

Better-Than- $\frac{4}{3}$ -Approximations for Leaf-to-Leaf Tree and Connectivity Augmentation*

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Abstract

The Connectivity Augmentation Problem (CAP) together with a well-known special case thereof known as the Tree Augmentation Problem (TAP) are among the most basic Network Design problems. There has been a surge of interest recently to find approximation algorithms with guarantees below 2 for both TAP and CAP, culminating in the currently best approximation factor for both problems of 1.393 through quite sophisticated techniques.

We present a new and arguably simple matching-based method for the well-known special case of leaf-to-leaf instances. Combining our work with prior techniques, we readily obtain a $(\frac{4}{3} + \varepsilon)$ -approximation for Leaf-to-Leaf CAP by returning the better of our solution and one of an existing method. Prior to our work, a $\frac{4}{3}$ -guarantee was only known for Leaf-to-Leaf TAP instances on trees of height 2. Moreover, when combining our technique with a recently introduced stack analysis approach, which is part of the above-mentioned 1.393-approximation, we can further improve the approximation factor to 1.29, obtaining for the first time a factor below $\frac{4}{3}$ for a nontrivial class of TAP/CAP instances.

1 Introduction

The Connectivity Augmentation Problem (CAP) is one of the most elementary Network Design problems. It asks to increase the edge-connectivity of a graph by one unit in the most economical way by adding edges/links from a given set. Formally, one is given a graph $G = (V, E)$ and an additional link set $L \subseteq \binom{V}{2}$, and the task is to determine a smallest size set of links $U \subseteq L$ such that the edge-connectivity of $(V, E \cup U)$ is strictly larger than that of G . A famous special case of CAP is the Tree Augmentation Problem (TAP), where the given graph G is a spanning tree. Already TAP is well-known to be APX-hard (see [KKL04], which presents an extension of a construction used in [FJ81] to prove NP-hardness), even on trees of diameter 5 with all links going between pairs of leaves, i.e., *leaf-to-leaf instances*. This motivated the search for strong constant-factor approximations. There has been extensive work during the last decades on the approximability of both TAP and CAP, and also their weighted counterparts where each link has a cost and instead of minimizing the size of U the goal is to minimize its cost.

For CAP (and therefore also TAP), multiple 2-approximations have been known for a long time, including through classical techniques like primal-dual algorithms and iterative rounding

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(see [FJ81, KT93, GGP⁺94, Jai01]). It was an important stepping stone to reach algorithms with an approximation guarantee below 2. During the last decades, a long line of research led to multiple approaches that beat the approximation factor 2 for TAP and CAP through the introduction of a rich set of techniques [Adj18, CG18b, CG18a, CKKK08, CN13, EFKN09, FGKS18, FJ81, GKZ18, KT93, KN16, KN18, Nag03, Nut20, GKZ18, CTZ20]. This led to the current state-of-the-art approximation factor of 1.393 [CTZ20], which is currently the best one for both TAP and CAP. The factor of 1.393 was obtained through a combination of quite sophisticated techniques, and the obtained factor is neither natural nor is it likely to be the “right” answer, i.e., it seems very likely that approximation algorithms with better approximation guarantees should exist.

A natural question we are interested in, is whether there may be a clean algorithm leading to a $4/3$ -approximation. This factor appears in other, related Network Design problems, in particular in the (unweighted) 2-Edge-Connected Spanning Subgraph problem (2-ECCS). In 2-ECCS, one starts with an empty graph G and the task is to pick a smallest number of links to obtain a 2-edge-connected graph spanning all vertices. Hence, instead of starting from a spanning tree to obtain a 2-edge-connected graph as in TAP, one starts with an empty graph. Recent advances on 2-ECCS led to the best-known approximation factor of $4/3$ [SV14, HVV19]. In the context of TAP and CAP, we still lack appropriate techniques to reach such factors, and progress along this line has only been achieved for quite restricted special cases. More precisely, for TAP, a factor of $4/3$ is known to be achievable if we are given an optimal solution to the natural LP relaxation, known as the *cut-LP*, that has the additional property of being half-integral [CJR99] or, more generally, fulfills that each non-zero entry is at least $1/2$ [IR18]. However, the *cut-LP* is in general not a half-integral LP [CJR99] and it may not contain any optimal point where each non-zero is at least $1/2$. Moreover, an approach for TAP was presented in [MN10] that leads to a $4/3$ -approximation for Leaf-to-Leaf TAP instances on trees of height at most 2. Finally, for CAP, we are unaware of any nontrivial class of instances where such factors, or even factors below the currently best 1.393-approximation, are known. Note that, even for TAP, natural special cases for which approximation factors below $4/3$ can be achieved are unknown to the best of our knowledge.

The goal of this paper is to make first progress in this regard for the case of Leaf-to-Leaf CAP (and therefore also TAP) instances. (We formally define Leaf-to-Leaf CAP instances in Section 1.1 and show their relation to Leaf-to-Leaf TAP and also Leaf-to-Leaf Cactus Augmentation, which is a well-known connection on which we heavily rely later on.) We think of Leaf-to-Leaf CAP instances as an appealing class because of the following reasons. First, they comprise a large family of nontrivial TAP/CAP instances, which have already been studied both in the context of TAP [MN10] and CAP [Nut21]. Second, the best known hardness results for TAP/CAP are based on leaf-to-leaf instances.

We also note that, for the weighted settings of TAP/CAP, any instance can be reduced in an approximation-preserving way to a weighted leaf-to-leaf one; hence, the difference between leaf-to-leaf and general instances vanishes in the weighted settings.¹

¹Consider a general weighted TAP instance $G = (V, E)$ with links $L \subseteq \binom{V}{2}$ and weights $w : L \rightarrow \mathbb{R}_{\geq 0}$. To transform it into a leaf-to-leaf instance, one can do the following operation for each vertex v : (i) Add two new vertices w_v^1, w_v^2 to the graph and connect each of them only to v ; hence, w_v^1 and w_v^2 are leaves; (ii) For each link $\ell \in L$ that has v as an endpoint, replace the endpoint v by w_v^1 ; (iii) Add a new link $\{w_v^1, w_v^2\}$ of cost 0. One can easily observe that the original instance is equivalent to the obtained leaf-to-leaf one, because, after including the newly added 0-cost links, one falls back to the original instance. An analogous construction works to reduce weighted CAP to weighted Leaf-to-Leaf CAP.

1.1 Preliminaries

Consider a CAP instance $(G = (V, E), L)$ on a graph G that is k -edge-connected (but not $(k + 1)$ -edge-connected). Hence, the goal is to add a smallest number of links $U \subseteq L$ such that $(V, E \cup U)$ is $(k + 1)$ -edge-connected. By Menger’s Theorem, a set $U \subseteq L$ leads to a $(k + 1)$ -connected graph $(V, E \cup U)$ if and only if each minimum cut in G is crossed by at least one link of U . (G, L) is a *Leaf-to-Leaf CAP* instance if, for each link $\{u, v\} \in L$, both u and v are contained in a minimal minimum cut. The *minimal minimum cuts* in G are minimum cuts $A \subseteq V$ for which no other minimum cut $B \subseteq V$ satisfies $B \subseteq A$. Classic uncrossing results imply that the minimal minimum cuts of a graph form a family of disjoint sets. Note that in case of a tree, these cuts are singleton cuts only containing a single leaf vertex. Thus, a leaf-to-leaf instance for a tree indeed maps to only having links with both endpoints being leaves.

For CAP, it is often significantly more convenient to first use a well-known reduction to the *Cactus Augmentation Problem* (CacAP), which is the special case of CAP where the underlying graph is a cactus, i.e., it is a connected graph with each edge being contained in a unique cycle. See Figure 1 for an example. This reduction from CAP to CacAP is approximation preserving and

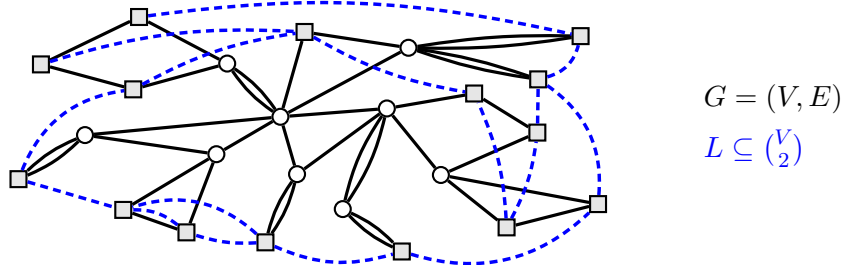


Figure 1: The black solid graph $G = (V, E)$ is a cactus. Its vertices of degree 2 are called *leaves* or *terminals* and are depicted as gray squares. Together with the dashed edges, which represent the links L and only go between leaves, we obtain a Leaf-to-Leaf CacAP instance.

an immediate consequence of the fact that the minimum cuts in a graph can be represented by a cactus (see [DKL76]). Vertices of degree 2 in a cactus are also called *terminals* or *leaves* and the reduction from CAP to CacAP implies that Leaf-to-Leaf CAP instances are transformed into Leaf-to-Leaf CacAP instances. (See Figure 1 for an example of a Leaf-to-Leaf CacAP instance.) Due to this equivalence, we therefore focus on the more structured Leaf-to-Leaf CacAP instances.

Note that any Leaf-to-Leaf TAP instance $(G = (V, E), L)$ can easily be cast as a Leaf-to-Leaf CacAP instance by adding for each edge $e \in E$ a parallel edge.

1.2 Our results

Our main result is the following.

Theorem 1. *There is a 1.29-approximation algorithm for Leaf-to-Leaf CAP (and therefore also Leaf-to-Leaf TAP).*

In the context of leaf-to-leaf instances, this improves on the 1.393-approximation of [CTZ20] (which also works for CAP instances that are not leaf-to-leaf) and also improves on (and is applicable to a much broader set of instances than) the $\frac{4}{3}$ -approximation for Leaf-to-Leaf TAP on trees of height 2 of [MN10].

Our main technical contribution, which is the central ingredient of our approach, is an arguably elegant technique to find a good CAP solution by first computing a maximum weight matching over the links with respect to judiciously chosen weights. This matching is then complemented through a simple LP to an actual CAP solution. We provide a detailed discussion of our matching-based approach in Section 2.

We highlight that [MN10], for the special case of TAP, also used an approach based on first computing a matching and extending it to a solution. Their approach to extend the matching to a solution uses a credit-based argument, whereas our matching-based approach relies on an LP.

1.3 Brief overview of main components

Similar to prior approaches in the field, our matching-based approach provides a guarantee that can be expressed in terms of different link types. To this end, given a CacAP instance $(G = (V, E), L)$, we can fix an arbitrary root $r \in V$ and define link types with respect to this root as follows. A link $\{u, v\} \in L$ is called *in-link*, if both u and v lie in the same connected component of $G - r$, where $G - r := G[V \setminus \{r\}]$ is the subgraph of G induced by $V \setminus \{r\}$, i.e., the graph obtained from G by removing the root and all edges incident with it. All other links are called *cross links*. (See Fig. 2.) We denote the sets of all in-links and cross-links by L_{in} and L_{cross} , respectively, and for any link set $U \subseteq L$, we use the shorthands $U_{\text{in}} := U \cap L_{\text{in}}$ and $U_{\text{cross}} := U \cap L_{\text{cross}}$.

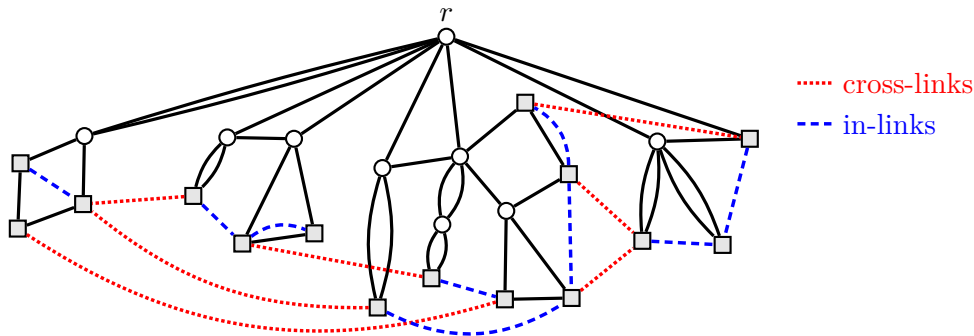


Figure 2: A Leaf-to-Leaf CacAP instance with root r on top. The leaves are drawn as gray squares. The dashed lines represent the links, which can be partitioned into cross-links (dotted in red) and in-links (dashed in blue).

Our matching-based approach leads to a combinatorial procedure that returns a solution with the guarantee stated below.

Theorem 2. *For any Leaf-to-Leaf CacAP instance (G, L) , we can efficiently compute a solution $F \subseteq L$ such that $|F| \leq |H| + \frac{1}{2}|H_{\text{in}}|$ for any solution H of the instance.*

In particular, the cardinality of the solution F we return is no larger than the cardinality of any optimal solution H plus half of the number of in-links of H . Clearly, this immediately implies a $3/2$ -approximation for Leaf-to-Leaf CacAP (and therefore also Leaf-to-Leaf CAP). The guarantees obtained through Theorem 2 are a strengthening, for the leaf-to-leaf case, of guarantees obtainable by prior methods, in particular one developed in [FGKS18] for TAP and extended in [CTZ20] to CacAP, which leads to a guarantee of $|F| \leq |H| + |H_{\text{in}}|$.

Our approach readily leads to better factors than $3/2$ when combined with prior techniques that have been developed recently and can be translated to the leaf-to-leaf setting. These approaches

are based on a reduction introduced in [Adj18], and later extended in [GKZ18, CTZ20], which first reduces the given instance to a better structured one, known as k -wide (for constant k).

Definition 3 (k -wide CacAP). *Let $k \in \mathbb{Z}_{\geq 1}$. A CacAP instance (G, L) is k -wide if there is a vertex r such that each connected component of $G - r$ contains at most k leaves of G . We call r a k -wide root of G , or simply a root. Moreover, for each vertex set $W \subseteq V \setminus \{r\}$ of a connected component of $G - r$, we call $G[W \cup \{r\}]$ an r -principal subcactus of G .*

In particular, the Leaf-to-Leaf CacAP instance highlighted in Fig. 2 is 6-wide, with respect to the indicated root r , and has 4 principal subcacti.

