta' y_{t-2} \\ \Delta_2 y_{t-2} \\ \Delta y_{t-3} \end{pmatrix} + \underbrace{\begin{pmatrix} \beta' I_2 & \beta' (I_2 + \Pi) \\ I_2 & I_2 + \Pi \\ 0 & (1, 0)' \end{pmatrix}}_{\tilde{B}} \begin{pmatrix} \nu_t \\ \nu_{t-1} \end{pmatrix} \\
\tilde{y}_t &= I_{\tilde{n}} x_t, \quad t \in 2\mathbb{Z}. \tag{A.7}
\end{aligned}$$

Finally, we investigate whether there are (different) parameter vectors such that the autocovariance function of  $(\tilde{y}_t)_{t \in 2\mathbb{Z}}$ , i.e.  $\tilde{\gamma}$ , is the same. (Recall that  $\theta^I := (\alpha_f^I, \alpha_s^I, 1, \beta_s, \sigma_{ff}^I, \sigma_{fs}^I, \sigma_{ss}^I)$  and  $\theta^{II} := (\alpha_f^{II}, \alpha_s^{II}, 1, \beta_s, \sigma_{ff}^{II}, \sigma_{fs}^{II}, \sigma_{ss}^{II})$  as defined in Section 2.2.) This is the case if  $\tilde{A}$ , and  $\tilde{B}\Sigma\tilde{B}'$  are equal though  $\theta^I \neq \theta^{II}$ . Consider  $\tilde{A}$ , for  $\alpha_f = 0$ , from the definition of  $M[\alpha, \beta]$  in equation (A.6), we can choose some  $\alpha_s^I \neq \alpha_s^{II}$  solving the quadratic equation  $2\alpha_s + \beta_s\alpha_s^2 - [M_{\alpha,\beta}]_{2,1} = 0$ , in which case  $\tilde{A}$  remains the same.

Next we consider  $\tilde{B}\Sigma\tilde{B}'$ . To get equality with some  $\theta^I, \theta^{II}$ , the covariances of the lagged fast variable have to be equal, this demands for  $\sigma_{ff}^I = \sigma_{ff}^{II}$ . Second, to get equality of the covariances of the cointegrating terms it is sufficient to look at the covariances of  $\Delta_N y_t$ . Direct calculations show that (with  $\alpha_f = 0, \sigma_{ff}^I = \sigma_{ff}^{II}$ )

$$\begin{aligned}
& [\tilde{B}\Sigma\tilde{B}']_{(2:3,2:3)} \\
&= \begin{pmatrix} \sigma_{ff} & \alpha_s\sigma_{ff} + (1 + \alpha_s\beta_s)\sigma_{sf} \\ \alpha_s\sigma_{ff} + (1 + \alpha_s\beta_s)\sigma_{sf} & \alpha_s(\alpha_s\sigma_{ff} + (1 + \alpha_s\beta_s)\sigma_{sf}) + (\alpha_s\sigma_{fs} + (1 + \alpha_s\beta_s)\sigma_{ss})(1 + \alpha_s\beta_s) \end{pmatrix}. \tag{A.8}
\end{aligned}$$

This yields  $\sigma_{sf}^{II} = \sigma_{sf}^I = \frac{\alpha_s^I\sigma_{ff} - \alpha_s^{II} + (1 + \alpha_s^I\beta_s)\sigma_{sf}^I}{(1 + \alpha_s^{II}\beta_s)}$ , where  $(1 + \alpha_s^{II}\beta_s) \neq 0$ . Finally, choose  $\sigma_{ss}^{II}$  such that the (2,2)-elements of (A.8) for parameters  $\theta^I$  and  $\theta^{II}$  become equal, this requires  $(1 + \alpha_s^I\beta_s) \neq 0$ . Hence, we have obtained a

pair  $\theta^I, \theta^{II}, \theta^I \neq \theta^{II}$ , where  $\tilde{A}$  and  $\tilde{B}\Sigma\tilde{B}'$  are the same. This implies by Lyapunov equations that the model parameters cannot be identified from  $\tilde{\gamma}$ . In this example we observed the following: If  $\alpha_f = 0$ ,  $(y_t^f)_{t \in \mathbb{Z}}$  is a random walk not affected by  $(y_t^s)_{t \in \mathbb{Z}}$ . Therefore, the high frequency variable  $y_t^f$  does not provide information about the parameter  $\alpha_s$ . By contrast, if  $\alpha_f \neq 0$  the fast variable  $y_t^f$  provides sufficient information to identify  $\alpha_s$ . To see this, if  $\alpha_f \neq 0$ ,  $\alpha_f$  follows from the (4,1)-element of  $\tilde{A}$ , while  $\alpha_s$  can be uniquely retrieved from the first coordinate of  $M_{\alpha,\beta}$ , that is from the equality  $2\alpha_f + \alpha_f^2 + \alpha_f\beta_s\alpha_s = [M_{\alpha,\beta}]_{(1,1)}$ .

## C Proof of Theorem 1

**Proof.** 1. ( $G \cap \Theta_1$  is dense.)

Suppose that  $\theta_0 \in \Theta_1$  does not satisfy at least one of the identifiability conditions. Let  $\varepsilon > 0$ , we show that there exists  $\theta \in G \cap \Theta_1$  such that  $\|\theta - \theta_0\| < \varepsilon$  by perturbing the eigenvalues / eigenvectors of the companion matrix  $A$  corresponding to  $\theta_0$ .

For this we define a mapping  $f_{\theta_0}$  that maps  $\mathcal{A}$  to a companion matrix  $\mathcal{A}^*$  with perturbed eigenvalues and eigenvectors such that  $\theta = \text{vec}(\mathcal{A}_1^* \cdots \mathcal{A}_p^*)$  is in  $G \cap \Theta_1$ :

1. Compute the Jordan decomposition of  $\mathcal{A} = Q\Lambda Q^{-1}$ .
2. Perturb the eigenvalues:

$$\bar{\mathcal{A}}^* = Q \underbrace{\left( \Lambda + \text{diag}(\underbrace{0, \dots, 0}_{n-r \text{ -times}}, \xi_1, \dots, \xi_{np-(n-r)}) \right)}_{\tilde{\Lambda}} Q^{-1}. \quad (\text{A.9})$$

3. We transform  $\bar{\mathcal{A}}^*$  to a similar matrix  $\mathcal{A}^*$  that has the companion structure by using the procedure from [Anderson et al. \(2016a\)](#):

$$\mathcal{A}^* = T\bar{\mathcal{A}}^*T^{-1}, \quad \text{hence} \quad \mathcal{A}^*T = \begin{pmatrix} \mathcal{A}_1^* & \cdots & \mathcal{A}_p^* \\ I_n & & 0 \\ & \ddots & \vdots \\ & & I_n & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_p \end{pmatrix} = T\bar{\mathcal{A}}^*,$$

where  $T_j$  for  $j = 1, \dots, p$  are the  $n \times np$  rowblocks of  $T$ . Now we set  $T_1 = [I_n \ 0 \ \dots \ 0]$  and solve the equation above:

$$\mathcal{A}_1^* T_1 + \dots + \mathcal{A}_p^* T_p = T_1 \bar{\mathcal{A}}^*, \quad T_1 = T_2 \bar{\mathcal{A}}^*, \quad \dots, \quad T_{p-1} = T_p \bar{\mathcal{A}}^*,$$

which yields

$$T_j = T_{j-1} \bar{\mathcal{A}}^{*-1} \quad \text{for } j = 2, \dots, p.$$

Clearly, the mapping  $f_{\theta_0} : \xi \mapsto \mathcal{A}^* \mapsto \theta$  for  $\xi = (\xi_{n-r+1}, \dots, \xi_{np})' \in \mathbb{R}^{np-(n-r)}$  is continuous at  $\theta_0$  and  $f_{\theta_0}(0) = \theta_0$  (as in this case  $T = I_{np}$ ). So for the  $\varepsilon$ -neighborhood around  $\theta_0$  denoted by  $U_\varepsilon(\theta_0)$  there exists a  $\delta > 0$ , such that for all  $\xi \in U_\delta(0)$  we have  $f_{\theta_0}(\xi) \in U_\varepsilon(\theta_0)$ , where  $U_\delta(0)$  is the open  $\delta$ -neighborhood around 0 in  $\mathbb{R}^{np-(n-r)}$ .

Now,  $\lambda^* := (1, \dots, 1, \lambda_{n-r+1} + \xi_1, \dots, \lambda_{np} + \xi_{np-n-r})$  are the eigenvalues of  $\mathcal{A}^*$  because they are the zeros of the characteristic polynomial of  $\tilde{\Lambda}$  in equation (A.9) from which we obtain  $\mathcal{A}^*$  by similarity transformation with  $TQ$ . For any  $\delta > 0$ , we can find a  $\xi \in U_\delta(0)$  such that the corresponding eigenvalues  $\lambda^*$  of  $\mathcal{A}^*$  satisfy the conditions (I1), (I3), (I4) and (I5). Analogously to equation (A.9), we can perturb the eigenvalues and eigenvectors of  $A$  to ensure conditions (I5) (second part) and (I6).

