

Representation of short distances in structurally sparse graphs*

Zdeněk Dvořák[†]

Abstract

A partial orientation \vec{H} of a graph G is a *weak r -guidance system* if for any two vertices at distance at most r in G , there exists a shortest path P between them such that \vec{H} directs all but one edge in P towards this edge. In case that \vec{H} has bounded maximum outdegree, this gives an efficient representation of shortest paths of length at most r in G . We show that graphs from many natural graph classes admit such weak guidance systems, and study the algorithmic aspects of this notion.

1 Introduction

We consider the following general question: Given an undirected unweighted graph G , can short distances in G be represented efficiently? More precisely, the setting that interests us is as follows:

- G is known to belong to some class \mathcal{G} of well-structured graphs (e.g., planar graphs, graphs of clique-width at most 6, ...)
- We are only interested in distances up to some fixed upper bound r .
- We are allowed to preprocess G in polynomial time; let D denote the resulting data structure.
- The data structure D should enable us to efficiently answer the queries of the following form:

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[†]Computer Science Institute, Charles University, Prague, Czech Republic. E-mail: rakdver@iuuk.mff.cuni.cz.

- Are two input vertices u and v at distance at most r in G ?

In case that the answer is positive, we may also want to determine the distance between u and v , and return a shortest path between them.

Note that we consider both \mathcal{G} and r to be fixed parameters. There are several criteria to consider:

- The time complexity of the preprocessing.
- The time complexity of the queries.
- The space complexity (the size of D).

Of course, there are some trade-offs between these criteria. E.g., D could store distances between all pairs of vertices, resulting in a relatively slow preprocessing time and space complexity $\Theta(|V(G)|^2)$, but constant query time. In this paper we consider a solution which still achieves constant query time (depending only on \mathcal{G} and r), but is memory efficient in the sense that storing D takes up about as much space as the graph G itself. To achieve this, D will only consist of an orientation of G .

An *orientation* of an undirected graph G is a directed graph \vec{H} such that for every $(u, v) \in E(\vec{H})$, we have $uv \in E(G)$, and for every $uv \in E(G)$, at least one of (u, v) and (v, u) is a directed edge of \vec{H} . Note that \vec{H} can contain both (u, v) and (v, u) , i.e., we allow an edge of G to be directed in both ways at the same time. Let $B_{\vec{H}}(v, a)$ denote the set of vertices reachable in \vec{H} from v by a directed path of length at most a . An *r -guidance system* is an orientation \vec{H} such that for any vertices $u, v \in V(G)$ at distance $\ell \leq r$ in G , there exist non-negative integers a and b such that $a + b = \ell$ and $B_{\vec{H}}(u, a) \cap B_{\vec{H}}(v, b) \neq \emptyset$; i.e., there is a shortest path between u and v in G whose edges are in \vec{H} directed towards one of its vertices. Note that if \vec{H} has maximum outdegree at most c , all such paths can be enumerated in time $O(c^\ell)$, and if c is small, this enables us to find a shortest path between a given pair of vertices (or verify that their distance is greater than r) efficiently.

The guidance systems were (without explicitly naming them) introduced by Kowalik and Kurowski [10], who proved that they can be used to represent short distances in planar graphs, and more generally for every F , in any graph avoiding F as a topological minor. As observed in [6], essentially the same argument shows that graphs from even more general graph classes, namely all classes with *bounded expansion* and more generally all *nowhere-dense* classes, admit guidance systems of bounded maximum outdegree. To state the result precisely, we need to introduce several definitions.

For a non-negative integer s , a graph H is an s -shallow minor of a graph G if H is obtained from G by contracting pairwise-disjoint subgraphs, each of radius at most s . For a class \mathcal{G} , let $\nabla_s \mathcal{G}$ denote the class of all graphs H that appear as s -shallow minors in graphs from \mathcal{G} . A class \mathcal{G} of graphs has *bounded expansion* if for every $s \geq 0$ there exists d_s such that every graph in $\nabla_s \mathcal{G}$ has average degree at most d_s . Even less restrictively, a class \mathcal{G} is *nowhere-dense* if for every $s \geq 0$ there exists d_s such that $K_{d_s} \notin \nabla_s \mathcal{G}$. Examples of classes of graphs with bounded expansion include planar graphs and more generally all proper minor-closed classes, graphs with bounded maximum degree and more generally all proper classes closed under topological minors, graphs drawn in the plane with $O(1)$ crossings on each edge, and many other classes of sparse graphs; see [12] for more details.

Theorem 1 (Dvořák and Lahiri [6]). *Let \mathcal{G} be a class of graphs and r a positive integer.*

- *If \mathcal{G} has bounded expansion, then there exists c such that every graph $G \in \mathcal{G}$ has an r -guidance system of maximum outdegree at most c . Moreover, such an r -guidance system can be found in time $O(|V(G)|)$.*
- *If \mathcal{G} is nowhere-dense, then for every $\varepsilon > 0$, there exists c such that every graph $G \in \mathcal{G}$ has an r -guidance system of maximum outdegree at most $c|G|^\varepsilon$. Moreover, such an r -guidance system can be found in time $O(|V(G)|^{1+\varepsilon})$.*

A graph with an orientation of maximum outdegree at most c necessarily has maximum average degree at most $2c$, and thus it is $(2c + 1)$ -degenerate. Hence, guidance systems of bounded maximum outdegree can only exist in sparse graphs. This brings us to the main topic of our paper: **Does there exist a variant of the notion useful for dense graphs?**

Note that representing distance one by a guidance system forces us to orient all edges. If we relax the notion to only represent distances $2, 3, \dots, r$, this may not be necessary. A *partial orientation* of a graph G is a spanning directed subgraph of an orientation of G (i.e., we allow some edges not to be oriented in either direction). An r^+ -guidance system is a partial orientation \vec{H} of a graph G such that for any vertices $u, v \in V(G)$ at distance ℓ in G , where $2 \leq \ell \leq r$, there exist non-negative integers a and b such that $a + b = \ell$ and $B_{\vec{H}}(u, a) \cap B_{\vec{H}}(v, b) \neq \emptyset$. Let us give a (trivial) example showing that there are dense graphs admitting r^+ -guidance systems.

Example 2. *Let G be a graph containing a universal vertex u , and let \vec{H} be the partial orientation obtained by directing all edges incident with u towards*

u. Observe that for any positive integer r , \vec{H} is an r^+ -guidance system in G of maximum outdegree one.

However, there are some quite simple graphs that do not admit r^+ -guidance systems of bounded outdegree. For a graph G and a positive integer k , let G^k denote the k -distance power of G , that is, the graph with vertex set $V(G)$ and two vertices adjacent if and only if the distance between them in G is at most k .

Example 3. *Let T be the graph obtained from $K_{1,n}$ by subdividing every edge exactly twice, let X be the set of its leaves, and let Y be the set of neighbors of the central vertex of degree n . Let $G = T^2$. Note that Y induces a clique in G , and any two vertices of X are joined by a unique path of length three using exactly one edge of this clique. This implies that in any 3^+ -guidance system for G , every edge of the clique on Y must be directed in at least one direction, and thus some vertex of Y has outdegree at least $(n - 1)/2$.*

This example highlights the fact that in dense graphs, we cannot afford to represent the shortest paths by having all of their edges oriented, and motivates another generalization of guidance systems.

Definition 4. *A weak r -guidance system is a partial orientation \vec{H} of G such that for any distinct vertices $u, v \in V(G)$ at distance $\ell \leq r$ in G , there exist non-negative integers a and b such that $a + b = \ell - 1$ and G contains an edge between $B_{\vec{H}}(u, a)$ and $B_{\vec{H}}(v, b)$; that is, there exists a shortest path between u and v in G such that all but one edge e of this path is directed in \vec{H} towards this exceptional edge e (which may or may not be directed).*

In particular, an r -guidance system (or an r^+ -guidance system) is also a weak r -guidance system. Note that if the graph G is represented so that we can in constant time test whether two vertices are adjacent, then a weak r -guidance system of maximum outdegree c makes it possible to find a shortest path between a given pair of vertices (or verify that their distance is greater than r) in time $O(c^{r-1})$.

The goal of this paper is to develop the theory of weak guidance systems; we show that several interesting graph classes admit weak guidance systems of small maximum outdegree (constant, or logarithmic in the number of vertices), address the algorithmic question of finding weak guidance systems efficiently, and describe an application of the notion in approximation of distance variants of the independence and domination number. On the negative side, we give examples of simple graph classes that do not admit weak guidance systems of small maximum outdegree.