The following statement shows that, to obtain approximation algorithms for Leaf-to-Leaf CacAP, it suffices to consider $O(1)$ -wide instances (up to an arbitrarily small error in the approximation factor).

Theorem 4. *Let $\alpha \geq 1$ and $\varepsilon > 0$. Given an α -approximation algorithm \mathcal{A} for any $O(1/\varepsilon^3)$ -wide Leaf-to-Leaf CacAP, there is an $\alpha \cdot (1 + \varepsilon)$ -approximation algorithm \mathcal{B} for (unrestricted) Leaf-to-Leaf CacAP that calls \mathcal{A} at most polynomially many times and performs further operations taking polynomial time.*

Theorem 4 has been proven for (non leaf-to-leaf) CacAP [CTZ20]. As we discuss later, a slight modification of the reduction used in [CTZ20] allows for translating this result to the leaf-to-leaf setting, where we want to make sure that the $O(1)$ -wide CacAP instance maintains the property of being also leaf-to-leaf if the original instance was leaf-to-leaf.

On $O(1)$ -wide CacAP instances (even weighted ones), we can leverage the following result from [CTZ20], which is a consequence of a technique in [BFG⁺14].

Lemma 5 ([CTZ20]). *For any weighted k -wide CacAP instance $(G = (V, E), L)$ with link costs $c \in \mathbb{R}_{\geq 0}^L$, we can compute in time $3^k \text{poly}(|V|)$ a CacAP solution $F \subseteq L$ with $c(F) \leq c(H) + c(H_{\text{cross}})$ for any solution H of the instance.*

The above statement is obtained by solving independently and optimally the CacAP problems on each principal subcactus of a given k -wide instance and then returning the union of all these solutions. This is the step that requires k to be constant to be efficiently executable.

Note how the guarantee given by Lemma 5 is complementary to the one we obtain through our matching-based procedure as described in Theorem 2. By returning the better of the two, we immediately obtain a $4/3$ -approximation for $O(1)$ -wide Leaf-to-Leaf CacAP.

Corollary 6. *Let $(G = (V, E), L)$ be an $O(1)$ -wide Leaf-to-Leaf CacAP instance. Computing a solution $F_1 \subseteq L$ as claimed by Theorem 2 and a solution $F_2 \subseteq L$ as claimed by Lemma 5 (with c being unit weights), and returning the one of smaller cardinality, leads to a $4/3$ -approximation.*

Proof. Let $\text{OPT} \subseteq L$ be an optimal solution of (G, L) . By Theorem 2, we have $|F_1| \leq |\text{OPT}| + \frac{1}{2}|\text{OPT}_{\text{in}}|$, and Lemma 5 provides $|F_2| \leq |\text{OPT}| + |\text{OPT}_{\text{cross}}|$. Hence, the better of the two has size

$$\min\{|F_1|, |F_2|\} \leq \frac{2}{3}|F_1| + \frac{1}{3}|F_2| \leq |\text{OPT}| + \frac{1}{3}|\text{OPT}_{\text{in}}| + \frac{1}{3}|\text{OPT}_{\text{cross}}| = \frac{4}{3}|\text{OPT}|,$$

as desired. □

Thus, Corollary 6 immediately implies, together with Theorem 4, that there is a $(4/3 + \varepsilon)$ -approximation for Leaf-to-Leaf CacAP. Finally, a recently introduced technique based on stack analysis [CTZ20] allows for obtaining approximation factors below $4/3$, and implies the claimed 1.29-approximation for Leaf-to-Leaf CacAP (and therefore also Leaf-to-Leaf CAP) as stated in Theorem 1.

1.4 Organization of paper

In [Section 2](#), we introduce our main new technical ingredient, namely a simple matching-based approach to derive Leaf-to-Leaf CacAP (or TAP) solutions, which leads to [Theorem 2](#). The reduction to $O(1)$ -wide Leaf-to-Leaf CacAP instances is discussed in [Section 3](#) with some additional explanation of how precisely we reuse results from prior work in [Appendix A](#). [Appendix B](#) discusses how the stack analysis approach of [\[CTZ20\]](#) allows for obtaining approximation factors below $4/3$ -factor, leading to the claimed 1.29-approximation for Leaf-to-Leaf CAP.

2 Our matching-based approach

We now describe our matching-based approach, which leads to [Theorem 2](#). In fact, we will show a slight generalization of [Theorem 2](#), which applies to a slightly larger problem class which we call *leaf-to-leaf+ instances*. This will be useful later on when we combine our matching-based approach with other algorithms to obtain our main result, [Theorem 1](#).

Definition 7 (Leaf-to-Leaf+ CacAP instance). *A CacAP instance $(G = (V, E), L)$ is a leaf-to-leaf+ instance if it has a root $r \in V$ such that every endpoint of a link in L is the root or a leaf of G .*

The main result of this section is the following, which immediately implies [Theorem 2](#).

Theorem 8. *For any Leaf-to-Leaf+ CacAP instance (G, L) , we can efficiently compute a solution $F \subseteq L$ such that $|F| \leq |H| + \frac{1}{2}|H_{\text{in}}|$ for any solution H of the instance.*

To describe our matching-based approach, consider a Leaf-to-Leaf+ CacAP instance $\mathcal{I} = (G = (V, E), L)$ together with a root $r \in V$. We denote by \mathcal{C} the set of 2-cuts of G , where, by convention, we only consider cuts not containing r , i.e.,

$$\mathcal{C} := \{C \subseteq V \setminus \{r\} : |\delta_E(C)| = 2\}.$$

Hence, the task is to find a smallest link set crossing all sets in \mathcal{C} .

We will first compute a large matching on the leaves T of G and then show that we can cheaply complete the matching to a solution F with the desired properties. On a high level, it seems intuitive that good solutions contain large matchings. In particular, a simple lower bound on the number of links needed in any solution is given by $|T|/2$, because each leaf needs to have a link incident with it. Any instance with an optimal solution close to this lower bound must thus contain a very large matching. However, simply computing a maximum cardinality matching and then completing it to a solution does not generally lead to strong solutions. See [Fig. 3](#) for an example.

As we formalize in the following, a key reason for a matching not to have a good completion is that it contains a certain type of links, which we call *bad*. For the special case of TAP, Maduel and Nutov [\[MN10\]](#) already identified these links as being undesirable and called them *redundant*.

Definition 9 (Bad link). *For a cut $C \in \mathcal{C}$, let $T_C \subseteq T \cap C$ be the set of leaves in the cut C that are endpoints of links covering the cut C . We say that a link $\ell = \{u, v\} \in L$ is a bad link if $T_C \subseteq \{u, v\} \subseteq C$ for some $C \in \mathcal{C}$.*

[Fig. 3](#) highlights an example of a bad link.

The key lemma in our matching-based approach is the following, which we show in [Section 2.2](#). In words, it says that large matchings without bad links can be cheaply augmented to a CacAP solution.

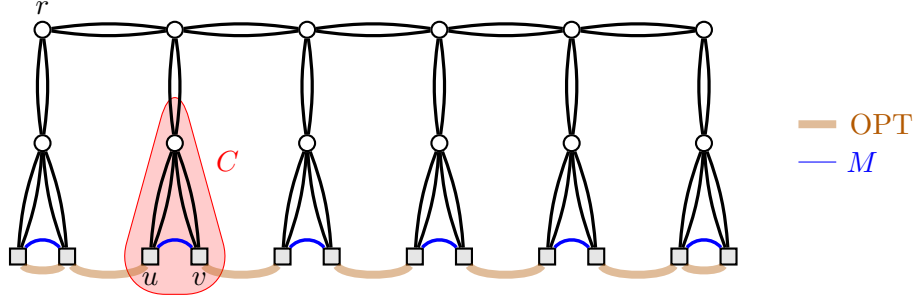


Figure 3: A Leaf-to-Leaf TAP instance, represented as a CacAP instance. The links are shown as blue lines and thick orange lines. The thick orange links show an optimal solution, which has cardinality 7. The blue are the unique maximum cardinality matching M on the link set. However, complementing the matching M to a solution requires at least 5 extra links (all orange/OPT ones except for the left-most and right-most one), leading to a solution of cardinality at least 12. By making the example wider, the ratio between the solution obtained by (optimally) complementing M and OPT approaches 2. The blue link $\{u, v\}$ is an example of a bad link, because there is a 2-cut, highlighted in red, such that each link crossing C has either u or v as one of its endpoints.

Lemma 10. *Given a feasible Leaf-to-Leaf+ CacAP instance (G, L) and a matching $M \subseteq L$ on the leaves of G without bad links, we can efficiently find a CacAP solution $F \subseteq L$ with $M \subseteq F$ such that:*

$$|F| \leq |M| + \frac{1}{2}|M_{\text{in}}| + (|T| - 2|M|). \quad (1)$$

Our matching-based algorithm now simply computes a matching without bad links that minimizes the right-hand side of (1), and completes it to a solution with the guarantee claimed by Lemma 10. Note that finding a matching without bad links that minimizes

$$|M| + \frac{1}{2}|M_{\text{in}}| + (|T| - 2|M|) = -|M_{\text{cross}}| - \frac{1}{2}|M_{\text{in}}| + |T|,$$

can easily be done by deleting all bad links and non leaf-to-leaf links and finding a maximum weight matching over the remaining links where each cross-link has a weight of 1 and each in-link has a weight of $1/2$.

To obtain that our matching-based procedure leads to a solution with the guarantees claimed by Theorem 2, we show that there exist matchings without bad links that lead to a cheap CacAP solution through Lemma 10.

Lemma 11. *For any Leaf-to-Leaf+ CacAP instance (G, L) , there exists a matching $M \subseteq L$ on the leaves of G without bad links such that*

$$|M| + \frac{1}{2}|M_{\text{in}}| + (|T| - 2|M|) \leq |H| + \frac{1}{2}|H_{\text{in}}|$$

for any solution H of the instance.

Finally, Theorem 2 is a straightforward consequence of the above statements.

Proof of Theorem 2. As discussed, we can efficiently find a matching M that minimizes the right-hand side of (1), and then use Lemma 10 to efficiently obtain a solution F . Lemma 11 implies that this solution has the desired guarantees. \square

We now provide details on the steps performed in our algorithm to obtain a solution $F \supseteq M$ from a leaf-to-leaf matching M without bad links as claimed in [Lemma 10](#). To extend M to a CacAP solution $M \cup U$, for some link set $U \subseteq L$, we need that U covers all 2-cuts not yet covered by M . Hence, $M \cup U$ is a CacAP solution if and only if $U \cap \delta_L(C) \neq \emptyset$ for all $C \in \mathcal{C}^M$, where

$$\mathcal{C}^M := \{C \subseteq V \setminus \{r\} : |\delta_E(C)| = 2 \text{ and } \delta_L(C) \cap M = \emptyset\}.$$

To find a good extension U , we use an LP based on directed links that is integral, which is an idea introduced in [\[CTZ20\]](#). More precisely, we use the following directed link set, which contains for each original link two antiparallel directed links:

$$\vec{L} := \bigcup_{\{u,v\} \in L} \{(u,v), (v,u)\},$$

and solve the following LP to find a good extension U :

$$\min \left\{ x(\vec{L}) : x \in \mathbb{R}_{\geq 0}^{\vec{L}}, x(\delta_{\vec{L}}^-(C)) \geq 1 \quad \forall C \in \mathcal{C}^M \right\}. \quad (\text{dir-LP})$$

In words, the LP requires one to “buy” directed links, and a cut is only counted as covered if a directed link is entering it.² Whereas (dir-LP) is thus not a relaxation of the problem of finding the best completion U for M , it can be shown to be integral with vertices being $\{0,1\}$ -vectors (see [Section 2.2](#)). Moreover, it has an optimal objective value that is no more than twice as expensive as the optimal completion. This follows from the observation that one can set, for each link $\{u,v\}$ in an optimal completion U^* , the values of x for both (u,v) and (v,u) to 1 to obtain a feasible solution for (dir-LP).

Our matching-based approach is summarized in [Algorithm 1](#).

Algorithm 1: Our matching-based approach

(1) Compute a leaf-to-leaf matching $M \subseteq L$ without bad links maximizing $w(M)$ for

$$w(\ell) := \begin{cases} 1 & \text{if } \ell \text{ is a cross-link} \\ \frac{1}{2} & \text{if } \ell \text{ is an in-link.} \end{cases}$$

(2) Compute an optimal vertex solution x^* of the LP below (x^* is a $\{0,1\}$ -vector):

$$\begin{aligned} \min \quad & x(\vec{L}) \\ & x(\delta_{\vec{L}}^-(C)) \geq 1 \quad \forall C \in \mathcal{C}^M \\ & x \in \mathbb{R}_{\geq 0}^{\vec{L}}. \end{aligned}$$

Let $U := \{\{u,v\} \in L : x^*((u,v)) = 1 \text{ or } x^*((v,u)) = 1\}$.

(3) Return $M \cup U$.

In the rest of this section, we provide the details to show that [Algorithm 1](#) indeed leads to a Leaf-to-Leaf+ CacAP solution $F = M \cup U$ satisfying $|F| \leq |H| + \frac{1}{2}|H_{\text{in}}|$ for any other solution H , which thus implies [Theorem 8](#). To this end, we first provide a proof for [Lemma 11](#) in [Section 2.1](#), thus showing that a good matching M exists. We then show in [Section 2.2](#) that our completion procedure used in [Algorithm 1](#) leads to a solution with the desired guarantees.

²We highlight that, in the context of TAP, (dir-LP) corresponds to the well-known integral up-link LP, which first replaces each link $\{u,v\}$ by at most two other links, one from each endpoint of the link to the vertex of the unique u - v path in the tree that lies closest to the root.

2.1 Existence of good matching (proof of Lemma 11)

We now provide a proof of Lemma 11, which shows that there is a good matching M .

Proof of Lemma 11. Let $H \subseteq L$ be any solution of the given Leaf-to-Leaf+ CacAP instance $(G = (V, E), L)$. Let $M^H \subseteq H$ be an inclusion-wise maximal matching consisting only of leaf-to-leaf links in H that are not bad. We denote by $T_{\text{cov}} \subseteq T$ the set of leaves covered by M^H . Notice that each link $\ell \in H \setminus M^H$ satisfies either

- (i) one endpoint of ℓ belongs to T_{cov} , or
- (ii) ℓ is a bad link.

Let $H^{\text{bad}} \subseteq H$ be the set of bad links in H for which (i) does not hold. The following claim upper bounds the number of links in H^{bad} .