We have to ensure that the image  $f_{\theta_0}(\xi)$  is real valued: Since  $\mathcal{A}$  is real valued, for any complex eigenvalue  $z = a + ib \in \mathbb{C} \setminus \mathbb{R}$ , the conjugate  $\bar{z} = a - ib$  is also an eigenvalue of  $\mathcal{A}$ . If the algebraic multiplicity of  $z$  is larger than 1,  $z$  has to be perturbed. As is easily shown, if we add to  $z$  and  $\bar{z}$  the same small real number, the resulting  $\bar{\mathcal{A}}^*$  (and therefore also  $\mathcal{A}^*$ ) is again real valued. Thus, we found  $\theta \in G$  close to  $\theta_0$  and are left with checking whether  $\theta$  is also in  $\Theta_1$ . (C3) is trivial.

For (C1), note that, still  $n - r$  eigenvalues of  $\mathcal{A}^*$  equal unity which ensures that  $\text{rk } \Pi = r$  (see Bauer and Wagner, 2012). Applying the procedures described above, we obtain the vector error correction representation corresponding to  $f_{\theta_0}(\xi) = \theta$ , say  $(\alpha(\xi), \beta(\xi), \Phi_1(\xi), \dots, \Phi_{p-1}(\xi))$ , and see that

$$g : \theta \mapsto \det \alpha_\perp(\xi)' (I_n - \sum_{j=1}^{p-1} \Phi_j(\xi)) \beta_\perp(\xi)$$

is continuous at  $\theta = \theta_0$ . We know that  $g(\theta_0) \neq 0$  since  $\theta_0 \in \Theta_1$ . So there exists  $\varepsilon_3 > 0$  such that the neighbourhood  $U_{\varepsilon_3}(g(\theta_0))$  is bounded away from

zero. By continuity there exists  $\varepsilon_2 > 0$  such that for all  $\theta \in U_{\varepsilon_3}(\theta_0)$ , we have  $g(\theta) \in U_{\varepsilon_2}(g(\theta_0))$ . For the same reasons as above we can find suitable  $\xi$  such that  $f_{\theta_0}(\xi) = \theta \in U_{\varepsilon}(\theta_0) \cap U_{\varepsilon_3}(\theta_0)$ . Hence  $\Theta \cap G$  is dense in  $\Theta$ .

2. ( $G \cap \Theta_1$  is open in  $(\Theta_1, d)$ ),

where  $d$  denotes the Euclidean metric. Suppose now for  $\theta^* \in G \cap \Theta_1$ , we have to show that there exists  $\varepsilon > 0$  such that  $U_{\varepsilon}(\theta_0) \subset G \cap \Theta_1$ . The eigenvalues are the zeros of the characteristic polynomial of  $A$  and therefore continuous functions at  $\theta^*$  (since as is well known, the zeros of any polynomial are continuous function of its coefficients). So the mapping

$$e : \theta \mapsto \mathcal{A} \mapsto (\lambda_{n-r+1} \ \cdots \ \lambda_{np}) = \lambda$$

is continuous in  $\theta^*$ . Clearly there is an open neighbourhood  $U \subset \mathbb{C}^{np-(n-r)}$  of  $\lambda^* = e(\theta^*)$  such that for all  $\lambda \in U$  the corresponding spectrum  $(1 \ \cdots \ 1 \ \lambda)'$  satisfies the identifiability conditions. The pre-image  $e^{-1}(U) \subset G$  is an open neighborhood of  $\theta_0$ . Analogously to the arguments applied above, we can establish **(C2)**.

**(I2)** follows from **(C4)** and **(I1)** which completes the proof. ■

## D Proof of Theorem 2

**Proof.** This follows from transforming a state-space system into prediction error form. See [Hannan and Deistler \(2012\)](#)[chapter 1] and [Gersing and Deistler \(2021\)](#). From [Johansen \(1995\)](#)[Proof of Theorem 4.2] it follows that the largest eigenvalue of  $A$  is in modulus smaller than one. Hence the system is stable. The linear expansion of the transfer function for a stable system is already the Wold representation as the inputs  $\tilde{v}_t$  are the innovations. Hence, the system is also miniphase (see, e.g., [Deistler and Scherrer, 2022](#), Chapters 2 and 7.3). ■

## E Proof of Theorem 3

**Proof.** By [Johansen \(1995\)](#)[Proof of Theorem 4.2] it follows that the eigenvalues of modulus smaller than 1 are the same, for  $\mathcal{A}$  and  $A$ .

*1.1 Observability for the Stock Case:* We use the PBH-Test (see, e.g., [Kailath, 1980](#), Section 2.4.3) to prove that the pair  $(A_b, C_b)$  is generically observable (note that the observability of  $(A_b, C_b)$  also implies the observability of

$(A_{b,c}, C_{b,c})$  since  $c$  is non-singular). For this, note that the eigenvectors of  $A_b$  are the same as the eigenvectors of  $A$ . Let  $\lambda$  be an eigenvalue of  $A$  and  $q = (q'_\beta \ q'_1 \ \cdots \ q'_{p-1})'$  the corresponding eigenvector. We write

$$Aq = \begin{bmatrix} \beta' \alpha + I_r & \beta' \Phi_1 & \cdots & \beta' \Phi_{p-1} \\ \alpha & \Phi_1 & \cdots & \Phi_{p-1} \\ 0 & I_n & & 0 \\ \vdots & & \ddots & \\ 0 & & & I_n & 0 \end{bmatrix} \begin{pmatrix} q_\beta \\ q_1 \\ \vdots \\ q_{p-1} \end{pmatrix} = \lambda \begin{pmatrix} q_\beta \\ q_1 \\ \vdots \\ q_{p-1} \end{pmatrix},$$

where  $q_\beta$  is  $r \times 1$  and  $q_i$  is  $n \times 1$  for  $i = 1, \dots, p-1$ . From this, we obtain the relations

$$(\beta' \alpha + I_r)q_\beta + \sum_{i=1}^{p-1} \beta' \Phi_i q_i = \lambda q_\beta \quad (\text{A.10})$$

$$\alpha q_\beta + \sum_{i=1}^{p-1} \Phi_i q_i = \lambda q_1 \quad (\text{A.11})$$

$$q_i = \lambda q_{i+1}, \quad i = 1, \dots, p-2. \quad (\text{A.12})$$

Since  $A$  is of full rank,  $\lambda \neq 0$  and  $q_1 = 0$  imply  $q = 0$ , which is a contradiction (noting that  $\alpha$  has rank  $r$ ). Now we look at

$$C_b q = \begin{pmatrix} I_r & 0 & \cdots & & & & 0 \\ 0 & I_n & \cdots & I_n & 0 & \cdots & 0 \\ 0 & 0 & (I_{n_f}, 0) & \cdots & (I_{n_f}, 0) & & \\ & & & \ddots & & \ddots & \vdots \\ & & & (I_{n_f}, 0) & \cdots & (I_{n_f}, 0) & 0 \end{pmatrix} \begin{pmatrix} \lambda^N q_\beta \\ \lambda^N q_1 \\ \vdots \\ \lambda^N q_{p-1} \end{pmatrix}, \quad (\text{A.13})$$

which is not equal to zero. If, for example,

$$\begin{aligned} \lambda^N q_1 + \cdots + \lambda^N q_N &= \lambda^N q_1 + \lambda^{N-1} q_1 + \cdots + q_1 = (1 + \lambda + \cdots + \lambda^N) q_1 \neq 0 \\ &\Leftrightarrow (1 + \lambda + \cdots + \lambda^N) \neq 0, \end{aligned}$$

which is generically the case (see Assumption 2). Recall that by  $q$  we denote eigenvectors of  $A$  and by  $v$  eigenvectors of  $\mathcal{A}$ , where both correspond to the same eigenvalue  $|\lambda| < 1$ . In Lemma 7, we show that

$$q_\beta = \frac{\lambda}{\lambda - 1} \beta' q_1 = \beta' v_1, \quad (\text{A.14})$$

so if we suppose that  $v_1$  is not in the right kernel of  $\beta'$ , we also get  $C_b q \neq 0$ .

