

MATRIX PRODUCT OPERATOR ALGEBRAS II: PHASES OF MATTER FOR 1D MIXED STATES

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ABSTRACT. The classification of topological phases of matter is fundamental to understand and characterize the properties of quantum materials. In this paper we study phases of matter in one-dimensional open quantum systems. We define two mixed states to be in the same phase if both states can be transformed into the other by a shallow circuit of local quantum channels. We aim to understand the emerging phase diagram of matrix product density operators that are renormalization fixed points. These states arise, for example, as boundaries of two-dimensional topologically ordered states. We first construct families of such states based on C^* -weak Hopf algebras, the algebras whose representations form a fusion category. More concretely, we provide explicit local fine-graining and local coarse-graining quantum channels for the renormalization procedure of these states. Finally, we prove that a subset of these states, those arising from C^* -Hopf algebras, are in the trivial phase.

1. INTRODUCTION

One of the main projects that quantum science is undertaking in the last decades is the understanding and classification of exotic topological phases of quantum matter. The approach to tackle this project is intrinsically connected to quantum information theory. On the one hand, topological phases of matter have been identified as valuable resources in quantum computing [21]. On the other, quantum information tools and ideas are playing a key role in the classification program.

Before going any further, it is important to define what it means that two systems belong to the same topological phase. Since topological properties have an inherent global nature, the key idea is that their ground states display similar global properties independently of their (possibly) different local features. For instance, a ferromagnetic state $|\uparrow\uparrow \cdots \uparrow\uparrow\rangle$ is topologically equivalent to an antiferromagnetic one $|\uparrow\downarrow \cdots \uparrow\downarrow\rangle$ since one can map locally one into the other, despite the fact that they have a very different magnetization behaviour.

A definition, motivated by quantum information, which tries to capture the global properties, is the existence of a short-depth geometrically local quantum

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circuit mapping one ground state into the other [11, 8]. Using Hastings-Wen’s quasi-adiabatic evolution [23] and Lieb-Robinson bounds [31] one can prove that this property is implied by the more standard definition of phase based on the existence of a gapped path of Hamiltonians connecting both systems [2].

The main advantage of the definition based on quantum circuits is that it focuses on states rather than on Hamiltonians, which is crucial to extend it to more general setups, like the one we are addressing here: open quantum systems. However, this approach poses an additional problem: one has to identify the relevant class of states to classify. For closed quantum systems this relevant class is precisely the set of ground states of gapped short-range Hamiltonians. Again quantum information theory provided us with a characterization of this set: ground states of short-ranged gapped Hamiltonians fulfill an area law for the entanglement between neighbouring regions, which implies that they are well approximated by “tensor network states”, in particular by matrix product states (MPS) and projected-entangled pair states (PEPS) [24, 1, 15].

A natural approach to classify phases is to first restrict the classification to “simple” states that nevertheless are representatives for each phase. Since topological properties are global, these representatives are taken to be insensitive to real space renormalization steps (being those a finite depth circuit), that is, they are renormalization fixed points (RFP). In 2D, for instance, the string-net models of Levin and Wen [29] are believed to provide a complete set of renormalization fixed points for non-chiral 2D topological phases.

The restriction to RFPs has two important benefits. On the one hand, RFPs in gapped phases have zero correlation length and thus they are *exactly* MPS and PEPS [15]; no approximation is needed. On the other hand, it is easier to identify the key global invariants and thus identify the different phases of RFP states.

These two points have been the crucial insights to successfully complete the classification of 1D phases with symmetries, the so-called symmetry protected topological (SPT) phases. Let us illustrate that this is the case by recalling the steps that led to the classification of 1D SPT phases. The first step was to prove that any MPS can be transformed into an RFP MPS in the same phase [42]. This restricts the classification problem to just RFP MPS. The second step was to identify the invariants of the phases using the set of RFP MPS. These invariants are a set of quantities which, on the one hand, are robust against short depth circuits and, on the other, are sufficient to identify each phase uniquely. For SPT phases with unique ground state, the invariants are the different equivalence classes of the second cohomology group of the symmetry group [12, 42]. For SPT phases with symmetry breaking and therefore degenerate ground states, the invariants are the different induced representations of the non-symmetry broken subgroup together with its second cohomology group [42]. The third step was to prove that any two RFP MPS that share the same invariants can be mapped into each other with a short depth quantum circuit. On top of that, a final and important step has been recently made: the breakthrough results of Ogata [37] show that one can even extend these arguments beyond the framework of MPS to cover all gapped short-range Hamiltonians.

All the previous results stand for closed quantum systems, where the object of interest is the ground state of a Hamiltonian. However, the question of classifying phases is far from being answered for open quantum systems, even in one dimension. Since isolation is never practically achieved, the characterization of those systems play a fundamental role in real applications.

In this manuscript, we take the first steps towards the classification of open quantum systems in 1D. A main difference between open and closed quantum systems is that evolutions in closed quantum systems (either Hamiltonian evolution or quantum circuits) are *reversible*, whereas this is no longer true in open quantum systems evolved under a Lindblad master equation. For instance, if one starts in a topologically ordered state, like the toric code [27], one cannot find a short depth quantum circuit mapping it into a product state. Short depth quantum circuits cannot create *or destroy* global correlations. However, local depolarizing noise can convert the toric code (and indeed any topologically ordered state, no matter how complex) into a product state in a short amount of time [16]. Destroying global correlations is therefore *easy* in the open quantum systems regime. Constructing global correlations is, on the other hand, still hard. In fact, local fast dissipative evolutions cannot create global correlations [28]. This shows that in the open quantum setting, phases should not be thought of as classes of an equivalence relation, but rather as a partial order given by the existence of a local fast dissipative evolution mapping one state into another one. This partial order can also be understood as the complexity present in the different topological phases. This proposal, due to [16], is the one we are taking here. Concretely, we will say that a mixed state ρ_1 is more complex than another one ρ_2 if there is a short-depth (geometrically local) circuit of quantum channels, i.e. completely positive trace-preserving linear maps, mapping ρ_1 into ρ_2 .

There are several subtleties to make this definition formal. First of all, ρ_1 and ρ_2 should be well defined for all system size n . Second, one should ask only for getting sufficiently close to ρ_2 , allowing for both $\text{polylog}(n)$ depth and $\text{polylog}(n)$ locality in the gates of the circuit. Finally, one could take either a discrete point of view, as here, or a continuous one, asking for a rapid mixing quasi-local Lindbladian evolution that approximates ρ_2 starting from ρ_1 . Since in this paper we are working only with RFP states, we will not need any of those subtleties here and we refer to [16] for a detailed analysis of those.

We notice that there are other definitions of phases in the open quantum system setting, like the works of Diehl et al. for Gaussian mixed states [17, 3, 4] and for quasi thermal states [22], where the authors generalize the notion of phases via gapped paths of Hamiltonians or via local unitary transformations respectively. We refer also to [16] for a detailed discussion about why the definition we are taking here seems more appropriate.

Encouraged by the successful classification of pure states sketched above, we will focus on RFP that are *gapped* mixed states, that is, mixed states which fulfill an area law for the mutual information. This is motivated by two facts. On the one hand, it is known that Gibbs states of short-range Hamiltonians fulfill an area law for the mutual information [44]. On the other hand, it is known that fixed points of rapidly mixing dissipative evolutions also fulfill an area law for the mutual information [9].

This naturally leads us to the set of RFP mixed states with a matrix product density operator (MPDO) representation. The structure of RFP MPDOs has been studied in detail in [14] where, up to minor technical conditions, the following is shown: (i) An MPDO is an RFP if there exist two quantum channels \mathfrak{T} and \mathfrak{S} that implement the local coarse graining and the local fine graining respectively, for which the given MPDO is a fixed point. (ii) The RFP condition for MPDOs is characterized operationally by the absence of length scales in the system; in particular by having zero correlation length and saturation of the area law. (iii) The existence of such \mathfrak{T} and \mathfrak{S} maps is equivalent to the fact that from the MPDO an MPO algebra can be constructed.

This result brings the classification of 1D mixed states into the understanding and classification of MPO algebras. Notably, MPO algebras are precisely the mathematical objects behind the classification of RFP 2D topologically ordered pure states in terms of PEPS [41, 10]. This is not a lucky coincidence, but a consequence of the remarkable bulk-boundary correspondence originated in the work of Li and Haldane [30]. In PEPS the bulk-boundary mapping is very explicit [13] and allows one to establish a dictionary between bulk and boundary properties [43, 26, 38]. Indeed, RFP MPDOs are expected to contain the set of boundary states associated to RFP 2D non-chiral topologically ordered systems [14].

A throughout study of MPO algebras is done in the first paper of this series [33]. There, it is shown that MPO algebras are closely related to representations of semisimple finite-dimensional weak Hopf algebras, which are, in turn, the algebraic description of fusion categories.

The paper is structured as follows. In Section 2 we recall the basic notions and results on weak Hopf algebras with a compatible C^* -structure, called C^* -weak Hopf algebras. In this setting, we introduce the canonical regular element, which is fundamental for our constructions. We also introduce the notion of biconnected C^* -WHA, whose representation categories are fusion categories. Moreover, we recall their characterization as matrix product operators with a boundary, as introduced in [33]. In Section 3 we recall the definition of RFP MPDOs given in [14] and provide the construction of a family of RFP MPDOs arising from any given biconnected C^* -weak Hopf algebra. In particular, we provide explicit constructions of the local coarse-graining and local fine-graining quantum channels \mathfrak{T} and \mathfrak{S} commented before. In Section 4 we describe the previous RFP MPDOs as the boundary states of topological 2D PEPS. In Section 5 we prove that the previous families of RFP MPDOs are in the trivial phase in the C^* -Hopf algebra case, in the sense that they can be obtained via a finite-depth and bounded-range circuit of quantum channels acting on the maximally mixed state. Moreover, we show that this result can be extended to the trivial sector of any biconnected C^* -weak Hopf algebra.

2. PRELIMINARIES

In this section we collect elementary notions on algebras, coalgebras and C^* -weak Hopf algebras, as well as their representation theory in terms of matrix product operators, recently developed in [33]. From now on, we assume that all vector spaces are finite dimensional and their ground field is the field of complex numbers \mathbb{C} . For any two vector spaces, we denote by $\mathfrak{L}(V, W)$ the set of \mathbb{C} -linear maps from V to W and let $\mathfrak{L}(V) := \mathfrak{L}(V, V)$. We denote by $V^* := \mathfrak{L}(V, \mathbb{C})$ the dual vector space and by $\langle \cdot, \cdot \rangle : V^* \times V \rightarrow \mathbb{C}$, $(f, x) \mapsto \langle f, x \rangle = f(x)$ the canonical pairing. An associative unital *algebra* is a vector space A endowed with an associative linear map $A \otimes A \rightarrow A$, called multiplication, denoted by juxtaposition, and an element $1 \in A$, called unit, satisfying $1x = x1 = x$ for all elements $x \in A$. A unital C^* -*algebra* is an algebra A with an anti-linear involutive algebra anti-homomorphism $(\cdot)^* : A \rightarrow A$, $x \mapsto x^*$, called **-operation*, and a compatible Banach space structure. In this context, positive elements of A are elements of the form $x = y^*y$ for some element $y \in A$. As usually, the multiplication, the unit element and the **-operation* of two C^* -algebras A and B are implicitly extended to their tensor product space $A \otimes B$ componentwise. Dually to the notion of algebra, a coassociative counital *coalgebra* is a vector space C endowed with a linear map $\Delta \in \mathfrak{L}(C, C \otimes C)$, called *comultiplication*, such that

$$(1) \quad (\text{Id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{Id}) \circ \Delta,$$

and a linear functional $\varepsilon \in C^*$, known as *counit*, compatible with the comultiplication in the sense that $(\text{Id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \text{Id}) \circ \Delta = \text{Id}$, where we have identified $C \otimes C \cong \mathbb{C} \otimes C \cong C$. Henceforth, we drop the words associative, unital, coassociative and counital. As usually done in the literature of coalgebras, we denote

$$(2) \quad \Delta^1 := \Delta \quad \text{and} \quad \Delta^{(n+1)} := (\Delta \otimes \text{Id}^{\otimes n}) \circ \Delta^{(n)}$$

for all $n \in \mathbb{N}$ and, complementarily, make use of Sweedler's notation

$$(3) \quad x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(n+1)} := \Delta^{(n)}(x)$$

for all $x \in C$ and all $n \in \mathbb{N}$, omitting the summation symbol and cumbersome indices. In this context, an element $x \in C$ is *cocentral* if

$$(4) \quad x_{(1)} \otimes x_{(2)} = x_{(2)} \otimes x_{(1)}.$$

i.e., its coproduct $\Delta(x) \in C \otimes C$ is invariant under the flip operator. In addition, an element $x \in C$ is *non-degenerate* if for all $y \in C$ there exist $\phi, \psi \in C^*$ such that

$$(5) \quad \langle \phi, x_{(1)} \rangle x_{(2)} = y = x_{(1)} \langle \psi, x_{(2)} \rangle,$$

roughly speaking, any element can be recovered from the coproduct $\Delta(x) \in C \otimes C$ by applying an appropriate linear functional on any of the cofactors.

In order to describe a sufficiently large family of renormalization fixed point mixed states, e.g. boundary states of 2D string-net models, we will introduce an algebraic construction that combines both structures of a C^* -algebra and a coalgebra. This is well-motivated from a representation theoretical point of view [36, 6] and may look odd from a purely algebraic one; thus, the subsequent axioms can be skipped in a first reading. The following definition is due to G. Böhm and K. Szlachányi.

Definition 2.1 (see [5, 36]). A *C^* -weak Hopf algebra* (C^* -WHA) is a vector space A simultaneously endowed with the structures of a C^* -algebra and a coalgebra for which the comultiplication is multiplicative, i.e.

$$(6) \quad (xy)_{(1)} \otimes (xy)_{(2)} = x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}$$

for all elements $x, y \in A$; the $*$ -operation $*$: $A \rightarrow A$ is comultiplicative, i.e.

$$(7) \quad (x^*)_{(1)} \otimes (x^*)_{(2)} = (x_{(1)})^* \otimes (x_{(2)})^*$$

for all elements $x \in A$; the counit $\varepsilon \in A^*$ is weakly comultiplicative, i.e.

$$(8) \quad \langle \varepsilon, xyz \rangle = \langle \varepsilon, xy_{(1)} \rangle \langle \varepsilon, y_{(2)}z \rangle = \langle \varepsilon, xy_{(2)} \rangle \langle \varepsilon, y_{(1)}z \rangle$$

for all elements $x, y, z \in A$; the unit $1 \in A$ is weakly comultiplicative, i.e.

$$(9) \quad 1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} = 1_{(1)} \otimes 1_{(2)}1_{(1')} \otimes 1_{(2')} = 1_{(1)} \otimes 1_{(1')}1_{(2)} \otimes 1_{(2')},$$

where the prime symbol distinguishes different coproducts of $1 \in A$, and there exists an anti-multiplicative linear map $S \in \mathfrak{L}(A)$ satisfying

$$(10) \quad S(x_{(1)})x_{(2)} = \langle \varepsilon, 1_{(1)}x \rangle 1_{(2)} \quad \text{and} \quad x_{(1)}S(x_{(2)}) = 1_{(1)} \langle \varepsilon, x1_{(2)} \rangle$$

for all elements $x \in A$, called antipode.

Remark 2.2 (see e.g. Subsection 2.1 in [6]). The previous axioms are self-dual in the sense that for any C^* -WHA A its dual vector space A^* can be canonically endowed with the structure of a C^* -WHA. For simplicity, let us denote all structure maps in the same way. First, the product of any two $\phi, \psi \in A^*$ is defined by the expression $\phi\psi := (\phi \otimes \psi) \circ \Delta$, the unit element of A^* is the counit $\varepsilon \in A^*$ of A and the $*$ -operation is given by $\langle \phi^*, x \rangle := \overline{\langle \phi, S(x)^* \rangle}$ for all $\phi \in A^*$ and all elements $x \in A$, where the bar denotes the complex conjugate. The coalgebra structure is given via the comultiplication $\langle \Delta(\phi), x \otimes y \rangle := \langle \phi, xy \rangle$ for all $\phi \in A^*$ and all

elements $x, y \in A$, and the counit is the map $A^* \rightarrow \mathbb{C}, \phi \mapsto \langle \phi, 1 \rangle$. Finally, the antipode is defined by $S(\phi) := \phi \circ S$ for all $\phi \in A^*$.