The rest of the paper is organized as follows.

- In Section 2, we give some basic properties of weak guidance systems, including the fact that they behave well under the distance power operation.
- In Section 3, we prove a result analogous to Theorem 1, showing that graphs from classes with structurally bounded expansion (i.e., definable in classes with bounded expansion by first-order logic formulas) admit weak guidance systems with bounded maximum outdegree. We also give an analogous result for structurally nowhere-dense graph classes.
- The results from Section 2 and 3 do not provide polynomial-time algorithms to find the weak guidance system if we are not provided with some additional information (e.g., in the case of graph powers, if we are only given the graph G^k , but not the graph G and its weak guidance system). In Section 4, we provide an approximation algorithm for this problem that for an n -vertex graph which admits a weak guidance system of maximum outdegree c returns one of maximum outdegree $O(c \log n)$. We also provide an algorithm that returns a weak guidance system of maximum outdegree $O(c \log c)$, assuming that certain set systems have bounded VC-dimension, which is in particular the case for classes with structurally bounded expansion studied in Section 3.
- In Section 5, we show an application of weak guidance systems in design of approximation algorithms for distance independence and domination number.
- In Section 6, we consider several graph classes that do not admit weak guidance systems of bounded maximum outdegree, specifically graphs of girth at least five and large average degree, split graphs, and graphs of bounded clique-width.

2 Basic properties of weak guidance systems

First, let us note that weak guidance systems enable us to circumvent the difficulty from Example 3.

Lemma 5. *Let G be a graph and let $k \geq 1$ and $c \geq 2$ be integers. For any positive integer r , if G has a weak kr -guidance system \vec{H} of maximum outdegree at most c , then G^k has a weak r -guidance system \vec{F} of maximum outdegree at most $2c^k$.*

Proof. Let \vec{F} be the partial orientation of G^k containing exactly the directed edges (u, v) such that $v \in B_{\vec{H}}(u, k)$. Note that \vec{F} has maximum outdegree at most

$$\sum_{\ell=1}^k c^\ell = c \cdot \frac{c^k - 1}{c - 1} < 2c^k.$$

Suppose that the distance between vertices x and y in G^k is $\ell \leq r$. Then the distance between x and y in G is between $(\ell - 1)k + 1$ and ℓk , and since \vec{H} is a weak kr -guidance system in G , there is a shortest path P between x and y in G oriented in \vec{H} towards an edge $e = x'y'$ of P , where x, x', y', y appear in P in order. Let $x_0 = x, x_1, \dots, x_a$ be the maximal sequence of vertices of P such that x_i is at distance ki from x in G and e is contained in the subpath of P between x_a and y . Analogously, let $y_0 = y, y_1, \dots, y_b$ be the maximal sequence of vertices of P such that y_i is at distance ki from y in G and e is contained in the subpath of P between y_b and x . Note that the distance between x_a and y_b in G is at most $2k - 1$, since each of them is at distance at most $k - 1$ from e . If the distance between x_a and y_b in G is greater than k , then note that $\ell = a + b + 2$ and $y' \neq y_b$; let $z = y'$ and let Q be the path $x_0 \dots x_a z y_b \dots y_0$. Otherwise, $\ell = a + b + 1$ and we let $z = y_b$ and $Q = x_0 \dots x_a y_b \dots y_0$. This gives a shortest path Q in G^k directed in \vec{F} towards its edge $x_a z$. \square

Let us remark that weak guidance systems are qualitatively different from guidance systems only in dense graphs, as in degenerate graphs, a weak guidance system can be completed to a guidance system by directing the rest of the edges while preserving the bounded maximum outdegree.

Observation 6. *If G admits a weak r -guidance system of maximum outdegree c and G is t -degenerate, then G also admits an r -guidance system of maximum outdegree at most $c + t$.*

Finally, we give the following description of weak r -guidance systems, which we use often in the rest of the paper. For vertices u and v of a graph G at distance ℓ , let $G(u \rightarrow v)$ be the set of neighbors of u at distance $\ell - 1$ from v ; i.e., $G(u \rightarrow v)$ consists of all possible second vertices of shortest paths from u to v .

Observation 7. *A partial orientation \vec{H} of a graph G is a weak r -guidance system if and only if the following claim holds for all $u, v \in V(G)$ at distance ℓ in G , where $2 \leq \ell \leq r$:*

- (\star) *Either u has an outneighbor in $G(u \rightarrow v)$, or v has an outneighbor in $G(v \rightarrow u)$.*

3 Weak guidance systems in structurally sparse graphs

The standard way of generalizing the concepts of bounded expansion and nowhere-density to dense graphs is through the notion of *first-order transductions*, see e.g. [7, 8, 2, 11]. For a positive integer k and a graph G , let kG denote the disjoint union of k copies of G . A *transduction* T consists of

- a positive integer k
- a binary predicate symbol M and unary predicate symbols U_1, \dots, U_s , and
- first-order formulas $\omega(x)$ and $\epsilon(x, y)$ with free variables x (resp. x and y) using these predicate symbols and the binary predicate symbol E .

For graphs H and G , we write $H \in T(G)$ if there exist sets $C_1, \dots, C_s \subseteq V(kG)$ such that $V(H)$ consists exactly of the vertices $v \in V(kG)$ satisfying

$$kG, U_1 := C_1, \dots, U_s := C_s \models \omega(v)$$

and $E(H)$ consists exactly of the pairs $u, v \in V(H)$ such that

$$kG, U_1 := C_1, \dots, U_s := C_s \models \epsilon(u, v),$$

where the predicate symbol E is interpreted as adjacency in kG and M is interpreted as the equivalence between the k copies of each vertex.

That is, a transduction allows us to blow up the graph by replicating each vertex a bounded number of times, then non-deterministically color some vertices (via the predicates U_1, \dots, U_s), and finally define the vertices and edges of the new graph by a first-order formula. As an example, if T is the transduction with $k = 1$, $s = 0$, $\omega(x) = \text{true}$ and

$$\epsilon(x, y) = (x \neq y) \wedge (\exists z)(z = x \vee E(x, z)) \wedge E(z, y),$$

then $H \in T(G)$ if and only if $H = G^2$. Hence, the transduction operation generalizes the graph power operations we considered in Lemma 5.

For a class of graphs \mathcal{G}' and a transduction T , let $T(\mathcal{G}')$ denote the class of all graphs G such that $G \in T(G')$ for some $G' \in \mathcal{G}'$. We say that a class of graphs \mathcal{G} has *structurally bounded expansion* (resp., is *structurally nowhere-dense*) if $\mathcal{G} \subseteq T(\mathcal{G}')$ for a transduction T and a graph class \mathcal{G}' of bounded expansion (resp., being nowhere-dense). The goal of this section is

to show that such graph classes admit weak guidance systems with bounded maximum outdegree.

In preparation for that, let us start by considering the graph classes with bounded *shrub-depth*. The notion of shrub-depth was defined by Ganian et al. [9] using the concept of *tree models*. For a positive integer m , an m -signature is a function $S : \mathbb{Z}^+ \rightarrow 2^{[m] \times [m]}$ assigning a symmetric relation $S(i)$ to each $i > 0$. For a positive integer d , an (m, d) -tree model of a graph G is a triple (T, φ, S) , where

- T is a rooted tree with leaf set $V(G)$ and such that the length of every root-leaf path is d ,
- $\varphi : V(G) \rightarrow [m]$ assigns one of m labels to each leaf,
- S is an m -signature, and
- for every $u, v \in V(G)$, if $2i$ is the distance between u and v in T (i.e., if i is the distance from u and v to their nearest common ancestor in T), then $uv \in E(G)$ if and only if $(\varphi(u), \varphi(v)) \in S(i)$.

A class \mathcal{G} of graphs has *shrub-depth at most d* if for some positive integer m , every graph in \mathcal{G} has an (m, d) -model.

Lemma 8. *For every class \mathcal{G} of graphs of bounded shrub-depth and every positive integer r , there exists a positive integer c such that every graph from \mathcal{G} has a weak r -guidance system of maximum outdegree at most c .*

Proof. Let m and d be positive integers such that every graph $G \in \mathcal{G}$ has an (m, d) -tree model (T, φ, S) . Let $c = r^3 m^r (d + 1)^{r^2} d$.