*1.2 Observability for the Flow Case:* The first part of the proof is analogous to the stock case. It remains to show that there exists no eigenvector that is in the right kernel of  $C_b$ , where  $C_b$  is now defined in (18). Now, analogously to the procedure in (A.13) we obtain, that an eigenvector of  $A_b$  is not in the rightkernel of  $C_b$  if e.g.

$$\begin{aligned} & \lambda^N q_1 + \dots + \lambda^N q_N - \lambda^N q_{N+1} - \dots - \lambda^N q_{2N} \\ &= \lambda^N q_1 + \lambda^{N-1} q_1 + \dots + \lambda q_1 - q_1 - \dots - \lambda^{-N+1} q_1 \\ &= \lambda^{N-1} (-1 - \lambda - \dots - \lambda^{N-1} + \lambda^N + \dots + \lambda^{2N-1}) q_1 \neq 0 \\ &\Leftrightarrow (-1 - \lambda - \dots - \lambda^{N-1} + \lambda^N + \dots + \lambda^{2N-1}) \neq 0, \end{aligned}$$

Also the second part is similar to the stock case: By Lemma 7,  $q_\beta = \frac{\lambda}{\lambda-1} \beta' q_1 = \beta' v_1$ . Assume that  $v_1$  is not in the right kernel of  $\beta'$  (as already done in the stock case). In addition, by considering the first  $r$  rows of the matrix  $c$  for the flow case, provided in (16), we get

$$\begin{aligned} & N I_r q_\beta - (N-1) \beta' q_1 - (N-2) \beta' q_2 - \dots - 2 \beta' q_{N-2} - \beta' q_{N-1} \\ &= N I_r \frac{\lambda}{\lambda-1} \beta' q_1 - \frac{(N-1)}{\lambda^0} \beta' q_1 - \frac{(N-2)}{\lambda} \beta' q_1 - \dots - \frac{2}{\lambda^{N-2}} \beta' q_1 - \frac{1}{\lambda^{N-1}} \beta' q_1 \\ &= \left( N \frac{\lambda}{\lambda-1} - \frac{(N-1)}{\lambda^0} - \frac{(N-2)}{\lambda} - \dots - \frac{2}{\lambda^{N-2}} - \frac{1}{\lambda^{N-1}} \right) \beta' q_1 \\ &= \frac{1}{\lambda^{N-1}} \left( N \frac{\lambda^N}{\lambda-1} - (N-1) \lambda^{N-1} - (N-2) \lambda^{N-2} - \dots - 2\lambda - 1 \right) \beta' q_1. \end{aligned}$$

Note that  $\lambda \neq 1$  and  $\lambda \neq 0$  by the model assumptions (recall that by Johansen (1995)[Proof of Theorem 4.2] it follows that the eigenvalues of modulus smaller than 1 are the same, for  $\mathcal{A}$  and  $A$ ). Hence, if  $v_1$  is not in the right kernel of  $\beta'$  and  $N \frac{\lambda^N}{\lambda-1} - (N-1) \lambda^{N-1} - (N-2) \lambda^{N-2} - \dots - 2\lambda - 1 \neq 0$  we also get that  $\tilde{C}_b q \neq 0$  for the flow case.

2. *Controllability:* It is enough to show that the matrix  $\mathbb{E} x_{t+1} (\tilde{y}'_t \quad \tilde{y}'_{t-N} \quad \dots)'$

has full rank. For  $k$  sufficiently large, we have

$$x_{t+N-1} = A_{b,c}^{k-1} x_{t-kN+1} + \sum_{j=0}^{k-1} A_{b,c}^j B_{b,c} \nu_{t-N-jN}^b$$

$$\Delta_N y_{t-kN} = \underbrace{\begin{bmatrix} 0_{n \times r} & I_n & 0 & \cdots & 0 \end{bmatrix}}_{S_{\Delta_N y}} x_{t-kN+1}$$

$$\begin{aligned} \mathbb{E} \Delta_N y_{t-kN} x'_{t-N+1} &= \mathbb{E} \left\{ S_{\Delta_N y} x_{t-kN+1} x'_{t-kN+1} A_{b,c}^{k-1'} + S_{\Delta_N y} x_{t-kN+1} \left( \sum_{j=0}^{k-2} A_{b,c}^j B_{b,c} \nu_{t-N-jN}^b \right)' \right\} \\ &= S_{\Delta_N y} \underbrace{c \Gamma_{rp} c'}_{\Gamma_{rp,c}} A_{b,c}^{k-1'}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{E} x_{t-N+1} \begin{pmatrix} \Delta_N y'_{t-kN} & \Delta_N y'_{t-(k+1)N} & \cdots & \Delta_N y'_{t-(k+p-1)N} \end{pmatrix} \\ = A_{b,c}^{k-1} \begin{bmatrix} \Gamma_{rp,c} S'_{\Delta_N y} & A_{b,c} \Gamma_{rp,c} S'_{\Delta_N y} & \cdots & A_{b,c}^{p-1} \Gamma_{rp,c} S'_{\Delta_N y} \end{bmatrix}, \end{aligned}$$

which has full rank if  $\Gamma_{rp} > 0$  as follows from the proof of Theorem 7 in [Anderson et al. \(2016a\)](#). By [Hannan and Deistler \(2012\)](#)[Theorem 2.3.3] controllability and observability imply that the system is minimal.  $\blacksquare$

**Lemma 7**

Suppose the [Assumption 1](#) and [2](#) hold. Then equation [\(A.14\)](#) holds.

**Proof.** Substracting  $\beta'$  times [\(A.11\)](#) from [\(A.10\)](#), we obtain

$$q_\beta = \lambda q_\beta - \lambda \beta' q_1 \text{ such that } q_\beta = \frac{\lambda}{\lambda - 1} \beta' q_1.$$

Next, we consider the eigenvector  $v = (v'_1 \cdots v'_p)'$  of  $\mathcal{A}$  corresponding to  $\lambda$  (recall that eigenvalues in modulus smaller than one of  $A$  and  $\mathcal{A}$  are the same). By using the relations of the parameters between the VECM and VAR representation, we get

$$\begin{aligned} \lambda v_1 &= (I_n + \alpha \beta') v_1 + \Phi_1 (v_1 - v_2) + \Phi_2 (v_2 - v_3) + \cdots + \Phi_{p-1} (v_{p-1} - v_p) \\ \alpha \beta' v_1 + \Phi_1 \frac{\lambda - 1}{\lambda} v_1 + \Phi_2 \frac{\lambda - 1}{\lambda^2} v_1 + \cdots + \Phi_{p-1} \frac{\lambda - 1}{\lambda^{p-1}} v_1 &= (\lambda - 1) v_1, \end{aligned}$$

where the last relation follows from  $v_i = \lambda v_{i+1}$  for  $i = 1, \dots, p-1$ , which results by the companion structure of  $A$ . Now, we see that  $q_1 = ((\lambda - 1)/\lambda) v_1$  solves [\(A.10\)](#) and [\(A.11\)](#).  $\blacksquare$

## F Proof of Theorem 4

**Proof.** Consider the stable, miniphase spectral factor  $\tilde{k}(\tilde{z})$ ,  $\tilde{z} := z^N$ , corresponding to the Wold representation of  $(\tilde{y}_t)_{t \in \mathbb{N}\mathbb{Z}}$ .

**Step 1:** We obtain an arbitrary minimal realisation  $(\bar{A}_{b,c}, \bar{B}_{b,c}, \bar{C}_{b,c})$  of  $\tilde{k}(\tilde{z})$ , e.g. by taking the echelon form, see [Hannan and Deistler \(2012\)](#)[Thm 2.5.2].

**Step 2:** (Obtain eigenvalues  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_{r+n(p-1)})$  and a linear combination of the eigenvectors of  $A$ , denoted  $q_i$ , from  $\bar{A}_{b,c}$ ).

By, e.g., [Hannan and Deistler \(2012\)](#)[Theorem 2.3.4] the parameter matrices of minimal systems relate via  $\bar{A}_{b,c} = T^{-1}A_{b,c}T$ ,  $\bar{C}_{b,c} = C_{b,c}T$  and  $\bar{B}_{b,c} = T^{-1}\tilde{B}_c$ , where  $T$  is a non-singular matrix.