Remark 2.3 (see Lemma 2.8 and Theorem 2.10 in [6]). In any C^* -WHA the antipode is unique, invertible, both an algebra and a coalgebra anti-homomorphism, and satisfies $S(x^*)^* = S^{-1}(x)$ for all elements $x \in A$.

Definition 2.4. A C^* -Hopf algebra (C^* -HA) A is a C^* -WHA for which the comultiplication $\Delta \in \mathfrak{L}(A, A \otimes A)$ is unit-preserving, i.e. $\Delta(1) = 1 \otimes 1$. Equivalently, the counit $\varepsilon \in A^*$ is an algebra homomorphism, i.e. $\langle \varepsilon, xy \rangle = \langle \varepsilon, x \rangle \langle \varepsilon, y \rangle$ for all elements $x, y \in A$ and $\langle \varepsilon, 1 \rangle = 1$.

Example 2.5. The group C^* -algebra $\mathbb{C}G$ of a finite group G is endowed with the structure of a C^* -HA by the linear extensions of the maps given by the expressions $\Delta(g) := g \otimes g$, $\langle \varepsilon, g \rangle := 1$ and $S(g) := g^* := g^{-1}$ for all elements $g \in G$.

Example 2.6. The dual vector space $(\mathbb{C}G)^*$ is again a C^* -algebra endowed with the multiplication $\langle \phi\psi, g \rangle := \langle \phi, g \rangle \langle \psi, g \rangle$, the unit element $g \mapsto 1$ and the $*$ -operation given by $\langle \phi^*, g \rangle := \overline{\langle \phi, g \rangle}$, for all $\phi, \psi \in (\mathbb{C}G)^*$ and all elements $g \in G$. Moreover, it becomes a C^* -HA too by virtue of the comultiplication $\langle \Delta(\phi), g \otimes h \rangle := \langle \phi, gh \rangle$, the counit $\langle \varepsilon, \phi \rangle := \langle \phi, 1 \rangle$, and the antipode $\langle S(\phi), g \rangle := \langle \phi, g^{-1} \rangle$ for all $\phi \in (\mathbb{C}G)^*$ and all elements $g, h \in G$.

The following example, due to G. I. Kac and V. G. Paljutkin, describes the smallest C^* -HA which is neither cocommutative, i.e. a group algebra, nor commutative, i.e. the dual of a group algebra.

Example 2.7 (see [25]). Let H_8 be the C^* -algebra generated by three elements x, y and z subject to the relations $x^2 = 1, y^2 = 1, z^2 = 2^{-1}(1 + x + y - xy), xy = yx, zx = yz, zy = xz, x^* = x, y^* = y$ and $z^* = z^{-1}$. It becomes a C^* -HA by means of $\Delta(x) := x \otimes x, \Delta(y) := y \otimes y, \Delta(z) := 2^{-1}(z \otimes z + yz \otimes z + z \otimes xz - yz \otimes xz), \langle \varepsilon, x \rangle = \langle \varepsilon, y \rangle = \langle \varepsilon, z \rangle = 1, S(x) := x, S(y) := y$ and $S(z) := z$.

The following example is the smallest proper C^* -WHA. It is known as the Lee-Yang C^* -WHA as it is reconstructed from the solutions of the pentagon equation arising from the Lee-Yang fusion rules.

Example 2.8 (cf. [5]). Let A_{LY} be the direct sum $\mathfrak{M}(2, \mathbb{C}) \oplus \mathfrak{M}(3, \mathbb{C})$ of full-matrix 2×2 and 3×3 C^* -algebras with complex coefficients, respectively. Let $\zeta \in \mathbb{R}$ be the unique positive solution to $z^4 + z^2 - 1 = 0$ and fix matrix units $e_1^{ij}, i, j = 1, 2$, in $\mathfrak{M}(2, \mathbb{C})$ and $e_2^{k\ell}, k, \ell = 1, 2, 3$, in $\mathfrak{M}(3, \mathbb{C})$. Then, the comultiplication of A_{LY} is defined by the expressions

$$\begin{aligned} \Delta(e_1^{11}) &:= e_1^{11} \otimes e_1^{11} + e_2^{11} \otimes e_2^{22}, \\ \Delta(e_1^{12}) &:= e_1^{12} \otimes e_1^{12} + \zeta^2 e_2^{12} \otimes e_2^{21} + \zeta e_2^{13} \otimes e_2^{23}, \\ \Delta(e_1^{22}) &:= e_1^{22} \otimes e_1^{22} + \zeta^4 e_2^{22} \otimes e_2^{11} + \\ &\quad \zeta^3 e_2^{23} \otimes e_2^{13} + \zeta^3 e_2^{32} \otimes e_2^{31} + \zeta^2 e_2^{33} \otimes e_2^{33}, \\ \Delta(e_2^{11}) &:= e_1^{11} \otimes e_2^{11} + e_2^{11} \otimes e_1^{22} + e_2^{11} \otimes e_2^{33}, \\ \Delta(e_2^{12}) &:= e_1^{12} \otimes e_2^{12} + e_2^{12} \otimes e_1^{21} + e_2^{13} \otimes e_2^{32}, \\ \Delta(e_2^{13}) &:= e_1^{12} \otimes e_2^{13} + e_2^{11} \otimes e_1^{22} + \zeta e_2^{12} \otimes e_2^{31} - \zeta^2 e_2^{13} \otimes e_2^{33}, \\ \Delta(e_2^{22}) &:= e_0^{22} \otimes e_2^{22} + e_2^{22} \otimes e_0^{11} + e_2^{33} \otimes e_2^{22}, \\ \Delta(e_2^{23}) &:= e_1^{22} \otimes e_2^{23} + e_2^{23} \otimes e_1^{21} + \zeta e_2^{32} \otimes e_2^{21} - \zeta^2 e_2^{33} \otimes e_2^{23}, \\ \Delta(e_2^{33}) &:= e_1^{22} \otimes e_2^{33} + e_2^{33} \otimes e_1^{22} + \zeta^2 e_2^{22} \otimes e_2^{11} - \\ &\quad \zeta^3 e_2^{23} \otimes e_2^{13} - \zeta^3 e_2^{32} \otimes e_2^{31} + \zeta^4 e_2^{33} \otimes e_2^{33} \end{aligned}$$

and the counit $\varepsilon \in (A_{LY})^*$ and the antipode $S \in \mathfrak{L}(A_{LY})$ are given by

$$\langle \varepsilon, e_1^{ij} \rangle = 1, \quad \langle \varepsilon, e_2^{k\ell} \rangle = 0, \quad S(e_1^{ij}) = e_1^{ji} \quad \text{and} \quad S(e_2^{k\ell}) = \zeta^{\ell-k} e_2^{\sigma(\ell)\sigma(k)}$$

for all $i, j \in \{1, 2\}$ and $k, \ell \in \{1, 2, 3\}$, where $\sigma(1) := 2$, $\sigma(2) := 1$, $\sigma(3) := 3$, endowing A_{LY} with the structure of a C^* -WHA. This specification has been slightly adapted from [5] as we will propose a tensor network description in [Example 3.5](#) consistent with its string-net model definition.

A *representation* of a C^* -WHA A is simply a representation of its underlying C^* -algebra, i.e. a couple (V, Φ) where V is a finite-dimensional complex vector space and $\Phi \in \mathfrak{L}(A, \mathfrak{L}(V))$ is an algebra homomorphism. If, in addition, V is a Hilbert space and $\Phi(x^*) = \Phi(x)^\dagger$ for all $x \in A$, it is said to be a $*$ -representation. A representation is *faithful* if the map Φ is injective. Two representations (V, Φ) and (W, Ψ) are *equivalent* if there is a bijective linear map $F \in \mathfrak{L}(V, W)$ such that $\Psi(x) = F \circ \Phi(x) \circ F^{-1}$ for all elements $x \in A$. Since A is, in particular, a finite dimensional C^* -algebra, the set $\text{Irr}(A)$ of equivalence classes of irreducible representations, also called *sectors*, is necessarily finite. In what follows, we fix a complete set $\{(V_\alpha, \Phi_\alpha) : \alpha \in \text{Irr}(A)\}$ of pairwise non-equivalent irreducible $*$ -representations of A and let $\text{Tr}_\alpha := \text{Tr} \circ \Phi_\alpha \in A^*$ stand for their corresponding characters. On account of self-duality, $\text{Irr}(A^*)$ are labels for the sectors of the dual C^* -WHA A^* , and we fix another complete set $\{(W_a, \Psi_a) : a \in \text{Irr}(A^*)\}$ of pairwise non-equivalent irreducible $*$ -representations of A^* . Let $\text{Tr}^a := \text{Tr} \circ \Psi_a \in A^{**} \cong A$ stand for their characters.

Remark 2.9 (see e.g. [18, 36]). The category of $*$ -representations of a C^* -WHA A has, by construction, the structure of a rigid monoidal category. The comultiplication $\Delta : A \rightarrow A \otimes A$ provides the monoidal product

$$V \boxtimes W := \{z \in V \otimes W : \Delta(1)z = z\}, \quad \Phi \boxtimes \Psi := (\Phi \otimes \Psi) \circ \Delta,$$

of any two $*$ -representations (V, Φ) and (W, Ψ) of A while the counit ensures the existence of a monoidal unit, called the trivial representation; see [6, 7].

The trivial representation has the unusual feature that it can be reducible. This motivates the following definition.

Definitions 2.10. A C^* -WHA is said to be *connected* if its trivial representation is irreducible, *coconnected* if the dual C^* -WHA is connected, and *biconnected* if it is both connected and coconnected.

For the sake of simplicity, we assume from now on that (V_1, Φ_1) (resp. (W_1, Ψ_1)) corresponds to the trivial representation of A if is connected (resp. coconnected).

Definitions 2.11. Let A be a C^* -WHA. Then,

$$\begin{aligned} A_L &:= \{x \in A : x_{(1)} \otimes x_{(2)} = x1_{(1)} \otimes 1_{(2)} = 1_{(1)}x \otimes 1_{(2)}\}, \\ A_R &:= \{y \in A : y_{(1)} \otimes y_{(2)} = 1_{(1)} \otimes y1_{(2)} = 1_{(1)} \otimes 1_{(2)}y\}, \end{aligned}$$

are two commuting $*$ -subalgebras of A , known as the target and source counital subalgebras of A , respectively. Moreover, $A_{\min} := A_L A_R \subseteq A$ is the minimal C^* -weak Hopf $*$ -algebra contained in A containing the unit element. It is said to be *minimal* if $A = A_{\min}$ and *regular* if the squared antipode restricted to A_{\min} is the identity, i.e. $S^2 \upharpoonright A_{\min} = \text{Id}$.

For any connected C^* -WHA A , its Grothendieck ring $\mathfrak{K}_0(A)$, i.e. the free \mathbb{Z} -module generated by the characters of representations of A with addition and multiplication defined accordingly, is a fusion ring [18]. In particular, the characters

$\{\mathrm{Tr}_\alpha \in A^* : \alpha \in \mathrm{Irr}(A)\}$ form a basis satisfying

$$\mathrm{Tr}_\alpha \cdot \mathrm{Tr}_\beta = \sum_\gamma N_{\alpha\beta}^\gamma \mathrm{Tr}_\gamma$$

for some $N_{ab}^c \in \mathbb{N} \cup \{0\}$, for all sectors $\alpha, \beta, \gamma \in \mathrm{Irr}(A)$. Hence, for any $*$ -representation (V, Φ) of A we can expand its character in the form $\mathrm{Tr}_V = \sum_\gamma \nu_\gamma \mathrm{Tr}_\gamma$, where $\nu_\alpha \in \mathbb{N} \cup \{0\}$ is the multiplicity of (V_α, Φ_α) within (V, Φ) . In this context, define the $|\mathrm{Irr}(A)| \times |\mathrm{Irr}(A)|$ matrix N_V with coefficients $(N_V)_{\beta\gamma} := \sum_\alpha \nu_\alpha N_{\alpha\beta}^\gamma$, for any two sectors $\beta, \gamma \in \mathrm{Irr}(A)$. Since $\mathfrak{K}_0(A)$ is, in particular, a transitive ring [20], N_V is a matrix with strictly positive entries. Thus, by virtue of the Frobenius-Perron theorem, the spectral radius of N_V , denoted d_V , is an algebraically simple positive eigenvalue, known as the Frobenius-Perron dimension of (V, Φ) . Though it is not needed, we remark that this notion coincides with the one of quantum dimension from the category of $*$ -representations; see [18, Proposition 8.23] for a rigorous statement. For simplicity of notation, let $d_\alpha := d_{V_\alpha}$ for all sectors $\alpha \in \mathrm{Irr}(A)$. Also, let $\mathfrak{D}^2 := \sum_\alpha d_\alpha^2$ denote the Frobenius-Perron dimension of the algebra. Dually, if A is coconnected, let $\{d_a : a \in \mathrm{Irr}(A^*)\}$ denote the dual Frobenius-Perron dimensions of A . It turns out that then $\sum_\alpha d_\alpha^2 = \sum_a d_a^2$ if A is a biconnected C^* -WHA [20].

Now we recall the following result, which proves the existence of a distinguished element satisfying a ‘‘pulling-through equation’’ in each connected C^* -WHA. These properties turn out to be enough to understand the properties of renormalization fixed points.

Theorem 2.12 (cf. [33]). *Let A be a biconnected C^* -WHA. Then,*

$$\Omega := \mathfrak{D}^{-2} \sum_a d_a \mathrm{Tr}^a \in A^{**} \cong A$$

is a cocentral non-degenerate positive idempotent, known as the canonical regular element of A . Moreover, there exists a unique linear map $T \in \mathfrak{L}(A)$ such that

$$(11) \quad T(x)\Omega_{(1)} \otimes \Omega_{(2)} = \Omega_{(1)} \otimes x\Omega_{(2)}$$

for all elements $x \in A$, usually referred to as pulling-through identity. In particular, T is an involutive algebra anti-homomorphism.

The canonical regular element is well-known in the literature of \mathbb{Z}_+ -rings and it satisfies an eigenvalue equation of the form

$$(12) \quad \Omega \cdot \mathrm{Tr}^a = \mathrm{Tr}^a \cdot \Omega = d_a \Omega$$

for all sectors $a \in \mathrm{Irr}(A^*)$. Equation 11 resembles the characterization of left integrals in C^* -WHAs; see e.g. [6]. However, these notions coincide if and only if A is a C^* -HA. In this context, $\Omega \in A$ is a well-known element in the literature, the Haar integral $h \in A$, and the linear map $T \in \mathfrak{L}(A)$ coincides with the antipode. In any case, it is convenient to rewrite Equation 11 as

$$(13) \quad T(\Omega_{(1)})x \otimes \Omega_{(2)} = T(\Omega_{(1)}) \otimes x\Omega_{(2)},$$

for all elements $x \in A$. Both identities will be used interchangeably.