Claim 12.

$$\sum_{v \in T \setminus T_{\text{cov}}} |\delta_H(v)| \geq |T \setminus T_{\text{cov}}| + |H^{\text{bad}}|.$$

Before proving Claim 12, we show that it implies the desired result. Assuming Claim 12, we get

$$\begin{aligned} |H| &= |M^H| + |H \setminus M^H| \geq |M^H| + \sum_{v \in T \setminus T_{\text{cov}}} |\delta_H(v)| - |H^{\text{bad}}| \geq |M^H| + |T \setminus T_{\text{cov}}| \\ &= |M^H| + (|T| - 2|M^H|), \end{aligned} \tag{2}$$

where the first inequality holds because all links in H that are incident to a vertex in $T \setminus T_{\text{cov}}$ are contained in $H \setminus M^H$ and only links in H^{bad} have both endpoints in $T \setminus T_{\text{cov}}$, and the second inequality follows from Claim 12. Hence, if $M \subseteq L$ is a matching using only leaf-to-leaf links that are not bad and such that it minimizes $|M| + \frac{1}{2}|M_{\text{in}}| + (|T| - 2|M|)$ among all such matchings, we obtain

$$|M| + \frac{1}{2}|M_{\text{in}}| + (|T| - 2|M|) \leq |M^H| + \frac{1}{2}|M_{\text{in}}^H| + (|T| - 2|M^H|) \leq |H| + \frac{1}{2}|H_{\text{in}}|,$$

where the first inequality follows from the fact that M has been chosen to minimize that expression and the second one is due to (2) and $\frac{1}{2}|M_{\text{in}}^H| \leq \frac{1}{2}|H_{\text{in}}|$, which holds because $M_{\text{in}}^H \subseteq H_{\text{in}}$.

It remains to prove Claim 12.

Proof of Claim 12. Let $G^{\text{bad}} = (T \setminus T_{\text{cov}}, H^{\text{bad}})$ be the graph induced by links in H^{bad} on $T \setminus T_{\text{cov}}$. Let $\mathcal{K} = (T_{\mathcal{K}}, H_{\mathcal{K}}^{\text{bad}})$ be one of its connected components and let $n_{\mathcal{K}} := |T_{\mathcal{K}}|$ and $m_{\mathcal{K}} := |H_{\mathcal{K}}^{\text{bad}}|$ denote the number of vertices and edges of \mathcal{K} , respectively. We will show

$$\sum_{v \in T_{\mathcal{K}}} |\delta_H(v)| \geq n_{\mathcal{K}} + m_{\mathcal{K}}. \tag{3}$$

The claim then follows by summing over all connected components of G^{bad} .

For each link $\ell \in H_{\mathcal{K}}^{\text{bad}}$, there exists a cut $C^\ell \in \mathcal{C}$ such that $\ell = \{u, v\}$ is a bad link with respect to C^ℓ , i.e., we have $T_{C^\ell} \subseteq \{u, v\} \subseteq C^\ell$. Let

$$C^{\mathcal{K}} := \bigcup_{\ell \in H_{\mathcal{K}}^{\text{bad}}} C^\ell. \tag{4}$$

Then $C^{\mathcal{K}}$ is also a 2-cut in G , which can be derived via well-known combinatorial uncrossing arguments as follows. Start with the family $\mathcal{F} := \{C^\ell : \ell \in H_{\mathcal{K}}^{\text{bad}}\}$. We maintain that \mathcal{F} is a family

of 2-cuts that are *connected* in the sense that the graph with vertex set \mathcal{F} that has an edge between $C_1 \in \mathcal{F}$ and $C_2 \in \mathcal{F}$ if $C_1 \cap C_2 \neq \emptyset$, is connected. Because, \mathcal{K} is connected, this holds for the starting \mathcal{F} . We then successively select any two sets $C_1, C_2 \in \mathcal{F}$ with $C_1 \cap C_2 \neq \emptyset$ and replace it by $C_1 \cup C_2$. Because C_1 and C_2 are 2-cuts that intersect (and both do not contain the root r), also $C_1 \cup C_2$ is a 2-cut. (This is the step implied by classic uncrossing techniques; see, e.g., Lemma 24 in [CTZ20].) At the end of this procedure we have the single cut $C^\mathcal{K}$ in our family, which is therefore also a 2-cut, as desired.

Because H is a Leaf-to-Leaf+ CacAP solution, there exists a link $\ell_\mathcal{K}$ in H that covers the 2-cut $C^\mathcal{K}$. Note that one endpoint of $\ell_\mathcal{K}$ must be contained in $T_\mathcal{K}$ because by (4) we have

$$\delta(C^\mathcal{K}) \subseteq \bigcup_{\ell \in H_\mathcal{K}^{\text{bad}}} \delta(C^\ell),$$

and the choice of C^ℓ together with the definition of bad links implies that for $\ell = \{v, w\} \in H^{\text{bad}}$, any link in $\delta(C^\ell)$ must have $u \in T_\mathcal{K}$ or $v \in T_\mathcal{K}$ as one of its endpoints. Moreover, one endpoint of $\ell_\mathcal{K}$ is not contained in $T_\mathcal{K}$, because $T_\mathcal{K} \subseteq C^\mathcal{K}$. Hence, in particular, $\ell_\mathcal{K}$ is not contained in the set $H_\mathcal{K}^{\text{bad}}$ of links of the component \mathcal{K} . Because each of the $m_\mathcal{K}$ many links in $H_\mathcal{K}^{\text{bad}}$ has both endpoints in $T_\mathcal{K}$ and the link $\ell_\mathcal{K}$ has only one endpoint in $T_\mathcal{K}$, we have

$$\sum_{v \in \mathcal{K}} |\delta_H(v)| \geq 2m_\mathcal{K} + 1 \geq n_\mathcal{K} + m_\mathcal{K},$$

because \mathcal{K} is connected. This shows (3). ■

□

2.2 Completing matchings to CacAP solutions

In this section we show that the completion U of M computed in step (2) of Algorithm 1 leads to a solution $F = U \cup M$ fulfilling the guarantees claimed by Lemma 10. To this end, we first observe that (dir-LP) is integral; actually, it only has $\{0, 1\}$ -vertices, i.e., vertices where each coordinate is either 0 or 1. This guarantees that step (2) of Algorithm 1 can indeed be performed as described.

We recall the definition of *residual instance* from [CTZ20, Definition 17], which will be useful when considering instances in which some links have been contracted.

Definition 13 (residual instance). *Let $\mathcal{I} = (G, L)$ be a CacAP instance and let $L' \subseteq L$. Let $L' = \{\ell_1, \dots, \ell_h\}$ be a numbering (ordering) of the links in L' . The residual instance of \mathcal{I} with respect to L' and this numbering is the instance that arises by performing the following contraction operation sequentially for each link $\ell = \ell_1$ up to $\ell = \ell_h$: contract all vertices that are on every u - v path in the cactus, where u and v are the endpoints of ℓ , into a single vertex.*

As proved in [CTZ20], any contraction order leads to the same outcome (Lemma 18 in [CTZ20]) and hence we will in the following simply talk about the residual instance of \mathcal{I} with respect to L' without specifying an order of the link in L' . Moreover, a residual instance with respect to some link set L' is a CacAP instance whose 2-cuts correspond precisely to the 2-cuts in G that have not been covered by L' (Lemma 19 in [CTZ20]). This implies that a link set F is a feasible solution for the residual instance of \mathcal{I} with respect to L' if and only if $F \cup L'$ is a feasible solution for the instance \mathcal{I} (Corollary 20 in [CTZ20]).

Lemma 14. *All vertices of the feasible region of (dir-LP) are within $\{0, 1\}^{\vec{L}}$.*

Proof. We consider the residual instance (G^M, L^M) with respect to the matching M . Then, as proved in [CTZ20, Lemma 19], one can observe that the 2-cuts \mathcal{C}_{G^M} of the residual instance correspond precisely to the cuts in \mathcal{C}^M . By Lemma 14 in [CTZ20], the LP

$$\min \left\{ x(\vec{L}) : x \in \mathbb{R}_{\geq 0}^{\vec{L}}, x(\delta_{\vec{L}}^-(C)) \geq 1 \text{ for all } C \in \mathcal{C}_{G^M} \right\}$$

is integral and hence also (dir-LP) is integral. Moreover, integrality of (dir-LP) readily implies that all vertices of the feasible region are $\{0, 1\}$ -vectors. To observe this, we show that any point $x \in \mathbb{R}_{\geq 0}^{\vec{L}}$ feasible for (dir-LP) with $x(\ell) > 1$ for some $\ell \in \vec{L}$ cannot be a vertex. Indeed, for such a point x , both increasing or decreasing $x(\ell)$ by a small quantity will lead to other feasible points in (dir-LP). This implies that x is not an extreme point of the feasible region of (dir-LP) and therefore not a vertex of it. \square

To show that Algorithm 1 returns a completion U satisfying $|U| \leq \frac{1}{2}|M_{\text{in}}| + (|T| - 2|M|)$, it remains to show that the optimal value of (dir-LP) is at most $\frac{1}{2}|M_{\text{in}}| + (|T| - 2|M|)$. We will prove this in two steps. First, we observe that if we only look at a subset of constraints consisting of a laminar subfamily $\mathcal{L} \subseteq \mathcal{C}^M$, then the statement holds. Note that this case includes Leaf-to-Leaf+TAP, where the set of all minimum cuts form a laminar family. In a second step, we leverage this result on laminar cuts to obtain the desired upper bound on (dir-LP) also for Leaf-to-Leaf+ CacAP instances, as desired.

Lemma 15. *Let $\mathcal{L} \subseteq \mathcal{C}^M$ be a laminar family. Then the optimum value of the LP*

$$\min \left\{ x(\vec{L}) : x \in \mathbb{R}_{\geq 0}^{\vec{L}}, x(\delta_{\vec{L}}^-(C)) \geq 1 \text{ for all } C \in \mathcal{L} \right\} \quad (5)$$

is at most $\frac{1}{2}|M_{\text{in}}| + (|T| - 2|M|)$.

Proof. We construct a feasible solution x^M for (5) with $x^M(L) \leq \frac{1}{2}|M_{\text{in}}| + (|T| - 2|M|)$. For a link $\ell \in \vec{L}$, let

$$\mathcal{C}^\ell := \{C \in \mathcal{L} : \ell \in \delta^-(C)\}$$

be the set of cuts in \mathcal{L} that are covered by ℓ . For a terminal $t \in T$, we call a link $(s, t) \in \vec{L}$ a *maximal link entering t* if the set $\mathcal{C}^{(s,t)}$ is inclusion-wise maximal among all links entering t . Similarly, for a link $\{v, w\} \in M$, we call $\ell \in \delta^-(\{v, w\})$ a *maximal link entering $\{v, w\}$* if the set \mathcal{C}^ℓ is inclusion-wise maximal among all links entering $\{v, w\}$. See Fig. 4 for an example.

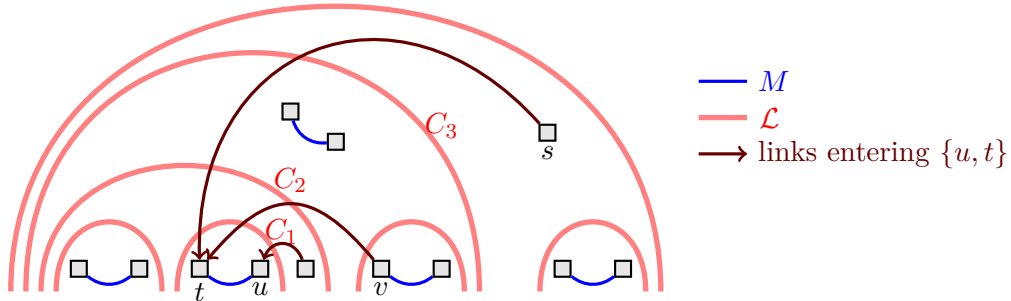


Figure 4: The link (s, t) is the maximal link entering t because $\mathcal{C}^{(s,t)} = \{C_1, C_2, C_3\} \supseteq \{C_1, C_2\} = \mathcal{C}^{(v,t)}$. Similarly, the link (s, t) is also the maximal link entering $\{t, u\}$.

Using that \mathcal{L} is a laminar family, we can observe the following.

Claim 16. *If ℓ is a link entering a terminal $t \in T$ and ℓ_{\max} is a maximal link entering t , then $\mathcal{C}^\ell \subseteq \mathcal{C}^{\ell_{\max}}$. Similarly, if ℓ is a link entering $\{v, w\} \in M$ and ℓ_{\max} is a maximal link entering $\{v, w\}$, then $\mathcal{C}^\ell \subseteq \mathcal{C}^{\ell_{\max}}$.*

Proof. Let ℓ be a link entering a terminal $t \in T$ and ℓ_{\max} a maximal link entering t . Because \mathcal{L} is a laminar family, the cuts of \mathcal{L} that contain t form a chain $C_1 \subsetneq C_2 \subsetneq \dots \subsetneq C_q$. Thus, \mathcal{C}^ℓ is a prefix of that chain, i.e., $\mathcal{C}^\ell = \{C_1, C_2, \dots, C_i\}$ for some $i \in [q]$. As ℓ_{\max} is a maximal link entering t , its prefix must be the largest one among all links, and thus $\mathcal{C}^{\ell_{\max}} \supseteq \mathcal{C}^\ell$, as desired.

The second part of the statement follows by an analogous reasoning. More precisely, let $\{v, w\} \in M$, let ℓ be a link entering $\{v, w\}$, and ℓ_{\max} be a maximal link entering $\{v, w\}$. Any cut $C \in \mathcal{L}$ fulfills either $\{v, w\} \subseteq C$ or $\{v, w\} \cap C = \emptyset$, because $\mathcal{L} \subseteq \mathcal{C}^M$ contains only cuts that are not covered by the matching M . Hence, all links in \mathcal{C}^ℓ are cuts of \mathcal{L} containing both v and w . As before, the family of all cuts in \mathcal{L} that contain both v and w form a chain because \mathcal{L} is laminar. Because ℓ_{\max} is a maximal link entering $\{v, w\}$, the chain $\mathcal{C}^{\ell_{\max}}$ must be the largest one among all those links. Thus, $\mathcal{C}^{\ell_{\max}} \supseteq \mathcal{C}^\ell$, as claimed. \blacksquare

We now explain how we construct a cheap feasible solution x^M for LP (5). Let $T^M \subseteq T$ be the set of leaves of G that are not covered by the matching M . Then $|T^M| = |T| - 2|M|$. We define the vector x^M as follows:

- for each leaf $t \in T^M$, we choose a maximal link $\ell \in \vec{L}$ entering t and set $x_\ell^M := 1$;
- for each in-link $\{v, w\} \in M_{\text{in}}$, we choose a maximal link $\ell \in \vec{L}$ entering $\{v, w\}$ and set $x_\ell^M := \frac{1}{2}$;
- for all other links in \vec{L} , we set $x_\ell^M := 0$.