Since  $\mathcal{A}$  (see equation (4)) is assumed to be diagonalizable (Assumption 2), the matrix  $A$  (see equation (10)); recall that by [Johansen \(1995\)](#)[Proof of Theorem 4.2] the of modulus smaller than 1 are the same, for  $\mathcal{A}$  and  $A$ ) can be expressed by means of  $A = Q\Lambda Q^{-1}$ , where  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_{r+n(p-1)})$  is the diagonal matrix of eigenvalues of  $A$  and  $Q = (q_1, \dots, q_{r+n(p-1)})$  contains the eigenvectors.  $A_b = A^N$ ,  $C_{b,c} = C_b c^{-1}$  and  $A_{b,c} = cA_b c^{-1}$ , such that  $\bar{A}_{b,c} = T^{-1}A_{b,c}T = T^{-1}cA_b c^{-1}T = (T^{-1}cQ)\Lambda^N(T^{-1}cQ)^{-1}$ ,  $\bar{C}_{b,c} = C_{b,c}T = C_b c^{-1}T$ . By the eigen-decomposition of  $\bar{A}_{b,c}$ , we obtain  $(T^{-1}cQ)$  and  $\Lambda^N$ . In addition,  $(0_{n \times r}, 0_{n \times n}, I_n \ 0 \dots 0)A^2 = (0_{n \times r}, I_n \ 0 \dots 0)A$  by the companion structure of  $A$ . Hence by (15), we have

$$\bar{C}_{b,c}T^{-1}cQ = C_b c^{-1}TT^{-1}cQ = C_b Q = \begin{pmatrix} (I_r \ 0 \ \dots \ 0)A^N \\ S_{n_f}^{(1)}A^N \\ S_{n_s}^{(1)}A^N \\ S_{n_f}^{(1)}A^{N-1} \\ S_{n_f}^{(1)}A^{N-2} \\ \vdots \\ S_{n_f}^{(1)}A \end{pmatrix} Q. \quad (\text{A.15})$$

Now we look at the last two rowblocks of  $C_b$  with the eigenvectors  $q_i$ ,  $1 \leq i \leq m$ . From assumption 2 (I4), it follows that the eigenvectors of  $A$  are the same as the eigenvectors of  $A^2$  (also as  $A^N$ ) (see [Felsenstein, 2014](#), Lemma 3.2.1), therefore we have

$$\begin{aligned} S_{n_f}^{(1)}A^2 q_i &= S_{n_f}^{(1)}\lambda_i^2 q_i \\ S_{n_f}^{(1)}A q_i &= S_{n_f}^{(1)}\lambda_i q_i, \end{aligned} \quad (\text{A.16})$$

and we can compute all eigenvalues not equal to one since  $S_{n_f}^{(1)}q_i \neq 0$  by assumption 2 (I6). The flow-case is analogous. Summing up, from  $\bar{A}_{b,c}$  we are

able to obtain  $T^{-1}cQ$ ,  $\Lambda^N = \text{diag}(\lambda_1^N, \dots, \lambda_{r+n(p-1)}^N)$  and  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_{r+n(p-1)})$ .

**Step 3:** (relate  $c^{-1}T$  to  $T$ )

To jointly treat the stock and the flow case, we write

$$c = \begin{pmatrix} c_{\beta\beta} & c_{\beta 1} & c_{\beta 2} & \cdots & c_{\beta N-1} & 0 & \cdots \\ 0 & c_{11} & c_{12} & \cdots & \cdots & c_{1N} & c_{1,N+1} & \cdots \\ \vdots & & & \ddots & & & & \ddots \end{pmatrix},$$

where  $c_{\beta 1}, \dots, c_{\beta N-1}$  and  $c_{N-1+j}$ ,  $j \geq 1$  are zero for the stock case (see equation (12)). For the flow case  $c_{11}, \dots, c_{N-1,1} = I_n$  and  $c_{1N}, \dots, c_{2N-1,1} = -I_n$  (see equation (16)). For the case of stock and flow variables the corresponding coordinates of  $c_{1N}, \dots, c_{2N-1,1}$  are zero for stock variables.

Let

$$A = \begin{pmatrix} A_\beta \\ A_1 \\ \vdots \\ A_{p-1} \end{pmatrix}, \quad T = \begin{pmatrix} T_\beta \\ T_1 \\ \vdots \\ T_{p-1} \end{pmatrix}, \quad \text{and} \quad R := c^{-1}T = \begin{pmatrix} R_\beta \\ R_1 \\ \vdots \\ R_{p-1} \end{pmatrix}. \quad (\text{A.17})$$

Observe that for the stock case (the flowcase is treated analogously)

$$\begin{aligned} \bar{C}_{b,c} \bar{A}_{b,c}^{-1} &= \left( \begin{pmatrix} I_r & c_{\beta 1} & \cdots & c_{\beta N-1} & 0 & \cdots & \cdots \\ 0 & I_n & I_n & \cdots & I_n & 0 & \cdots \end{pmatrix} A^N \right) c^{-1} T \underbrace{T^{-1} c A^{-N} c^{-1} T}_{\bar{A}_{b,c}^{-1}} \\ &= \begin{pmatrix} I_r & c_{\beta 1} & \cdots & c_{\beta N-1} & 0 & \cdots & \cdots \\ 0 & I_n & I_n & \cdots & I_n & 0 & \cdots \end{pmatrix} c^{-1} T = \begin{pmatrix} [c]_{(1:n+r,1:m)} \\ \# \end{pmatrix} c^{-1} T = \begin{pmatrix} T_\beta \\ T_1 \\ \# \end{pmatrix} \end{aligned} \quad (\text{A.18})$$

where “ $\#$ ” denotes some matrix entries which are not important here. Note that  $\bar{A}_{b,c} = T^{-1}cA_b c^{-1}T$ . From Steps 1 and 2, we obtain  $\bar{A}_c := T^{-1}cAc^{-1}T = T^{-1}cQ\Lambda Q^{-1}c^{-1}T$ .  $Ac^{-1}T = c^{-1}T\bar{A}_c$  and

$$AR = A \begin{pmatrix} R_\beta \\ R_1 \\ \vdots \\ R_{p-1} \end{pmatrix} = \begin{pmatrix} (I_r + \beta'\alpha)R_\beta + \beta'\Phi_1 R_1 + \cdots + \beta'\Phi_{p-1} R_{p-1} \\ \alpha R_\beta + \Phi_1 R_1 + \cdots + \Phi_{p-1} R_{p-1} \\ R_1 \\ \vdots \\ R_{p-2} \end{pmatrix} = \begin{pmatrix} R_\beta \bar{A}_c \\ R_1 \bar{A}_c \\ R_2 \bar{A}_c \\ \vdots \\ R_{p-1} \bar{A}_c \end{pmatrix} = R \bar{A}_c \quad (\text{A.19})$$

$$AR = R \bar{A}_c = c^{-1} \begin{pmatrix} T_\beta \bar{A}_c \\ T_1 \bar{A}_c \\ T_2 \bar{A}_c \\ \vdots \\ T_p \bar{A}_c \end{pmatrix} = c^{-1} \begin{pmatrix} T_\beta T^{-1} c A c^{-1} T \\ T_1 T^{-1} c A c^{-1} T \\ T_2 T^{-1} c A c^{-1} T \\ \vdots \\ T_p T^{-1} c A c^{-1} T \end{pmatrix}. \quad (\text{A.20})$$

Now,  $R_\beta = c_\beta^{-1}T$  and  $R_1 = c_1^{-1}T$ , where  $c_\beta^{-1} := [c^{-1}]_{(1:r,1:m)}$  and  $c_1^{-1} := [c^{-1}]_{(r+1:r+n,1:m)}$ . Therefore, we receive  $R_i$  for  $i = 2, \dots, p-1$ , given  $R_1 = c_1^{-1}T_1$  from the recursion  $R_{i+1} = R_i \bar{A}_c^{-1}$ , for  $i = 1, \dots, p-2$ .

**Step 4:** (obtain  $R = c^{-1}T$ ,  $T$  and  $\beta, \Phi_1, \dots, \Phi_{p-1}$ )

To retrieve  $T$  and  $R$  we proceed as follows: By means of (A.18) and (A.20), and the assumption  $p \geq 2N$  we derive

$$T = cR = \begin{pmatrix} c_{\beta\beta}R_\beta + c_{\beta 1}R_1 + c_{\beta 2}R_2 + \dots + c_{\beta N-1}R_{N-1} \\ 0R_\beta + c_{11}R_1 + c_{12}R_2 + \dots + c_{1N}R_N + c_{1,N+1}R_{N+1} + \dots + c_{1,2N}R_{2N} \\ 0R_\beta + 0R_1 + R_2 + R_3 + \dots + R_{N+1} \\ \vdots \\ R_N + R_{N+1} + \dots + R_{2N-1} \\ \hline I_n R_{N+1} \\ \vdots \\ I_n R_{p-1} \end{pmatrix} \quad (\text{A.21})$$

Recall that for stock case  $c_{1j} = I_n$ ,  $j = 1, \dots, N$ ,  $c_{1j} = 0$ ,  $j > N$ ,  $c_{\beta j} = 0$ , for  $j \geq 1$ , while for the flow case  $c_{1j} = I_n$ ,  $j = 1, \dots, N$ ,  $c_{1j} = -I_n$ ,  $j = N+1, \dots, 2N$ , and  $c_{\beta j} = -(N-j)\beta'$ , for  $j = 1, \dots, N-1$ .