Let us interpret now representations of a given C^* -WHA in terms of tensor networks; see [33] for an exhaustive discussion. As usually done in the tensor network literature we employ a graphical notation, briefly described in Appendix A. Consider any sequence $\{(V_{[i]}, \Phi_{[i]}) : i \in \mathbb{N}\}$ of representations of a C^* -WHA A . It turns out that the endomorphisms $(\Phi_{[1]} \otimes \cdots \otimes \Phi_{[k]}) \circ \Delta^{(k-1)}(x)$ can be described in terms of matrix product operators, for all elements $x \in A$. More concretely, there

exist a Hilbert space W and tensors $A_{[i]} \in \mathfrak{L}(W) \otimes \mathfrak{L}(V_{[i]})$, $i \in \mathbb{N}$, independent of $x \in A$, such that one can write

$$(14) \quad \begin{array}{c} \text{Diagram: A circle with a horizontal line through its center. The top and bottom arcs are red, and the left and right arcs are black. Four black dots are on the horizontal line. A red dot is on the left arc. A red dot is on the top arc. A red dot is on the bottom arc. A red dot is on the right arc. Labels: } b(x) \text{ (left), } A_{[1]} \text{ (top), } A_{[2]} \text{ (right), } A_{[k]} \text{ (bottom).} \\ \equiv (\Phi_{[1]} \otimes \cdots \otimes \Phi_{[k]}) \circ \Delta^{(k-1)}(x) \end{array}$$

for some linear map $b \in \mathfrak{L}(A, W \otimes W^*)$, for all $k \in \mathbb{N}$. We will usually restrict to the translation-invariant case, for which $\Phi_{[1]} = \Phi_{[2]} = \cdots =: \Phi$ and $A_{[1]} = A_{[2]} = \cdots$. Notice that the physical indices, associated to Hilbert spaces V and V^* , are depicted by black lines, while the virtual indices, associated to Hilbert spaces W and W^* , are depicted by red lines. Thus, from now on, we will drop the labels, since no misunderstanding can arise. For instance, we can express the multiplicativity of the coproduct (see [Definition 2.1](#)) with this simplified graphical notation as

$$(15) \quad \begin{array}{c} \text{Diagram: A circle with a horizontal line through its center. The top and bottom arcs are red, and the left and right arcs are black. Four black dots are on the horizontal line. A red dot is on the left arc. A red dot is on the top arc. A red dot is on the bottom arc. A red dot is on the right arc. Labels: } b(x) \text{ (left), } b(y) \text{ (top), } b(xy) \text{ (right).} \\ = \text{Diagram: A circle with a horizontal line through its center. The top and bottom arcs are red, and the left and right arcs are black. Four black dots are on the horizontal line. A red dot is on the left arc. A red dot is on the top arc. A red dot is on the bottom arc. A red dot is on the right arc. Label: } b(xy) \text{ (right).} \end{array}$$

for all elements $x, y \in A$.

We finish this section by reinterpreting the different properties of the canonical regular element in graphical notation. First, it is easy to check by induction on $n \in \mathbb{N}$ that the fact that $\Omega \in A$ is a cocentral element implies

$$(16) \quad \Omega_{(1)} \otimes \Omega_{(2)} \otimes \cdots \otimes \Omega_{(n)} = \Omega_{(\sigma(1))} \otimes \Omega_{(\sigma(2))} \otimes \cdots \otimes \Omega_{(\sigma(n))}$$

for any shift permutation σ of $\{1, \dots, n\}$. In turn, this can be rephrased as the translation-invariance of the associated MPOs:

$$(17) \quad \begin{array}{c} \text{Diagram: A circle with a horizontal line through its center. The top and bottom arcs are red, and the left and right arcs are black. Four black dots are on the horizontal line. A red dot is on the left arc. A red dot is on the top arc. A red dot is on the bottom arc. A red dot is on the right arc. Label: } b(\Omega) \text{ (left).} \\ = \text{Diagram: A circle with a horizontal line through its center. The top and bottom arcs are red, and the left and right arcs are black. Four black dots are on the horizontal line. A red dot is on the top arc. A red dot is on the bottom arc. A red dot is on the right arc. Label: } b(\Omega) \text{ (top).} \\ = \cdots = \text{Diagram: A circle with a horizontal line through its center. The top and bottom arcs are red, and the left and right arcs are black. Four black dots are on the horizontal line. A red dot is on the bottom arc. A red dot is on the right arc. Label: } b(\Omega) \text{ (bottom).} \end{array}$$

In order to interpret the action of the linear map $T \in \mathfrak{L}(A)$, first note that the linear map $A \rightarrow \mathfrak{L}(V^*)$, $x \mapsto (\Phi \circ T(x))^t$ defines a representation of A , where $(\cdot)^t$ stands for the transpose operation. As discussed below, it is not necessarily a $*$ -representation. By [Equation 14](#), one can depict, e.g.

$$(18) \quad \begin{array}{c} \text{Diagram: A circle with a horizontal line through its center. The top and bottom arcs are red, and the left and right arcs are black. Four black dots are on the horizontal line. A red dot is on the left arc. A red dot is on the top arc. A red dot is on the bottom arc. A red dot is on the right arc. Label: } b(x) \text{ (left).} \\ \equiv \Phi^{\otimes 3} \circ (T \otimes \text{id} \otimes \text{id}) \circ \Delta^{(2)}(x) \end{array}$$

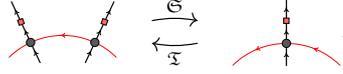
for all $x \in A$, for some white rank-four tensor, where all physical spaces in the picture are V . For the sake of clarity, we have reversed the direction of the physical arrows corresponding to the new tensor, since $T \in \mathfrak{L}(A)$ is an anti-multiplicative map. With this notation, [Equations 11](#) and [13](#) can be interpreted respectively as follows:

$$(19) \quad \begin{array}{c} \text{Diagram: A circle with a horizontal line through its center. The top and bottom arcs are red, and the left and right arcs are black. Four black dots are on the horizontal line. A red dot is on the left arc. A red dot is on the top arc. A red dot is on the bottom arc. A red dot is on the right arc. Label: } b(\Omega) \text{ (left).} \\ = \text{Diagram: A circle with a horizontal line through its center. The top and bottom arcs are red, and the left and right arcs are black. Four black dots are on the horizontal line. A red dot is on the top arc. A red dot is on the bottom arc. A red dot is on the right arc. Label: } b(\Omega) \text{ (top).} \\ \text{and} \\ \text{Diagram: A circle with a horizontal line through its center. The top and bottom arcs are red, and the left and right arcs are black. Four black dots are on the horizontal line. A red dot is on the left arc. A red dot is on the top arc. A red dot is on the bottom arc. A red dot is on the right arc. Label: } b(\Omega) \text{ (left).} \\ = \text{Diagram: A circle with a horizontal line through its center. The top and bottom arcs are red, and the left and right arcs are black. Four black dots are on the horizontal line. A red dot is on the top arc. A red dot is on the bottom arc. A red dot is on the right arc. Label: } b(\Omega) \text{ (top).} \end{array}$$

Note that, since $x \in A$ is arbitrary, we have dropped the tensor $b(x)$ and reexpressed the identities using open boundary MPOs instead; see [33].

3. RENORMALIZATION FIXED POINT MPDOs

In this section we define a distinguished family of MPOs starting from a biconnected C^* -WHA and show that they are RFP MPDOs, as defined in [14]. More concretely, we provide explicit expressions for both local coarse-graining and local fine-graining quantum channels \mathfrak{T} and \mathfrak{S} for which the generating rank-four tensor is a fixed point under the corresponding induced flows, very much in the spirit of standard renormalization spirit. The generating tensor of the RFP MPDOs is obtained here by appropriately weighting the tensor from the original MPO algebra, described in the previous section, obtaining:



This weighting is done by means of the canonical regular element of the dual C^* -WHA. To this end, let us examine first the properties of this linear functional, which formally plays the role in C^* -WHAs of the character of the usual left-regular representation.

Lemma 3.1. *Let A be a connected C^* -WHA. Then, the canonical regular element $\omega \in A^*$ of the dual C^* -WHA A^* is the unique trace-like, faithful, positive linear functional on A that is idempotent, i.e. $(\omega \otimes \omega) \circ \Delta = \omega$.*

Proof. Recall [Theorem 2.12](#) and [Remark 2.2](#). It is easy to check that $\omega \in A^*$ is a trace-like linear functional since it is a cocentral element of A^* . Also, it is a faithful and positive linear functional by construction. In addition, it satisfies the eigenvalue equation $\text{Tr}_\alpha \cdot \omega = \omega \cdot \text{Tr}_\alpha = d_\alpha \omega$ for all sectors $\alpha \in \text{Irr}(A)$; see [20, Section 3] for a proof. We note that this implies, in particular, that $\omega \in A^*$ is idempotent. Assume now that $f \in A^*$ is any linear functional satisfying the properties above. Since it is trace-like, it can be expanded in the form $f = \sum_\alpha f_\alpha \text{Tr}_\alpha$ for some numbers $f_\alpha \in \mathbb{C}$, $\alpha \in \text{Irr}(A)$. By evaluating f on the primitive central idempotents of A it is easy to check that $f_\alpha > 0$ for all $\alpha \in \text{Irr}(A)$, since f is assumed to be also a faithful positive linear functional. Define the $|\text{Irr}(A)| \times |\text{Irr}(A)|$ matrix N_f with complex coefficients $(N_f)_{\beta\gamma} := \sum_\alpha f_\alpha N_{\alpha\beta}^\gamma$, which implements the left-multiplication by $f \in A^*$ in the basis $\{\text{Tr}_\alpha : \alpha \in \text{Irr}(A)\}$, i.e. it satisfies $N_f \psi = f\psi$ for all $\psi \in A^*$. Then, $N_f f = f^2 = f$ and $N_f \omega = f\omega = \sum_\alpha f_\alpha d_\alpha \omega \propto \omega$, where the first equation holds since $f \in A^*$ is idempotent by hypothesis and the second equation follows from the eigenvalue equation. Since A is connected, the Grothendieck ring $\mathfrak{K}_0(A)$ is, in particular, a transitive ring, and hence N_f has strictly positive entries; see e.g. [35] and [20, Section 3]. By virtue of the Frobenius-Perron theorem, it has only one eigenvector with strictly positive entries, up to a constant. Therefore, $f = \omega$, since both are positive idempotents. \square

Now, given a faithful $*$ -representation of the C^* -WHA, we define the appropriate weight extending the previous linear functional to the representation space.

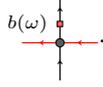
Remark 3.2. Let A be a connected C^* -WHA and let (V, Φ) be a faithful $*$ -representation of A . Let $b(f)$ denote the boundary weight for the matrix product operators arising from the dual C^* -WHA A^* , for all $f \in A^*$. It turns out that $b(\omega) = \Phi(c_\omega)$ for some strictly positive central element $c_\omega \in A$. It provides an extension of $\omega \in A^*$ to the representation space $\mathfrak{L}(V)$ in the sense that

$$\text{Tr}(b(\omega)\Phi(x)) = \text{Tr}(\Phi(c_\omega x)) = \langle \omega, x \rangle$$

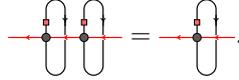
for all elements $x \in A$.

Proof. For all sectors $\alpha \in \text{Irr}(A)$, let $e_\alpha \in A$ be the corresponding primitive central idempotent of A and let $\nu_\alpha \in \mathbb{C}$ denote the multiplicity of (V_α, Φ_α) within (V, Φ) . Then, define the element $c_\omega := \mathfrak{D}^{-2} \sum_\alpha d_\alpha \nu_\alpha^{-1} e_\alpha \in A$. Trivially, it is a central invertible positive element and satisfies $\text{Tr}(\Phi(c_\omega x)) = \mathfrak{D}^{-2} \sum_\alpha \nu_\alpha^{-1} d_\alpha \text{Tr}(\Phi(x e_\alpha)) = \mathfrak{D}^{-2} \sum_\alpha \nu_\alpha^{-1} d_\alpha \nu_\alpha \text{Tr}_\alpha(x) = \langle \omega, x \rangle$ for all elements $x \in A$, as we wanted to prove. \square

Let us now consider the tensor obtained by multiplying the MPO tensor in Equation 14 by $b(\omega) = \Phi(c_\omega)$ in the physical space:



Idempotence of $\omega \in A^*$ implies that this tensor generates an MPO with zero correlation length; see [14]:



It is clear that computations of correlation functions using the MPDOs generated by the previous tensor will be length-independent. In particular, it induces the following family of mixed states:

Theorem 3.3. *Let A be a biconnected C^* -WHA and let (V, Φ) be a faithful $*$ -representation of A . Then, the operators*

$$\rho(x, n) := \langle \omega, x \rangle^{-1} b(\omega)^{\otimes n} \Phi^{\otimes n}(\Delta^{(n-1)}(x)) \in \mathfrak{L}(V^{\otimes n})$$

are RFP MPDOs for all positive non-zero elements $x \in A$ and all $n \in \mathbb{N}$. Specifically, there are quantum channels $\mathfrak{T} : \mathfrak{L}(V) \rightarrow \mathfrak{L}(V \otimes V)$ and $\mathfrak{S} : \mathfrak{L}(V \otimes V) \rightarrow \mathfrak{L}(V)$, known as local fine-graining and coarse-graining maps, respectively, such that

$$\mathfrak{T}(\rho(x, 1)) = \rho(x, 2) \quad \text{and} \quad \mathfrak{S}(\rho(x, 2)) = \rho(x, 1)$$

for all positive non-zero elements $x \in A$ and all $n \in \mathbb{N}$.

Let us illustrate the construction with an extremely modest example.

Example 3.4. Let $A := \mathbb{C}\mathbb{Z}_2$ be the C^* -HA arising from the group $G := \mathbb{Z}_2$ generated by $g \in G$; see Example 2.5. It possesses only two sectors, namely the equivalence classes of the trivial representation and the sign representation, each one-dimensional. Consider that both physical and virtual spaces are $V := W := \mathbb{C}^2$, with basis elements $|1\rangle, |2\rangle$, and consider the faithful $*$ -representation of A $\Phi \in \mathfrak{L}(A, \mathfrak{L}(\mathbb{C}^2))$ defined by $\Phi(g) := \sigma_z$, the usual Pauli-Z matrix. It is easy to see that both Frobenius-Perron dimensions are 1 and hence the canonical regular elements of A and A^* are given by $\Omega = 2^{-1}(e + g)$ and $\omega(x) = (x, e)_V$, for all $x \in A$, respectively. A tensor generating the corresponding MPOs is specified by the non-zero coefficients

$$\begin{array}{c} 1 \\ | \\ 1 \text{---} \bullet \text{---} 1 \\ | \\ 1 \end{array} = \begin{array}{c} 2 \\ | \\ 1 \text{---} \bullet \text{---} 1 \\ | \\ 2 \end{array} = \begin{array}{c} 1 \\ | \\ 2 \text{---} \bullet \text{---} 2 \\ | \\ 1 \end{array} = - \begin{array}{c} 2 \\ | \\ 2 \text{---} \bullet \text{---} 2 \\ | \\ 2 \end{array} = 1.$$

Moreover, in this case the weight is trivially given by $c_\omega = 2^{-1}e$ and thus

$$\rho(x, n) = \frac{1}{2^n} (\mathbf{1}^{\otimes n} + \frac{(x, g)_V}{(x, e)_V} \sigma_z^{\otimes n}),$$

are the induced RFP MPDOs, for all positive non-zero $x \in A$. In particular, $\rho(\Omega, n) = 2^{-n}(\mathbf{1}^{\otimes n} + \sigma_z^{\otimes n})$ is the boundary state of the toric code; see [14].

Example 3.5. Let A_{LY} be the Lee-Yang C^* -WHA from Example 2.8. It possesses only two sectors, denoted 1 and τ , for which it is easy to check that $d_1 = 1$ and $d_\tau = \zeta^{-2} = 2^{-1}(1 + \sqrt{5})$, respectively. Consider that both physical and virtual spaces are $V := W := \mathbb{C}^5$ and let $\Phi \in \mathfrak{L}(A_{\text{LY}}, \mathfrak{L}(\mathbb{C}^5))$ be the faithful $*$ -representation

geometries described by square lattices, although the proof works for any 2D PEPS defined on any directed pseudo-graph. In this case, we will consider the 2D PEPS tensor depicted as follows:

$$(20) \quad \begin{array}{c} \text{Diagram of a square tensor with four legs} \\ \text{:=} \\ \text{Diagram of a tensor with four legs, including weights } b(\omega)^{\frac{1}{4}}, b(\Omega), \text{ and } \Psi(\hat{\xi}_L) \end{array}$$

Here, the physical space is given by the tensor product $V \otimes V \otimes V^* \otimes V^*$ and there are four virtual indices, each of them corresponding to V or V^* . If read in clockwise direction starting from the virtual weight $b(\Omega) \in \mathfrak{L}(W)$, the tensor is algebraically described by the expression

$$(b(\omega)^{\frac{1}{4}})^{\otimes 4} (\Phi(\Omega_{(1)}) \otimes \Phi(\Omega_{(2)}) \otimes \langle \hat{\xi}_L, \Omega_{(3)} \rangle \Phi(\xi^{\frac{1}{2}} T(\Omega_{(4)})) \otimes \Phi(\xi^{\frac{1}{2}} T(\Omega_{(5)})))$$

as an operator from physical to virtual spaces. Recall that $b(\omega) \in \mathfrak{L}(V)$ is an invertible positive central operator and hence it can be freely moved along the physical indices. Let us now show that the boundary operator is the desired operator.