Clearly, $x^M(L) = |T^M| + \frac{1}{2}|M_{\text{in}}| = \frac{1}{2}|M_{\text{in}}| + (|T| - 2|M|)$. Thus, it only remains to prove $x^M(\delta_{\vec{L}}^-(C)) \geq 1$ for all cuts $C \in \mathcal{L}$.

To this end, fix a cut $C \in \mathcal{L}$. Recall that $T_C \subseteq T \cap C$ is the set of terminals in C that have an incident link covering C . We first observe that if T_C contains the head of a cross-link $\ell \in M_{\text{cross}}$, then, because $x_\ell^M = 1$ and any cross-link covers all cuts containing its head, we have $x^M(\delta_{\vec{L}}^-(C)) \geq 1$ as desired.

We now distinguish two cases. First, suppose T_C contains a terminal $t \in T^M$ that is not covered by the matching M . Then we have $x_{\ell_{\max}} = 1$ for a maximal link $\ell_{\max} \in \vec{L}$ entering t . Because $t \in T_C$, there exists a link $\ell \in \vec{L}$ that enters t and covers C . By Claim 16, the maximality of ℓ_{\max} implies that also ℓ_{\max} covers C . Hence, $x^M(\delta^-(C)) \geq x_{\ell_{\max}}^M = 1$.

We now consider the remaining case where all terminals in T^C are covered by in-links in the matching M . Let $t_1 \in T^C \subseteq C$. (Note that T_C is not empty because our CacAP instance is feasible.) In the matching M , the terminal t_1 is covered by an in-link $\{s_1, t_1\} \in M_{\text{in}}$. Because the link $\{s_1, t_1\}$ is not bad, there exists a terminal $t_2 \in T_C \setminus \{s_1, t_1\}$, which is covered in the matching M by an in-link $\{s_2, t_2\} \in M_{\text{in}}$. For each $i \in \{1, 2\}$, we chose a maximal link $\ell_i \in \vec{L}$ entering $\{s_i, t_i\}$ and defined $x_{\ell_i} := \frac{1}{2}$. Because $t_i \in T_C$, there exists a link $\ell \in \vec{L}$ that enters t_i and covers C . By Claim 16, this implies that also the link ℓ_i covers the cut C . Hence, we have $x^M(\delta^-(C)) \geq x_{\ell_1}^M + x_{\ell_2}^M = \frac{1}{2} + \frac{1}{2} = 1$. \square

We now use Lemma 15 to show that (dir-LP) has a cheap feasible solution even if \mathcal{C}^M is not necessarily laminar. This will complete the proof of Lemma 10.

Lemma 17. *The optimum value of (dir-LP) is at most $\frac{1}{2}|M_{\text{in}}| + |T| - 2|M|$.*

Proof. We consider the dual of (dir-LP), which is

$$\max \left\{ \sum_{C \in \mathcal{C}^M} \lambda_C : \lambda \in \mathbb{R}_{\geq 0}^{\mathcal{C}^M}, \sum_{C \in \mathcal{C}^M} \lambda_C \cdot \chi_{\vec{L}}^{\delta^-(C)} \leq \chi_{\vec{L}} \right\}. \quad (6)$$

Suppose by the sake of deriving a contradiction that the optimum value of the primal LP (dir-LP) is strictly greater than $\frac{1}{2}|M_{\text{in}}| + |T| - 2|M|$. Then, by strong duality, there exists a feasible solution λ^* to the dual LP (6) with

$$\sum_{C \in \mathcal{C}^M} \lambda_C^* > \frac{1}{2}|M_{\text{in}}| + |T| - 2|M|.$$

We will show by a standard combinatorial uncrossing argument that we may assume the support $\{C \in \mathcal{C}^M : \lambda_C^* > 0\}$ of λ^* to be a laminar family $\mathcal{L} \subseteq \mathcal{C}^M$. Then by restricting λ^* to its support, we obtain a feasible solution to

$$\max \left\{ \sum_{C \in \mathcal{L}} \lambda_C : \lambda \in \mathbb{R}_{\geq 0}^{\mathcal{L}}, \sum_{C \in \mathcal{L}} \lambda_C \cdot \chi_{\vec{L}}^{\delta^-(C)} \leq \chi_{\vec{L}} \right\} \quad (7)$$

with value $\sum_{C \in \mathcal{L}} \lambda_C^* > \frac{1}{2}|M_{\text{in}}| + |T| - 2|M|$. This implies, again by duality, that the optimum value of the LP

$$\min \left\{ x(\vec{L}) : x \in \mathbb{R}_{\geq 0}^{\vec{L}}, x(\delta_{\vec{L}}^-(C)) \geq 1 \text{ for all } C \in \mathcal{L} \right\}$$

is strictly greater than $\frac{1}{2}|M_{\text{in}}| + |T| - 2|M|$, contradicting Lemma 15.

It remains to show that we may assume the support of λ^* to be laminar. Let $S, T \in \mathcal{C}^M$ be two crossing sets. Then

$$\chi^{\delta^-(S)} + \chi^{\delta^-(T)} \geq \chi^{\delta^-(S \cap T)} + \chi^{\delta^-(S \cup T)}.$$

Hence, if S and T are in the support of λ^* , we can decrease λ_S^* and λ_T^* by some sufficiently small $\varepsilon > 0$ and at the same time increase $\lambda_{S \cap T}^*$ and $\lambda_{S \cup T}^*$ by ε , such that the vector λ^* remains feasible for (6) and the value of $\sum_{C \in \mathcal{C}^M} \lambda_C^*$ does not change. However, the value of $\sum_{C \in \mathcal{C}^M} \lambda_C^* \cdot |C|^2$ increases. Therefore, if we choose $\lambda \in \mathbb{R}_{\geq 0}^{\mathcal{C}^M}$ to be a feasible solution to (6) with $\sum_{C \in \mathcal{C}^M} \lambda_C = \sum_{C \in \mathcal{C}^M} \lambda_C^*$ that maximizes $\sum_{C \in \mathcal{C}^M} \lambda_C \cdot |C|^2$ among all such vectors λ , then the support of λ is a laminar family $\mathcal{L} \subseteq \mathcal{C}^M$. \square

3 Reducing to $O(1)$ -wide instances

Recall that [CTZ20] gave a reduction from general instances of CacAP to $O(1)$ -wide instances (Definition 3). In this section we show that this reduction can be adapted for the case of leaf-to-leaf CacAP instances: we show that, at the expense of an arbitrarily small constant loss in the approximation factor, it suffices to consider $O(1)$ -wide leaf-to-leaf CacAP instances.

Theorem 18. *Let $\alpha \geq 1$, $\varepsilon > 0$, and $k := \frac{64(8+3\varepsilon)}{\varepsilon^3}$. Given an α -approximation algorithm \mathcal{A} for k -wide Leaf-to-Leaf CacAP instances, there is an $\alpha \cdot (1 + 2\varepsilon)$ -approximation algorithm \mathcal{B} for (unrestricted) Leaf-to-Leaf CacAP that calls \mathcal{A} at most polynomially many times and performs further operations taking polynomial time.*

The reason why we cannot use the reduction to $O(1)$ -wide instances given in [CTZ20] as a black box is that in our setting we have to make sure that we maintain the property that all links are leaf-to-leaf links. The k -wide instances that result from the reduction given in [CTZ20] are obtained from the original instance by (possibly repeatedly) deleting links and applying the following two types of operations, which we call *splitting* and *contraction*.

Definition 19 (splitting). Let $C \subseteq V$ with $|\delta_E(C)| = 2$. Splitting at C leads to two sub-instances \mathcal{I}_C and $\mathcal{I}_{V \setminus C}$, where \mathcal{I}_C is the instance obtained from \mathcal{I} by contracting all vertices except for those in C , and $\mathcal{I}_{V \setminus C}$ is the instance obtained from \mathcal{I} by contracting C .

Definition 20 (contraction). Let $\mathcal{I} = (G = (V, E), L)$ be a CacAP instance and let $\ell \in L$. To contract $\ell = \{u, v\}$ means that we contract the set of vertices that are contained in every u - v path in G .

We observe that splitting operations maintain the leaf-to-leaf property.

Lemma 21. Let $\mathcal{I} = (G = (V, E), L)$ be an instance of CacAP and let $C \subseteq V$ with $|\delta_E(C)| = 2$. If \mathcal{I} is an instance of Leaf-to-Leaf CacAP, then also the CacAP instances \mathcal{I}_C and $\mathcal{I}_{V \setminus C}$ that result from splitting \mathcal{I} at C are instances of Leaf-to-Leaf CacAP.

More generally, if p is the number of vertices in \mathcal{I} that are endpoints of a link in L , but are not leaves of G , then the total number of such non-leaf endpoints of links in \mathcal{I}_C and $\mathcal{I}_{V \setminus C}$ is also p .

Proof. Let p_C and $p_{V \setminus C}$ be the number of non-leaf endpoints of links in \mathcal{I} that are contained in C and $V \setminus C$, respectively. Then $p_C + p_{V \setminus C} = p$. In the sub-instance \mathcal{I}_C , the set $V \setminus C$ is contracted into a single vertex $v_{V \setminus C}$ that is a leaf for the new instance, because $|\delta_E(C)| = |\delta_E(v_{V \setminus C})| = 2$. For each link $\ell \in \delta_E(C)$, one of its endpoints is replaced by the leaf $v_{V \setminus C}$ while the other endpoint does not change. As the endpoints of other links are not changed and all leaves of G in C remain leaves after the contraction of $V \setminus C$, the number of non-leaf endpoints of links in \mathcal{I}_C is p_C . By symmetry, it follows that the number of non-leaf endpoints of links in $\mathcal{I}_{V \setminus C}$ is $p_{V \setminus C}$. \square

Unfortunately, contractions of links do not maintain the leaf-to-leaf property. See Fig. 5. However, we will use that such contractions are not applied too often.

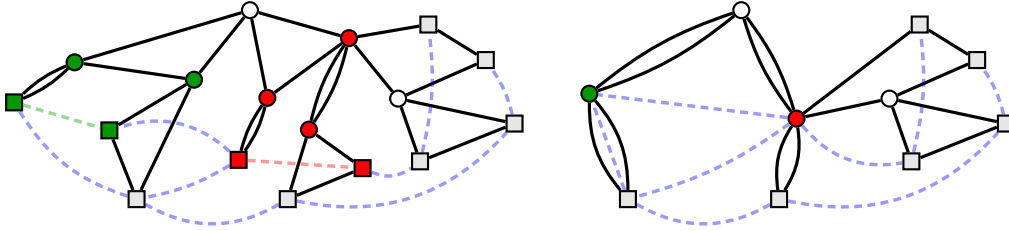


Figure 5: The left picture shows a leaf-to-leaf instance where two links to be contracted are highlighted in green and red, respectively, together with the vertices to be contracted. The right picture shows the resulting instance after the two links have been contracted. Notice that after these contraction operations there exist both leaf-to-non-leaf links and non-leaf-to-non-leaf links.

The following definition formally captures the type of decompositions that we obtain from the reduction from [CTZ20].

Definition 22. Let $\mathcal{I} = (G = (V, E), L)$ be a CacAP instance. A sequence of splitting operations, contraction operations, and deletions of links is called a (γ, k) -splitting of (G, L) if the following two properties are satisfied.

- All instances obtained after applying the sequence of operations are k -wide; we call these instances the split-minors of the (γ, k) -splitting.
- The number of contraction operations is at most γ .

The proof from [CTZ20] yields the following general reduction statement.

Theorem 23 ([CTZ20]). *Let $f : \{\mathcal{I} : \mathcal{I} \text{ CacAP instance}\} \rightarrow \mathbb{R}_{\geq 0}$ be an efficiently computable function and let $\varepsilon > 0$. Suppose we are given an efficient algorithm \mathcal{A} that for any feasible k -wide CacAP instance \mathcal{I} computes a solution with at most $\alpha \cdot \text{OPT}(\mathcal{I}) + f(\mathcal{I})$ many links. Then there is an algorithm \mathcal{B} that for any feasible CacAP instance \mathcal{I} computes a solution with at most*

$$\alpha \cdot (1 + \varepsilon) \cdot \text{OPT}(\mathcal{I}) + \max_{\substack{\mathcal{S} \text{ an } (\varepsilon|\text{OPT}(\mathcal{I})|, k)\text{-} \\ \text{splitting of } \mathcal{I}}} \sum_{\mathcal{J} \text{ split minor of } \mathcal{S}} f(\mathcal{J})$$

many links, while calling \mathcal{A} at most polynomially many times and performing further operations taking polynomial time.

Because [CTZ20] does not formally state Theorem 23 in this form, we provide details on how the theorem follows from [CTZ20] in Appendix A. Observe that when $f \equiv 0$, the statement of Theorem 23 gives a reduction to k -wide instances similar to Theorem 18, but for general (not necessarily leaf-to-leaf) CacAP instances. In order to prove Theorem 18, we will apply Theorem 23 with the function f being defined by

$$f((G = (V, E), L)) := |\{v \in V : v \text{ is not a leaf of } G, \text{ but an endpoint of a link in } L\}|. \quad (8)$$

The next lemma shows why this choice of the function f is useful for proving Theorem 18.

Lemma 24. *Let the function f be defined by (8). Then, for any Leaf-to-Leaf CacAP instance \mathcal{I} , we have*

$$\max_{\substack{\mathcal{S} \text{ an } (\gamma, k)\text{-} \\ \text{splitting of } \mathcal{I}}} \sum_{\mathcal{J} \text{ split minor of } \mathcal{S}} f(\mathcal{J}) \leq \gamma.$$

Proof. Contracting a single link increases the number of non-leaf endpoints of links by at most one. Moreover, the deletion of links does not increase the total number of non-leaf endpoints and by Lemma 21 the same holds for splitting operations. Thus, for every (γ, k) -splitting \mathcal{S} , the total number $\sum_{\mathcal{J} \text{ split minor of } \mathcal{S}} f(\mathcal{J})$ of non-leaf endpoints is upper bounded by the number of contraction operations in the splitting \mathcal{S} , which is at most γ by Definition 22. \square

In order to be able to prove Theorem 18 using Theorem 23, we will show that using an approximation algorithm for Leaf-to-Leaf CacAP, we can give an approximation algorithm for CacAP with few non-leaf endpoints of links. More precisely, we show the following lemma.