For a positive integer k , a k -type is a pair (f, g) of functions $f : [k] \rightarrow [m]$ and $g : [k]^2 \rightarrow \{0\} \cup [d]$. The *type* of a k -tuple (v_1, \dots, v_k) of vertices of G is the k -type (f, g) such that $f(i) = \varphi(v_i)$ for $i \in [k]$ and $g(i, j)$ is half of the distance between v_i and v_j in T . For each vertex $x \in V(T)$, each positive integer $k \leq r$, and each k -type t , if there exist a k -tuple (v_1, \dots, v_k) of leaves of T with ancestor x and of type t , fix such a k -tuple $Q(x, t) = (v_1, \dots, v_k)$ arbitrarily and let $A(x, t) = \{v_1, \dots, v_k\}$; otherwise, let $A(x, t) = \emptyset$. For each non-leaf vertex $y \in V(T)$, if y has more than r children x such that $A(x, t) \neq \emptyset$, then let $R(y, t)$ be a set of $r + 1$ of them chosen arbitrarily; otherwise let $R(y, t)$ be the set of all children x of y such that $A(x, t) \neq \emptyset$. Let $B(y, t) = \bigcup_{x \in R(y, t)} A(x, t)$, and let $B(y)$ be the union of $B(y, t)$ over all k -types t with $k \leq r$.

Let \vec{H} be the partial orientation of G containing exactly the edges (u, v) such that $uv \in E(G)$ and $v \in B(y)$ for some ancestor y of u in T . Clearly,

\vec{H} has maximum outdegree at most c . Let us now argue that \vec{H} is a weak r -guidance system.

Consider any vertices $u, v \in V(G)$ at distance ℓ in G , where $2 \leq \ell \leq r$, and let $P = u_0 u_1 \dots u_\ell$, where $u_0 = u$ and $u_\ell = v$, be a shortest path from u to v in G . We will show that the condition (\star) from Observation 7 is satisfied for u and v . Let y be the nearest common ancestor of u and u_1 in T , let X be the set of children of y that have a descendant belonging to $V(P)$, and let x_1 be the child of y whose descendant is u_1 . Suppose first that v is not a descendant of x_1 . Let Q be the tuple of vertices of P that are descendants of x_1 (in any order) and let t be its type. Since $|X| \leq r + 1$ and $A(x_1, t) \neq \emptyset$, there exists $x'_1 \in R(y, t) \setminus (X \setminus \{x_1\})$. Let $Q' = Q(x'_1, t)$ and let P' be obtained from P by replacing the vertices of Q by the vertices of Q' . Observe that since Q and Q' have the same type and the same common ancestors with the other vertices of P , P' is also a shortest path from u to v in G . Moreover, the construction of \vec{H} implies that the first edge of P' is directed away from u , establishing the validity of the condition (\star) from Observation 7.

Hence, suppose that v is a descendant of x_1 . In particular, this implies that y is also the nearest common ancestor of u and v . Let x_2 be the child of y whose descendant is u . By symmetry, we can assume that $u_{\ell-1}$ is a descendant of x_2 as well. Let $Q_1 = (u_1, u_2, \dots, u_k)$ be the maximal initial segment of $P - u$ consisting of descendants of x_1 ; we have $k < \ell - 1$. Let t_1 be the type of Q_1 . Since $|X| \leq r + 1$ and $A(x_1, t_1) \neq \emptyset$, there exists $x'_1 \in R(y, t_1) \setminus (X \setminus \{x_1\})$. Let $Q'_1 = Q(x'_1, t_1)$ and let P'_1 be obtained from P by replacing the vertices of Q_1 by the vertices of Q'_1 . Observe that since Q and Q' have the same type and the same common ancestors with u and u_{k+1} , P'_1 is also a shortest path from u to v in G . Moreover, the construction of \vec{H} implies that the first edge of P'_1 is directed away from u , establishing the validity of the condition (\star) from Observation 7.

We conclude that \vec{H} is a weak r -guidance system. \square

Crucially, the notions of structurally bounded expansion and structural nowhere-density can be characterized in terms of *bounded shrub-depth covers*. A *cover* of a graph G is a system of subsets of $V(G)$. Let a be a positive integer. A cover \mathcal{C} of G is *a-generic* if for every subset $A \subseteq V(G)$ of size at most a , there exists $C \in \mathcal{C}$ such that $A \subseteq C$. An *a-generic bounded shrub-depth cover assignment* for a graph class \mathcal{G} is a function \mathcal{C} that to each graph $G \in \mathcal{G}$ assigns an a -generic cover $\mathcal{C}(G)$ such that the class

$$\mathcal{C}(\mathcal{G}) = \{G[C] : G \in \mathcal{G}, C \in \mathcal{C}(G)\}$$

has bounded shrub-depth.

Theorem 9 (Gajarský et al. [8] and Dreier et al. [3]). *Let \mathcal{G} be a class of graphs and let a be a positive integer.*

- *If \mathcal{G} has structurally bounded expansion, then for some positive integer k , \mathcal{G} has an a -generic bounded shrub-depth cover assignment \mathcal{C} such that $|\mathcal{C}(G)| \leq k$ for every $G \in \mathcal{G}$.*
- *If \mathcal{G} is structurally nowhere-dense and $\varepsilon > 0$, then for some positive integer k , \mathcal{G} has an a -generic bounded shrub-depth cover assignment \mathcal{C} such that $|\mathcal{C}(G)| \leq k|V(G)|^\varepsilon$ for every $G \in \mathcal{G}$.*

Together with Lemma 8, this gives the main result of this section.

Corollary 10. *Let \mathcal{G} be a class of graphs and let r be a positive integer.*

- *If \mathcal{G} has structurally bounded expansion, then for some positive integer c , every graph in \mathcal{G} has a weak r -guidance system of maximum outdegree at most c .*
- *If \mathcal{G} is structurally nowhere-dense and $\varepsilon > 0$, then for some positive integer c , every graph in \mathcal{G} has a weak r -guidance system of maximum outdegree at most $c|V(G)|^\varepsilon$.*

Proof. Let \mathcal{C} be an $(r + 1)$ -generic bounded shrub-depth cover assignment and k the corresponding constant from Theorem 9. Let c_0 be the constant from Lemma 8 for the class $\mathcal{C}(\mathcal{G})$. Let $c = kc_0$.

For any graph $G \in \mathcal{G}$, let \vec{H} be the union of the weak r -guidance systems of the subgraphs $G[C]$ for $C \in \mathcal{C}(G)$ obtained using Lemma 8. Clearly, the maximum outdegree of \vec{H} is at most c if \mathcal{G} has structurally bounded expansion and at most $c|V(G)|^\varepsilon$ if \mathcal{G} is structurally nowhere-dense. Moreover, consider any vertices u and v at distance at most r in G , and let P be a shortest path between them. Since the cover $\mathcal{C}(G)$ is $(r + 1)$ -generic, there exists $C \in \mathcal{C}(G)$ such that $G[C]$ contains P . Since \vec{H} restricted to C is a weak r -guidance system in $G[C]$, there exists a shortest path between u and v in $G[C]$ (and thus also in G) directed by \vec{H} towards one of its edges. We conclude that \vec{H} is a weak r -guidance system in G . \square

Let us remark that r -guidance systems can be used to characterize bounded expansion and nowhere-density.

Lemma 11. *Let \mathcal{G} be a class of graphs closed under induced subgraphs.*

- If there exists $c : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ such that for every positive integer r , every $G \in \mathcal{G}$ has an r -guidance system of maximum outdegree at most $c(r)$, then \mathcal{G} has bounded expansion.
- If there exists $c : \mathbb{Z}^+ \times \mathbb{R}^+ \rightarrow \mathbb{Z}^+$ such that for every positive integer r and for every $\varepsilon > 0$, every $G \in \mathcal{G}$ has an r -guidance system of maximum outdegree at most $c(r, \varepsilon)|V(G)|^\varepsilon$, then \mathcal{G} is nowhere-dense.

Proof. Suppose for a contradiction that \mathcal{G} is not nowhere-dense. By assumptions, for every $\varepsilon > 0$, every graph $G \in \mathcal{G}$ has an orientation with maximum outdegree at most $c(1, \varepsilon)|V(G)|^\varepsilon$, and thus the maximum average degree of subgraphs of G is at most $2c(1, \varepsilon)|V(G)|^\varepsilon$. By [5, Theorem 6], there exists $r \geq 2$, a graph $G \in \mathcal{G}$, and a graph H of average degree $d > 2c(r, \varepsilon)|V(G)|^\varepsilon$ such that G contains the graph H' obtained from H by subdividing each edge exactly $r - 1$ times as an induced subgraph. Since \mathcal{G} is closed under induced subgraphs, we can assume $G = H'$. Suppose \vec{H} is an r -guidance system in G . Then for every $uv \in E(H)$, the corresponding path P_{uv} of length r in G contains an edge directed away from u or from v , and thus the average outdegree of the vertices of H in G is at least $|E(H)|/|V(H)| = d/2 > c(r, \varepsilon)|V(G)|^\varepsilon$. This contradicts the assumptions.