Step 1. Let us first simplify the transfer operator associated to the previous 2D PEPS tensor, $\mathbb{E} = \mathbb{E}(A, V, \Phi) \in \mathfrak{L}(V \otimes V \otimes V^* \otimes V^*)$ obtained by contracting the physical indices of the 2D PEPS tensor and its corresponding conjugate transpose if regarded as an operator. Algebraically, it is given by the expression

$$\begin{aligned} \mathbb{E} = & (b(\omega)^{\frac{1}{2}})^{\otimes 4} (\Phi(\Omega_{(1)}) \Phi(\Omega_{(1')})^\dagger \otimes \Phi(\Omega_{(2)}) \Phi(\Omega_{(2')})^\dagger \otimes \overline{\langle \hat{\xi}_L^{-1}, \Omega_{(3')} \rangle \langle \hat{\xi}_L^{-1}, \Omega_{(3')} \rangle} \\ & \cdot \Phi(T(\Omega_{(4')}))^\dagger \Phi(\xi) \Phi(T(\Omega_{(4)})) \otimes \Phi(T(\Omega_{(5')}))^\dagger \Phi(\xi) \Phi(T(\Omega_{(5)}))), \end{aligned}$$

where we have employed that $b(\omega) \in \mathfrak{L}(V)$ is positive and central and $\Phi(\xi) \in \mathfrak{L}(V)$ is positive, since $\Phi \in \mathfrak{L}(A, \mathfrak{L}(V))$ is a $*$ -representation and $\xi \in A$ is positive. Note that the order of composition is reversed for the terms associated to white tensors. In order to fully describe \mathbb{E} in terms of tensor networks, note that

$$\overline{\langle \hat{\xi}_L^{-1}, x \rangle} = \langle (\hat{\xi}_L^{-1})^*, S(x)^* \rangle = \langle \hat{\xi}_L^{-1}, S(x)^* \rangle = \langle \hat{\xi}_L^{-1}, S^{-1}(x^*) \rangle = \langle \hat{\xi}_R^{-1}, x^* \rangle$$

for all $x \in A$, where the first equality is due to [Remark 2.2](#), the second equality follows from the positivity of $\hat{\xi}_L \in A^*$, the third equality is due to [Remark 2.3](#) and the fourth equality follows from the definition of $\hat{\xi}_R \in A^*$, see [Lemma 3.6](#). In addition, recall that $\Phi \in \mathfrak{L}(A, \mathfrak{L}(V))$ is a $*$ -representation and $T(x)^* \xi = \xi T(x^*)$ for all $x \in A$, see [Lemma 3.6](#). Therefore:

$$\begin{aligned} \mathbb{E} = & (b(\omega)^{\frac{1}{2}})^{\otimes 4} (\Phi(\Omega_{(1)}) \Omega_{(1')}^* \otimes \Phi(\Omega_{(2)}) \Omega_{(2')}^*) \\ & \otimes \langle \hat{\xi}_R^{-1}, \Omega_{(3')}^* \rangle \langle \hat{\xi}_L^{-1}, \Omega_{(3)} \rangle \Phi(\xi T(\Omega_{(4')}^*) T(\Omega_{(4)})) \otimes \Phi(\xi T(\Omega_{(5')}^*) T(\Omega_{(5)})). \end{aligned}$$

Hence, the transfer operator can be represented graphically as follows:

$$\mathbb{E} = \begin{array}{c} \text{Diagram of the transfer operator } \mathbb{E} \text{ with weights } b(\Omega^*), b(\omega)^{\frac{1}{2}}, \Phi(\xi), \Psi(\hat{\xi}_R^{-1}), \text{ and } \Psi(\hat{\xi}_L^{-1}) \end{array} = \begin{array}{c} \text{Diagram of the transfer operator } \mathbb{E} \text{ with weights } b(\Omega^*), b(\omega)^{\frac{1}{2}}, \Phi(\xi), \Psi(\hat{\xi}_R^{-1}), \text{ and } \Psi(\hat{\xi}_L^{-1}) \end{array}$$

On the other hand, $\Psi(\hat{\xi}_L^{-1})$ and $\Psi(\hat{\xi}_R^{-1})$ can be “moved” from the virtual to the physical spaces using the following identities:

$$x_{(1)}\langle \hat{\xi}_L^{-1}, x_{(2)} \rangle = \xi_L^{-1}x \quad \text{and} \quad x_{(1)}\langle \hat{\xi}_R^{-1}, x_{(2)} \rangle = x\xi_L^{-1}$$

for all elements $x \in A$; see [Lemma 3.6](#). In graphical notation, the previous formulas are rephrased in the following form:

$$(21) \quad \begin{array}{c} \Psi(\hat{\xi}_L^{-1}) \\ \downarrow \\ \text{---} \end{array} = \begin{array}{c} \Phi(\xi_L^{-1}) \\ \downarrow \\ \text{---} \end{array} \quad \text{and} \quad \begin{array}{c} \Psi(\hat{\xi}_R^{-1}) \\ \downarrow \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \downarrow \\ \Phi(\xi_L^{-1}) \end{array} .$$

By virtue of these identities, the fact that $T \in \mathfrak{L}(A)$ is an algebra anti-homomorphism and $\Omega^* = \Omega = \Omega^2$ by [Theorem 2.12](#), it follows that

$$\begin{array}{c} b(\Omega^*) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} b(\omega)^{\frac{1}{2}} \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Psi(\hat{\xi}_R^{-1}) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Phi(\xi) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} b(\Omega) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Psi(\hat{\xi}_L^{-1}) \\ \downarrow \\ \text{---} \end{array} = \begin{array}{c} b(\Omega^*) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} b(\omega)^{\frac{1}{2}} \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Phi(\xi_L^{-1}) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Phi(\xi) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} b(\Omega) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Phi(\xi_L^{-1}) \\ \downarrow \\ \text{---} \end{array} = \begin{array}{c} b(\omega)^{\frac{1}{2}} \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Phi(\xi) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Phi(\xi_L^{-1}) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} b(\Omega) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Phi(\xi_L^{-1}) \\ \downarrow \\ \text{---} \end{array} .$$

Applying again [Equation 21](#), we conclude that the transfer operator takes the form:

$$\mathbb{E} = \begin{array}{c} b(\omega)^{\frac{1}{2}} \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} b(\omega)^{\frac{1}{2}} \Phi(\xi) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Phi(\xi_L^{-1}) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} b(\Omega) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Phi(\xi_L^{-1}) \\ \downarrow \\ \text{---} \end{array} = \begin{array}{c} b(\omega)^{\frac{1}{2}} \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} b(\omega)^{\frac{1}{2}} \Phi(\xi) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Psi(\hat{\xi}^{-1}) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} b(\Omega) \\ \downarrow \\ \text{---} \end{array} .$$

since $\Psi \in \mathfrak{L}(A^*, \mathfrak{L}(W))$ is a $*$ -representation and $\xi^{-1} = \xi_L^{-1}\xi_R^{-1}$.

Step 2. Let us consider the concatenation of two transfer operators. By virtue of the pulling-through identity [Equation 11](#),

$$(22) \quad \begin{array}{c} b(\omega)^{\frac{1}{2}} \Phi(\xi) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} b(\omega) \Phi(\xi) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Phi(\xi_L^{-1}) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} b(\Omega) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Phi(\xi_L^{-1}) \\ \downarrow \\ \text{---} \end{array} = \begin{array}{c} b(\omega)^{\frac{1}{2}} \Phi(\xi) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Psi(\hat{\xi}^{-1}) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} b(\Omega) \\ \downarrow \\ \text{---} \end{array} .$$

Step 3. Let us consider the concatenation of four transfer operators around the vertices of a plaquette. In particular, we prove that the ansatz 2D PEPS tensors gives rise to a *normalized* PEPS and its boundary state is the RFP MPDO defined in [Theorem 3.3](#). Consider, from a top view, the procedure of simplifying the concatenation of transfer operators that form a whole plaquette:

$$\begin{array}{c} b(\omega) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Phi(\xi) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Psi(\hat{\xi}^{-1}) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Psi(\hat{\xi}^{-1}) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} b(\Omega) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Phi(\xi) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} b(\omega) \\ \downarrow \\ \text{---} \end{array} = \begin{array}{c} b(\omega) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Phi(\xi) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Psi(\hat{\xi}^{-1}) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Psi(\hat{\xi}^{-1}) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} b(\Omega) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Phi(\xi) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} b(\omega) \\ \downarrow \\ \text{---} \end{array} = \dots = \begin{array}{c} b(\omega) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Phi(\xi) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Psi(\hat{\xi}^{-1}) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Psi(\hat{\xi}^{-1}) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} b(\Omega) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \Phi(\xi) \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} b(\omega) \\ \downarrow \\ \text{---} \end{array} .$$

For the purpose of closing the plaquette, recall [Lemma 3.7](#) and [Lemma 3.6](#):

$$\text{Diagram} = \sum_a \text{Diagram} \propto \text{Diagram}$$

Note that in the previous equations the inner circle representing $\text{Tr}^a(\hat{\xi}^{-1})$ is not independent of the outer shape and hence it gives rise to possibly different constant in each sector, as it is a sum over all sectors $a \in \text{Irr}(A)$. As showed in [Lemma 3.6](#), these are precisely the Frobenius-Perron dimensions which define, in each sector, the canonical regular element $\Omega \in A$. Therefore we can rewrite it in terms of the weight $b(\Omega) \in \mathcal{L}(W)$, as done in the last equality. Iterating this procedure for each plaquette of the lattice proves that matrix product density operators defined in the previous section arise naturally as boundary states of topological 2D PEPS.

Note also that [Equation 22](#) is nothing but a natural 2D generalization of the renormalization fixed point condition for MPS defined in [\[14\]](#). In that sense, we can conclude that the RFP MPDOs considered in this paper are boundary states of PEPS fulfilling this renormalization fixed point property. \square

5. CLASSIFICATION VIA SHALLOW CIRCUITS OF QUANTUM CHANNELS

In this section we prove that RFP MPDOs arising from C^* -HAs belong to the trivial phase. Namely, we provide explicit definitions of depth-two circuits of finite-range quantum channels that map the maximally mixed state to these RFP MPDOs. Finally, we show that our construction cannot be extended to arbitrary biconnected C^* -WHAs, which lead us to the conjecture that there are non-trivial phases in this context.

In order to deepen the intuition towards the general case of C^* -HAs, let first examine the simplest non-trivial example.

Example 5.1. RFP MPDOs arising from the group C^* -HA $A := \mathbb{C}\mathbb{Z}_2$, introduced in [Example 2.5](#) and [Example 3.4](#), are in the trivial phase. Specifically, we build

$$\rho(\Omega, n) = \frac{1}{2^n}(\mathbf{1}^{\otimes n} + \sigma_z^{\otimes n})$$

via a depth-two circuit of range-two quantum channels from the maximally mixed state $\text{Tr}(\mathbf{1})^{-n}\mathbf{1}^{\otimes n}$. We assume without loss of generality that $n \in \mathbb{N}$ is even and propose the following procedure:

Step 1 (“initialization”). We first construct $n/2$ copies $(\rho_2)^{\otimes n/2}$ of the mixed state ρ_2 between pairs of nearest neighbors by replacing the product states separately. This is easily done by means of the quantum channel $\mathfrak{N} : X \otimes Y \mapsto \text{Tr}(X \otimes Y)\rho_2$. In the Choi-Jamiołkowski picture, this process can be depicted as follows:

$$\text{Diagram} = \text{Diagram}$$

When the system size is an odd natural number simply replace three of them with the mixed state ρ_3 , for example.

Step 2 (“gluing”). Now, we “glue” together all these copies of ρ_2 in order to obtain the target mixed state ρ_n . This is done inductively by means of the following quantum channel, called from now on *gluing* map:

$$(23) \quad \mathfrak{G} : X \otimes Y \mapsto \frac{1}{2^2}(\text{Tr}(X \otimes Y)\mathbf{1} \otimes \mathbf{1} + \text{Tr}(X\sigma_z \otimes Y\sigma_z)\sigma_z \otimes \sigma_z)$$

$\rho(\Omega, 2) \otimes \cdots \otimes \rho(\Omega, 2)$ as previously done. Now, by virtue of [Lemma 5.2](#), let $\text{Id} \otimes \mathfrak{G}_\Omega \otimes \cdots \otimes \mathfrak{G}_\Omega \otimes \mathfrak{G}_x \otimes \text{Id}$ be the second layer of quantum channels, where all subindices are $\Omega \in A$ except for one, which is $x \in A$. This second layer of channels then glues together all local MPDOs into the single MPDO $\rho(x, n)$. \square

For general RFP MPDOs constructed from biconnected C^* -WHAs a straightforward generalization of the previous procedure is not possible anymore, since the comultiplication is no longer unit-preserving.

Remark 5.4. There are no trace-preserving gluing maps for general biconnected C^* -WHAs such that [Equation 24](#) holds for all elements $x \in A$.

See [Appendix F](#) for a proof. Unfortunately, the description of the phases in this general case is still an open problem. Nevertheless, some evidence indicate the existence of non-trivial phases, as we conjecture here.

Conjecture 5.5. *RFP MPDOs arising from the Lee-Yang C^* -WHA of [Example 2.8](#) do not belong to the trivial phase.*

However, these obstructions can be circumvented if one restricts to the trivial sector. The following result establishes the existence of an special gluing map, motivated by the characterization of simple RFP MPDO tensors in [\[14\]](#).

Lemma 5.6. *Let A be a biconnected C^* -WHA and let (V, Φ) be a faithful $*$ -representation of A . There is a quantum channel $\mathfrak{G}_1 \in \mathfrak{L}(\mathfrak{L}(V \otimes V))$, called “gluing” map, such that*

$$(25) \quad (\text{Id}^{\otimes m-1} \otimes \mathfrak{G}_1 \otimes \text{Id}^{\otimes n-1})(\rho(1, m) \otimes \rho(1, n)) = \rho(1, m+n)$$

for all $m, n \in \mathbb{N}$.

A proof is given in [Appendix F](#). As an immediate corollary, similar to the case of C^* -HAs, we obtain the following result.