Lemma 25. *Let $\alpha \geq 1$ and the function f be defined by (8). Suppose we are given an α -approximation algorithm \mathcal{A} for k -wide Leaf-to-Leaf CacAP instances. Then there is an algorithm \mathcal{A}' that for any feasible k -wide CacAP instance \mathcal{I} computes a solution with at most $\alpha \cdot \text{OPT}(\mathcal{I}) + f(\mathcal{I})$ many links, while calling \mathcal{A} at most once and performing further operations taking polynomial time.*

Proof. Given a feasible k -wide CacAP instance $\mathcal{I} = (G = (V, E), L)$, we will transform it into a k -wide leaf-to-leaf instance. As a first step, we will compute a set X of at most $f(\mathcal{I})$ many links that we can contract to make sure that every non-leaf endpoint of a link in \mathcal{I} is merged with a leaf of G or has no incident links anymore. In a second step, we will then modify the resulting CacAP instance \mathcal{I}^X such that we obtain a k -wide leaf-to-leaf instance $\tilde{\mathcal{I}}^X$ with the following properties:

- (a) $|\text{OPT}(\tilde{\mathcal{I}}^X)| \leq |\text{OPT}(\mathcal{I})|$, and
- (b) for every solution F of $\tilde{\mathcal{I}}^X$, the link set $F \cup X$ is a feasible solution for the instance \mathcal{I} .

Finally, we use the given algorithm \mathcal{A} to compute an α -approximate solution F for the k -wide Leaf-to-Leaf CacAP instance $\tilde{\mathcal{I}}^X$. Then, using $|X| \leq f(\mathcal{I})$ and properties (a) and (b), we can conclude that $F \cup X$ is a solution for \mathcal{I} with

$$|F \cup X| \leq \alpha \cdot \left| \text{OPT}(\tilde{\mathcal{I}}^X) \right| + |X| \leq \alpha \cdot |\text{OPT}(\mathcal{I})| + f(\mathcal{I}).$$

Let us now explain how we choose the set $X \subseteq L$ and construct the instance \mathcal{I}^X . We will choose a set $X \subseteq L$ with $|X| \leq f(\mathcal{I})$; then we define \mathcal{I}^X to be the residual instance of \mathcal{I} with respect to X (see Definition 13). It is well known that $F \subseteq L$ is a feasible solution to the residual instance \mathcal{I}^X if and only if $F \cup X$ is a feasible solution to the original instance (see for example Corollary 20 in [CTZ20]). We will call a node of the residual instance that arose from the contraction of several vertices of the original instance \mathcal{I} , a *supernode*. We say that the vertices that were contracted to obtain a supernode s belong to s .

Let $B := \{v \in V : v \text{ is not a leaf of } G, \text{ but an endpoint of a link in } L\}$. Then $f(\mathcal{I}) = |B|$. The following claim describes how we choose the set X of links that we will contract.

Claim 26. *We can efficiently find a set $X \subseteq L$ with $|X| \leq f(\mathcal{I})$ such that the residual instance with respect to X has the following properties:*

- (i) every vertex $v \in B$ belongs to a supernode, and
- (ii) for every supernode s
 - there is a leaf of G that belongs to s , or
 - s has no incident link in the residual instance \mathcal{I}^X .

Proof. We construct the set X by the following algorithm.

1. Initialize $X = \emptyset$.
2. As long as (i) or (ii) is not fulfilled, iterate the following:
 - Choose v to be either a vertex in B violating (i) or a supernode violating (ii).
 - Choose an arbitrary link $\ell \in \delta(v)$ and add ℓ to X .
 - Apply the contraction operation for ℓ (Definition 20).

Note that a link $\ell \in \delta(v)$ exists in every iteration of the algorithm. If $v \in B$, this follows from the definition of B , and if v is a supernode, this follows from the fact that v violates (ii). At the end of the algorithm, the instance \mathcal{I}^X fulfills (i) and (ii) by construction. In order to prove $|X| \leq f(\mathcal{I}) = |B|$, we show that the number of vertices violating (i) or (ii) strictly decreases in every iteration of the algorithm. This will conclude the proof because at the beginning of the algorithm (i) and (ii) are violated only by the vertices in B (as no supernodes exist). Now consider an iteration in which we chose a vertex v violating (i) or (ii) and contracted $\ell = \{v, w\}$. If w is a leaf of G or a supernode with a leaf of G belonging to w , then the supernode arising from the contraction of ℓ does not violate (ii). Otherwise, by the definition of B , the vertex w must be either an element of B (violating (i)) or a supernode violating (ii). When contracting ℓ , the two vertices v and w violating (i) or (ii) get merged into a single supernode. In any of these cases the number of vertices violating (i) or (ii) decreases strictly. ■

It remains to show that we can transform the residual instance \mathcal{I}^X into a k -wide instance of Leaf-to-Leaf CacAP with properties (a) and (b). We construct the instance $\tilde{\mathcal{I}}^X$ from \mathcal{I}^X as follows. For every supernode s that has at least one incident link in \mathcal{I}^X , we add a new auxiliary vertex t_s and two copies of the edge $\{s, t_s\}$ to the cactus. Then we still have a cactus and t_s is one of its leaves. Moreover, we replace every link $\{s, v\}$ by the link $\{t_s, v\}$. The leaf-to-leaf instance that results from applying this transformation for all supernodes with at least one incident link is the instance

$\tilde{\mathcal{I}}^X = (\tilde{G}, \tilde{L})$. We view \tilde{L} as a subset of L by identifying each link in \tilde{L} with the corresponding link in L from which it arose in the construction. We now show that $\tilde{\mathcal{I}}^X$ has the desired properties.

Claim 27. *For every solution F of $\tilde{\mathcal{I}}^X$, the link set $F \cup X$ is a feasible solution for the instance \mathcal{I} .*

Proof. Let $(G^X = (V^X, E^X), L^X) := \mathcal{I}^X$ and let F be a solution of $\tilde{\mathcal{I}}^X = (\tilde{G} = (\tilde{V}, \tilde{E}), \tilde{L})$. Then the graph $(\tilde{V}, \tilde{E} \cup F)$ is 3-edge-connected. Because $(V^X, E^X \cup F)$ arises from this graph by contracting the pair $\{s, t_s\}$ for every supernode $s \in V^X$ for which we added a vertex t_s , also the graph $(V^X, E^X \cup F)$ is 3-edge connected, i.e., F is a feasible solution for \mathcal{I}^X . Because \mathcal{I}^X is the residual instance of \mathcal{I} with respect to X , we indeed have that $F \cup X$ is a solution for \mathcal{I} (see for example [CTZ20, Corollary 20]). ■

Claim 28. *The instance $\tilde{\mathcal{I}}^X$ is feasible and we have $|\text{OPT}(\tilde{\mathcal{I}}^X)| \leq |\text{OPT}(\mathcal{I})|$.*

Proof. We construct from $\text{OPT}(\mathcal{I})$ a solution F for $\tilde{\mathcal{I}}^X$ of at most the same cardinality. The solution F that we construct contains for every link $\ell = \{v, w\} \in \text{OPT}(\mathcal{I})$

- (i) the link ℓ if v and w do not belong to the same supernode, and
- (ii) an arbitrary link incident to t_s if v and w belong to the same supernode s and s has an incident link in \mathcal{I}^X . (Note that in this case t_s exists and has an incident link in $\tilde{\mathcal{I}}^X$.)

By construction $|F| \leq |\text{OPT}(\mathcal{I})|$. Because $\text{OPT}(\mathcal{I})$ is a feasible solution for \mathcal{I} and hence for \mathcal{I}^X , every violated cut, i.e., every 2-cut of the cactus \tilde{G} that does not contain a link from F , must be of the form $\delta(t_s)$ for some supernode s .

Suppose such a violated cut exists. The construction of $\tilde{\mathcal{I}}^X$ implies that there exists a link that is incident to t_s . Moreover, by Claim 26 this implies that there is a leaf v of G that belongs to the supernode s . The solution $\text{OPT}(\mathcal{I})$ must contain a link ℓ that is incident to v . If both endpoints of ℓ belong to the supernode s , then F contains a link in $\delta(t_s)$ by (ii). Otherwise, $\ell \in F \cap \delta(t_s)$, because we replaced the endpoint s of ℓ by t_s in the construction of \mathcal{I}^X . Thus, $\delta(t_s)$ is not a violated cut, a contradiction. ■

Claim 29. *The instance $\tilde{\mathcal{I}}^X$ is k -wide.*

Proof. Let r be a k -wide center of G and let \tilde{r} be either r or the supernode in \tilde{G} to which r belongs. We show that \tilde{r} is a k -wide center for \tilde{G} . To this end, let us consider the vertex set \tilde{W} of a connected component of $\tilde{G} - \tilde{r}$. Then the vertices in V that are contained in \tilde{W} or belong to a supernode contained in \tilde{W} are all part of the same connected component of $G - r$. Hence, at most k of these vertices are leaves of G . To complete the proof, we will show that every leaf of \tilde{G} in \tilde{W} is either a leaf of G or an auxiliary vertex t_s for a supernode s with a leaf of G belonging to s .

Let $\tilde{t} \in \tilde{W}$ be a leaf of \tilde{G} . If \tilde{t} is a vertex in V , i.e., it is neither a supernode nor an auxiliary vertex, then its degree in \tilde{G} is the same as in G , implying that \tilde{t} is a leaf of G . If \tilde{t} is an auxiliary vertex t_s for a supernode s , then by Claim 26, there exists a leaf of G that belongs to s . Finally, suppose \tilde{t} is a supernode. Then \tilde{t} cannot have any incident links as otherwise we would have added the leaf t_s incident to \tilde{t} . However, because \tilde{t} is a leaf of \tilde{G} , this contradicts the feasibility of the instance $\tilde{\mathcal{I}}^X$ (Claim 28). ■

Claim 27, Claim 28, and Claim 29 imply that $\tilde{\mathcal{I}}^X$ indeed has the desired properties. □

Theorem 18 now follows directly from Theorem 23, Lemma 24, and Lemma 25.

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A Details of the reduction to $O(1)$ -wide instances from [CTZ20]

In this Section we provide details on how [Theorem 23](#) follows from [CTZ20]. [Theorem 5](#) in [CTZ20] is the same as [Theorem 23](#) with $f \equiv 0$. The same proof can be extended to the case of a general efficiently computable function f . We will first provide a brief outline of the overall proof and then provide adapted versions of those lemmas and proofs from [CTZ20] that require (small) changes in order to obtain [Theorem 23](#).

Given an instance $\mathcal{I} = (G, L)$, let $P_{\text{CacAP}}(\mathcal{I})$ be the convex hull of all actual solutions, namely

$$P_{\text{CacAP}}(\mathcal{I}) := \text{conv}(\{\chi^F : F \subseteq L, (V, E \cup F) \text{ is 3-edge-connected}\}) .$$

The proof of Theorem 23 is based on the round-or-cut framework. We will show that there exists an efficient algorithm that, given a point $x \in [0, 1]^L$, returns either

(i) a CacAP solution with at most

$$\alpha \cdot (1 + \varepsilon) \cdot \text{OPT}(\mathcal{I}) \quad + \quad \max_{\substack{\mathcal{S} \text{ an } (\varepsilon \cdot x(L), k)\text{-} \\ \text{splitting of } \mathcal{I}}} \sum_{\mathcal{J} \text{ split minor of } \mathcal{S}} f(\mathcal{J})$$

many links, or

(ii) a vector $w \in \mathbb{R}^L$ such that $w^T x < w^T x'$ for all $x' \in P_{\text{CacAP}}(G, L)$.

Having such an algorithm, we guess the number $|\text{OPT}(\mathcal{I})|$ of links in an optimum solution and run the Ellipsoid method to find a point in the polytope $\{x \in P_{\text{CacAP}}(\mathcal{I}) : x(L) = |\text{OPT}(\mathcal{I})|\}$. As a separation oracle, we can return a separating hyperplane if the given point $x \in [0, 1]^L$ does not fulfill $x(L) = |\text{OPT}(\mathcal{I})|$ and otherwise apply the algorithm that returns either a CacAP solution as in (i) or a separating hyperplane as in (ii). If we obtain a CacAP solution as in (i), this solution has the desired properties (as required in Theorem 23) because $x(L) = |\text{OPT}(\mathcal{I})|$; in this case we are done and we terminate the Ellipsoid method. Otherwise, we have obtained the separating hyperplane that we need to continue with the Ellipsoid method. Because the Ellipsoid terminates in polynomial time, it indeed suffices to provide an algorithm that given a point $x \in [0, 1]^L$ either returns (i) or (ii). Polynomial-time termination of the Ellipsoid Method follows by classical results (see [GLS93, Theorem (6.4.1)]) because the polytope $\{x \in P_{\text{CacAP}}(\mathcal{I}) : x(L) = |\text{OPT}(\mathcal{I})|\}$, over which we run the Ellipsoid Method, has facet complexity that is bounded polynomially in $|L|$, because it is a $\{0, 1\}$ -polytope in $[0, 1]^L$.

We now give a brief outline of the algorithm used to round $x \in [0, 1]^L$ or separate x from $P_{\text{CacAP}}(G, L)$. The algorithm proceeds in three steps:

- The first step, called *heavy cut covering*, computes a set $L_H \subseteq L$ of links with $|L_H| \leq \frac{\varepsilon}{2}x(L)$ that covers all x -heavy cuts, i.e., we have $L_H \cap \delta(C) \neq \emptyset$ for all 2-cuts $C \in \mathcal{C}$ with $x(\delta_L(C)) > \frac{16}{\varepsilon}$. Then we consider the residual instance with respect to L_H . [CTZ20, Lemma 22] shows that this residual instance has no heavy cuts. Moreover, if we take any solution F for the residual instance, then $F \cup L_H$ is a feasible solution for \mathcal{I} (by [CTZ20, Corollary 20]).
- In the second step, we split the instance into k -wide instances using the splitting operations from Definition 19.
- Finally, for every split minor arising from the splitting procedure we either find a CacAP solution or we obtain a separating hyperplane. If we find a CacAP solution for all of the split minors, we merge them to a CacAP solution of the overall instance.