The argument for the bounded expansion case is analogous, using [5, Theorem 5] instead of [5, Theorem 6]. \square

Note that the assumption of being closed under induced subgraphs is needed, as seen by Example 2: This example together with Observation 6 shows that the class of graphs formed from cliques by subdividing each edge once and adding a universal vertex afterwards admits an r -guidance system of maximum outdegree at most 4 for every $r \geq 1$; but this class is not nowhere-dense.

It is tempting to ask whether weak r -guidance systems do not similarly characterize structurally bounded expansion or structural nowhere-density. However, this is not the case. We define a *weak ∞ -guidance system* to be a partial orientation that is a weak r -guidance system for every positive integer r . *Interval graphs* are the intersection graphs of sets of open intervals in the real line.

Example 12. Consider any interval graph G . Let \vec{H} be the partial orientation of G obtained as follows. For each $u \in V(G)$, let v_1 and v_2 be the neighbors of u such that the right endpoint of the interval of v_1 is maximum among all neighbors of u , and the left endpoint of the interval of v_2 is minimum among them. Include in \vec{H} the edges (u, v_1) and (u, v_2) . Then \vec{H} is a weak ∞ -guidance system in G of maximum outdegree at most two.

The class of interval graphs is closed under induced subgraphs, but it is well-known not to be structurally nowhere-dense.

4 Algorithmic aspects

Note that Theorem 9 only gives a polynomial-time algorithm to obtain the covers if we are given a graph G' such that $G \in T(G')$, where G' belongs to a bounded expansion/nowhere dense graph class. If only G is provided, it is currently not known how to obtain the covers efficiently. Consequently, Corollary 10 does not give an efficient algorithm to obtain weak guidance systems. In this section, we address this issue, giving a polynomial-time algorithm that given an n -vertex graph returns a weak guidance system whose maximum outdegree is worse than optimal only by an $O(\log n)$ factor, and an improved approximation algorithm in case certain relevant set systems have bounded VC-dimension.

First, let us introduce one more relaxation of the guidance system notion. A *fractional orientation* of a graph G is a function p that assigns a non-negative real number $p(u, v)$ to each pair (u, v) of adjacent vertices of G . The *outdegree* $d_p^+(u)$ of a vertex u in the fractional orientation p is $\sum_{v:uv \in E(G)} p(u, v)$. We say that p is a *fractional r -guidance system* if for every $u, v \in V(G)$ at distance ℓ , where $2 \leq \ell \leq r$, we have

$$\sum_{y \in G(u \rightarrow v)} p(u, y) + \sum_{y \in G(v \rightarrow u)} p(v, y) \geq 1. \quad (1)$$

By Observation 7, weak guidance systems can naturally be interpreted as fractional guidance systems.

Observation 13. *Suppose \vec{H} is a weak r -guidance system in a graph G , of maximum outdegree c . Let us define $p(u, v) = 1$ for every $(u, v) \in E(\vec{H})$ and $p(u, v) = 0$ for every $uv \in E(G)$ such that $(u, v) \notin E(\vec{H})$. Then p is a fractional r -guidance system of maximum outdegree c .*

Moreover, an optimal fractional guidance system can be constructed through linear programming.

Lemma 14. *If a graph G has a weak r -guidance system of maximum outdegree c_0 , we can find a fractional r -guidance system of maximum outdegree at most c_0 in G in polynomial time.*

Proof. Let p be an optimal solution to the following linear program.

$$\begin{aligned}
& p(u, v) \geq 0 && \text{for every } (u, v) \text{ s.t. } uv \in E(G) \\
& \sum_{v:uv \in E(G)} p(u, v) \leq c && \text{for every } u \in V(G) \\
& \sum_{y \in G(u \rightarrow v)} p(u, y) + \sum_{y \in G(v \rightarrow u)} p(v, y) \geq 1 && \text{for every } u, v \in V(G) \text{ at distance between 2 and } r \\
& \text{minimize } c
\end{aligned}$$

Then p is a fractional r -guidance system of maximum outdegree at most c in G , and $c \leq c_0$ by Observation 13.

Note that the sets $G(u \rightarrow v)$ can be computed in polynomial time by first computing the distances between all pairs of vertices of G . The linear program has $O(n^2)$ variables and constraints, and thus its optimal solution can be found in polynomial time. \square

Fractional r -guidance systems can be directly used to test presence of shortest paths, with a small probability of error. Let p be a fractional r -guidance system in a graph G . If u is a non-isolated vertex of G , then by a p -random neighbor of u , we mean a neighbor of u selected at random, with the probability that a neighbor v is selected being $p(u, v)/d_p^+(u)$; if $d_p^+(u) = 0$, the probability is $1/\deg u$, instead. For distinct vertices u and v and a positive integer r , a $\text{random } (p, r)\text{-exploration}$ between u and v is a random pair of walks (P_u, P_v) from u and v selected as follows:

- If $uv \in E(G)$, then $P_u = uv$ and $P_v = v$.
- Otherwise, if $r = 1$ or u or v is an isolated vertex, then $P_u = u$ and $P_v = v$.
- Otherwise, let $x \in \{u, v\}$ be selected uniformly at random, and let y be a p -random neighbor of x ;
 - if $x = u$, then select a random $(p, r - 1)$ -exploration (P_y, P_v) between y and v and let P_u be the concatenation of uy and P_y , and
 - if $x = v$, then select a random $(p, r - 1)$ -exploration (P_u, P_y) between u and y and let P_v be the concatenation of vy and P_y .

Observation 15. *Suppose p is a fractional r -guidance system in a graph G , of maximum outdegree c . Let u and v be distinct vertices of G at distance*

at most r , and let (P_u, P_v) be a random (p, r) -exploration between u and v . The probability that $P_u \cup P_v$ is a shortest path between u and v in G is at least $(4c)^{-(r-1)}$.

Proof. We prove the claim by induction on the distance ℓ between u and v that the probability is at least $(4c)^{-(\ell-1)}$. If $\ell = 1$, then $P_u = uv$ and $P_v = v$ with probability 1. Hence, suppose that $\ell \geq 2$. By (1) and symmetry, we can assume that

$$\sum_{y \in G(u \rightarrow v)} p(u, y) \geq \frac{1}{2},$$

and thus

$$\frac{\sum_{y \in G(u \rightarrow v)} p(u, y)}{d_p^+(u)} \geq \frac{1}{2c}.$$

Hence, with probability at least $\frac{1}{4c}$, when choosing (P_u, P_v) , we pick $x = u$ and $y \in G(u \rightarrow v)$, so that the distance between y and v is $\ell - 1$. By the induction hypothesis, the probability that $P_y \cup P_v$ is a shortest path between y and v , and thus $P_u \cup P_v$ is a shortest path between u and v , is then at least $(4c)^{-(\ell-2)}$. The result follows by multiplying these probabilities. \square

Note that for Observation 15 to be practically useful, we would need a representation of p that enables us to choose a p -random neighbor efficiently; in that case, we could iterate $k(4c)^{r-1}$ times the procedure from Observation 15 to find the shortest path between u and v (or decide that the distance between them is greater than r) with error probability at most e^{-k} . More interestingly, we can turn a fractional r -guidance system to a weak r -guidance system with a logarithmic loss in the maximum degree.

Lemma 16. *Let c be a positive real number, let n be a positive integer, and let $m = \lceil 4c \log n \rceil$. Suppose p is a fractional r -guidance system in a graph G , with maximum outdegree at most c . There exists an algorithm that in polynomial time returns a weak r -guidance system \vec{H} in G with maximum outdegree at most m .*

Proof. Let us say that pair $\{u, v\}$ of vertices is *dissatisfied* by a partial orientation \vec{F} if the distance ℓ between u and v satisfies $2 \leq \ell \leq r$ and \vec{F} contains neither an edge from u to $G(u \rightarrow v)$ nor an edge from v to $G(v \rightarrow u)$. By Observation 7, \vec{F} is a weak r -guidance system if and only if there are no dissatisfied pairs.