Theorem 5.7. *Let A be a biconnected C^* -WHA and let (V, Φ) be a faithful $*$ -representation of A . Then, for all $n \in \mathbb{N}$ there exist two depth-two circuits of bounded-range quantum channels that map $\text{Tr}(\mathbf{1})^{-n} \mathbf{1}^{\otimes n}$ into $\rho(1, n)$ and $\rho(\text{Tr}^1, n)$. That is, the sequences $(\rho(1, n))_{n=1}^\infty$ and $(\rho(\text{Tr}^1, n))_{n=1}^\infty$ are in the trivial phase.*

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APPENDIX A. GRAPHICAL NOTATION FOR TENSOR NETWORKS

In this appendix we introduce a slightly adapted version of the usual graphical notation from the literature of tensor networks. From now on, let V be a finite dimensional complex vector space. First, a vector $v \in V$ is depicted by a shape (e.g. a circle) and a line sticking out of it, associated to the vector space and labeled accordingly; we convey to draw the arrow outgoing from the shape. Second, any

element $f \in V^*$ is a vector from V^* or a linear functional on V , and we alternatively draw it with an arrow ingoing to the shape. Pictorially,

$$v = \begin{array}{c} v \\ \bullet \longrightarrow \end{array} \quad \text{and} \quad f = \begin{array}{c} V^* \\ \longleftarrow \bullet \end{array} = \begin{array}{c} V \\ \longrightarrow \bullet \end{array} f.$$

Note that, although we have represented the previous elements using straight horizontal lines, no reading order has been prescribed. Here, we identify each (finite dimensional complex) vector space V and its double dual V^{**} , which are canonically isomorphic. Therefore, one can regard any vector $v \in V$ as a linear functional on V^* and thus, exchange the orientation of any line by labeling the dual:

$$\begin{array}{c} v \\ \bullet \longrightarrow \end{array} \begin{array}{c} V \\ \longrightarrow \end{array} = \begin{array}{c} v \\ \bullet \longrightarrow \end{array} \begin{array}{c} V^{**} \\ \longrightarrow \end{array} = \begin{array}{c} v \\ \bullet \longleftarrow \end{array} \begin{array}{c} V^* \\ \longrightarrow \end{array}.$$

Now, let V and W be two (finite dimensional complex) vector spaces. Any vector $v \in V \otimes W$ in the tensor product is depicted by a shape with two lines, e.g.:

$$\begin{array}{c} V \quad v \quad W \\ \longrightarrow \bullet \longleftarrow \end{array}.$$

By virtue of the previous identification, one can rewrite

$$\begin{array}{c} V \quad v \quad W \\ \longrightarrow \bullet \longleftarrow \end{array} = \begin{array}{c} V^* \quad v \quad W \\ \longleftarrow \bullet \longrightarrow \end{array} = \begin{array}{c} V^* \quad v \quad W^* \\ \longleftarrow \bullet \longrightarrow \end{array} = \begin{array}{c} V \quad v \quad W^* \\ \longrightarrow \bullet \longrightarrow \end{array}.$$

In addition, the tensor product of two vectors $v \in V$ and $w \in W$ is depicted by placing both representations in the same picture, next to each other:

$$\begin{array}{c} V \quad v \otimes w \quad W \\ \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \end{array} = \begin{array}{c} V \quad v \quad w \quad W \\ \longrightarrow \bullet \bullet \longrightarrow \end{array}.$$

Note that, although the labels V and W should allow to identify the corresponding indices, even if $V = W$, it may not be enough. One solution, used in this paper, consists on prescribing colors to each index. Additionally, note that we have implicitly identified $V \otimes W$ and $W \otimes V$, since no order is prescribed in the previous picture. However, these are again canonically isomorphic and no preference is considered here. In this context, it is natural to represent the action of the canonical pairing by joining the corresponding lines, i.e.

$$f \otimes v = \begin{array}{c} v \quad V \\ \bullet \longrightarrow \end{array} \begin{array}{c} V \quad f \\ \longrightarrow \bullet \end{array} \mapsto \begin{array}{c} v \quad V \quad f \\ \bullet \longrightarrow \bullet \end{array} = \langle f, v \rangle.$$

On the other hand, since $\mathfrak{L}(V, W)$ is canonically isomorphic to $V^* \otimes W$, we can also represent any linear map $F \in \mathfrak{L}(V, W)$ in the following form

$$\begin{array}{c} V^* \quad F \quad W \\ \longrightarrow \bullet \longrightarrow \end{array} = \begin{array}{c} V \quad F \quad W \\ \longrightarrow \bullet \longrightarrow \end{array}.$$

Remarkably, the distinction between $F \in \mathfrak{L}(V, W)$ and its transpose $F^t \in \mathfrak{L}(W^*, V^*)$, $W^* \ni g \mapsto g \circ F \in V^*$, is only reflected in the diagram by the arrows and their labels.

APPENDIX B. THE CANONICAL REGULAR ELEMENT

Here we recall additional results on the framework of C*-WHA not introduced in the main text. As a matter of fact, we are interested in describing the canonical regular element in terms of these. First, it is well-known that in any C*-WHA A there exists a unique non-degenerate two-sided normalized integral $h \in A$, known as the *Haar integral of A* ; see Definition 3.24 and Theorem 4.5 in [6]. In particular,

$$(26) \quad h^2 = h^* = h = S(h).$$

By self-duality, let $\hat{h} \in A^*$ denote the Haar integral of the dual C*-WHA. We also recall the existence of $\Lambda \in A$, known as the *dual left-integral of \hat{h}* , such that

$$(27) \quad \langle \hat{h}, \Lambda_{(1)} \rangle \Lambda_{(2)} = 1 \quad \text{and} \quad S(\Lambda_{(1)}) \otimes \Lambda_{(2)} = \Lambda_{(2)} \otimes \Lambda_{(1)};$$

see e.g. Theorem 3.18 and Lemma 3.20 in [6]. Second, there is a unique positive element $g \in A$ implementing the antipode squared as an inner automorphism, i.e.

$$(28) \quad S^2(x) = gxg^{-1}$$

for all elements $x \in A$, among other properties, known as the *canonical group-like element of A* ; see Proposition 4.9 in [6]. As its name implies, it is a *group-like element*, i.e. it satisfies the following property:

$$(29) \quad g_{(1)} \otimes g_{(2)} = g1_{(1)} \otimes g1_{(2)} = 1_{(1)}g \otimes 1_{(2)}g.$$

Moreover, it can be decomposed in the form $g = g_L g_R^{-1}$ for two $g_L, g_R > 0$ given by

$$(30) \quad g_L := (\langle \hat{h}, h_{(1)} \rangle h_{(2)})^{\frac{1}{2}} \in A_L \quad \text{and} \quad g_R := S(g_L) = S^{-1}(g_L) \in A_R.$$

By self-duality, we denote by $\hat{g} \in A^*$ the canonical group-like element of the dual C*-WHA. Finally, let us recall the following formula.

Proposition B.1. *For any C*-WHA,*

$$x_{(1)} \langle \hat{g}, x_{(2)} \rangle = g_R x g_R^{-1} \quad \text{and} \quad \langle \hat{g}, x_{(1)} \rangle x_{(2)} = g_L x g_L^{-1}$$

for all elements $x \in A$. In particular,

$$1_{(1)} \langle \hat{g}, 1_{(2)} \rangle = 1 = \langle \hat{g}, 1_{(1)} \rangle 1_{(2)}.$$

Proof. See Scholium 2.7 and Lemma 4.13 in [6] for a proof. \square

Proposition B.2 (see [35]). *For any connected C*-WHA A ,*

$$\text{Tr}_\alpha(g) = \langle \varepsilon, 1 \rangle d_\alpha$$

for all sectors $\alpha \in \text{Irr}(A)$.

Proposition B.3. *In any connected C*-WHA*

$$\langle \omega, x \rangle = \mathfrak{D}^{-2} \varepsilon(1)^{-1} \langle \hat{h}, g_L^{-1} g_R^{-1} x \rangle = \mathfrak{D}^{-2} \varepsilon(1)^{-1} \langle \hat{h}, x g_L^{-1} g_R^{-1} \rangle$$

for all elements $x \in A$. Equivalently, for any coconnected C*-WHA,

$$\Omega = \mathfrak{D}^{-2} \varepsilon(1)^{-1} \Lambda_{(1)} \langle \hat{g}^{-1}, \Lambda_{(2)} \rangle = \mathfrak{D}^{-2} \varepsilon(1)^{-1} \langle \hat{g}^{-1}, \Lambda_{(1)} \rangle \Lambda_{(2)}.$$

Proof. Assume first that A is a connected C*-WHA. There exists a well-known element, called the S -invariant trace of A , see [6], given by the expression $\sum_\alpha \text{Tr}_\alpha(g) \text{Tr}_\alpha$. By virtue of Theorem 2.12 and Proposition B.2, one easily checks that both elements are proportional. \square

Remark B.4 (see [33]). The linear map $T \in \mathfrak{L}(A)$ in Theorem 2.12 is given by

$$T(x) = S(x_{(1)}) \langle \hat{g}, x_{(2)} \rangle = \langle \hat{g}, x_{(1)} \rangle S^{-1}(x_{(2)})$$

Remark B.5 (see [33]). In any coconnected C*-WHA, $\omega \circ T = \omega \circ S = \omega$.

Finally, let us particularize the previous notions and results in the context of C*-Hopf algebras. We refer the reader to [34] for more details.

Proposition B.6. *Let A be a C*-HA. Then:*

- (1) $S^2 = \text{Id}$ and the canonical grouplike element is $g = 1$;
- (2) $d_\alpha = \dim_{\mathbb{C}}(V_\alpha)$ for all sectors $\alpha \in \text{Irr}(A)$;
- (3) the dual left integral of the Haar measure $\hat{h} \in A^*$ is $t = \mathfrak{D}^2 \Omega$;
- (4) the canonical regular element and the Haar integral coincide, i.e. $\Omega = h$;
- (5) the linear map $T \in \mathfrak{L}(A)$ coincides with the antipode $S \in \mathfrak{L}(A)$;

$$(6) \quad g_L = g_R = \mathfrak{D}^{-1}1.$$

Proof. (1) It is well-known that for any C*-HA it holds that $S^2 = \text{Id}$ [39, 40]. Since the unit element $1 \in A$ satisfies the defining properties of the canonical group-like element too, which is unique, we can conclude that $g = 1$. (2) Consider that $\varepsilon(1) = 1$ by [Definition 2.4](#) and hence [Proposition B.2](#) proves that $\dim_{\mathbb{C}}(V_{\alpha}) = \text{Tr}_{\alpha}(1) = \text{Tr}_{\alpha}(g) = \varepsilon(1)d_{\alpha} = d_{\alpha}$ for all sectors $\alpha \in \text{Irr}(A)$. (3) Since the axioms of C*-HAs are self-dual $\hat{g} = \varepsilon$ and hence $\Omega = \mathfrak{D}^{-2}\varepsilon(1)^{-1}t = \mathfrak{D}^{-2}t$, where the first expression follows from [Proposition B.3](#). (4) Every C*-HA is *unimodular*, see [34], i.e. every left integral is a two-sided integral, and the subspace of two-sided integrals is one-dimensional. Hence $t = \eta h$ for some $\eta \in \mathbb{C}$. Since $\Omega^2 = \Omega$ and $h^2 = h$, the only possibility left is $\eta = \mathfrak{D}^2$. (5) This follows trivially as a consequence of [Remark B.4](#) since $\hat{g} = \varepsilon$. (6) Recall the definition of g_L and g_R in [Equation 30](#) and consider both steps (3) and (4). \square

APPENDIX C. PROOF OF [LEMMA 3.6](#)

Here we restate and prove the following result.

Lemma 3.6. *Let A be a biconnected C*-WHA. There exists a unique element $\xi \in A$ such that $\langle \omega, \xi T(\Omega_{(1)}) \rangle \Omega_{(2)} = 1$. Furthermore, it satisfies the following properties:*

- (1) *it is positive, invertible and $\xi^{-1} = \langle \omega, \Omega_{(1)} \rangle \Omega_{(2)} = \langle \omega, T(\Omega_{(1)}) \rangle \Omega_{(2)}$;*
- (2) *it is invariant under $T \in \mathfrak{L}(A)$, i.e. $T(\xi) = \xi$;*
- (3) *it satisfies the relation $T(x)^* = \xi T(x^*) \xi^{-1}$ for all elements $x \in A$;*
- (4) *$\text{Tr}_{\alpha}(\xi^{-1}) = d_{\alpha} \langle \omega, \Omega \rangle$ for all sectors $\alpha \in \text{Irr}(A)$;*
- (5) *it can be decomposed as $\xi = \xi_L \xi_R$ for two positive elements $\xi_L \in A_L$ and $\xi_R = S(\xi_L) = S^{-1}(\xi_L) \in A_R$;*

Dually, if we denote $\hat{\xi} = \hat{\xi}_L \hat{\xi}_R \in A^$, then:*

- (6) *$x_{(1)} \langle \hat{\xi}_L, x_{(2)} \rangle = \xi_L^{-1} x$ and $x_{(1)} \langle \hat{\xi}_R^{-1}, x_{(2)} \rangle = x \xi_L^{-1}$ for all $x \in A$.*

Finally, if A is a C-HA, then $\xi_L^2 = \xi_R^2 = \xi = \mathfrak{D}^2 \varepsilon(1) 1 = \langle \omega, \Omega \rangle^{-1} 1$.*

Proof. Since $\Omega \in A$ is non-degenerate there exists a linear functional $f \in A^*$ such that $\langle f, \Omega_{(1)} \rangle \Omega_{(2)} = 1$; see [Equation 5](#) and [Theorem 2.12](#). On the other hand, since $\omega \in A^*$ is non-degenerate, there exists an element $\xi \in A$ such that $\langle \omega, \xi x \rangle = \langle f \circ T, x \rangle$ for all elements $x \in A$. Therefore,

$$\langle \omega, \xi T(\Omega_{(1)}) \rangle \Omega_{(2)} = \langle f \circ T, T(\Omega_{(1)}) \rangle \Omega_{(2)} = \langle f, \Omega_{(1)} \rangle \Omega_{(2)} = 1,$$

where in the second equality we have used that $T \in \mathfrak{L}(A)$ is involutive, i.e. $T \circ T = \text{Id}$. Recall that $\omega \in A^*$ is cocentral, it is a trace-like linear functional of A^* , see [Remark 2.2](#). It follows by the pulling-through identity in [Equation 11](#) that

$$1 = \langle \omega, \xi T(\Omega_{(1)}) \rangle \Omega_{(2)} = \langle \omega, T(\Omega_{(1)}) \xi \rangle \Omega_{(2)} = \langle \omega, T(\Omega_{(1)}) \rangle \xi \Omega_{(2)},$$

and hence $\xi \in A$ is invertible. Its inverse is then trivially given by the expression

$$\xi^{-1} = \langle \omega, T(\Omega_{(1)}) \rangle \Omega_{(2)} = \langle \omega, \Omega_{(1)} \rangle \Omega_{(2)}.$$

where the last equality follows from [Remark B.5](#). Let us prove now (4). By virtue of [Proposition B.3](#), it is easy to conclude by its defining property $\langle \omega, \xi T(\Omega_{(1)}) \rangle \Omega_{(2)} = 1$ that $\xi \in A$ is necessarily given by the expression

$$(31) \quad \xi = \mathfrak{D}^4 \varepsilon(1)^2 g_L g_R.$$

Consequently, a natural choice of positive elements $\xi_L \in A_L$ and $\xi_R \in A_R$ is

$$(32) \quad \xi_L := \mathfrak{D}^2 \varepsilon(1) g_L \quad \text{and} \quad \xi_R := \mathfrak{D}^2 \varepsilon(1) g_R = S(\xi_L).$$