Let us now recap the main statement from [CTZ20] that we use for merging the CacAP solutions for the split minors to a solution of the overall instance. Recall that splitting an instance \mathcal{I} at a cut $C \in \mathcal{C}$ leads to two sub-instances \mathcal{I}_C and $\mathcal{I}_{V \setminus C}$, where \mathcal{I}_C is the instance obtained from \mathcal{I} by contracting all vertices except for those in C , and $\mathcal{I}_{V \setminus C}$ is the instance obtained from \mathcal{I} by contracting C . If we want to merge solutions $F_C, F_{V \setminus C} \subseteq L$ to the sub-instances \mathcal{I}_C and $\mathcal{I}_{V \setminus C}$, respectively, to a solution of \mathcal{I} , it may be not enough to simply take the union of F_C and $F_{V \setminus C}$. However, we can bound the number of links that need to be added to $F_C \cup F_{V \setminus C}$ to obtain a solution to \mathcal{I} as shown in [CTZ20].

Proposition 30 (Proposition 11, [CTZ20]). *Given a feasible CacAP instance $\mathcal{I} = (G = (V, E), L)$, a 2-cut $C \in \mathcal{C}$, and solutions $F_C, F_{V \setminus C} \subseteq L$ to \mathcal{I}_C and $\mathcal{I}_{V \setminus C}$, respectively, one can efficiently compute a link set $F \subseteq L$ such that*

- (i) $F_C \cup F_{V \setminus C} \cup F$ is a CacAP solution to \mathcal{I} , and
- (ii) $|F| \leq |\delta_L(C) \cap F_C| - 1$.

We now explain in detail those parts of the proof from [CTZ20] that need to be adapted to obtain Theorem 23. First, we provide a generalization of Lemma 32 from [CTZ20], which yields an algorithm for k -wide instances that either returns a cheaply mergeable solution or a separating hyperplane. In this algorithm we will use the fact that the function f is efficiently computable. We need the following definition.

Definition 31 (λ -contraction minor). *For an instance $\mathcal{I} = (G, L)$, a λ -contraction minor is an instance \mathcal{I}' that can be obtained by \mathcal{I} by*

- deleting links, and
- applying contraction operations (see Definition 20) for at most λ many links.

Now we can generalize Lemma 32 from [CTZ20] as follows.³

Lemma 32. *Let $\varepsilon \geq 0$ and $\varepsilon' = \frac{\varepsilon}{4}$. Suppose there is an efficient algorithm \mathcal{A} that for any k -wide CacAP instance \mathcal{J} returns a solution of cost at most $\alpha \cdot \text{OPT}(\mathcal{J}) + f(\mathcal{J})$, where f is a function that is efficiently computable. Then there is a polynomial-time algorithm that, given a k -wide CacAP instance $\mathcal{I} = (G = (V, E), L)$, a vector $x \in [0, 1]^L$, and a vertex s of G with $|\delta_E(s)| = 2$ and $x(\delta_L(s)) \leq \frac{16}{\varepsilon}$, either returns*

- a CacAP solution $F \subseteq L$ with

$$|F| + |\delta_F(s)| \leq (1 + \varepsilon') \cdot \alpha \cdot (x(L) + x(\delta_L(s))) + \max\{f(\mathcal{I}'): \mathcal{I}' \text{ is a } \lambda\text{-contraction minor of } \mathcal{I}\},$$

where $\lambda = \frac{1+\varepsilon'}{\varepsilon'} \cdot \frac{16}{\varepsilon}$, or

- a vector $w \in \mathbb{R}^L$ such that $w^T x < w^T x'$ for all $x' \in P_{\text{CacAP}}(\mathcal{I})$.

Proof. The proof of this lemma follows the proof of Lemma 32 in [CTZ20]. Let $r \in V$ be a k -wide root. If $x(\delta_L(s)) < 1$, we have $x(\delta_L(s)) < x'(\delta_L(s))$ for all $x' \in P_{\text{CacAP}}(\mathcal{I})$ and can thus return $w = \chi^{\delta_L(s)}$. Otherwise, we proceed as follows. For every set $S \subseteq \delta_L(s)$ with $|S| \leq \lambda$, we apply the given algorithm \mathcal{A} for k -wide instances to the residual instance \mathcal{I}' of $(G, L \setminus \delta_L(s))$ with respect to S . This instance \mathcal{I}' arises from (G, L) by applying contraction operations for the links in S after deleting all links in $\delta_L(s)$. Thus, \mathcal{I}' is a λ -contraction minor and it is k -wide by [CTZ20, Lemma 31].

If for some such set S , the algorithm \mathcal{A} returns a solution F' with $|F'| + 2|S| \leq (1 + \varepsilon') \cdot \alpha \cdot (x(L) + x(\delta_L(s))) + f(\mathcal{I}')$, we return $F = F' \cup S$. Here we use the fact that f is an efficiently computable function; hence, we can efficiently check whether we are in this case. Otherwise, we define $\mu := 1 + \frac{\varepsilon'(x(L) + x(\delta_L(s)))}{x(\delta_L(s))}$ and claim that

$$x'(L) + \mu \cdot x'(\delta_L(s)) > x(L) + \mu \cdot x(\delta_L(s)) \quad \forall x' \in P_{\text{CacAP}}(\mathcal{I}), \quad (9)$$

which leads to a vector $w \in \mathbb{R}^L$ as desired. Suppose that (9) does not hold. Then there exists a solution F^* of \mathcal{I} with

$$|F^*| + \mu \cdot |\delta_{F^*}(s)| \leq x(L) + \mu \cdot x(\delta_L(s)) = (1 + \varepsilon') \cdot (x(L) + x(\delta_L(s))). \quad (10)$$

³Lemma 32 from [CTZ20] applies to so-called unsplittable instances, but these instances are k -wide by [CTZ20, Lemma 12] and hence Lemma 32 is indeed a generalization of Lemma 32 from [CTZ20]. We also remark that the reason why we bound $|F| + |\delta_F(s)|$ instead of $|F|$ in Lemma 32 is that this is needed to guarantee that F is cheaply mergeable.

For $S = \delta_{F^*}(s)$ this implies

$$\frac{\varepsilon' (x(L) + x(\delta_L(s)))}{x(\delta_L(s))} \cdot |S| \leq \mu \cdot |S| \leq (1 + \varepsilon') \cdot (x(L) + x(\delta_L(s)))$$

and hence

$$|S| \leq \frac{1 + \varepsilon'}{\varepsilon'} x(\delta_L(s)) \leq \lambda.$$

Thus, we considered the set S in our algorithm described above. Let F' be the output of the α -approximation algorithm \mathcal{A} applied to the residual instance \mathcal{I}' of $(G, L \setminus \delta_L(s))$ with respect to S . Then $|F'| \leq \alpha \cdot |F^* \setminus S| + f(\mathcal{I}')$ because the set $F^* \setminus S$ is a feasible solution of this residual instance (see [CTZ20, Corollary 20]). Therefore, using $\alpha \geq 1$, $S = \delta_{F^*}(s)$, and (10), we get

$$\begin{aligned} |F'| + 2|S| &\leq \alpha \cdot |F^* \setminus S| + f(\mathcal{I}') + 2|S| \\ &\leq \alpha \cdot (|F^*| + |S|) + f(\mathcal{I}') \\ &\leq \alpha \cdot (|F^*| + \mu|\delta_{F^*}(s)|) + f(\mathcal{I}') \\ &\leq (1 + \varepsilon') \cdot \alpha \cdot (x(L) + x(\delta_L(s))) + f(\mathcal{I}'), \end{aligned}$$

contradicting the fact that we did not return $F' \cup S$. \square

The next lemma provides a generalization of Lemma 33 from [CTZ20] (except that we choose k slightly differently). This lemma extends the algorithm that, given $x \in [0, 1]^L$, either rounds it to a cheap solution or provides a separating hyperplane from k -wide instances to general instances without x -heavy cuts.

Lemma 33. *Let $0 \leq \varepsilon \leq 1$ and $k := \frac{64(8+3\varepsilon)}{\varepsilon^3}$. Suppose there exist an efficient algorithm \mathcal{A} that for any k -wide CacAP instance \mathcal{J} returns a solution of cost at most $\alpha \cdot \text{OPT}(\mathcal{J}) + f(\mathcal{J})$, where f is a function that is efficiently computable. Then, for any CacAP instance $\mathcal{I} = (G = (V, E), L)$ and $x \in [0, 1]^L$ such that no cut is x -heavy, there is a polynomial-time algorithm that computes either*

- a CacAP solution L' with

$$|L'| \leq \alpha \cdot \left(1 + \frac{\varepsilon}{2}\right) \cdot x(L) + \max_{\substack{\mathcal{S} \text{ a } (\gamma, k)\text{-} \\ \text{splitting of } \mathcal{I}}} \sum_{\mathcal{J} \text{ split minor of } \mathcal{S}} f(\mathcal{J}),$$

where $\gamma = \frac{\varepsilon}{4} \cdot |T|$ with T being the set of leaves of G , or

- a vector $w \in \mathbb{R}^L$ such that $w^T x < w^T x'$ for all $x' \in P_{\text{CacAP}}(\mathcal{I})$.

Proof. The proof of this lemma follows from Lemma 33 in [CTZ20] and proceeds by induction on the number of vertices. We fix an arbitrary root $r \in V$. If there is no 2-cut $C \subsetneq V \setminus \{r\}$ such that $|C \cap T| > \frac{k}{2}$, then, following the terminology of [CTZ20], the instance is called *unsplittable*. (More precisely, for an instance to be unsplittable no such cut C must exist that is not x -heavy; however, as no x -heavy cuts exist in our instance by assumption, this condition can be dropped here.) By [CTZ20, Lemma 12], every unsplittable instance is k -wide, and we can thus apply algorithm \mathcal{A} to \mathcal{I} to obtain a feasible solution L' . If $|L'| \leq \alpha \cdot x(L) + f(\mathcal{I})$, then

$$|L'| \leq \alpha \cdot x(L) + f(\mathcal{I}) \leq \alpha \cdot \left(1 + \frac{\varepsilon}{2}\right) \cdot x(L) + \max_{\substack{\mathcal{S} \text{ a } (\gamma, k)\text{-} \\ \text{splitting of } \mathcal{I}}} \sum_{\mathcal{J} \text{ split minor of } \mathcal{S}} f(\mathcal{J}),$$

because \mathcal{I} itself is a (γ, k) -split-minor. In this case we return L' . Otherwise, we return $w = \chi^L$, which fulfills $w^T x < w^T x'$ for all $x' \in P_{\text{CacAP}}(\mathcal{I})$ because \mathcal{A} returns a solution of cost at most $\alpha \cdot \text{OPT}(\mathcal{J}) + f(\mathcal{J})$.

Hence, we may assume that \mathcal{I} is splittable at some cut C , i.e., there is a cut C that fulfills $|T \cap C| > \frac{k}{2}$ and $C \subsetneq V \setminus \{r\}$. We choose $C \in \mathcal{C}$ to be a minimal 2-cut with these properties.

We denote by $\mathcal{I}_C = (G_C, L_C)$ the instance arising from \mathcal{I} by contracting $V \setminus C$ and we choose as new root the vertex s arising from the contraction of $V \setminus C$. (Note that $\{r\} \subsetneq V \setminus C$.) Because C was chosen minimally, the instance \mathcal{I}_C with root s is k -wide (this is again a consequence of [CTZ20, Lemma 12]). We apply Lemma 32 to \mathcal{I}_C to either obtain a CacAP solution F_C for \mathcal{I}_C with

$$|F_C| + |\delta_{F_C}(s)| \leq (1 + \frac{\varepsilon}{4}) \cdot \alpha \cdot (x(L_C) + x(\delta_L(s))) + \max\{f(\mathcal{I}'): \mathcal{I}' \text{ is a } \lambda\text{-contraction minor for } \mathcal{I}_C\}, \quad (11)$$

where $\lambda = \frac{1+\varepsilon'}{\varepsilon'} \cdot \frac{16}{\varepsilon}$, or a vector $w_C \in \mathbb{R}^{L_C}$ satisfying $w_C^T x_C < w_C^T x'$ for all $x' \in P_{\text{CacAP}}(\mathcal{I}_C)$, where x_C is the restriction of the vector x to the links in L_C .

Moreover, we denote by $\mathcal{I}_{V \setminus C} = (G_{V \setminus C}, L_{V \setminus C})$ the CacAP instance arising from \mathcal{I} by contracting C . By induction, we can obtain a CacAP solution $F_{V \setminus C}$ for $\mathcal{I}_{V \setminus C}$ such that

$$|F_{V \setminus C}| \leq \alpha \cdot (1 + \frac{\varepsilon}{2}) \cdot x(L_{V \setminus C}) + \max_{\substack{\mathcal{S} \text{ a } (\gamma', k)\text{-} \\ \text{splitting of } \mathcal{I}_{V \setminus C}}} \sum_{\mathcal{J} \text{ split minor of } \mathcal{S}} f(\mathcal{J}), \quad (12)$$

where $\gamma' = \frac{\varepsilon}{4} \cdot |T_{V \setminus C}|$ with $T_{V \setminus C}$ being the set of leaves of $G_{V \setminus C}$, or a vector $w_{\bar{C}} \in \mathbb{R}^{V \setminus C}$ such that $w_{\bar{C}}^T x_{\bar{C}} < w_{\bar{C}}^T x'$ for all $x' \in P_{\text{CacAP}}(\mathcal{I}_{V \setminus C})$, where $x_{\bar{C}}$ is the vector x restricted to the entries corresponding to links in $L_{V \setminus C}$.

If we obtained a vector $w_C \in \mathbb{R}^{L_C}$ such that $w_C^T x_C < w_C^T x'$ for all $x' \in P_{\text{CacAP}}(\mathcal{I}_C)$, we can extend it to a vector $w \in \mathbb{R}^L$ such that $w^T x < w^T x'$ for all $x' \in P_{\text{CacAP}}(\mathcal{I})$ by setting $w_\ell = 0$ for all $\ell \in L \setminus L_C$. Here we use that restricting a CacAP solution F of \mathcal{I} to $F \cap L_C$ yields a CacAP solution for \mathcal{I}_C . We can proceed analogously if we have a vector $w_{\bar{C}} \in \mathbb{R}^{L_{V \setminus C}}$ such that $w_{\bar{C}}^T x_{\bar{C}} < w_{\bar{C}}^T x'$ for all $x' \in P_{\text{CacAP}}(\mathcal{I}_{V \setminus C})$.