From the above considerations  $T_1$  can be obtained from (A.18). Since  $R_{i+1} = R_i \bar{A}_c^{-1}$ , equation (A.21) yields

$$T_1 = \begin{cases} R_1 + R_2 + \dots + R_N & , \text{ for the stock case,} \\ R_1 + R_2 + \dots + R_N - R_{N+1} - \dots - R_{2N} & , \text{ for the flow case.} \end{cases} \quad (\text{A.22})$$

In the above Step 3, we obtained  $R_{i+1} = R_i \bar{A}_c^{-1}$ , which results in

$$T_1 = \begin{cases} R_1 + R_2 + \dots + R_N & , \text{ for the stock case,} \\ R_1 + R_2 + \dots + R_N - (R_1 + \dots + R_N) \bar{A}_c^{-N} & , \text{ for the flow case,} \end{cases} \quad (\text{A.23})$$

such that  $R_1 + \dots + R_N = T_1$  for the stock and  $R_1 + \dots + R_N = T_1 (I_m - \bar{A}_c^{-N})^{-1}$  for the flow case. As already obtained above,  $R_{i+1} = R_i \bar{A}_c^{-1}$ . This yields  $R_1 + \dots + R_N = R_1 \sum_{j=1}^N \bar{A}_c^{-j+1}$ . Since  $R_1 + \dots + R_N$  follows from (A.23) we are also able to derive  $R_1$  and therefore  $R_{i+1}$  by the recursion  $R_{i+1} = R_i \bar{A}_c^{-1}$ ,  $i = 2, \dots, p-1$ . Finally, we observe

$$\begin{aligned} T_2 &= R_2 + R_3 + \dots + R_{N+1} = (R_1 + R_2 + \dots + R_N) \bar{A}_c^{-1} \\ &\vdots \\ T_N &= R_N + R_{N+1} + \dots + R_{N+N-1} = \dots \\ T_{N+1} &= R_{N+1} \\ &\vdots \\ T_{p-1} &= R_{p-1} . \end{aligned} \quad (\text{A.24})$$

Hence  $T_i$ ,  $i = 2, \dots, p-1$ , are provided by (A.23). Recall that  $T_\beta$  and  $T_1$  follow from (A.18).

**Step 5:** (Obtain  $\Sigma_\nu$ )

Let

$$\begin{aligned}\gamma_{\Delta_{Ny}}(\kappa - \ell) &:= \mathbb{E} \Delta_{Ny} y_{t-\ell} \Delta_{Ny} y'_{t-\kappa}, \\ \gamma_\beta(\kappa - \ell) &:= \mathbb{E} \beta' y_{t-\ell} (\beta' y_{t-\kappa})',\end{aligned}\tag{A.25}$$

$$\gamma_{\beta, \Delta_{Ny}}(\kappa - \ell) := \mathbb{E} \beta' y_{t-\ell} \Delta_{Ny} y'_{t-\kappa} = (\mathbb{E} \Delta_{Ny} y_{t-\kappa} (\beta' y_{t-\ell})')' = \gamma_{\Delta_{Ny}, \beta}(\ell - \kappa)', \text{ and}$$

$$\begin{aligned}\Gamma_{rp} &:= \mathbb{E} \underline{x}_{t+1} \underline{x}'_{t+1} \in \mathbb{R}^{m \times m} \\ &= \begin{pmatrix} \gamma_\beta(0) & \gamma_{\beta, \Delta y}(0) & \gamma_{\beta, \Delta y}(1) & \dots & \gamma_{\beta, \Delta y}(p-2) \\ \gamma_{\Delta y, \beta}(0) & \gamma_{\Delta y}(0) & \gamma_{\Delta y}(1) & \dots & \gamma_{\Delta y}(p-2) \\ \gamma_{\Delta y, \beta}(-1) & \gamma_{\Delta y}(-1) & \gamma_{\Delta y}(0) & \dots & \gamma_{\Delta y}(p-3) \\ & & & \ddots & \\ \gamma_{\Delta y, \beta}(-p+2) & \gamma_{\Delta y}(-p+2) & \gamma_{\Delta y}(-p+3) & \dots & \gamma_{\Delta y}(0) \end{pmatrix} \\ &= \begin{pmatrix} \gamma_\beta(0) & \gamma_{\beta, \Delta y}(0) & \gamma_{\Delta y, \beta}(-1)' & \dots & \gamma_{\Delta y, \beta}(p-2)' \\ \gamma_{\Delta y, \beta}(0) & \gamma_{\Delta y}(0) & \gamma_{\Delta y}(-1)' & \dots & \gamma_{\Delta y}(-p+2)' \\ \gamma_{\Delta y, \beta}(-1) & \gamma_{\Delta y}(-1) & \gamma_{\Delta y}(0) & \dots & \gamma_{\Delta y}(-p+3)' \\ & & & \ddots & \\ \gamma_{\Delta y, \beta}(-p+2) & \gamma_{\Delta y}(-p+2) & \gamma_{\Delta y}(-p+3) & \dots & \gamma_{\Delta y}(0) \end{pmatrix},\end{aligned}\tag{A.26}$$

where  $\underline{x}_t$  was defined in (10),(11). The last step follows from the fact that  $(\underline{x}_t)_{t \in \mathbb{Z}}$  is stationary, such that  $\Gamma_{rp}$  has to be symmetric.

Let  $S_\beta := (I_{r \times r}, 0_{r \times n}, \dots, 0) \in \mathbb{R}^{r \times m}$ , and  $S_{\Delta_{Ny}} := (0_{n \times r}, I_n, 0, \dots, 0) \in \mathbb{R}^{n \times m}$ . Then (10) and (11) result in

$$\begin{aligned}\gamma_w(-hN) &:= \mathbb{E} u_{t-hN} u_t' = S_\beta c A^{hN} c^{-1} c \Gamma_{rp} c' S_\beta' = S_\beta A_{b,c}^h c \Gamma_{rp} c' S_\beta', \\ \gamma_{w, \Delta_{Ny}}(-hN) &:= \mathbb{E} u_{t-hN} \Delta_{Ny} y_t' = S_\beta c A^{hN} c^{-1} c \Gamma_{rp} c' S_{\Delta_{Ny}}' = S_\beta A_{b,c}^h c \Gamma_{rp} c' S_{\Delta_{Ny}}', \\ \gamma_{\Delta_{Ny}}(-hN) &= S_{\Delta_{Ny}} c A^{hN} c^{-1} c \Gamma_{rp} c' S_{\Delta_{Ny}}' = S_{\Delta_{Ny}} A_{b,c}^h c \Gamma_{rp} c' S_{\Delta_{Ny}}', \text{ and}\end{aligned}$$

$$\underbrace{\begin{pmatrix} \gamma_w(0) & \gamma_{w, \Delta_{Ny}}(0) \\ \gamma_{w, \Delta_{Ny}}(0) & \gamma_{\Delta_{Ny}}(0) \\ \gamma_{w, \Delta_{Ny}}(N) & \gamma_{\Delta_{Ny}}(N) \\ \vdots & \\ \gamma_{w, \Delta_{Ny}}((np-2)N) & \gamma_{\Delta_{Ny}}((np-2)N) \end{pmatrix}}_{\Gamma_{\beta \Delta_{Ny}}} = \underbrace{\begin{pmatrix} S_\beta \\ S_{\Delta_{Ny}} \\ S_{\Delta_{Ny}} A_{b,c}^N \\ S_{\Delta_{Ny}} A_{b,c}^{2N} \\ \vdots \\ S_{\Delta_{Ny}} A_{b,c}^{N(np-2)} \end{pmatrix}}_{\mathcal{O}_N} c \Gamma_{rp} c' \begin{pmatrix} S_\beta' \\ S_{\Delta_{Ny}}' \end{pmatrix}.$$

Note that  $\mathcal{O}_N A_{b,c}^{-N} = \mathcal{O}$ , where  $\mathcal{O}$  is defined in (A.30). The matrix  $\mathcal{O}$  has full column rank, as will be shown in Lemma 8, such that also  $\mathcal{O}_N$  has full

rank. Thus we obtain the first two column blocks of  $\Gamma_{rp,c}$ . Now looking at the specific structure of