Since $g_L, g_R > 0$, ξ is strictly positive, as we wanted to prove. We now prove (2), i.e. that $T(\xi) = \xi$, note by the previous expressions that it turns out to be enough to

check that $T(g_L) = g_R$ and $T(g_R) = g_L$. We refer to [Equations 43a](#) and [43b](#) below for elementary proofs of these facts. In addition, note that (4) is straightforward by the eigenvalue equation [Equation 12](#). See [Scholium 2.7](#) and [Lemma 4.13](#) from [\[6\]](#) for a proof of (6). Let us now move to the proof of (3). For simplicity, we prove the equivalent formula $\xi T(x)\xi^{-1} = T(x^*)^*$ for all $x \in A$. To this end, we recall first that

$$(33) \quad \xi y \xi^{-1} = g_L g_R y g_L^{-1} g_R^{-1} = \langle \hat{g}, y_{(1)} \rangle y_{(2)} \langle \hat{g}, y_{(3)} \rangle$$

for all elements $y \in A$, see [Proposition B.1](#). On the other hand, by virtue of the the fact that $S^{-1}(\hat{g}) = \hat{g}^{-1}$ and the positivity of $\hat{g} \in A^*$,

$$(34) \quad \langle \hat{g}^{-1}, y \rangle = \langle \hat{g}, S^{-1}(y) \rangle = \langle \hat{g}^*, S^{-1}(y) \rangle = \overline{\langle \hat{g}, S(S^{-1}(y))^* \rangle} = \overline{\langle \hat{g}, y^* \rangle}$$

for all elements $y \in A$. Thus,

$$\begin{aligned} \xi T(x)\xi^{-1} &= \xi S(x_{(1)})\xi^{-1} \langle \hat{g}, x_{(2)} \rangle && \text{by [Remark B.4](#)} \\ &= \langle \hat{g}, S(x_{(1)})_{(1)} \rangle S(x_{(1)})_{(2)} \langle \hat{g}, S(x_{(1)})_{(3)} \rangle \langle \hat{g}, x_{(2)} \rangle && \text{by [Equation 33](#)} \\ &= \langle \hat{g}, S(x_{(3)}) \rangle S(x_{(2)}) \langle \hat{g}, S(x_{(1)}) \rangle \langle \hat{g}, x_{(4)} \rangle && \text{by [Remark 2.3](#)} \\ &= \langle \hat{g}^{-1}, x_{(1)} \rangle S(x_{(2)}) \langle \hat{g}^{-1}, x_{(3)} \rangle \langle \hat{g}, x_{(4)} \rangle && \text{by [Equation 30](#)} \\ &= \langle \hat{g}^{-1}, x_{(1)} \rangle S(x_{(2)}) && \text{by [Remark 2.2](#)} \\ &= \langle \hat{g}^{-1}, x_{(1)} \rangle S^{-1}(S^2(x_{(2)})) \\ &= \langle \hat{g}^{-1}, x_{(1)} \rangle \langle \hat{g}, x_{(2)} \rangle S^{-1}(x_{(3)}) \langle \hat{g}^{-1}, x_{(4)} \rangle && \text{by [Equation 28](#)} \\ &= S^{-1}(x_{(1)}) \langle \hat{g}^{-1}, x_{(2)} \rangle && \text{by [Remark 2.2](#)} \\ &= S^{-1}(x_{(1)}) \overline{\langle \hat{g}, x_{(2)}^* \rangle} && \text{by [Equation 34](#)} \\ &= S(x_{(1)}^*)^* \overline{\langle \hat{g}, x_{(2)}^* \rangle} && \text{by [Remark 2.3](#)} \\ &= S((x^*)_{(1)})^* \overline{\langle \hat{g}, (x^*)_{(2)} \rangle} && \text{by [Definition 2.1](#)} \\ &= T(x^*)^*, && \text{by [Remark B.4](#)} \end{aligned}$$

for all elements $x \in A$, as we wanted to prove. Finally, if A is a C*-HA, it is already known by [Proposition B.6](#) that $g_L = g_R = \mathfrak{D}^{-1}1$. This, together with the definition of $\xi \in A$ in [Lemma 3.6](#) and the fact that $\varepsilon(1) = 1$, leads to the expressions $\xi_L = \xi_R = \mathfrak{D}1$ and $\xi = \mathfrak{D}^21$, as we wanted to prove. \square

APPENDIX D. PROOF OF [THEOREM 3.3](#)

We now provide algebraic explicit expressions for both local coarse-graining and fine-graining quantum channels. We restate and prove the following theorem now.

Theorem 3.3. *Let A be a biconnected C*-WHA and let (V, Φ) be a faithful *-representation of A . Then, the operators*

$$\rho(x, n) := \langle \omega, x \rangle^{-1} b(\omega)^{\otimes n} \Phi^{\otimes n}(\Delta^{(n-1)}(x)) \in \mathfrak{L}(V^{\otimes n})$$

are RFP MPDOs for all positive non-zero elements $x \in A$ and all $n \in \mathbb{N}$. Specifically, there are quantum channels $\mathfrak{T} : \mathfrak{L}(V) \rightarrow \mathfrak{L}(V \otimes V)$ and $\mathfrak{S} : \mathfrak{L}(V \otimes V) \rightarrow \mathfrak{L}(V)$, known as local fine-graining and coarse-graining maps, respectively, such that

$$\mathfrak{T}(\rho(x, 1)) = \rho(x, 2) \quad \text{and} \quad \mathfrak{S}(\rho(x, 2)) = \rho(x, 1)$$

for all positive non-zero elements $x \in A$ and all $n \in \mathbb{N}$.

Proof. As previously done, let us define the local coarse-graining quantum channel

$$(35) \quad \mathfrak{T}(X) := \text{Tr}(\Phi(\xi T(\Omega_{(1)}))X)\Phi(c_\omega\Omega_{(2)}) \otimes \Phi(c_\omega\Omega_{(3)})$$

for all $X \in \mathfrak{L}(V)$. First, let us check that $\mathfrak{T}(\rho(x, 1)) = \rho(x, 2)$ for all positive non-zero $x \in A$. Indeed,

$$\begin{aligned} \mathfrak{T}(\rho(x, 1)) &= \frac{1}{\langle \omega, x \rangle} \text{Tr}(\Phi(\xi T(\Omega_{(1)})c_\omega x))\Phi(c_\omega\Omega_{(2)}) \otimes \Phi(c_\omega\Omega_{(3)}) \\ &= \frac{1}{\langle \omega, x \rangle} \langle \omega, \xi T(\Omega_{(1)})x \rangle \Phi(c_\omega\Omega_{(2)}) \otimes \Phi(c_\omega\Omega_{(3)}) && \text{by Remark 3.2} \\ &= \frac{1}{\langle \omega, x \rangle} \langle \omega, \xi T(\Omega_{(1)}) \rangle \Phi(c_\omega x_{(1)}\Omega_{(2)}) \otimes \Phi(c_\omega x_{(2)}\Omega_{(3)}) && \text{by Equation 13} \\ &= \frac{1}{\langle \omega, x \rangle} \Phi(c_\omega x_{(1)}\mathbf{1}_{(1)}) \otimes \Phi(c_\omega x_{(2)}\mathbf{1}_{(2)}) && \text{by Lemma 3.6} \\ &= \frac{1}{\langle \omega, x \rangle} \Phi(c_\omega x_{(1)}) \otimes \Phi(c_\omega x_{(2)}) = \rho(x, 2) && \text{by Definition 2.1} \end{aligned}$$

Second, this map is trace-preserving:

$$\begin{aligned} \text{Tr}(\mathfrak{T}(X)) &= \text{Tr}(\Phi(\xi T(\Omega_{(1)}))X)\text{Tr}(\Phi(c_\omega\Omega_{(2)}))\text{Tr}(\Phi(c_\omega\Omega_{(3)})) \\ &= \text{Tr}(\Phi(\xi T(\Omega_{(1)}))X)\langle \omega, \Omega_{(2)} \rangle \langle \omega, \Omega_{(3)} \rangle && \text{by Remark 3.2} \\ &= \text{Tr}(\Phi(\xi T(\Omega_{(1)}))X)\langle \omega, \Omega_{(2)} \rangle && \text{by Lemma 3.1} \\ &= \text{Tr}(\Phi(\xi T(\xi^{-1}))X) && \text{by Lemma 3.6} \\ &= \text{Tr}(\Phi(\xi\xi^{-1})X)\text{Tr}(X) && \text{by Lemma 3.6} \end{aligned}$$

Finally, since $\Omega = \Omega^2 = \Omega\Omega^*$ (in fact, only positivity of Ω is needed), we can rewrite the map in the following form:

$$\begin{aligned} \mathfrak{T}(X) &= \text{Tr}(\Phi(\xi T(\Omega_{(1)}(\Omega^*)_{(1')})X)\Phi^{\otimes 2}(c_\omega^{\otimes 2}\Delta(\Omega_{(2)}(\Omega^*)_{(2')}))) \\ &= \text{Tr}(\Phi(\xi T(\Omega_{(1)}(\Omega^*)_{(1')})X)\Phi^{\otimes 2}(c_\omega^{\otimes 2}\Delta(\Omega_{(2)})\Delta((\Omega^*)_{(2')}))) \\ &= \text{Tr}(\Phi(\xi T(\Omega_{(1)}\Omega^*_{(1')})X)\Phi^{\otimes 2}(c_\omega^{\otimes 2}\Delta(\Omega_{(2)})\Delta(\Omega_{(2')})*)) \\ &= \text{Tr}(\Phi(\xi T(\Omega_{(1)}^*)T(\Omega_{(1)}))X)\Phi^{\otimes 2}(c_\omega^{\otimes 2}\Delta(\Omega_{(2)})\Delta(\Omega_{(2')})*)) \\ &= \text{Tr}(\Phi(T(\Omega_{(1')})^*\xi T(\Omega_{(1)}))X)\Phi^{\otimes 2}(c_\omega^{\otimes 2}\Delta(\Omega_{(2)})\Delta(\Omega_{(2')})*)) \\ &= \text{Tr}(\Phi(T(\Omega_{(1')})^*\xi^{\frac{1}{2}}\xi^{\frac{1}{2}}T(\Omega_{(1)}))X)\Phi^{\otimes 2}((c_\omega^{\frac{1}{2}})^{\otimes 2}\Delta(\Omega_{(2)})\Delta(\Omega_{(2')})* (c_\omega^{\frac{1}{2}})^{\otimes 2}) \\ &= \text{Tr}(\Phi(T(\Omega_{(1')})^*\xi^{\frac{1}{2}})\Phi(\xi^{\frac{1}{2}}T(\Omega_{(1)}))X)\Phi^{\otimes 2}((c_\omega^{\frac{1}{2}})^{\otimes 2}\Delta(\Omega_{(2)}))\Phi^{\otimes 2}(\Delta(\Omega_{(2')})* (c_\omega^{\frac{1}{2}})^{\otimes 2}) \\ &= \text{Tr}(\Phi(\xi^{\frac{1}{2}}T(\Omega_{(1)}))X)\Phi(\xi^{\frac{1}{2}}T(\Omega_{(1')})^\dagger)\Phi^{\otimes 2}((c_\omega^{\frac{1}{2}})^{\otimes 2}\Delta(\Omega_{(2)}))\Phi^{\otimes 2}((c_\omega^{\frac{1}{2}})^{\otimes 2}\Delta(\Omega_{(2')}))^\dagger \\ &= (\text{Tr} \otimes \text{Id} \otimes \text{Id})(Q(X \otimes \mathbf{1} \otimes \mathbf{1})Q^\dagger) \end{aligned}$$

where

$$Q := \Phi^{\otimes 3}(\xi^{\frac{1}{2}}T(\Omega_{(1)}) \otimes c_\omega^{\frac{1}{2}}\Omega_{(2)} \otimes c_\omega^{\frac{1}{2}}\Omega_{(3)})$$

Thus, \mathfrak{T} is completely positive. Now, let us define a local fine-graining quantum channel \mathfrak{S} . Consider first the following hermitian projectors

$$(36) \quad P := \Phi^{\otimes 2}(\Delta(1)), \quad P^\perp := \Phi^{\otimes 2}(1 \otimes 1 - \Delta(1)), \quad P + P^\perp = \mathbf{1} \otimes \mathbf{1}$$

and let $\rho_0 \in \mathfrak{L}(V)$ be any mixed state. Define

$$(37) \quad \mathfrak{S}(X) := \text{Tr}(\Phi(\Delta(\xi T(\Omega_{(1)})))X)\Phi(c_\omega\Omega_{(2)}) + \text{Tr}(P^\perp X)\rho_0$$

for all elements $X \in \mathfrak{L}(V \otimes V)$. We first check that it satisfies $\mathfrak{S}(\rho(x, 2)) = \rho(x, 1)$ for all positive non-zero $x \in A$. Notice that the second summand in the right-hand side of Equation 37 simply vanishes, i.e. $P^\perp \rho(x, 2) = 0$, since $\rho(x, 2)$ is supported on the orthogonal subspace $P \cdot \mathfrak{L}(V^{\otimes 2})$. Thus,

$$\begin{aligned} \mathfrak{S}(\rho(x, 2)) &= \frac{1}{\langle \omega, x \rangle} \text{Tr}(\Phi^{\otimes 2}(c_\omega^{\otimes 2}\Delta(\xi T(\Omega_{(1)})x))\Phi(c_\omega\Omega_{(2)})) && \text{by Definition 2.1} \\ &= \frac{1}{\langle \omega, x \rangle} \langle \omega \otimes \omega, \Delta(\xi T(\Omega_{(1)})x) \rangle \Phi(c_\omega\Omega_{(2)}) && \text{by Remark 3.2} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\langle \omega, x \rangle} \langle \omega, \xi T(\Omega_{(1)})x \rangle \Phi(c_\omega \Omega_{(2)}) && \text{by Lemma 3.1} \\
&= \frac{1}{\langle \omega, x \rangle} \langle \omega, \xi T(\Omega_{(1)}) \rangle \Phi(c_\omega x \Omega_{(2)}) && \text{by Equation 11} \\
&= \frac{1}{\langle \omega, x \rangle} \Phi(c_\omega x) = \rho(x, 1) && \text{by Lemma 3.6}
\end{aligned}$$

for all positive non-zero elements $x \in A$, as we wanted to prove. Secondly, let us check that it is trace-preserving:

$$\begin{aligned}
\text{Tr}(\mathfrak{S}(X)) &= \text{Tr}(\Phi^{\otimes 2}(\Delta(\xi T(\Omega_{(1)})))X) \text{Tr}(\Phi(c_\omega \Omega_{(2)})) + \text{Tr}(P^\perp X) \\
&= \text{Tr}(\Phi^{\otimes 2}(\Delta(\xi T(\Omega_{(1)})))X) \langle \omega, \Omega_{(2)} \rangle + \text{Tr}(P^\perp X) && \text{by Remark 3.2} \\
&= \text{Tr}(\Phi^{\otimes 2}(\Delta(\xi T(\xi^{-1})))X) + \text{Tr}(P^\perp X) && \text{by Lemma 3.6} \\
&= \text{Tr}(\Phi^{\otimes 2}(\Delta(\xi \xi^{-1}))X) + \text{Tr}(P^\perp X) && \text{by Lemma 3.6} \\
&= \text{Tr}(PX) + \text{Tr}(P^\perp X) = \text{Tr}((P + P^\perp)X) = \text{Tr}(X) && \text{by Equation 36}
\end{aligned}$$

for all $X \in \mathfrak{L}(V \otimes V)$. That \mathfrak{S} is completely positive can be proved analogously and we do not include it here: simply notice that the second summand in Equation by Equation 37 is clearly a completely positive map, and a similar argument to that for \mathfrak{T} applies to the first summand. \square

APPENDIX E. PROOF OF LEMMA 5.2

In this appendix we derive a proof of Lemma 5.2. We first provide the following auxiliary result, related to the trace-preserving condition of the gluing map.

Lemma E.1. *Let A be a C^* -HA. Then,*

$$x_{(1)} \otimes \langle \omega, x_{(2)} \rangle x_{(3)} = \langle \omega, x \rangle 1 \otimes 1$$

for all elements $x \in A$.

Proof. Fix any $x \in A$. Since $\Omega \in A$ is non-degenerate, there exists $f \in A^*$ such that

$$(38) \quad x = \Omega_{(1)} \langle f, \Omega_{(2)} \rangle.$$

As an immediate consequence,

$$(39) \quad \langle \omega, x \rangle = \langle \omega, \Omega_{(1)} \rangle \langle f, \Omega_{(2)} \rangle = \mathfrak{D}^{-2} \langle f, 1 \rangle,$$

where the last equality follows from Lemma 3.6. Then, it is easy to conclude that

$$\begin{aligned}
x_{(1)} \langle \omega, x_{(2)} \rangle \otimes x_{(3)} &= \Omega_{(1)} \otimes \langle \omega, \Omega_{(2)} \rangle \Omega_{(3)} \langle f, \Omega_{(4)} \rangle && \text{by Equation 38} \\
&= \Omega_{(4)} \otimes \langle \omega, \Omega_{(1)} \rangle \Omega_{(2)} \langle f, \Omega_{(3)} \rangle && \text{by Theorem 2.12} \\
&= \mathfrak{D}^{-2} 1_{(3)} \otimes 1_{(1)} \langle f, 1_{(2)} \rangle && \text{by Lemma 3.6} \\
&= \mathfrak{D}^{-2} \langle f, 1 \rangle 1 \otimes 1 && \text{by Definition 2.4} \\
&= \langle \omega, x \rangle 1 \otimes 1, && \text{by Equation 39}
\end{aligned}$$

as we wanted to prove. \square

Lemma 5.2. *Let A be a C^* -HA and let (V, Φ) be a faithful $*$ -representation of A . Then, for all positive non-zero elements $x \in A$ there exists a quantum channel $\mathfrak{G}_x \in \mathfrak{L}(\mathfrak{L}(V \otimes V))$, called “gluing” map, such that*

$$(24) \quad (\text{Id}^{\otimes m-1} \otimes \mathfrak{G}_x \otimes \text{Id}^{\otimes n-1})(\rho(\Omega, m) \otimes \rho(\Omega, n)) = \rho(x, m+n)$$

for all $m, n \in \mathbb{N}$.