Otherwise, we obtained solutions F_C and $F_{V \setminus C}$ as discussed above and apply Proposition 30. This yields a set $F \subseteq L$ such that $F \cup F_C \cup F_{V \setminus C}$ is a CacAP solution for \mathcal{I} and $|F| \leq |F_C \cap \delta_L(C)| = |\delta_{F_C}(s)|$. Thus, combining (11) and (12), we obtain

$$\begin{aligned} |F \cup F_C \cup F_{V \setminus C}| &\leq |F_{V \setminus C}| + |F_C| + |\delta_{F_C}(s)| \\ &\leq \alpha \cdot (1 + \frac{\varepsilon}{2}) \cdot x(L_{V \setminus C}) + \max_{\substack{\mathcal{S} \text{ a } (\gamma', k)\text{-} \\ \text{splitting of } \mathcal{I}_{V \setminus C}}} \sum_{\mathcal{J} \text{ split minor of } \mathcal{S}} f(\mathcal{J}) \\ &\quad + (1 + \frac{\varepsilon}{4}) \cdot \alpha \cdot (x(L_C) + x(\delta_L(s))) \\ &\quad + \max\{f(\mathcal{I}'): \mathcal{I}' \text{ is a } \lambda\text{-contraction minor for } \mathcal{I}_C\} \\ &= \alpha (1 + \frac{\varepsilon}{2}) x(L_{V \setminus C}) + \alpha (1 + \frac{\varepsilon}{2}) x(L_C) - \alpha \frac{\varepsilon}{4} \cdot x(L_C) \\ &\quad + \alpha (1 + \frac{\varepsilon}{4}) \cdot x(\delta_L(C)) \\ &\quad + \max_{\substack{\mathcal{S} \text{ a } (\gamma', k)\text{-} \\ \text{splitting of } \mathcal{I}_{V \setminus C}}} \sum_{\mathcal{J} \text{ split minor of } \mathcal{S}} f(\mathcal{J}) \\ &\quad + \max\{f(\mathcal{I}'): \mathcal{I}' \text{ is a } \lambda\text{-contraction minor for } \mathcal{I}_C\} \\ &= \alpha (1 + \frac{\varepsilon}{2}) \cdot x(L) + \alpha \left(2 + \frac{3\varepsilon}{4}\right) \cdot x(\delta_L(C)) - \alpha \frac{\varepsilon}{4} \cdot x(L_C) \\ &\quad + \max_{\substack{\mathcal{S} \text{ a } (\gamma', k)\text{-} \\ \text{splitting of } \mathcal{I}_{V \setminus C}}} \sum_{\mathcal{J} \text{ split minor of } \mathcal{S}} f(\mathcal{J}) \end{aligned} \quad (13)$$

$$+ \max\{f(\mathcal{I}'): \mathcal{I}' \text{ is a } \lambda\text{-contraction minor for } \mathcal{I}_C\},$$

where we used $L_C \cup L_{V \setminus C} = L$ and $L_C \cap L_{V \setminus C} = \delta_L(C)$. Because $|C \cap T| > \frac{k}{2}$, we have $x(L_C) \geq \frac{k}{4} = 16 \cdot \frac{8+3\varepsilon}{\varepsilon^3}$. Moreover, $x(\delta_L(C)) \leq \frac{16}{\varepsilon}$ because $\delta_L(C)$ is not x -heavy. This implies

$$\left(2 + \frac{3\varepsilon}{4}\right) \cdot x(\delta_L(C)) \leq \left(2 + \frac{3\varepsilon}{4}\right) \cdot \frac{16}{\varepsilon} \leq \frac{\varepsilon}{4} \cdot 16 \cdot \frac{8+3\varepsilon}{\varepsilon^3} \leq \frac{\varepsilon}{4} \cdot x(L_C).$$

Together with (13), we obtain

$$\begin{aligned} |F \cup F_C \cup F_{V \setminus C}| &\leq \alpha \cdot \left(1 + \frac{\varepsilon}{2}\right) \cdot x(L) \\ &\quad + \max\{f(\mathcal{I}'): \mathcal{I}' \text{ is a } \lambda\text{-contraction minor for } \mathcal{I}_C\} \\ &\quad + \max_{\substack{\mathcal{S} \text{ a } (\gamma', k)\text{-} \\ \text{splitting of } \mathcal{I}_{V \setminus C}}} \sum_{\mathcal{J} \text{ split minor of } \mathcal{S}} f(\mathcal{J}). \end{aligned} \tag{14}$$

Let $\mathcal{I}'_C \in \operatorname{argmax}\{f(\mathcal{I}'): \mathcal{I}' \text{ is a } \lambda\text{-contraction minor for } \mathcal{I}_C\}$. It remains to prove the following claim.

Claim 34.

$$f(\mathcal{I}'_C) + \max_{\substack{\mathcal{S} \text{ a } (\gamma', k)\text{-} \\ \text{splitting of } \mathcal{I}_{V \setminus C}}} \sum_{\mathcal{J} \text{ split minor of } \mathcal{S}} f(\mathcal{J}) \leq \max_{\substack{\mathcal{S} \text{ a } (\gamma, k)\text{-} \\ \text{splitting of } \mathcal{I}}} \sum_{\mathcal{J} \text{ split minor of } \mathcal{S}} f(\mathcal{J}).$$

Proof. Let $\mathcal{J}_1, \dots, \mathcal{J}_N$ be the split minors of a (γ, k) -splitting of $\mathcal{I}_{V \setminus C}$ that maximize

$$\max_{\substack{\mathcal{S} \text{ a } (\gamma', k)\text{-} \\ \text{splitting of } \mathcal{I}_{V \setminus C}}} \sum_{\mathcal{J} \text{ split minor of } \mathcal{S}} f(\mathcal{J}).$$

We finish the proof by showing that $\mathcal{I}'_C, \mathcal{J}_1, \dots, \mathcal{J}_N$ are the split minors of a (γ, k) -splitting of \mathcal{I} . By construction, all $\mathcal{I}'_C, \mathcal{J}_1, \dots, \mathcal{J}_N$ are obtained after applying a sequence of splitting and contraction operations and deletions of links. Moreover, they are all k -wide. We only need to prove that the number of contractions is at most $\gamma = \frac{\varepsilon}{4} \cdot |T|$.

Recall we choose the cut C to split at such that $|C \cap T| \geq \frac{k}{2}$, implying $|T_{V \setminus C}| \leq |T| - \frac{k}{2} + 1$. Therefore, because $\mathcal{J}_1, \dots, \mathcal{J}_N$ are the split minors of a (γ', k) -splitting of $\mathcal{I}_{V \setminus C}$, the number of contractions applied to obtain $\mathcal{J}_1, \dots, \mathcal{J}_N$ from $\mathcal{I}_{V \setminus C}$ (by splitting, contraction, and deletion of links) is at most $\gamma' = \frac{\varepsilon}{4} \cdot |T_{V \setminus C}| \leq \frac{\varepsilon}{4} \cdot (|T| - \frac{k}{2} + 1)$. The total number of contractions to obtain $\mathcal{I}'_C, \mathcal{J}_1, \dots, \mathcal{J}_N$ is thus at most

$$\lambda + \frac{\varepsilon}{4} \cdot (|T| - \frac{k}{2} + 1) \leq \frac{\varepsilon}{4} \cdot |T| = \gamma$$

because

$$\lambda + \frac{\varepsilon}{4} = \frac{1 + \varepsilon'}{\varepsilon'} \cdot \frac{16}{\varepsilon} + \frac{\varepsilon}{4} = \frac{1 + \varepsilon/4}{\varepsilon/4} \cdot \frac{16}{\varepsilon} + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{8} \cdot \frac{64(8 + 3\varepsilon)}{\varepsilon^3} = \frac{\varepsilon}{4} \cdot \frac{k}{2}.$$

■

□

In order to obtain an algorithm that, given a point $x \in [0, 1]^L$, returns either a CacAP solution as in (i) or a vector $w \in \mathbb{R}^L$ as in (ii), we proceed as follows. First, we observe that if $x(L) \leq |T|/2$ we can return a vector w as in (ii) because $x'(L) \geq |T|/2$ for every vector $x' \in P_{\text{CacAP}}(\mathcal{I})$. Hence, we will assume in the following that this is not the case.

Given $x \in [0, 1]^L$ and $\mathcal{W} = \{C \in \mathcal{C} : C \text{ is } x\text{-heavy}\}$, we apply [CTZ20, Theorem 23] to obtain a cheap heavy cut covering, i.e., a set $L_H \subseteq L$ of links covering all heavy cuts with $|L_H| \leq \frac{\varepsilon}{2} \cdot x(L)$. Then we consider the residual instance \mathcal{I}_H of \mathcal{I} with respect to L_H (see Definition 13) and apply Lemma 33. Because \mathcal{I}_H arises from \mathcal{I} by the contraction of $|L_H|$ many links, every (γ, k) -splitting of \mathcal{I}_H is a $(\gamma + |L_H|, k)$ -splitting of \mathcal{I} . Hence, the application of Lemma 33 yields either a separating hyperplane or a CacAP solution F for \mathcal{I}_H with

$$|F| \leq \alpha \cdot \left(1 + \frac{\varepsilon}{2}\right) \cdot x(L) + \max_{\mathcal{S} \text{ an } (\varepsilon \cdot x(L), k)\text{-splitting of } \mathcal{I}} \sum_{\mathcal{J} \text{ split minor of } \mathcal{S}} f(\mathcal{J}),$$

where we used $\gamma + |L_H| \leq \frac{\varepsilon}{4} \cdot |T| + \frac{\varepsilon}{2} \cdot x(L) \leq \varepsilon \cdot x(L)$. Then [CTZ20, Corollary 20] implies that $F \cup L_H$ is a solution for the instance \mathcal{I} with the desired guarantee (i).

B Combining our matching-based approach with the stack analysis (Proof of Theorem 1)

To obtain approximation factors below $\frac{4}{3}$, we use the stack analysis approach from [CTZ20], which strengthens the procedure guaranteed by Lemma 5. In this section we explain how we can adapt this stack analysis approach from [CTZ20] for our purposes and we prove that it leads to the claimed approximation ratio of 1.29.

Let us first recall the definition of *shadows* and *minimality* of links from [CTZ20].

Definition 35. Let (G, L) be a CacAP instance and let $\{u, v\}$ be a link. Then $\{\bar{u}, \bar{v}\} \in \binom{V}{2}$ is a shadow of the link $\{u, v\}$ if \bar{u} and \bar{v} are vertices that lie on every u - v path in G . A shadow $\bar{\ell}$ of link ℓ is a strict shadow of ℓ if $\bar{\ell} \neq \ell$.

Following [CTZ20], we say that a link $\ell_1 \in L$ is *minimal with respect to* $\ell_2 \in L$ if for any strict shadow $\bar{\ell}_1 \in \binom{V}{2}$ of ℓ_1 , the 2-cuts covered by $\{\bar{\ell}_1, \ell_2\}$ are a strict subset of those covered by $\{\ell_1, \ell_2\}$ and the 2-cuts covered by $\{\ell_2\}$ are a strict subset of those covered by $\{\ell_1, \ell_2\}$; or formally, for any strict shadow $\bar{\ell}_1$ of ℓ_1 ,

$$\begin{aligned} \{C \in \mathcal{C} : \{\bar{\ell}_1, \ell_2\} \cap \delta(C) \neq \emptyset\} &\subsetneq \{C \in \mathcal{C} : \{\ell_1, \ell_2\} \cap \delta(C) \neq \emptyset\}, \text{ and} \\ \{C \in \mathcal{C} : \{\ell_2\} \cap \delta(C) \neq \emptyset\} &\subsetneq \{C \in \mathcal{C} : \{\ell_1, \ell_2\} \cap \delta(C) \neq \emptyset\}. \end{aligned}$$

We remark that the second of the two conditions above is implied by the first one whenever ℓ_1 admits a strict shadow. However, for the case where ℓ_1 does not admit a strict shadow, which corresponds to both endpoints of ℓ_1 lying in the same cycle of the cactus, the second condition is necessary.

We now introduce the notion of *weakly L_{cross} -minimality*. A similar notion, called *L_{cross} -minimality*, has been introduced in [CTZ20]. The reason why we work with the notion of weak L_{cross} -minimality is that we will need this later to combine the stack analysis approach with our matching-based approach from Section 2.

Definition 36. A set $F \subseteq L$ is called *weakly L_{cross} -minimal* if for every two distinct links $\ell, \ell' \in F_{\text{cross}}$ the link ℓ is minimal with respect to ℓ' .

The property of L_{cross} -minimality introduced in [CTZ20] requires in addition that every cross-link $\ell_1 \in F_{\text{cross}}$ is minimal with respect to every in-link $\ell_2 \in F_{\text{in}}$. However, for leaf-to-leaf instances, the notion of weak L_{cross} -minimality will be sufficient for the stack analysis approach.

The stack analysis approach first solves a linear program and then rounds the LP solution to a CacAP solution. We will next introduce the polytope over which this linear program optimizes.

For a k -wide CacAP instance (G, L) with root r let G_1, \dots, G_p denote the principal subcacti. Moreover, for $i \in \{1, \dots, p\}$, we define $L_i \subseteq L$ to be the set of links that have at least one endpoint in G_i that is distinct from the root r . We define

$$P^{\min}(G_i, L_i) := \text{conv} \left(\{ \chi^F : F \subseteq L_i \text{ is a weakly } L_{\text{cross}}\text{-minimal feasible solution for } G_i \} \right)$$

to be the convex hull of incidence vectors of weakly L_{cross} -minimal feasible CacAP solutions for G_i , where a feasible CacAP solution for G_i is a set of links that covers every $C \in \mathcal{C}$ that is a subset of the vertices of G_i . The same proof as for [CTZ20, Lemma 42] shows that we can optimize over $P^{\min}(G_i)$ in polynomial time when k is constant. Indeed, we can first observe that any weakly L_{cross} -minimal solution for (G_i, L_i) contains at most k cross-links and we can “guess” those in polynomial time by enumerating all possible choices. Then we can complete this set of cross-links in a cheapest possible way, using that instances of CacAP with a constant number of terminals are efficiently solvable [BFG⁺14].