$$\Gamma_{rp,c} = \left( \begin{array}{cc|cccc} \gamma_{u^*}^{(0)} & \gamma_{u^*,\Delta Ny}^{(0)} & \gamma_{u^*,\Delta Ny}^{(1)} & \gamma_{u^*,\Delta Ny}^{(2)} & \cdots & \gamma_{u^*,\Delta Ny}^{(N-1)} \\ \gamma_{\Delta Ny,u^*}^{(0)} & \gamma_{\Delta Ny}^{(0)} & \gamma_{\Delta Ny}^{(1)} & \gamma_{\Delta Ny}^{(2)} & \cdots & \gamma_{\Delta Ny}^{(N-1)} \\ \gamma_{\Delta Ny,u^*}^{(-1)} & \gamma_{\Delta Ny}^{(-1)} & \gamma_{\Delta Ny}^{(0)} & \gamma_{\Delta Ny}^{(1)} & \cdots & \gamma_{\Delta Ny}^{(N-2)} \\ \gamma_{\Delta Ny,u^*}^{(-2)} & \gamma_{\Delta Ny}^{(-2)} & \gamma_{\Delta Ny}^{(-1)} & \gamma_{\Delta Ny}^{(0)} & \cdots & \gamma_{\Delta Ny}^{(N-3)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{\Delta Ny,u^*}^{(-(N-1))} & \gamma_{\Delta Ny}^{(-(N-1))} & \gamma_{\Delta Ny}^{(-(N-2))} & \gamma_{\Delta Ny}^{(-(N-3))} & \cdots & \gamma_{\Delta Ny}^{(0)} \\ \gamma_{\Delta y,u^*}^{(-N)} & \gamma_{\Delta y,\Delta Ny}^{(-N)} & \gamma_{\Delta y,\Delta Ny}^{(-(N-1))} & \gamma_{\Delta y,\Delta Ny}^{(-(N-2))} & \cdots & \gamma_{\Delta y,\Delta Ny}^{(-1)} \end{array} \right) \cdot \left( \begin{array}{cccc} \gamma_{u^*,\Delta y}^{(N)} & \gamma_{u^*,\Delta y}^{(N+1)} & \cdots & \gamma_{u^*,\Delta Ny}^{(p-2)} \\ \gamma_{\Delta Ny,\Delta y}^{(N)} & \gamma_{\Delta Ny,\Delta y}^{(N+1)} & \cdots & \gamma_{\Delta Ny,\Delta Ny}^{(p-2)} \\ \gamma_{\Delta Ny,\Delta y}^{(N-1)} & \gamma_{\Delta Ny,\Delta y}^{(N)} & \cdots & \gamma_{\Delta Ny,\Delta Ny}^{(p-3)} \\ \gamma_{\Delta Ny,\Delta y}^{(N-2)} & \gamma_{\Delta Ny,\Delta y}^{(N-1)} & \cdots & \gamma_{\Delta Ny,\Delta Ny}^{(p-4)} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{\Delta Ny,\Delta y}^{(1)} & \gamma_{\Delta Ny,\Delta y}^{(2)} & \cdots & \gamma_{\Delta Ny,\Delta Ny}^{(p+2-(N-1))} \\ \gamma_{\Delta y}^{(0)} & \gamma_{\Delta y}^{(1)} & \cdots & \gamma_{\Delta y}^{(p+2-N)} \end{array} \right), \quad (\text{A.27})$$

we see the following relations

$$\begin{aligned} \Gamma_{rp,c}^{(2+h)} &= \Gamma_{rp,c}^{(2)}(h) & \text{for } h = 1, \dots, m-2 \\ \Gamma_{rp,c}(h) &= A_c^h \Gamma_{rp,c} & \text{for } h = 1, 2, \dots \\ \Gamma_{rp,c}(2+h) &= A_c^h \Gamma_{rp,c}^{(2)}, \end{aligned} \quad (\text{A.28})$$

where by  $\Gamma_{rp,c}^{(j)}(h)$ , we denote the  $j$ -th column block of  $\Gamma_{rp,c}(h)$ . The first equation follows from the structure of the autocovariances of the states, i.e.  $\Gamma_{rp,c}(h) = \mathbb{E}x_{t+h}x_t'$  for  $h \in \mathbb{N}_0$ , the second equation follows from the Lyapunov equations. Hence, we receive all columns of  $\Gamma_{rp,c}$  by using the recursions in (A.28) and therefore of  $\Gamma_{rp} = c^{-1}\Gamma_{rp,c}c^{-1'}$ . Finally, again by using the Lyapunov equations we have all second moments of  $(\Delta y_t)_{t \in \mathbb{Z}}$  and  $(u_t^S)_{t \in \mathbb{Z}}$ . Now  $\Sigma_\nu$  retained by using the ‘‘high frequency Yule-Walker type equations’’, that is,

$$\begin{aligned} \Delta y_t - \alpha\beta^t y_{t-1} - \Phi_1 \Delta y_{t-1} - \cdots - \Phi_{p-1} \Delta y_{t-p+1} &= \nu_t \\ \Delta y_t \Delta y_t' - \alpha\beta^t y_{t-1} \Delta y_t' - \Phi_1 \Delta y_{t-1} \Delta y_t' - \cdots - \Phi_{p-1} \Delta y_{t-p+1} \Delta y_t' &= \nu_t \Delta y_t' \\ \underbrace{\mathbb{E} \Delta y_t \Delta y_t'}_{\gamma_{\Delta y}^{(0)}} - \alpha \underbrace{\mathbb{E} \beta^t y_{t-1} \Delta y_t'}_{\gamma_{\beta y \Delta y}^{(1)}} - \Phi_1 \underbrace{\mathbb{E} \Delta y_{t-1} \Delta y_t'}_{\gamma_{\Delta y}^{(1)}} - \cdots - \Phi_{p-1} \underbrace{\mathbb{E} \Delta y_{t-p+1} \Delta y_t'}_{\gamma_{\Delta y}^{(p-1)}} &= \underbrace{\mathbb{E} \nu_t \Delta y_t'}_{\Sigma_\nu}. \end{aligned} \quad (\text{A.29})$$

Hence, also generic identifiability of  $\Sigma_\nu$  is established.

Finally we prove continuity of  $\pi^{-1}$ . This involves two steps: 1. The continuity of the mapping from the observed second moments to the parameters of a canonical minimal realisation  $(\bar{A}_{b,c}, \bar{B}_{b,c}, \bar{C}_{b,c})$  (say the echelon form): Recall that the set of transfer functions with McMillan-degree  $m$ , call it

$\tilde{M}(m)$ , can be decomposed in disjoint pieces corresponding to different Kronecker indices summing up to  $m$ . The set of transfer functions where the first  $m$  rows of the Hankel matrix are a basis of the row space of the Hankel matrix is generic in  $\tilde{M}(m)$  (w.r.t. the pointwise topology for  $\tilde{M}(m)$  (see [Hannan and Deistler, 2012](#), p. 65)). This set is also called the “generic neighbourhood”. As has been shown in Step 5 above,  $\Gamma_{r,pc}$  from equation (A.27) has full rank  $m$ . We know that the linear dependencies in the Hankel matrix of the transfer function, say  $\tilde{\mathcal{H}}$ , and the Hankel matrix of the second moments, say  $\tilde{\mathcal{H}}_\gamma$ , are the same (for the definitions see [Anderson et al., 2016a](#)). Now since  $\Gamma_{rp,c}$  is the upper left  $m \times m$  block of  $\tilde{\mathcal{H}}_\gamma$ , we know that the first  $m$  rows of  $\tilde{\mathcal{H}}$  are a basis of the row space of  $\tilde{\mathcal{H}}$ . Therefore  $\Theta_I$  is a subset of the generic neighbourhood.

2. Note that from a given minimal realisation  $(\bar{A}_{b,c}, \bar{B}_{b,c}, \bar{C}_{b,c})$  of  $\theta \in \Theta_I$  all transformations involved in the retrieval algorithm described above are continuous.  $\blacksquare$

### Lemma 8

Suppose that Assumptions 1 and 2 hold. The matrix

$$\mathcal{O} = \begin{pmatrix} S_\beta A_{b,c} \\ S_{\Delta_{Ny}} A_{b,c} \\ S_{\Delta_{Ny}} A_{b,c}^2 \\ \vdots \\ S_{\Delta_{Ny}} A_{b,c}^{n(p-1)} \end{pmatrix} \quad (\text{A.30})$$

is of full column rank  $m = r + n(p - 1)$ .

**Proof.** The proof is very similar to the proof that the observability matrix is of full rank in [Anderson et al. \(2016a\)](#)[Proof of Theorem 7, page 823]. Since the matrix  $c$  is of full rank  $m$  we are allowed to consider  $A^N$  and  $A$ . To see this, let  $\tilde{q}_i$  now denote an eigenvector of  $cAc^{-1}$  with eigenvalue  $\lambda_i$ , then  $(cA^Nc^{-1})\tilde{q}_i = cA^{N-1}c^{-1}cAc^{-1}\tilde{q}_i = \lambda_i cA^{N-1}c^{-1}\tilde{q}_i = \lambda_i^N \tilde{q}_i$ . In addition, if  $q_i$  is an eigenvector of  $A$ , then  $\tilde{q}_i = cq_i$  is an eigenvector of  $A_{b,c}$ . Moreover,  $A_{b,c}^j \tilde{q}_i = cA^{jN}c^{-1}cq_i = \lambda_i^{Nj} cq_i$ . The eigenvalues of  $A$  are such that  $\lambda_i \neq \lambda_j$  implies  $\lambda_i^2 \neq \lambda_j^2$ , the eigenvectors of  $A$  and  $A^2$  coincide. To see this, let  $q_i \in \mathbb{R}^m$  and  $\lambda_i \in \mathbb{R}$  denote an eigenvector and an eigenvalue of the matrix  $A$ . Then,  $Aq_i = \lambda_i q_i$  and  $A^2q_i = AAq_i = \lambda_i Aq_i = \lambda_i^2 q_i$ ; for  $N > 2$  this works in the same way. Therefore it is sufficient to look at the eigenvectors and eigenvalues of the matrix  $A$ . Similar to [Anderson et al. \(2016a\)](#)[Lemma 2]

we have shown in the proof of the above Theorem 3 that the first  $r + n$  components of an eigenvector of  $A$  or  $cAc^{-1}$  are not equal to a vector of zeros. Therefore, by the Popov-Belevitch-Hautus (PBH)-eigenvector test (see, e.g., Kailath, 1980, page 135), the matrix  $\mathcal{O}$  has full *column* rank  $r + n(p - 1)$ . That is,

$$\begin{pmatrix} (A^N - \lambda_i^N I_m) \\ \begin{pmatrix} S_\beta \\ S_{\Delta y} \end{pmatrix} A^N \end{pmatrix} q_i = \begin{pmatrix} 0_{m \times 1} \\ \lambda_i^N [q_i]_{1:n+r} \neq 0_{n+r \times 1} \end{pmatrix}. \quad (\text{A.31})$$

■

## G Proof of Theorem 5

**Proof.** The proof is constructed as follows: For each of the identifiability conditions in Assumption 2, we suppose that **(Ij)** is violated for  $j = 1, \dots, 6$  and show that there exists no “observationally equivalent”  $\theta \in \Theta_I$ .