Let X be any set of pairs of vertices of G at distance between 2 and r . Let \vec{F} be a random partial orientation of G obtained by, for each non-isolated

vertex z of G , choosing a random p -neighbor z' and adding the edge (z, z') . Clearly, \vec{F} has maximum outdegree at most one. Moreover, consider any $\{u, v\} \in X$. By (1) and symmetry, we can assume that

$$\sum_{y \in G(u \rightarrow v)} p(u, y) \geq 1/2.$$

Hence, the probability that $u' \in G(u \rightarrow v)$ (and thus $\{u, v\}$ is not dissatisfied in \vec{F}) is at least $\frac{1}{2c}$. By the linearity of expectation, the expected number of dissatisfied pairs in X is at most $(1 - \frac{1}{2c})|X|$.

Moreover, we can use the method of conditional probabilities to derandomize this procedure and to deterministically construct a partial orientation \vec{F} of G of maximum outdegree at most one such that the number of pairs in X dissatisfied by \vec{F} is at most $(1 - \frac{1}{2c})|X|$. Indeed, we can select the outneighbors one by one, always maintaining the invariant (initially satisfied by the computation from the previous paragraph) that the expected number of pairs in X dissatisfied by the orientation obtained by choosing the remaining outneighbors as random p -neighbors is at most $(1 - \frac{1}{2c})|X|$. To do so, when processing a vertex u , we only need to be able to compute this expected number after each possible choice of the outneighbor of u , which is straightforward due to the linearity of expectation.

Now, to obtain \vec{H} , we let X_0 be the set of all pairs of vertices whose distance is between 2 and r in G . Then, for $i = 1, \dots, m$, we use the procedure described in the previous paragraph to find a partial orientation \vec{F}_i of maximum outdegree at most one so that the set X_i of pairs from X_{i-1} dissatisfied by \vec{F}_i has size at most $(1 - \frac{1}{2c})|X_{i-1}|$. Note that

$$|X_m| \leq (1 - \frac{1}{2c})^m |X_0| \leq \frac{|X_0|}{n^2} < 1,$$

and thus $X_m = \emptyset$. Consequently, no pair is dissatisfied by

$$\vec{H} = \bigcup_{i=1}^m \vec{F}_i,$$

and thus \vec{H} is the desired weak r -guidance system in G . \square

Combining Lemmas 14 and 16, we obtain the following claim.

Corollary 17. *There exists an algorithm that, for an input n -vertex graph G that admits a weak r -guidance system of maximum outdegree at most c , outputs in polynomial time a weak r -guidance system of maximum outdegree $O(c \log n)$.*

Let us remark that the logarithmic loss in Corollary 17 cannot be avoided in general. For positive integers a and k , let $m = k2^{k+1}$ and let $G_{a,k}$ be the random graph obtained as follows. We start with a random bipartite graph with parts L of size a and R of size ma , with each vertex of L being adjacent to each vertex of R independently with probability $1/2$. We then divide R into m parts R_1, \dots, R_m of size a arbitrarily, and for $i = 1, \dots, m$, we add a vertex x_i adjacent to all vertices of R_i .

Lemma 18. *There exists an integer a_0 such that for every $a \geq a_0$ and $k \leq \log a$, with positive probability*

- $G_{a,k}$ has a fractional 2-guidance system with maximum outdegree at most 3, and
- $G_{a,k}$ does not have a weak 2-guidance system with maximum outdegree at most k .

Proof. Let us use the notation from the definition of the graph $G_{a,k}$. Note that

- for $i = 1, \dots, m$ and $v \in L$, the expected number of neighbors of v in R_i is $a/2$, and by Chernoff inequality, the probability that v has less than $a/3$ neighbors in R_i is less than $\exp(-a/36)$.
- for distinct vertices $u, v \in R$, the expected number of common neighbors of u and v in L is $a/4$, and by Chernoff inequality, the probability that u and v have less than $a/5$ common neighbors in L is less than $\exp(-a/200)$,
- for distinct $u, v \in L$, the probability that u and v have less than $a/5$ common neighbors in R_1 is also less than $\exp(-a/200)$, and
- for $i \in 1, \dots, m$ and a k -tuple K of vertices of R_i , the expected number of vertices of L with no neighbor in K is $2^{-k}a$, and by Chernoff inequality, the probability that the number of such vertices is at most $2^{-k-1}a$ is at most $\exp(-2^{-k-3}a)$.

Hence, the probability that any of these events occurs is less than

$$ma \cdot \exp(-a/36) + (m^2 + 1)a^2 \cdot \exp(-a/200) + ma^k \cdot \exp(-2^{-k-3}a) < 1$$

if a is sufficiently large (and using the assumption that $k \leq \log a$; note that the basis of the logarithm is e , and thus $2^k \leq a^{\log 2} \ll a$). Hence, with positive probability,

- for $i = 1, \dots, m$, each vertex $v \in L$ has at least $a/3$ neighbors in R_i ,
- any distinct vertices $u, v \in R$ have at least $a/5$ common neighbors in L ,
- any distinct vertices $u, v \in L$ have at least $a/5$ common neighbors in R_1 , and
- for $i \in 1, \dots, m$ and for every k -tuple K of vertices of R_i , more than $2^{-k-1}a$ vertices of L have no neighbor in K .

Let us define a fractional orientation p of $G_{a,k}$ as follows:

- For $i = 1, \dots, m$ and $v \in R_i$, we set $p(x_i, v) = 3/a$,
- for each adjacent $u \in R$ and $z \in L$, we set $p(u, z) = 2.5/a$, and
- for each adjacent $z \in L$ and $u \in R_1$, we set $p(z, u) = 2.5/a$;

p is 0 everywhere else. Note that this fractional orientation has maximum outdegree at most 3, since $\deg x_i = |R_i| = a$, the number of neighbors of $u \in R$ in L is at most $|L| = a$, and the number of neighbors of $z \in L$ in R_1 is at most $|R_1| = a$. Consider now any vertices $x, y \in V(G_{a,k})$ at distance exactly two from one another. Note that $G_{a,k}$ is bipartite, and thus either $x, y \in R$, or $x, y \in V(G_{a,k}) \setminus R$. There are the following cases:

- One of x and y belongs to $\{x_1, \dots, x_m\}$, say $x = x_i$. Then y necessarily belongs to L , and y has at least $a/3$ neighbors in R_i . Hence, $|G_{a,k}(x \rightarrow y)| \geq a/3$ and

$$\sum_{z \in G_{a,k}(x \rightarrow y)} p(x, z) \geq a/3 \cdot 3/a = 1.$$

- Both x and y belong to L . Since x and y have at least $a/5$ common neighbors in R_1 , we have $|G_{a,k}(x \rightarrow y) \cap R_1| = |G_{a,k}(y \rightarrow x) \cap R_1| \geq a/5$, and

$$\sum_{z \in G_{a,k}(x \rightarrow y)} p(x, z) + \sum_{z \in G_{a,k}(y \rightarrow x)} p(y, z) \geq 2 \cdot a/5 \cdot 2.5/a = 1.$$

- Similarly, if $x, y \in R$, then x and y have at least $a/5$ common neighbors in L , and

$$\sum_{z \in G_{a,k}(x \rightarrow y)} p(x, z) + \sum_{z \in G_{a,k}(y \rightarrow x)} p(y, z) \geq 2 \cdot a/5 \cdot 2.5/a = 1.$$

Therefore, p is a fractional 2-guidance system for $G_{a,k}$.

Consider now any partial orientation \vec{H} of $G_{a,k}$ with maximum outdegree at most k . Then each vertex $v \in L$ has an outneighbor in R_i for at most k choices of i , and thus there exists $i \in \{1, \dots, m\}$ such that at least $(1 - k/m)a = (1 - 2^{-k-1})a$ vertices of L have no outneighbor in R_i . Let K be a k -tuple of vertices of R_i containing all outneighbors of x_i . More than $2^{-k-1}a$ vertices of L have no neighbor in K , and thus there exists a vertex $v \in L$ with no outneighbor in R_i and no neighbor in K . However, x_i and v are at distance 2, yet neither x_i nor v has an outneighbor in $G_{a,k}(x_i \rightarrow v) = G_{a,k}(v \rightarrow x_i) \subseteq R_i \setminus K$. Hence \vec{H} is not a weak 2-guidance system. Consequently, every weak 2-guidance system for $G_{a,k}$ must have maximum outdegree greater than k . \square

Note that if we set $k = \lfloor \log a \rfloor$, we have

$$n = |V(G_{a,k})| \leq (m+1)(a+1) \leq (k2^{k+1} + 1) \cdot (\exp(k+1) + 1) \leq \exp(O(k)),$$

and thus Lemma 18 gives examples of graphs with an arbitrarily large number n of vertices and a fractional 2-guidance system of maximum outdegree at most 3 such that every weak 2-guidance system has maximum outdegree $\Omega(\log n)$.