This implies that we can also optimize efficiently over the polytope

$$P_{\text{bundle}}^{\min}(G, L) := \left\{ x \in [0, 1]^L : x|_{L_i} \in P^{\min}(G_i, L_i) \text{ for all } i \in \{1, \dots, p\} \right\}$$

because the fact that we can efficiently optimize over $P^{\min}(G_i, L_i)$ implies that we can separate over $P^{\min}(G_i, L_i)$ and thus over $P_{\text{bundle}}^{\min}(G, L)$. For a more detailed description of how we can optimize over $P_{\text{bundle}}^{\min}(G, L)$ we refer to [CTZ20] where an analogous reasoning has been used.

[CTZ20] implies that we can round vectors in $P_{\text{bundle}}^{\min}(G, L)$ for k -wide instances of Leaf-to-Leaf+ CacAP with a certain guarantee on the cost of the CacAP. Formally, [CTZ20] considered general k -wide instances of CacAP and defined $P_{\text{bundle}}^{\min}(G, L)$ in a slightly different way, requiring that the vectors $x|_{L_i}$ for $i \in \{1, \dots, p\}$ are convex combinations of incidence vectors of L_{cross} -minimal solutions (instead of weakly L_{cross} -minimal solutions). However, this stronger property is used only in a single place of the proof, namely the proof of Lemma 52 in [CTZ20], which is a trivial statement for leaf-to-leaf+ instances.⁴ Therefore, the guarantees from [CTZ20] still apply in our setting despite the slightly different definition of $P_{\text{bundle}}^{\min}(G, L)$.

The precise guarantee on the solution that we obtain by rounding a point $x \in P_{\text{bundle}}^{\min}(G, L)$ is given by the optimal value of some optimization problem, which we will state later in this section (in the proof of Theorem 1).

Note that in general, $P_{\text{bundle}}^{\min}(G, L)$ is not a relaxation of the CacAP problem and it might even be the case that the polytope $P_{\text{bundle}}^{\min}(G, L)$ is empty even though the instance (G, L) is feasible. The reason is that in general, not every instance (G, L) has a weakly L_{cross} -minimal solution. However, we will show that every *root-shadow complete* instance of Leaf-to-Leaf+ CacAP has a weakly L_{cross} -minimal optimum solution (see Lemma 38 below).

Definition 37. *An instance (G, L) of Leaf-to-Leaf+ CacAP is root-shadow complete if for every cross link $\{u, v\} \in L_{\text{cross}}$, both $\{u, r\}$ and $\{r, v\}$ are contained in L . Then $\{u, r\}$ and $\{r, v\}$ are the root shadows of $\{u, v\}$.*

⁴Lemma 52 from [CTZ20] provides a lower bound on $x(L_{\text{up}})$ where $L_{\text{up}} \subseteq L$. This lower bound is 0 for every leaf-to-leaf+ instance and thus the statement of Lemma 52 from [CTZ20] trivially holds in our setting.

We may always assume that the leaf-to-leaf+ instance that we are given is root-shadow complete because, given an arbitrary leaf-to-leaf+ instance (G, L) , we can consider its *root-shadow completion*

$$(G, L \cup \{\{u, r\}, \{v, r\} : \{u, v\} \in L_{\text{cross}}\})$$

that we obtain by adding all root-shadows of cross-links. Given any solution to the root-shadow completion we can always turn it into a solution of the original instance with the same number of link by replacing every root shadow by the original link. We remark that the root-shadow completion of a leaf-to-leaf+ instance is again a leaf-to-leaf+ instance. However, the root-shadow completion of a pure leaf-to-leaf instance is not a leaf-to-leaf instance and this is the reason why we work with leaf-to-leaf+ instances in this paper.

Lemma 38. *Every root-shadow complete instance of Leaf-to-Leaf+ CacAP has a weakly L_{cross} -minimal optimum solution.*

Proof. Let OPT be a an optimum solution of a Leaf-to-Leaf+ CacAP instance with a minimum number of cross-links among all optimum solutions. We claim that OPT is weakly L_{cross} -minimal. Suppose this is not the case. Then there exist cross-links $\ell_1, \ell_2 \in \text{OPT}_{\text{cross}}$ such that ℓ_1 is not minimal with respect to ℓ_2 . Let $\ell_1 = \{s, t\}$ and note that any strict shadow of ℓ_1 , i.e., any shadow $\bar{\ell}_1 \neq \ell_1$ of ℓ_1 , covers at most one of the 2-cuts $\{s\}, \{t\} \in \mathcal{C}$. Thus, one of the endpoints of ℓ_1 , say t , must be also an endpoint of ℓ_2 , because otherwise for every strict shadow $\bar{\ell}_1$ of ℓ_1 at least one of the 2-cuts $\{s\}, \{t\} \in \mathcal{C}$ would be uncovered by $\{\bar{\ell}_1, \ell_2\}$, even though it was covered by $\{\ell_1, \ell_2\}$. Because $\ell_1 = \{s, t\}$ and ℓ_2 are both cross-links and have the common endpoint t , the set of 2-cuts in \mathcal{C} covered by $\{s, r\}$ and ℓ_2 is the same as the set of 2-cuts covered by ℓ_1 and ℓ_2 . Hence, we can replace the link ℓ_1 in OPT by its root-shadow $\{s, r\}$ and maintain an optimum solution. However, this replacement decreased the number of cross-links, contradicting our choice of OPT. \square

Lemma 38 implies that for any root-shadow complete instance (G, L) of CacAP, we have $\min\{x(L) : x \in P_{\text{bundle}}^{\min}(G, L)\} \leq |\text{OPT}|$ because $\chi^{\text{OPT}} \in P_{\text{bundle}}^{\min}(G, L)$ for any weakly L_{cross} -minimal optimum solution OPT.

We now show how we combine our matching-based approach from Section 2 with the stack analysis approach from [CTZ20] to prove Theorem 1.

Theorem 1. *There is a 1.29-approximation algorithm for Leaf-to-Leaf CAP (and therefore also Leaf-to-Leaf TAP).*

Proof. By Theorem 18, it suffices to show that there is a ρ -approximation algorithm for $O(1)$ -wide instances of Leaf-to-Leaf CacAP for some $\rho < 1.29$.

Let us now describe such a ρ -approximation algorithm for $O(1)$ -wide instances of Leaf-to-Leaf CacAP. For a k -wide instance of Leaf-to-Leaf CacAP, we consider its root-shadow completion (G, L) , which is a Leaf-to-Leaf+ CacAP instance. Let OPT be a weakly L_{cross} -minimal optimum solution of the instance (G, L) , which exists by Lemma 38.

We compute a matching $M \subseteq L$ on the leaves of G without bad links that minimizes $|M| + \frac{1}{2}|M_{\text{in}}| + (|T| - 2|M|)$. By Lemma 10, we have

$$|M| + \frac{1}{2}|M_{\text{in}}| + (|T| - 2|M|) \leq |\text{OPT}| + \frac{1}{2}|\text{OPT}_{\text{in}}|.$$

Because OPT is weakly L_{cross} -minimal, this implies that the incidence vector χ^{OPT} of OPT is a

feasible solution to the following linear program:

$$\begin{aligned}
& \min && x(L) \\
& \text{s.t.} && |M| + \frac{1}{2}|M_{\text{in}}| + (|T| - 2|M|) \leq x(L) + \frac{1}{2}x(L_{\text{in}}) \\
& && x \in P_{\text{bundle}}^{\min}(G, L).
\end{aligned} \tag{15}$$

We compute an optimum solution x to the LP (15). Because χ^{OPT} is a feasible solution to (15), we have $x(L) \leq |\text{OPT}|$.

By Lemma 10, we can compute a feasible CacAP solution F_1 with

$$|F_1| \leq |M| + \frac{1}{2}|M_{\text{in}}| + (|T| - 2|M|) \leq x(L) + \frac{1}{2}x(L_{\text{in}}). \tag{16}$$

Finally, we apply the rounding procedure from the stack analysis approach from [CTZ20] to obtain a solution F_2 and return the cheaper of the two solutions F_1 and F_2 .

In order to state the guarantee on $|F_2|$ that we obtain from [CTZ20], we need the following definitions. We define the function $g : [0, 1] \rightarrow \mathbb{R}_{\geq 0}$ as

$$g(\lambda) := \lambda \cdot (1 - e^{-\lambda})$$

and the function $\text{gain} : [0, 1]^2 \rightarrow \mathbb{R}$ by

$$\text{gain}(\lambda, \eta) := \begin{cases} \lambda(e^{-\eta} - 1 + \eta) \cdot e^{-\lambda + \eta} & \text{if } \eta > \frac{1}{2}\lambda \\ \lambda(e^{-\eta} - 1 + \eta) \cdot (1 - \lambda + \eta) & \text{otherwise.} \end{cases}$$

Moreover, for a leaf $t \in T$, we define $\lambda_t^0 := x(\delta(t) \cap L_{\text{cross}})$. By [CTZ20],⁵ we can round any vector $x \in P_{\text{bundle}}^{\min}(G, L)$ to a CacAP solution F_2 such that

$$|F_2| \leq x(L_{\text{in}}) + 2 \cdot x(L_{\text{cross}}) - b \cdot \sum_{t \in T} g(\lambda_t^0) \tag{17}$$

if $b \in [\frac{5}{12}, \frac{1}{2}]$ is such that the following expression is non-negative for all $v, w \in T$

$$\begin{aligned}
& \frac{b}{\lambda_w^0 - \eta_w^{c_v}} \cdot \text{gain}(\lambda_w^0, \eta_w^{c_v}) \\
& - s_{vw} \cdot \left(b - \frac{1}{3}\right) - (\eta_w^{c_v} - s_{vw}) \cdot \left(2 \left(b - \frac{2}{5}\right) - \frac{1}{30}\right) \\
& + \max\{0, x(S_v) - \eta_w^{c_v}\} \cdot \left(\frac{1}{2} - b\right) + \max\{0, 1 - x(S_v) - \eta_w^{c_v} + s_{vw}\} \cdot (1 - b),
\end{aligned} \tag{18}$$

for all s_{vw} ($v, w \in T$) and $\eta_w^{c_v}$ ($v, w \in T$) that fulfill

$$\begin{aligned}
0 & \leq s_{vw} \leq \eta_w^{c_v} \leq \lambda_w^0 \leq 1 \\
0 & \leq s_{vw} \leq \lambda_v^0 \leq 1.
\end{aligned}$$

⁵More precisely, the statement we use here follows essentially from Lemma 58 in [CTZ20]. Note that Condition (18) is for leaf-to-leaf+ instances equivalent to equation (27) in [CTZ20]. The condition $b \in [\frac{5}{12}, \frac{1}{2}]$ is equivalent to the condition $z_1 \leq z_2 \leq 0$ below equation (18) in [CTZ20]. We remark that in a leaf-to-leaf+ instance, the values λ_t^1 , μ_t^0 , and μ_t^1 appearing in [CTZ20] are all equal to zero by definition of these values and thus we eliminated their occurrences here. In particular, our definitions of the functions g and gain then coincide with those from [CTZ20]. Finally, we again highlight that we work with a slightly different definition of $P_{\text{bundle}}^{\min}(G, L)$ than [CTZ20] but the only place where the stronger formulation in [CTZ20] is needed is in the proof of Lemma 52 in [CTZ20], which is trivial for leaf-to-leaf+ instances.

Using a computer program, we can verify that Condition (18) is fulfilled for $b := 0.452$.

Let $\alpha := \frac{x(L_{\text{cross}})}{x(L)}$. Then, by (17), the CacAP solution F_2 obtained from the stack analysis approach fulfills

$$\begin{aligned} |F_2| &\leq x(L_{\text{in}}) + 2 \cdot x(L_{\text{cross}}) - b \cdot \sum_{v \in T} g(\lambda_v^0) \\ &= \left(1 + \alpha - \frac{b}{x(L)} \sum_{v \in T} g(\lambda_v^0) \right) \cdot x(L) \\ &= \left(1 + \alpha - \frac{b \cdot 2\alpha}{\sum_{v \in T} \lambda_v^0} \sum_{v \in T} g(\lambda_v^0) \right) \cdot x(L). \end{aligned} \quad (19)$$

The solution we return has $\min\{|F_1|, |F_2|\}$ many links and thus by combining (16), $x(L_{\text{in}}) = (1 - \alpha) \cdot x(L)$, and (19), we obtain an approximation ratio of

$$\max \left\{ \min \left\{ \frac{3}{2} - \frac{1}{2}\alpha, 1 + \alpha - \frac{b \cdot 2\alpha}{\sum_{v \in T} \lambda_v^0} \sum_{v \in T} g(\lambda_v^0) \right\} : \alpha \in [0, 1], \lambda_v^0 \in [0, 1] \forall v \in T, \right. \\ \left. \alpha \cdot |T| \leq \sum_{v \in T} \lambda_v^0 \leq |T| \right\}.$$

Because the function g is convex in λ , replacing λ_v^0 by its average value over all of the leaves $v \in T$, does not increase the value of $\sum_{v \in T} g(\lambda_v^0)$. Moreover, this replacement does not change $\sum_{v \in T} \lambda_v^0$. Thus, we can simplify the optimization problem and conclude that we obtain an approximation ratio of

$$\rho := \max \left\{ \min \left\{ \frac{3}{2} - \frac{1}{2}\alpha, 1 + \alpha - \frac{b \cdot 2\alpha}{\lambda^0} \cdot g(\lambda^0) \right\} : 0 \leq \alpha \leq \lambda^0 \leq 1 \right\}.$$

As $g(\lambda) = \lambda(1 - e^{-\lambda})$ and $b = 0.452$, this yields

$$\rho = \max \left\{ \min \left\{ \frac{3}{2} - \frac{1}{2}\alpha, 1 + \alpha - 0.904\alpha \cdot (1 - e^{-\lambda^0}) \right\} : 0 \leq \alpha \leq \lambda^0 \leq 1 \right\}. \quad (20)$$

The optimum value of this optimization problem is attained for $\alpha = \lambda^0$ being the unique solution to $6\alpha + 9\alpha e^{-\alpha} = 5$. This yields $\alpha = \lambda^0 = 0.4202$ and $\rho < 1.29$. \square