Suppose **(I1)** or **(I2)** are violated for  $\theta_{-I}$ , then it follows that the McMillan degree of  $\tilde{k}(\tilde{z})$  is less than  $m$ . Hence there exists no  $\theta \in \Theta_I$  with the same auto-covariance function  $\tilde{\gamma}$ , which is granted by  $\tilde{K}(0) = I_{\tilde{n}}$ .

Suppose **(I3)** or **(I4)** are violated, then the minimal realisation of  $\bar{A}_{b,c}$ , which is directly obtained from  $\tilde{\gamma}$  has eigenvalues  $\lambda_i^N = \lambda_j^N$  for some  $i \neq j$ , and thus **(I4)** is violated.

Suppose that neither of the conditions in **(I5)** hold, then by equations (A.13), we have  $C_b q = 0$  and the system is not observable (and therefore of McMillan degree smaller than  $m$ ).

Suppose that condition **(I6)** is not satisfied, then after going through steps Steps 1 and 2 of the retrieval algorithm in the proof of Theorem 4, we obtain in equation (A.16) that  $S_{n_f}^{(1)} q_i = 0$  for some  $i$  and therefore we are outside of  $\Theta_I$  already. ■

## H Proof of Theorem 6

By considering the Granger-Representation-Theorem for the solution on  $\mathbb{Z}$  in Bauer and Wagner (2012) [equation (26)], (see also Johansen, 1995, The-

orem 4.3), we obtain (see also Hansen and Johansen, 1998, Exercise 4.5)

$$\mathbb{E} \Delta y_t = C(\mu_0 + \mu_1 t) + M_c \mu_1, \quad (\text{A.32})$$

$$\text{with } C := \beta_{\perp} \left( \alpha'_{\perp} \left( I_n - \sum_{j=1}^{p-1} \Phi_j \right) \beta_{\perp} \right)^{-1} \alpha'_{\perp}, \quad (\text{A.33})$$

and  $M_c$  is the limit of the stable part of the impulse responses in the particular solution defined in Bauer and Wagner (2012) [equation (8)] (this is  $k_{\bullet}(1)$  in Bauer and Wagner (2012) equation (26)).

Let  $\bar{\alpha}' = (\alpha' \alpha)^{-1} \alpha'$ , then

$$\begin{aligned} \mathbb{E} \beta' y_t = \bar{\alpha}' & \left[ C(\mu_0 + \mu_1 t) + M_c \mu_1 - \mu_0 - \mu_1 t, \right. \\ & \left. - \sum_{j=1}^{p-1} \Phi_j (C(\mu_0 + \mu_1(t-j)) + M_c \mu_1) \right]. \quad (\text{A.34}) \end{aligned}$$

For  $N = 1$ , both moments in equations (A.32) and (A.34) are observable (or consistently estimable from observed data) for the stock as well as for the flow case. In addition, for  $N > 1$  we get

$$\begin{aligned} \mathbb{E} \beta' y_{t-\ell} = \bar{\alpha}' & \left[ C(\mu_0 + \mu_1(t-\ell)) + M_c \mu_1 - \mu_0 - \mu_1(t-\ell) \right. \\ & \left. - \sum_{j=1}^{p-1} \Phi_j (C(\mu_0 + \mu_1(t-j)) + M_c \mu_1) \right] \text{ for the stock case, and} \end{aligned} \quad (\text{A.35})$$

$$\mathbb{E} \beta' w_t = \mathbb{E} \sum_{\ell=0}^{N-1} \beta' y_{t-\ell}, \quad \text{for the flow case.} \quad (\text{A.36})$$

$$\begin{aligned}
\mathbb{E} \Delta_N y_t &= \sum_{j=0}^{N-1} (C(\mu_0 + \mu_1(t-j)) + M_c \mu_1) \\
&= NC\mu_0 + C\mu_1 \left( \sum_{j=0}^{N-1} (t-j) \right) + NCM_c\mu_1, \quad \text{for the stock case, and}
\end{aligned} \tag{A.37}$$

$$\begin{aligned}
\mathbb{E} \Delta_N^\Sigma y_t &= \sum_{\ell=0}^{N-1} \mathbb{E} \Delta_N y_{t-\ell} = \sum_{\ell=0}^{N-1} \sum_{j=\ell}^{\ell+N-1} (C(\mu_0 + \mu_1(t-j)) + M_c \mu_1) \\
&= N^2 C \mu_0 + C \mu_1 \left( \sum_{\ell=0}^{N-1} \sum_{j=\ell}^{\ell+N-1} (t-j) \right) + N^2 M_c \mu_1, \quad \text{for the flow case.}
\end{aligned} \tag{A.38}$$

As already stated in Section 4, we first remove the deterministic trends from  $\mathbb{E} \beta' y_t$  and  $\mathbb{E} \Delta_N y_t$  for the stock case or  $\mathbb{E} \beta' w_t = \mathbb{E} \sum_{j=0}^{N-1} \beta' y_{t-j}$  and  $\mathbb{E} \Delta_N^\Sigma y_t = \mathbb{E} \sum_{j=0}^{N-1} \Delta_N y_{t-j}$  for the flow case.  $\beta$  follows from mixed frequency data as demonstrated in Chambers (2020). Then the parameters of the model without deterministic terms are generically identified as shown in the main text. For the five cases arising from model (21) we get:

**Case  $H_2(r)$ :** ( $\mu_0 = \mu_1 = 0$ ) No deterministic terms. This case was shown above.

**Case  $H_1(r)$ :** ( $\mu_1 = 0$  and  $\mu_0 \neq 0$ .) We have a linear trend in  $\mathbb{E} y_t$ , and a constant in  $\mathbb{E} \Delta y_t$ , and a constant in  $\mathbb{E} \beta' y_t$ . Let the matrix  $S_C$  select  $n-r$  basis rows of  $C$  (e.g.,  $S_C = \alpha'_\perp$  can be used). In this case it holds that

$$\begin{aligned}
\mathbb{E} \begin{pmatrix} \beta' y_t \\ S_C \Delta_N y_t \end{pmatrix} &= \begin{pmatrix} \bar{\alpha}'((I_n - \sum_{j=1}^{p-1} \Phi_j)C - I_n) \\ N S_C C \end{pmatrix} \mu_0, \quad \text{for the stock case, and} \\
\mathbb{E} \begin{pmatrix} \beta' w_t \\ S_C \Delta_N^\Sigma y_t \end{pmatrix} &= \begin{pmatrix} N \bar{\alpha}'((I_n - \sum_{j=1}^{p-1} \Phi_j)C - I_n) \\ N^2 S_C C \end{pmatrix} \mu_0, \quad \text{for the flow case,}
\end{aligned}$$

where the matrix on the LHS before  $\mu_0$  is of rank  $n$  since the first rowblock has rank  $r$  and  $C$  has rank  $n-r$  and both are mutually orthogonal as  $\alpha'_\perp \alpha = 0$  and therefore (with  $\alpha'_\perp$  the last term of  $C$ )

$$\alpha'_\perp \left( (I_n - \sum_{j=1}^{p-1} \Phi_j) C - I_n \right)' \bar{\alpha} = 0.$$

From this we can compute  $\mu_0$ .