However, we can do better in case the VC-dimension of relevant systems is bounded. Recall that a system \mathcal{S} of subsets of a set X *shatters* a set $A \subseteq X$ if $\{A \cap S : S \in \mathcal{S}\}$ contains all subsets of A , and that the *VC-dimension* of \mathcal{S} is the size of the largest subset of X shattered by \mathcal{S} . The key property of systems with bounded VC-dimension is that they admit efficient (randomized) approximation for smallest hitting set in terms of the size of the smallest fractional hitting set (a *hitting set* for \mathcal{S} is a subset of X intersecting all elements of \mathcal{S} , and a *fractional hitting set* is a function $w : X \rightarrow \mathbb{R}_0^+$ such that, defining $w(A) = \sum_{x \in A} w(x)$ for each subset A of X , each element $S \in \mathcal{S}$ satisfies $w(S) \geq 1$; the size of the fractional hitting set w is $w(X)$). For the following standard result, see e.g. [13].

Theorem 19. *There exists a polynomial-time randomized algorithm that, given a system \mathcal{S} of subsets of a set X of VC-dimension at most d and a fractional hitting set w of size s , with probability at least $1/2$ returns a hitting set for \mathcal{S} of size $O(ds \log s)$.*

For a graph G , integer $r \geq 2$, and vertex $u \in V(G)$, let $\text{VC}(G, r, u)$ denote the VC-dimension of the system

$$\{G(u \rightarrow v) : v \in V(G), 2 \leq d_G(u, v) \leq r\},$$

and let $\text{VC}(G, r) = \max_{u \in V(G)} \text{VC}(G, r, u)$.

Theorem 20. *There exists a polynomial-time randomized algorithm that, for an input n -vertex graph G that admits a weak r -guidance system of maximum outdegree at most c , with probability at least $1/2$ outputs a weak r -guidance system of maximum outdegree $O(\text{VC}(G, r) \cdot c \log c)$.*

Proof. Let p be a fractional r -guidance system of maximum outdegree at most c in G found using Lemma 14. For each $u \in V(G)$, let R_u be the set of vertices $v \in V(G)$ such that $2 \leq d_G(u, v) \leq r$ and

$$\sum_{z \in G(u \rightarrow v)} p(u, z) \geq 1/2.$$

Since p is a fractional r -guidance system, for each $u, v \in V(G)$ such that $2 \leq d_G(u, v) \leq r$, we have $v \in R_u$ or $u \in R_v$.

Let \mathcal{S}_u be the system $\{G(u \rightarrow v) : v \in R_u\}$ of subsets of the set $N_G(u)$ of neighbors of u . For $z \in N_G(u)$, let us define $w(z) = 2p(u, z)$. By the choice of R_u , we have $w(S) \geq 1$ for each $S \in \mathcal{S}_u$, and thus w is a fractional hitting set for \mathcal{S}_u . Moreover, $w(N_G(u)) \leq 2c$, since the maximum outdegree of p is at most c . The VC-dimension of \mathcal{S}_u is at most $\text{VC}(G, r, u) \leq \text{VC}(G, r)$, and thus we can by Theorem 19 find a hitting set $H_u \subseteq N_G(u)$ for \mathcal{S}_u of size $O(\text{VC}(G, r) \cdot c \log c)$; note that we iterate the algorithm $\Omega(|V(G)|)$ times to make the probability of error less than $\frac{1}{2^{|V(G)|}}$, and thus we find a valid hitting set for each $u \in V(G)$ with probability at least $1/2$.

Let us now define a partial orientation \vec{G} of G by, for each $u \in V(G)$, directing the edges from u to H_u . Clearly, \vec{G} has maximum outdegree $O(\text{VC}(G, r) \cdot c \log c)$. Moreover, consider any $u, v \in V(G)$ such that $2 \leq d_G(u, v) \leq r$. By symmetry, we can assume that $v \in R_u$, and thus H_u intersects the set $G(u \rightarrow v) \in \mathcal{S}_u$. Hence, u has an outneighbor in $G(u \rightarrow v)$. By Observation 7, we conclude that \vec{G} is a weak r -guidance system for G . \square

In particular, this is useful for structurally nowhere-dense classes (and in particular for classes with structurally bounded expansion), as follows from the fact that first-order definable sets in graphs from these classes have bounded VC-dimension. More precisely, for a first-order formula $\psi(\vec{x}, \vec{y})$ with two groups \vec{x} and \vec{y} of free variables, a graph G , and a $|\vec{x}|$ -tuple \vec{u} of vertices of G , let $S_{\psi, G}(\vec{u})$ be the set of $|\vec{y}|$ -tuples \vec{v} of vertices of G such that $G \models \psi(\vec{u}, \vec{v})$, and let $\mathcal{S}_{\psi, G}$ be the system

$$\{S_{\psi, G}(\vec{u}) : \vec{u} \in V(G)^{|\vec{x}|}\}$$

of sets of $|\vec{y}|$ -tuples of vertices of G . The following bound follows from the results of Adler and Adler [1], see also [14] for a more precise bounds and the discussion of the possibility to introduce vertex and edge colors (unary and binary predicates from the statement of the theorem).

Theorem 21. *For every nowhere-dense graph class \mathcal{G} and a first-order formula $\psi(\vec{x}, \vec{y})$ using unary predicate symbols U_1, \dots, U_s and binary predicate symbols E_1, \dots, E_t , there exists a constant d such that the following claim holds. Consider any graph $G \in \mathcal{G}$, and interpret U_i for $i \in \{1, \dots, s\}$ as a subset of $V(G)$ and E_j for $j \in \{1, \dots, t\}$ as a subset of $E(G)$. Then the system $\mathcal{S}_{\psi, G}$ has VC-dimension at most d .*

This easily gives the following consequence.

Lemma 22. *For every structurally nowhere-dense class \mathcal{G} of graphs and every integer $r \geq 2$, there exists a constant d such that $\text{VC}(G, r) \leq d$ for every graph $G \in \mathcal{G}$.*

Proof. Since \mathcal{G} is structurally nowhere-dense, there exists a nowhere-dense class \mathcal{G}_0 and a transduction $T = (k, M, U_1, \dots, U_s, \omega, \epsilon)$ such that for each $G \in \mathcal{G}$ there exists a graph $H \in \mathcal{G}_0$ such that $G \in T(H)$; let C_1^G, \dots, C_s^G be the corresponding subsets of $V(H)$ used to interpret U_1, \dots, U_s .

For $H \in \mathcal{G}_0$, let $(kH)'$ be the graph obtained from the disjoint union of k copies of G by adding a clique on each k -tuple of vertices corresponding to the same vertex of H , and let M_H be the set of the edges of these cliques. Also, let E_H be the set of edges of kH . Let $\mathcal{G}_1 = \{(kH)' : H \in \mathcal{G}_0\}$. Since $(kH)'$ is a subgraph of the lexicographic product of H with a clique of bounded size and \mathcal{G}_0 is nowhere-dense, the class \mathcal{G}_1 is nowhere-dense as well [12].

Note that there exists a first-order formula $\psi_r(x_1, x_2, y)$ with three free variables such that for each $u, v \in V(G)$ satisfying $2 \leq d_G(u, v) \leq r$ and $z \in V(G)$, $G \models \psi_r(u, v, z)$ if and only if $z \in G(u \rightarrow v)$. Let ψ'_r be the formula obtained from ψ_r by restricting the quantification to vertices satisfying ω and replacing each usage of the adjacency predicate by ϵ . Clearly, if $G \in T(H)$, then

$$G \models \psi_r(u, v, z) \text{ iff } (kH)', U_1 := C_1^G, \dots, U_s := C_s^G, E := E_H, M := M_H \models \psi_r(u, v, z).$$

Therefore, with the interpretation of the unary and binary symbols as above, $\mathcal{S}_{\psi_r, G}$ is a subset of $\{S \cap V(G) : S \in \mathcal{S}_{\psi'_r, (kH)'}\}$, and thus the VC-dimension of $\mathcal{S}_{\psi_r, G}$ is at most as large as the VC-dimension of $\mathcal{S}_{\psi'_r, (kH)'}$. Since $(kH)' \in \mathcal{G}_1$ and \mathcal{G}_1 is nowhere-dense, Theorem 21 implies that this VC-dimension is bounded. \square

Hence, Theorem 20 gives the following algorithmic form of Corollary 10.