Proof. Fix any positive non-zero element $x \in A$. We recall first the definition of the gluing map previously given in [Section 5](#). For simplicity, let $\mathfrak{G}_x := \mathfrak{T} \circ \mathfrak{G}$ for the linear map $\mathfrak{G} \in \mathfrak{L}(\mathfrak{L}(V \otimes V), \mathfrak{L}(V))$ defined by the expression

$$(40) \quad \mathfrak{G}(X \otimes Y) := \frac{1}{\langle \omega, x \rangle} \text{Tr}(\Phi(S(x_{(1)}))X) \Phi(c_\omega x_{(2)}) \text{Tr}(\Phi(S(x_{(3)}))Y)$$

for all $X, Y \in \mathfrak{L}(V)$. Then, it is enough to check that $\mathfrak{G}(\rho(\Omega, 2) \otimes \rho(\Omega, 2)) = \rho(\Omega, 3)$. To this end, let us recall that, in the case of C*-HAs,

$$(41) \quad \langle \omega, \Omega_{(1)} \rangle \Omega_{(2)} = \frac{1}{\varepsilon^2} \mathbf{1} = \langle \omega, \Omega \rangle \mathbf{1},$$

where the first equality is stated in [Lemma 3.6](#) and the second equality follows by applying the counit in the first one, since $\varepsilon(1) = 1$. Then,

$$\begin{aligned} (\text{Id} \otimes \mathfrak{G} \otimes \text{Id})(\rho(\Omega, 2) \otimes \rho(\Omega, 2)) &= \\ &= \frac{1}{\langle \omega, x \rangle} \frac{1}{\langle \omega, \Omega \rangle^2} \Phi(c_\omega \Omega_{(1)}) \otimes \langle \omega, S(x_{(1)}) \Omega_{(2)} \rangle \Phi(c_\omega x_{(2)}) \langle \omega, S(x_{(3)}) \Omega_{(1')} \rangle \otimes \Phi(c_\omega \Omega_{(2')}) \\ &= \frac{1}{\langle \omega, x \rangle} \frac{1}{\langle \omega, \Omega \rangle^2} \Phi(c_\omega x_{(1)} \Omega_{(1)}) \otimes \langle \omega, \Omega_{(2)} \rangle \Phi(c_\omega x_{(2)}) \langle \omega, S(x_{(3)}) \Omega_{(1')} \rangle \otimes \Phi(c_\omega \Omega_{(2')}) \\ &= \frac{1}{\langle \omega, x \rangle} \frac{1}{\langle \omega, \Omega \rangle^2} \Phi(c_\omega x_{(1)} \Omega_{(1)}) \otimes \langle \omega, \Omega_{(2)} \rangle \Phi(c_\omega x_{(2)}) \langle \omega, \Omega_{(1')} \rangle \otimes \Phi(c_\omega x_{(3)} \Omega_{(2')}) \\ &= \frac{1}{\langle \omega, x \rangle} \Phi(c_\omega x_{(1)} \mathbf{1}) \otimes \Phi(c_\omega x_{(2)}) \otimes \Phi(c_\omega x_{(3)} \mathbf{1}) = \rho(x, 3). \end{aligned}$$

This calculation can be explained as follows. In the first place, we have replaced the trace with the canonical regular element $\omega \in A^*$ since by [Remark 3.2](#) the weight $c_\omega \in A$, which is central, defines a linear extension of ω to the representation space. In the second and third steps we have applied the pulling-through identity; see [Equation 11](#). Finally, we apply twice [Equation 41](#) to get rid of Ω and the coefficients $\langle \omega, \Omega \rangle^{-1}$. As an aside, note that $\langle \omega, \Omega_{(1)} \rangle \Omega_{(2)} = \Omega_{(1)} \langle \omega, \Omega_{(2)} \rangle$ since Ω is cocentral; see [Equation 16](#). Since \mathfrak{T} is a quantum channel it only remains to prove that \mathfrak{G} is also a quantum channel. On the one hand, that \mathfrak{G} is trace-preserving is a straightforward consequence of [Lemma E.1](#):

$$\begin{aligned} \text{Tr}(\mathfrak{G}(X \otimes Y)) &= \frac{1}{\langle \omega, x \rangle} \text{Tr}(\Phi(S(x_{(1)}))X) \langle \omega, x_{(2)} \rangle \text{Tr}(\Phi(S(x_{(3)}))Y) \quad \text{by Remark 3.2} \\ &= \frac{1}{\langle \omega, x \rangle} \langle \omega, x \rangle \text{Tr}(\Phi(S(1))X) \text{Tr}(\Phi(S(1))Y) \quad \text{by Lemma E.1} \\ &= \text{Tr}(X) \text{Tr}(Y) = \text{Tr}(X \otimes Y) \quad \text{by Remark 2.3} \end{aligned}$$

for all $X, Y \in \mathfrak{L}(V)$. On the other hand, in order to prove that \mathfrak{G} is completely positive, let $x = yy^*$ for some element $y \in A$. Then, we can rewrite it as follows

$$\begin{aligned} \mathfrak{G}(X \otimes Y) &= \\ &= \frac{1}{\langle \omega, x \rangle} \text{Tr}(\Phi(S((yy^*)_{(1)}))X) \Phi(c_\omega (yy^*)_{(2)}) \text{Tr}(\Phi(S(yy^*)_{(3)})Y) \\ &= \frac{1}{\langle \omega, x \rangle} \text{Tr}(\Phi(S(y_{(1)}y_{(1')}^*))X) \Phi(c_\omega y_{(2)}y_{(2')}^*) \text{Tr}(\Phi(S(y_{(3)}y_{(3')}^*))Y) \\ &= \frac{1}{\langle \omega, x \rangle} \text{Tr}(\Phi(S(y_{(1')}^*)S(y_{(1)}))X) \Phi(c_\omega y_{(2)}y_{(2')}^*) \text{Tr}(\Phi(S(y_{(3')}^*)S(y_{(3)}))Y) \\ &= \frac{1}{\langle \omega, x \rangle} \text{Tr}(\Phi(S(y_{(1')}^*)^*S(y_{(1)}))X) \Phi(c_\omega y_{(2)}y_{(2')}^*) \text{Tr}(\Phi(S(y_{(3')}^*)^*S(y_{(3)}))Y) \\ &= \frac{1}{\langle \omega, x \rangle} \text{Tr}(\Phi(S(y_{(1)}))X \Phi(S(y_{(1')}^*)) \Phi(c_\omega y_{(2)}y_{(2')}^*) \text{Tr}(\Phi(S(y_{(3)}))Y \Phi(S(y_{(3')}^*)) \\ &= (\text{Tr} \otimes \text{Id} \otimes \text{Tr})(Q(X \otimes \mathbf{1} \otimes Y)Q^\dagger) \end{aligned}$$

where we have defined

$$(42) \quad Q := \frac{1}{\langle \omega, x \rangle^{1/2}} \Phi^{\otimes 4}(S(y_{(1)}) \otimes c_\omega^{1/2} y_{(2)} \otimes \otimes S(y_{(3)})).$$

Therefore, \mathfrak{G} is completely positive. Indeed, in the first step we have applied that the comultiplication is multiplicative and the $*$ -operation is a coalgebra homomorphism; see [Definition 2.1](#). In the second and third steps we have used that $S \in \mathfrak{L}(A)$ is an algebra anti-homomorphism and the relation between the antipode and the $*$ -operation; see [Remark 2.3](#). Note that, for C*-HAs, $S = S^{-1}$; see [Proposition B.6](#).

The fourth step is a simple consequence of the fact that Φ is a $*$ -representation and the cyclic property of the trace. Finally, the middle term can be rewritten in the form $\Phi(c_\omega y_{(2)} y_{(2')}^*) = \Phi(c_\omega^{1/2} y_{(2)}) \Phi(c_\omega^{1/2} y_{(2')})^\dagger$ since $c_\omega \in A$ is positive central element and Φ is a $*$ -representation. \square

APPENDIX F. PROOF OF LEMMA 5.6

In this appendix we prove Lemma 5.6. In order to perform an analogous construction of this gluing map to the one given in the C^* -HA case, we first derive an appropriate version of the usual pulling-through identity in Equation 11 to the trivial sector.

Lemma F.1. *Let A be a biconnected C^* -WHA. Then,*

$$x_L S(1_{(1)}) \otimes 1_{(2)} \otimes S(1_{(3)}) y_R = S(1_{(1)}) \otimes y_R 1_{(2)} x_L \otimes S(1_{(3)})$$

for all elements $x_L \in A_L$ and $y_R \in A_R$.

Proof. First, recall Equations 2.31a and 2.31b from [6]:

$$\begin{aligned} x_L S(1_{(1)}) \otimes 1_{(2)} &= S(1_{(1)}) \otimes 1_{(2)} x_L, \\ y_R 1_{(1)} \otimes S(1_{(2)}) &= 1_{(1)} \otimes S(1_{(2)}) y_R. \end{aligned}$$

for all $x_L \in A_L$ and $y_R \in A_R$. This, together with Definitions 2.11, leads by taking coproducts accordingly to the following identities:

$$\begin{aligned} x_L S(1_{(1)}) \otimes 1_{(2)} \otimes 1_{(3)} &= S(1_{(1)}) \otimes 1_{(2)} x_L \otimes 1_{(3)}, \\ 1_{(1)} \otimes y_R 1_{(2)} \otimes S(1_{(3)}) &= 1_{(1)} \otimes 1_{(2)} \otimes S(1_{(3)}) y_R, \end{aligned}$$

respectively, for all elements $x_L \in A_L$ and $y_R \in A_R$. Finally, since A_L and A_R commute, we conclude the result by combining both identities. \square

In addition, we adapt slightly Lemma 3.6 to the trivial sector, which is a key property concerning complete positivity of the gluing map in Lemma 5.6. The following result solves this problem.

Lemma F.2. *Let A be a biconnected C^* -WHA. Then,*

$$\xi_R S(x_L^*) = S(x_L)^* \xi_R \quad \text{and} \quad S(y_R) \xi_L = \xi_L S(y_R^*)^*$$

for all elements $x_L \in A_L$ and $y_R \in A_R$.

Proof. In the first place, note that $T \in \mathfrak{L}(A)$ coincides with S and S^{-1} restricted to A_L and A_R , respectively. Indeed, by virtue of Remark B.4, Proposition B.1 and Remark 2.3,

$$(43a) \quad T(x_L) = S(x_L 1_{(1)}) \langle \hat{g}, 1_{(2)} \rangle = S(x_L),$$

$$(43b) \quad T(y_R) = \langle \hat{g}, 1_{(1)} \rangle S^{-1}(1_{(2)} y_R) = S^{-1}(y_R) = S(y_R^*)^*,$$

for all $x_L \in A_L$ and $y_R \in A_R$. Then, recall Lemma 3.6 to conclude that

$$S(x_L^*) = T(x_L^*) = \xi_L^{-1} \xi_R^{-1} T(x_L)^* \xi_L \xi_R = \xi_L^{-1} \xi_R^{-1} S(x_L)^* \xi_L \xi_R = \xi_R^{-1} S(x_L)^* \xi_R,$$

where in the last step we have used that $S(x_L) \in A_R$ and A_L and A_R commute. The remaining identity is proved similarly. \square

The following auxiliary results arise naturally in the course of the derivation of the properties of the gluing map.

Lemma F.3. *Let A be a biconnected C^* -WHA. Then,*

$$\langle \hat{h}, \Omega_{(1)} \rangle \Omega_{(2)} = \mathfrak{D}^{-2} \varepsilon(1)^{-1} 1.$$

Proof. It is easy to check that

$$\mathfrak{D}^2\varepsilon(1)\langle\hat{h}, \Omega_{(1)}\rangle\Omega_{(2)} = \langle\hat{h}, t_{(1)}\rangle t_{(2)}\langle\hat{g}, t_{(3)}\rangle = 1_{(1)}\langle\hat{g}, 1_{(2)}\rangle = 1,$$

where the first step is a consequence of the characterization of $\Omega \in A$ in [Proposition B.3](#), the second follows from the definition of dual left integral in [Equation 27](#) and the third equality is due to [Proposition B.1](#). \square

Lemma F.4. *Let A be a biconnected C^* -WHA. Then,*

$$(44) \quad 1_{(1)}\langle\hat{h}, 1_{(2)}\rangle \otimes 1_{(3)} = \varepsilon(1)^{-1}1 \otimes 1.$$

Proof. Equivalently, we will check that

$$\langle\phi\hat{h}\psi, 1\rangle = \varepsilon(1)^{-1}\langle\phi, 1\rangle\langle\psi, 1\rangle$$

for all $\phi, \psi \in A^*$. Recall that $\hat{h} \in A^*$ is a one-dimensional projector supported on the trivial sector [[6](#), [Lemma 4.8](#)]. Hence,

$$(45) \quad \langle\phi\hat{h}\psi\hat{h}, \text{Tr}^1\rangle = \langle\phi\hat{h}, \text{Tr}^1\rangle\langle\psi\hat{h}, \text{Tr}^1\rangle \quad \text{and} \quad \langle\phi\hat{h}, \text{Tr}^a\rangle = \delta_a$$

for all $\phi, \psi \in A^*$ and all sectors $a \in \text{Irr}(A^*)$. In particular

$$(46) \quad (f\hat{h})(\text{Tr}^1) = (f\hat{h})\left(\sum_a \hat{d}_a \text{Tr}^a\right) = \mathfrak{D}^2(f\hat{h})(\Omega) = \varepsilon(1)^{-1}\langle f, 1\rangle$$

for all $f \in A^*$. Thus, we conclude that:

$$\varepsilon(1)^{-1}\langle\phi\hat{h}\psi, 1\rangle = \langle\phi\hat{h}\psi\hat{h}, \text{Tr}^1\rangle = \langle\phi\hat{h}, \text{Tr}^1\rangle\langle\psi\hat{h}, \text{Tr}^1\rangle = \varepsilon(1)^{-2}\langle\phi, 1\rangle\langle\psi, 1\rangle,$$

where the first equality follows from [Equation 46](#) using $f := \phi\hat{h}\psi$, the second equality is simply [Equation 45](#) and the third equality follows from [Equation 46](#) considering $f := \phi, \psi$. \square

Lemma F.5. *Let A be a biconnected C^* -WHA. Then,*

$$1_{(1)} \otimes \langle\omega, 1_{(2)}\rangle 1_{(3)} = \mathfrak{D}^2\xi_R^{-1} \otimes \xi_L^{-1}.$$

Proof. Note by the definition of A_L and A_R in [Definitions 2.11](#) and the decomposition $\xi^{-1} = \xi_L^{-1}\xi_R^{-1}$ in [Lemma 3.6](#), that

$$(47) \quad (\xi^{-1})_{(1)} \otimes (\xi^{-1})_{(2)} \otimes (\xi^{-1})_{(3)} = \xi_L^{-1}1_{(1)} \otimes 1_{(2)} \otimes \xi_R^{-1}1_{(3)}$$

Then, the statement follows from the following calculation:

$$\begin{aligned} 1_{(1)} \otimes \langle\omega, 1_{(2)}\rangle 1_{(3)} &= \mathfrak{D}^2\varepsilon(1)\langle\hat{h}, \Omega_{(1)}\rangle\Omega_{(2)} \otimes \langle\omega, \Omega_{(3)}\rangle\Omega_{(4)} && \text{by Lemma F.3} \\ &= \mathfrak{D}^2\varepsilon(1)\langle\hat{h}, \Omega_{(3)}\rangle\Omega_{(4)} \otimes \langle\omega, \Omega_{(1)}\rangle\Omega_{(2)} && \text{by Equation 16} \\ &= \mathfrak{D}^2\varepsilon(1)\langle\hat{h}, (\xi^{-1})_{(2)}\rangle(\xi^{-1})_{(3)} \otimes (\xi^{-1})_{(1)} && \text{by Lemma 3.6} \\ &= \mathfrak{D}^2\varepsilon(1)\langle\hat{h}, 1_{(2)}\rangle\xi_R^{-1}1_{(3)} \otimes \xi_L^{-1}1_{(1)} && \text{by Equation 47} \\ &= \mathfrak{D}^2\xi_R^{-1} \otimes \xi_L^{-1} && \text{by Lemma F.4} \end{aligned}$$

as we wanted to prove. \square

Lemma F.6. *Let A be a biconnected C^* -WHA. Then,*

$$(48) \quad \langle\omega, 1_{(1)}\rangle 1_{(2)}\langle\omega, 1_{(3)}\rangle = \mathfrak{D}^2\langle\omega, 1\rangle\xi^{-1}.$$