**Case  $H_1^*(r)$ :** ( $\mu_1 = 0$ ,  $\mu_0 \neq 0$  and  $\alpha'_\perp \mu_0 = 0$ ) no linear trend but constant in  $\mathbb{E} y_t$ , constant in  $\mathbb{E} \beta' y_{t-1}$ , and  $\mathbb{E} \Delta y_t = 0$ . This results in  $r$  parameters in the cointegration equation. In this case we write  $\mu_t = \alpha \rho_0$ , where  $\rho_0 \in \mathbb{R}^r$ . Note that  $C\mu_t = C\alpha\rho_0 = 0$ , which results in

$$\begin{aligned} \mathbb{E} \begin{pmatrix} \beta' y_t \\ \Delta_N y_t \end{pmatrix} &= \begin{pmatrix} \bar{\alpha}'((I_n - \sum_{j=1}^{p-1} \Phi_j)C - I_n) \\ NC \end{pmatrix} \alpha \rho_0 \\ &= \begin{pmatrix} -\bar{\alpha}' \alpha \rho_0 \\ N0 \end{pmatrix} = \begin{pmatrix} -\rho_0 \\ 0 \end{pmatrix}, \quad \text{for the stock case.} \end{aligned}$$

For the flow case we get

$$\begin{aligned} \mathbb{E} \begin{pmatrix} \beta' w_t \\ \Delta_N^\Sigma y_t \end{pmatrix} &= \begin{pmatrix} N\bar{\alpha}'((I_n - \sum_{j=1}^{p-1} \Phi_j)C - I_n) \\ N^2 C \end{pmatrix} \alpha \rho_0 \\ &= \begin{pmatrix} -N\bar{\alpha}' \alpha \rho_0 \\ N^2 0 \end{pmatrix} = \begin{pmatrix} -N\rho_0 \\ 0 \end{pmatrix}. \end{aligned}$$

This allows to uniquely retrieve the unknown parameter  $\rho_0$  from  $\mathbb{E} \beta' y_{t-1}$ .

**Case  $H(r)$ :** ( $\mu_0 \in \mathbb{R}^n$  and  $\mu_1 \neq 0$ ) quadratic trend in  $\mathbb{E} y_t$ , linear trend in  $\mathbb{E} \beta' y_t$ , linear trend in  $\mathbb{E} \Delta y_t$ . For this the stock case we get the system of linear equations

$$\mathbb{E} \begin{pmatrix} \beta' y_t \\ S_C \Delta_N^\Sigma y_t \\ \beta' y_{t-N} \\ S_C \Delta_N^\Sigma y_{t-N} \end{pmatrix} = M_\mu \begin{pmatrix} \mu_0 \\ \mu_1 \end{pmatrix},$$

where

$$\begin{aligned} M_\mu^S &:= \begin{pmatrix} \bar{\alpha}'(C - I_n - \sum_{j=1}^{p-1} C\Phi_j) & | & \bar{\alpha}'[Ct + M_c - tI_n - \sum_{j=1}^{p-1} \Phi_j(C(t-j) - M_c)] \\ NS_C C & | & S_C C (\sum_{\ell=0}^{N-1} (t-\ell) + NM_c) \\ \bar{\alpha}'(C - I_n - \sum_{j=1}^{p-1} C\Phi_j) & | & \bar{\alpha}'[C(t-N) + M_c - (t-N)I_n - \sum_{j=1}^{p-1} \Phi_j(C(t-N-j) - M_c)] \\ NS_C C & | & S_C C (\sum_{\ell=0}^{N-1} (t-N-\ell) + NM_c) \end{pmatrix} \\ &= \begin{pmatrix} M_{\mu 11}^S & M_{\mu 12}^S \\ M_{\mu 21}^S & M_{\mu 22}^S \end{pmatrix}. \end{aligned}$$

For the flow case we have

$$\mathbb{E} \begin{pmatrix} \beta' w_t \\ S_C \Delta_N^\Sigma y_t \\ \beta' w_{t-N} \\ S_C \Delta_N^\Sigma y_{t-N} \end{pmatrix} = M_\mu \begin{pmatrix} \mu_0 \\ \mu_1 \end{pmatrix},$$

where

$$M_{\mu}^{\mathcal{F}} := \left( \begin{array}{c|c} \frac{N\bar{\alpha}'(C - I_n - \sum_{j=1}^{p-1} C\Phi_j)}{N^2 S_C C} & \\ \hline \frac{N\bar{\alpha}'(C - I_n - \sum_{j=1}^{p-1} C\Phi_j)}{N^2 S_C C} & \end{array} \right) \frac{\alpha' \left[ C \sum_{\ell=0}^{N-1} (t - \ell) + NM_c - \left( \sum_{\ell=0}^{N-1} (t - \ell) \right) I_n - \sum_{j=1}^{p-1} \Phi_j \left( C \sum_{\ell=0}^{N-1} (t - \ell) - NM_c \right) \right]}{S_C \left( \left( \sum_{\ell=0}^{N-1} \sum_{j=0}^{N-1} (t - j - \ell) \right) C + N^2 M_c \right)} \frac{\bar{\alpha}' \left[ C \sum_{\ell=0}^{N-1} (t - N - \ell) + NM_c - \sum_{\ell=0}^{N-1} (t - \ell - N) I_n - \sum_{j=1}^{p-1} \Phi_j \left( C \left( \sum_{\ell=0}^{N-1} (t - N - j - \ell) \right) - NM_c \right) \right]}{S_C \left( \left( \sum_{\ell=0}^{N-1} \sum_{j=0}^{N-1} (t - N - \ell - j) \right) C + N^2 M_c \right)} \right)}{\left( \begin{array}{cc} M_{\mu 11}^{\mathcal{F}} & M_{\mu 12}^{\mathcal{F}} \\ M_{\mu 21}^{\mathcal{F}} & M_{\mu 22}^{\mathcal{F}} \end{array} \right)}.$$

The matrix  $S_C$  selects  $n - r$  basis rows of  $C$  (e.g.,  $S_C = \alpha'_{\perp}$  can be used) and  $M_{\mu 11}^{\mathcal{F}} = M_{\mu 21}^{\mathcal{F}}$ . The matrix  $M_{\mu 11}^{\mathcal{F}}$  is of full column rank  $n$  by the above calculations where we set  $\mu_1 = 0$ . The determinant of a blocked matrix is given by

$$\det M_{\mu} = \det M_{\mu 11} \det \left( M_{\mu 22} - \underbrace{M_{\mu 21} M_{\mu 11}^{-1}}_{I_n} M_{\mu 12} \right) = \det M_{\mu 11} \det (M_{\mu 22} - M_{\mu 12}).$$

Note that  $M_{\mu 22}^{\mathcal{S}} - M_{\mu 12}^{\mathcal{S}} = \left( \frac{N\bar{\alpha}'(C - I_n - \sum_{j=1}^{p-1} C\Phi_j)}{N^2 S_C C} \right) = NM_{\mu 11}^{\mathcal{S}}$  for the stock case, while for the flow case  $M_{\mu 22}^{\mathcal{F}} - M_{\mu 12}^{\mathcal{F}} = \left( \frac{N^2 \bar{\alpha}'(C - I_n - \sum_{j=1}^{p-1} C\Phi_j)}{N^3 S_C C} \right) = NM_{\mu 11}^{\mathcal{F}}$ . Therefore,  $\det M_{\mu} \neq 0$ . This allows to uniquely solve for  $\mu_0$  and  $\mu_1$ .

**Case  $H^*(r)$ :** ( $\mu_0 \in \mathbb{R}^n$ ,  $\mu_1 \neq 0$ , with  $\alpha'_{\perp} \mu_1 = 0$ ) linear trend in  $\mathbb{E}y_t$ , linear trend in  $\mathbb{E}\beta' y_t$ , and constant in  $\mathbb{E}\Delta y_t$ .  $\mu_0$  contains  $n$  free parameters, while  $\mu_t = \mu_0 + \alpha \rho_1 t$ , where  $\rho_1 \in \mathbb{R}^r$ . Hence,  $C\mu_t = C\mu_0 + C\alpha \rho_1 t = C\mu_0$ . This results in

$$\begin{aligned} \mathbb{E} \begin{pmatrix} \beta' y_t \\ S_C \Delta_N y_t \\ \beta' y_{t-N} \end{pmatrix} &= [M_{\mu}^{\mathcal{S}}]_{1:r+n, 1:2(r+m)} \begin{pmatrix} \mu_0 \\ \alpha \rho_1 \end{pmatrix} \\ &= \underbrace{[M_{\mu}^{\mathcal{S}}]_{1:r+n, 1:2(r+n)}}_{M_{\mu, H_1^*(r)}^{\mathcal{S}}} \begin{pmatrix} I_n & 0_{n \times r} \\ 0_{n \times n} & \alpha \end{pmatrix} \begin{pmatrix} \mu_0 \\ \rho_1 \end{pmatrix}. \end{aligned}$$

Since  $M_\mu^{\mathcal{F}}$  has full rank  $2n$ , the matrix  $M_{\mu H_1^*(r)}^{\mathcal{F}}$  has rank  $r + n$ . This allows to uniquely obtain  $\mu_0$  and  $\rho_1$ . For the flow case this works equivalently.