Corollary 23. *Let \mathcal{G} be a class of graphs and let r be a positive integer.*

- *If \mathcal{G} has structurally bounded expansion, then there exists c and a randomized algorithm that for an input n -vertex graph $G \in \mathcal{G}$ outputs in polynomial time with probability at least $1/2$ a weak r -guidance system of maximum outdegree at most c .*
- *If \mathcal{G} is structurally nowhere-dense and $\varepsilon > 0$, then there exists c and a randomized algorithm that for an input n -vertex graph $G \in \mathcal{G}$ outputs in polynomial time with probability at least $1/2$ a weak r -guidance system of maximum outdegree at most cn^ε .*

5 Distance domination and independence number

For a positive integer r , a set S of vertices of a graph G is *r -dominating* if every vertex of G is at distance at most r from S , and *r -independent* if distinct vertices of S are at distance greater than r from one another. Let $\gamma_r(G)$ denote the smallest size of an r -dominating set in G , and $\alpha_r(G)$ the largest size of an r -independent set in G . Observe that if D is an r -dominating and A a $2r$ -independent set in G , then every vertex of D is at distance at most r from at most one vertex of A , and since every vertex of A is at distance at most r from D , we have $|A| \leq |D|$. Consequently, $\alpha_{2r}(G) \leq \gamma_r(G)$. In general, the converse inequality does not hold and it is not even possible to bound $\gamma_r(G)$ by a function of $\alpha_{2r}(G)$; however, as shown in [4], if G is from a class of graphs with bounded expansion, then $\gamma_r(G) = O(\alpha_{2r}(G))$. A small variation of the argument gives the following stronger claim.

Lemma 24. *For all positive integers c and r , there exists a linear-time algorithm that, given a graph G together with its $2r$ -guidance system of maximum outdegree less than c , returns an r -dominating set D and a $2r$ -independent set A in G such that $|D| \leq c^2|A|$.*

Note that this implies that $\gamma_r(G) \leq |D| \leq c^2\gamma_r(G)$ and $\frac{1}{c^2}\alpha_{2r}(G) \leq |A| \leq \alpha_{2r}(G)$, and thus this gives a linear-time algorithm to approximate both the r -domination and the $2r$ -independence number of G within the constant factor c^2 . The presence of a weak $2r$ -guidance system of bounded outdegree is not by itself sufficient to ensure a similar result.

Example 25. Let \vec{K} be a random orientation of the clique with vertex set $\{1, \dots, n\}$ (for each edge, choose direction uniformly independently at random). Let G be the graph obtained from \vec{K} as follows: We have $V(G) = \{v_1, \dots, v_n, u_1, \dots, u_n, z\}$, where for each $i \in \{1, \dots, n\}$, u_i is adjacent to z , v_i , and all vertices v_j such that $(i, j) \in E(\vec{K})$. Let \vec{H} be the partial orientation of G where for $i \in \{1, \dots, n\}$, the edge $v_i u_i$ for $i \in \{1, \dots, n\}$ is directed towards u_i , and the edge $u_i z$ is directed towards z . Note that for any distinct $i, j \in \{1, \dots, n\}$, we have $(i, j) \in E(\vec{K})$ or $(j, i) \in E(\vec{K})$, and thus the path $v_i u_i v_j$ or $v_j u_j v_i$ has the first edge directed towards its middle vertex. Consequently, \vec{H} is a weak 2-guidance system for G of maximum outdegree one. Moreover, any 2-independent set in G contains at most one of the vertices $\{v_1, \dots, v_n, z\}$ and at most one of the vertices $\{u_1, \dots, u_n\}$, and thus $\alpha_2(G) \leq 2$. On the other hand, we have $\gamma_1(G) = \Omega(\log n)$: By replacing each vertex v_i by u_i in an optimal dominating set and possibly adding z , we obtain a dominating set D of size at most $\gamma_1(G) + 1$ containing none of the vertices v_1, \dots, v_n , and to dominate these vertices, observe that with high probability D needs to contain $\Omega(\log n)$ of the vertices u_1, \dots, u_n .

However, we can solve this issue by adding an additional obstruction. An (r, k) -halfgraph in a graph G is a sequence $u_1, \dots, u_k, v_1, \dots, v_k$ of vertices of G such that for every $i, j \in \{1, \dots, k\}$,

- if $j < i$, then the distance between u_i and v_j in G is greater than r , and
- if $j \geq i$, then the distance between u_i and v_j in G is exactly r .

We say that a graph is (r, k) -stable if it does not contain any (r, k) -halfgraph.

Theorem 26. For all positive integers r, k , and $c \geq 2$, there exists a constant b and a polynomial-time algorithm that, given an (r, k) -stable graph G together with its weak $2r$ -guidance system \vec{H} of maximum outdegree at most c , returns an r -dominating set D and a $2r$ -independent set A in G such that $|D| \leq b|A|$.

Proof. Let D and A' be the sets of vertices of G obtained as follows. We initialize $D := \emptyset$ and $A' := \emptyset$. As long as D is not an r -dominating set, we choose a vertex x at distance greater than r from D arbitrarily, we add x to A' , and we add x and all vertices reachable in \vec{H} from x by directed paths of length at most r to D . At the end, D is an r -dominating set and $|D| \leq c^{r+1}|A'|$.

Let \prec be the linear ordering on vertices of A' such that $x \prec y$ when x was added to A' before y . The algorithm above enforces the following property

(†): If $x \prec y$, then every vertex reachable from x by a directed path in \vec{H} of length at most r is at distance greater than r from y .

Let $\sigma(1) = 0$ and for $p = 2, \dots, k$, let $\sigma(p) = c^{2r+1}(\sigma(p-1) + 1)$. The set A' is not necessarily $2r$ -independent, however it has the following property: If $S \subseteq A'$ consists of vertices pairwise at distance at most $2r$ from one another, then $|S| \leq \sigma(k+1)$. To prove this, we will show a stronger claim. For a positive integer $p \leq k+1$, a p -halfgraph extension of S is a sequence $u_p, \dots, u_k, v_p, \dots, v_k$ of vertices of G such that for $i = p, \dots, k$,

- (i) $u_i \in A'$, $u_i \prec u_{i+1}$ if $i < k$, and $s \prec u_i$ for every $s \in S$.
- (ii) \vec{H} contains a directed path from u_i to v_i of length exactly r ,
- (iii) the distance between u_i and v_j in G is exactly r for every $j \in \{i, \dots, k\}$, and
- (iv) the distance between v_i and s is exactly r for every $s \in S$.

We will prove by induction on p that if there exists a p -halfgraph extension of S , then $|S| \leq \sigma(p)$; $|S| \leq \sigma(k+1)$ then follows, since an empty sequence trivially forms a $(k+1)$ -halfgraph extension of S . For $p = 1$, note that if $1 \leq j < i \leq k$, then $u_j \prec u_i$ by (i), and (ii) and (†) imply that the distance between v_j and u_i in G is greater than r . Together with (iii), this implies that G contains an (r, k) -halfgraph, which is a contradiction. That is, the case $p = 1$ can never occur and the conclusion $|S| \leq \sigma(1)$ holds trivially.

Suppose now that $p \geq 2$ and that the claim holds for $p-1$. If $S = \emptyset$, then $|S| \leq \sigma(p)$ holds. Otherwise, let u_{p-1} be the last vertex of S in the ordering \prec . Since the distance between any vertices of S is at most $2r$ and \vec{H} is a weak $2r$ -guidance system, for each $s \in S \setminus \{u_{p-1}\}$, there exists a shortest path P_s in G between u_{p-1} and s directed in \vec{H} towards one of its edges. Let Q_s be the longest initial segment of P_s directed away from u_{p-1} . By the choice of u_{p-1} , we have $s \prec u_{p-1}$, and thus (†) implies that the part of P_s directed away from s has length at most $r-1$, and consequently $|E(Q_s)| \geq r$.