Proof. First, it will be useful to compute the constant $\langle\omega, 1\rangle$ in a more operative way. The following calculation is a direct consequence of [Proposition B.3](#) and [Equation 32](#):

$$(49) \quad \langle\omega, 1\rangle = \frac{1}{\mathfrak{D}^2\varepsilon(1)}\langle\hat{h}, g_L^{-1}g_R^{-1}\rangle = \frac{\mathfrak{D}^4\varepsilon(1)^2}{\mathfrak{D}^2\varepsilon(1)}\langle\hat{h}, \xi_L^{-1}\xi_R^{-1}\rangle = \mathfrak{D}^2\varepsilon(1)\langle\hat{h}, \xi_L^{-1}\xi_R^{-1}\rangle.$$

Now, by an analogous reasoning as in the previous proof:

$$\langle\omega, 1_{(1)}\rangle 1_{(2)}\langle\omega, 1_{(3)}\rangle = \mathfrak{D}^2\varepsilon(1)\langle\hat{h}, \Omega_{(1)}\rangle\langle\omega, \Omega_{(2)}\rangle\Omega_{(3)}\langle\omega, \Omega_{(4)}\rangle \quad \text{by Lemma F.3}$$

$$\begin{aligned}
&= \mathfrak{D}^2 \varepsilon(1) \langle \hat{h}, \Omega_{(4)} \rangle \langle \omega, \Omega_{(1)} \rangle \Omega_{(2)} \langle \omega, \Omega_{(3)} \rangle && \text{by Equation 16} \\
&= \mathfrak{D}^2 \varepsilon(1) \langle \hat{h}, \xi_R^{-1} 1_{(3)} \rangle \xi_L^{-1} 1_{(1)} \langle \omega, 1_{(2)} \rangle && \text{by Lemma 3.6} \\
&= \mathfrak{D}^4 \varepsilon(1) \langle \hat{h}, \xi_R^{-1} \xi_L^{-1} \rangle \xi_L^{-1} \xi_R^{-1} && \text{by Lemma F.5} \\
&= \mathfrak{D}^2 \langle \omega, 1 \rangle \xi_L^{-1} \xi_R^{-1} = \mathfrak{D}^2 \langle \omega, 1 \rangle \xi^{-1} && \text{by Equation 49}
\end{aligned}$$

as we wanted to prove. \square

Remark 5.4. There are no trace-preserving gluing maps for general biconnected C^* -WHAs such that Equation 24 holds for all elements $x \in A$.

Proof. Suppose by contradiction that there exists a trace-preserving linear map $\mathfrak{G} \in \mathfrak{L}(\mathfrak{L}(V \otimes V))$ that is a “gluing map”. In particular,

$$(\text{Id} \otimes \mathfrak{G} \otimes \text{Id})(\rho(\Omega, 2) \otimes \rho(\Omega, 2)) = \rho(\Omega, 4).$$

On the one hand, after performing a partial trace on the second and third subsystems, the left-hand side would be trivially given by the product state

$$\begin{aligned}
(\text{Id} \otimes \text{Tr} \otimes \text{Tr} \otimes \text{Id})(\rho(\Omega, 2) \otimes \rho(\Omega, 2)) &= \langle \omega, \Omega \rangle^{-2} \Phi(c_\omega \Omega_{(1)}) \langle \omega, \Omega_{(2)} \rangle \otimes \langle \omega, \Omega_{(1')} \rangle \Phi(c_\omega \Omega_{(2')}) \\
&= \langle \omega, \Omega \rangle^{-2} \Phi(c_\omega \xi^{-1}) \otimes \Phi(c_\omega \xi^{-1})
\end{aligned}$$

by virtue of Remark 3.2 and Lemma 3.6. However, the right-hand side would take the following form:

$$\begin{aligned}
(\text{Id} \otimes \text{Tr} \otimes \text{Tr} \otimes \text{Id})(\rho(\Omega, 4)) &= \langle \omega, \Omega \rangle^{-2} \Phi(c_\omega \Omega_{(1)}) \otimes \langle \omega, \Omega_{(2)} \rangle \langle \omega, \Omega_{(3)} \rangle \Phi(c_\omega \Omega_{(4)}) && \text{by Remark 3.2} \\
&= \langle \omega, \Omega \rangle^{-2} \Phi(c_\omega \Omega_{(1)}) \otimes \langle \omega, \Omega_{(2)} \rangle \Phi(c_\omega \Omega_{(3)}) && \text{by Lemma 3.1} \\
&= \langle \omega, \Omega \rangle^{-2} \Phi(c_\omega \Omega_{(2)}) \otimes \langle \omega, \Omega_{(3)} \rangle \Phi(c_\omega \Omega_{(1)}) && \text{by Equation 16} \\
&= \langle \omega, \Omega \rangle^{-2} \Phi(c_\omega \xi_R^{-1} 1_{(2)}) \otimes \Phi(c_\omega \xi_L^{-1} 1_{(1)}) && \text{by Lemma 3.6}
\end{aligned}$$

which is not a product state. This contradicts the previous equation. \square

Lemma 5.6. Let A be a biconnected C^* -WHA and let (V, Φ) be a faithful $*$ -representation of A . There is a quantum channel $\mathfrak{G}_1 \in \mathfrak{L}(\mathfrak{L}(V \otimes V))$, called “gluing” map, such that

$$(25) \quad (\text{Id}^{\otimes m-1} \otimes \mathfrak{G}_1 \otimes \text{Id}^{\otimes n-1})(\rho(1, m) \otimes \rho(1, n)) = \rho(1, m+n)$$

for all $m, n \in \mathbb{N}$.

Proof. For simplicity, let $\mathfrak{G}_1 := \mathfrak{T} \circ \mathfrak{G}$, where $\mathfrak{T} : \mathfrak{L}(V) \rightarrow \mathfrak{L}(V \otimes V)$ stands for the local coarse-graining quantum channel from Section 3 and $\mathfrak{G} : \mathfrak{L}(V \otimes V) \rightarrow \mathfrak{L}(V)$ is given by

$$\mathfrak{G}(X \otimes Y) := \frac{1}{\mathfrak{D}^2} \text{Tr}(\Phi(S(1_{(1)}) \xi_L) X) \Phi(c_\omega 1_{(2)}) \text{Tr}(\Phi(\xi_R S(1_{(3)})) Y)$$

for all $X, Y \in \mathfrak{L}(V)$. First, assume that $m = n = 2$ without loss of generality and let us check that it fulfills $\mathfrak{G}(\rho(1, 2) \otimes \rho(1, 2)) = \rho(1, 3)$. To this end, it turns out to be enough to prove:

$$(50) \quad \mathfrak{G}(\Phi(c_\omega x_L) \otimes \Phi(c_\omega x_R)) = \langle \omega, 1 \rangle \Phi(c_\omega x_L x_R)$$

for all $x_L \in A_L$ and $x_R \in A_R$. Indeed, in that case,

$$\begin{aligned}
(\text{Id} \otimes \mathfrak{G} \otimes \text{Id})(\rho(1, 2)^{\otimes 2}) &= \frac{1}{\langle \omega, 1 \rangle^2} \Phi(c_\omega 1_{(1)}) \otimes \mathfrak{G}(\Phi(c_\omega 1_{(2)}) \otimes \Phi(c_\omega 1_{(1')})) \otimes \Phi(c_\omega 1_{(2')}) \\
&= \frac{1}{\langle \omega, 1 \rangle} \Phi(c_\omega 1_{(1)}) \otimes \Phi(c_\omega 1_{(2)} 1_{(1')}) \otimes \Phi(c_\omega 1_{(2')}) \\
&= \frac{1}{\langle \omega, 1 \rangle} \Phi(c_\omega 1_{(1)}) \otimes \Phi(c_\omega 1_{(2)}) \otimes \Phi(c_\omega 1_{(3)}) = \rho(1, 3).
\end{aligned}$$

by the weak comultiplicativity of the counit and the fact that $1_{(1)} \otimes 1_{(2)} \in A_R \otimes A_L$; see Definition 2.1 and [6]. Thus, let us move to the proof of Equation 50:

$$\mathfrak{G}(\Phi(c_\omega x_L) \otimes \Phi(c_\omega x_R)) = \frac{1}{\mathfrak{D}^2} \langle \omega, S(1_{(1)}) \xi_L x_L \rangle \Phi(c_\omega 1_{(2)}) \langle \omega, \xi_R S(1_{(3)}) x_R \rangle \quad \text{by Remark 3.2}$$

$$\begin{aligned}
&= \frac{1}{\mathfrak{D}^2} \langle \omega, \xi_L x_L S(1_{(1)}) \rangle \Phi(c_\omega 1_{(2)}) \langle \omega, S(1_{(3)}) x_R \xi_R \rangle \\
&= \frac{1}{\mathfrak{D}^2} \langle \omega, S(1_{(1)}) \rangle \Phi(c_\omega x_R \xi_R 1_{(2)} \xi_L x_L) \langle \omega, S(1_{(3)}) \rangle \quad \text{by Lemma F.1} \\
&= \frac{1}{\mathfrak{D}^2} \langle \omega, 1_{(1)} \rangle \Phi(c_\omega x_R \xi_R 1_{(2)} \xi_L x_L) \langle \omega, 1_{(3)} \rangle \quad \text{by Remark B.5} \\
&= \langle \omega, 1 \rangle \Phi(c_\omega x_R \xi_R \xi_R^{-1} \xi_L^{-1} \xi_L x_L) \quad \text{by Equation 48} \\
&= \langle \omega, 1 \rangle \Phi(c_\omega x_R x_L)
\end{aligned}$$

as we wanted to prove. Additionally, \mathfrak{G} is trace-preserving as an immediate consequence of [Lemma F.5](#):

$$\begin{aligned}
\text{Tr}(\mathfrak{G}(X \otimes Y)) &= \frac{1}{\mathfrak{D}^2} \text{Tr}(\Phi(S(1_{(1)}) \xi_L) X) \langle \omega, 1_{(2)} \rangle \text{Tr}(\Phi(\xi_R S(1_{(3)})) Y) \quad \text{by Remark 3.2} \\
&= \text{Tr}(\Phi(S(\xi_R^{-1}) \xi_L) X) \text{Tr}(\Phi(\xi_R S(\xi_L^{-1})) Y) \quad \text{by Lemma F.5} \\
&= \text{Tr}(\Phi(\xi_L^{-1} \xi_L) X) \text{Tr}(\Phi(\xi_R \xi_R^{-1}) Y) \quad \text{by Equation 32} \\
&= \text{Tr}(X) \text{Tr}(Y) = \text{Tr}(X \otimes Y).
\end{aligned}$$

Finally, in order to check that \mathfrak{G} is a completely positive linear map, let us first consider the following two calculations:

$$\begin{aligned}
\text{Tr}(\Phi(S(x_R y_R^*) \xi_L) X) &= \text{Tr}(\Phi(S(y_R^*) S(x_R) \xi_L) X) \quad \text{by Remark 2.3} \\
&= \text{Tr}(\Phi(S(y_R^*) \xi_L S(x_R^*)^*) X) \quad \text{by Lemma F.2} \\
&= \text{Tr}(\Phi(S(y_R^*) \xi_L^{\frac{1}{2}} \xi_L^{\frac{1}{2}} S(x_R^*)^*) X) \quad \text{by Equation 32} \\
&= \text{Tr}(\Phi(S(y_R^*) \xi_L^{\frac{1}{2}}) \Phi(\xi_L^{\frac{1}{2}} S(x_R^*)^*) X) \\
&= \text{Tr}(\Phi(\xi_L^{\frac{1}{2}} S(x_R^*)^*) X \Phi(S(y_R^*) \xi_L^{\frac{1}{2}})) \\
&= \text{Tr}(\Phi(\xi_L^{\frac{1}{2}} S(x_R^*)^*) X \Phi((\xi_L^{\frac{1}{2}} S(y_R^*))^*)) \quad \text{by Equation 32} \\
&= \text{Tr}(\Phi(\xi_L^{\frac{1}{2}} S(x_R^*)^*) X \Phi(\xi_L^{\frac{1}{2}} S(y_R^*)^*)^\dagger)
\end{aligned}$$

for all $x_R, y_R \in A_R$ and, analogously,

$$\begin{aligned}
\text{Tr}(\Phi(\xi_R S(x_L y_L^*)) Y) &= \text{Tr}(\Phi(\xi_R S(y_L^*) S(x_L)) Y) \quad \text{by Remark 2.3} \\
&= \text{Tr}(\Phi(S(y_L^*)^* \xi_R S(x_L)) Y) \quad \text{by Lemma F.2} \\
&= \text{Tr}(\Phi(S(y_L^*)^* \xi_R^{\frac{1}{2}} \xi_R^{\frac{1}{2}} S(x_L)) Y) \quad \text{by Equation 32} \\
&= \text{Tr}(\Phi(S(y_L^*)^* \xi_R^{\frac{1}{2}}) \Phi(\xi_R^{\frac{1}{2}} S(x_L)) Y) \\
&= \text{Tr}(\Phi(\xi_R^{\frac{1}{2}} S(x_L)) Y \Phi(S(y_L^*)^* \xi_R^{\frac{1}{2}})) \\
&= \text{Tr}(\Phi(\xi_R^{\frac{1}{2}} S(x_L)) Y \Phi((\xi_R^{\frac{1}{2}} S(y_L^*))^*)) \quad \text{by Equation 32} \\
&= \text{Tr}(\Phi(\xi_R^{\frac{1}{2}} S(x_L)) Y \Phi(\xi_R^{\frac{1}{2}} S(y_L^*)^*)^\dagger)
\end{aligned}$$

for all $x_L, y_L \in A_L$. Now, recall that $1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} \in A_R \otimes A \otimes A_L$; see [\[6\]](#). This allows us to rewrite \mathfrak{G} in the following form:

$$\begin{aligned}
\mathfrak{G}(X \otimes Y) &= \frac{1}{\mathfrak{D}^2} \text{Tr}(\Phi(S((1 \cdot 1^*)_{(1)}) \xi_L) X) \Phi(c_\omega (1 \cdot 1^*)_{(2)}) \text{Tr}(\Phi(\xi_R S((1 \cdot 1^*)_{(3)})) Y) \\
&= \frac{1}{\mathfrak{D}^2} \text{Tr}(\Phi(S(1_{(1)} (1^*)_{(1')}) \xi_L) X) \Phi(c_\omega 1_{(2)} (1^*)_{(2')}) \text{Tr}(\Phi(\xi_R S(1_{(3)} 1_{(3')})) Y) \\
&= \frac{1}{\mathfrak{D}^2} \text{Tr}(\Phi(S(1_{(1)} 1_{(1')}) \xi_L) X) \Phi(c_\omega 1_{(2)} 1_{(2')}) \text{Tr}(\Phi(\xi_R S(1_{(3)} 1_{(3')})) Y) \\
&= (\text{Tr} \otimes \text{Id} \otimes \text{Id})(Q(X \otimes \mathbf{1} \otimes Y) Q^\dagger)
\end{aligned}$$

where the last step follows from the previous calculations, and we have defined

$$(51) \quad Q := \frac{1}{\mathfrak{D}} \Phi^{\otimes 3}(\xi_L^{\frac{1}{2}} S(1_{(1)}^*) \otimes c_\omega^{\frac{1}{2}} 1_{(2)} \otimes \xi_R^{\frac{1}{2}} S(1_{(3)})).$$

This concludes the proof. □

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