For any directed path Q in \vec{H} starting in u_{p-1} of length between r and $2r$, let S_Q be the set of vertices $s \in S \setminus \{u_{p-1}\}$ such that $Q_s = Q$. The preceding argument shows that S is the union of the sets S_Q over all such paths, and thus we can fix Q such that $|S_Q| \geq |S|/c^{2r+1}$. If $|S_Q| \leq 1$, then $|S| \leq 2^{2r+1} \leq \sigma(p)$, as required. Hence, suppose that $|S_Q| \geq 2$. Let v_{p-1} be the final vertex of Q and let s_Q be the first vertex of S_Q in the ordering \prec . Consider any vertex $s' \in S_Q \setminus \{s_Q\}$. Note that G contains a path of length at most $2r - |E(Q)| \leq r$ from s_Q to v_{p-1} with all but possibly the

last edge directed away from s_Q in \vec{H} , and since $s_Q \prec s'$ by the choice of s_Q , (\dagger) implies that s' is at distance at least r from v_{p-1} . Since s' is also at distance at most $2r$ from u_{p-1} through a shortest path whose initial segment is Q , s' is at distance at most $2r - |E(Q)| \leq r$ from v_{p-1} . We conclude that $|E(Q)| = r$ and all vertices of $S_Q \setminus \{s_Q\}$ are at distance exactly r from v_{p-1} . Therefore, $u_{p-1}, \dots, u_k, v_{p-1}, \dots, v_k$ is a $(p-1)$ -halfgraph extension of $S_Q \setminus \{s_Q\}$, and $|S_Q \setminus \{s_Q\}| \leq \sigma(p-1)$ by the induction hypothesis. But then $|S| \leq c^{2r+1}|S_Q| \leq c^{2r+1}(\sigma(p-1) + 1) = \sigma(p)$.

Let F be the auxiliary graph with $V(F) = A'$ and with distinct vertices $u, v \in A'$ adjacent if the distance between them in G is at most $2r$. We claim that each vertex of F has at most $c^{2r+1}\sigma(k+1)$ neighbors that precede it in the ordering \prec . Indeed, let N be the set of such neighbors of a vertex $u \in A'$, and for each directed path Q in \vec{H} starting in u of length between r and $2r$, let N_Q consist of the vertices $v \in N$ such that Q is the maximal initial directed segment of a shortest path from u to v in G which is directed towards one of its edges by \vec{H} . As in the preceding part of the proof, note that (\dagger) and the fact that \vec{H} is a weak $2r$ -guidance system implies that N is the union of the sets N_Q over such paths, and thus we can fix such a path Q for which $|N_Q| \geq |N|/c^{2r+1}$. However, the vertices of N_Q are at distance at most $2r - |E(Q)| \leq r$ from the final vertex of Q , and thus they are pairwise at distance at most $2r$ from one another. Consequently, $|N_Q| \leq \sigma(k+1)$, and $|N| \leq c^{2r+1}\sigma(k+1)$.

We conclude that F is $c^{2r+1}\sigma(k+1)$ -degenerate, and thus it is $(c^{2r+1}\sigma(k+1) + 1)$ -colorable and has an independent set A of size at least

$$\frac{|A'|}{c^{2r+1}\sigma(k+1) + 1} \geq \frac{|D|}{c^{r+1}(c^{2r+1}\sigma(k+1) + 1)}.$$

By the construction of F , A is a $2r$ -independent set in G . Therefore, the theorem holds with $b = c^{r+1}(c^{2r+1}\sigma(k+1) + 1)$. \square

By the results of Adler and Adler [1], for any structurally nowhere-dense graph class \mathcal{G} and every r , there exists k so that all graphs in \mathcal{G} are (r, k) -stable. In combination with Corollary 23, we have the following consequence.

Corollary 27. *For any class \mathcal{G} with structurally bounded expansion and for any positive integer r , there exists a constant b and a polynomial-time randomized algorithm that, given a graph $G \in \mathcal{G}$ with probability at least $1/2$ returns an r -dominating set D and a $2r$ -independent set A in G such that $|D| \leq b|A|$.*

6 Graph classes without bounded outdegree weak guidance systems

To better understand obstructions to the existence of weak r -guidance systems of bounded maximum outdegree, it is natural to consider the dual of the linear program from the proof of Lemma 14, which can be reformulated as follows. For $uz \in E(G)$, let $R_r(u, z)$ be the set of vertices $v \in V(G)$ such that the distance between u and v is between 2 and r and z lies on a shortest path from u to v in G ; i.e., $z \in G(u \rightarrow v)$.

Lemma 28. *Let G be a graph and let r be a positive integer. Let c be the solution to the following optimization problem:*

$$\begin{aligned}
 & y_{uv} \geq 0 && \text{for every } u, v \in V(G) \text{ at distance between 2 and } r \\
 & x_u = \max_{z:uz \in E(G)} \sum_{v \in R_r(u,z)} y_{uv} && \text{for every } u \in V(G) \\
 \text{maximize } & \frac{\sum_{uv:2 \leq d_G(u,v) \leq r} y_{uv}}{\sum_{v \in V(G)} x_v}
 \end{aligned}$$

Then every fractional or weak r -guidance system in G has maximum outdegree at least c .

Proof. The dual of the linear program from the proof of Lemma 14 is

$$\begin{aligned}
 & x_u \geq 0 && \text{for every } u \in V(G) \\
 & y_{uv} \geq 0 && \text{for every } u, v \in V(G) \text{ at distance between 2 and } r \\
 & \sum_{u \in V(G)} x_u = 1 \\
 & \sum_{v \in R_r(u,z)} y_{uv} \leq x_u && \text{for every } (u, z) \text{ s.t. } uz \in E(G) \\
 \text{maximize } & \sum_{uv:2 \leq d_G(u,v) \leq r} y_{uv}
 \end{aligned}$$

This is equivalent to the optimization problem from the statement of the lemma. Hence, its solution c provides a lower bound on the maximum outdegree of a fractional r -guidance system in G , and by Observation 13 also a lower bound on the maximum outdegree of a weak r -guidance system in G . \square

As an example, this easily shows that no good weak guidance systems exist for graphs of girth at least five and large maximum average degree (the *maximum average degree* of a graph is the maximum of the average degrees of its subgraphs).

Lemma 29. *Let G be a graph of girth at least five and maximum average degree $d \geq 2$. Every fractional or weak 2-guidance system in G has maximum outdegree at least $d/2$.*

Proof. Let $Z \subseteq V(G)$ be a smallest set such that $G[Z]$ has average degree d . Since $d \geq 2$, every vertex of $G[Z]$ has degree at least two, since deleting vertices of degree at most one would not decrease the average degree.

Since G has girth at least 5, any vertices $u, v \in Z$ at distance two in $G[Z]$ have a unique common neighbor $z \in Z$; we define

$$y_{uv} = \frac{1}{\deg_{G[Z]} z - 1}.$$

For any pair $u, v \in V(G)$ of vertices at distance two in G such that $\{u, v\} \not\subseteq Z$ or the common neighbor of u and v does not belong to Z , we define $y_{uv} = 0$. For any edge uz of G , if $\{u, z\} \subseteq Z$, then we have $|R_2(u, z) \cap Z| = \deg_{G[Z]} z - 1$, and thus

$$\sum_{v \in R_2(u, z)} y_{uv} = 1;$$

while if $\{u, z\} \not\subseteq Z$, then

$$\sum_{v \in R_2(u, z)} y_{uv} = 0.$$

Therefore,

$$x_u = \max_{z: uz \in E(G)} \sum_{v \in R_2(u, z)} y_{uv} = 1$$

for $u \in Z$ and $x_u = 0$ for $u \in V(G) \setminus Z$, and

$$\frac{\sum_{uv: d_G(u, v) = 2} y_{uv}}{\sum_{u \in V(G)} x_u} = \frac{\frac{1}{2} \cdot \sum_{u \in Z} \sum_{z: uz \in E(G[Z])} \sum_{v \in R_2(u, z)} y_{uv}}{|Z|} = \frac{|E(G[Z])|}{|Z|} = d/2.$$

The claim now follows from Lemma 28. \square

This shows that weak guidance systems can be qualitatively different from guidance systems only in graphs of girth at most four.

Corollary 30. *Let G be a graph of girth at least five. For any $r \geq 2$, if G admits a weak r -guidance system of maximum outdegree at most c , then G also admits an r -guidance system of maximum outdegree at most $3c$.*

Proof. By Lemma 29, G has maximum average degree at most $2c$, and thus G is $2c$ -degenerate. The claim then follows by Observation 6. \square

Next, we consider the class of *split graphs*. A graph G is a split graph if there exists a partition (A, B) of its vertex set where A is a clique and B is an independent set.

Lemma 31. *For every n such that n is a power of a prime, there exists a split graph G_n with $2(n^2 + n + 1)$ vertices such that every fractional or weak 2-guidance system in G has maximum outdegree at least $(n + 1)/2$.*

Proof. It is well-known that whenever n is a power of prime, there exists a finite projective plane B of order n , i.e., a system of $n^2 + n + 1$ subsets of the set $A = [n^2 + n + 1]$ with the property that

- (i) $|p_1 \cap p_2| = 1$ for every distinct $p_1, p_2 \in B$ and
- (ii) every element of A belongs to exactly $n + 1$ sets from B .

Let G_n be the graph with vertex set $A \cup B$, vertices in A forming a clique, vertices in B forming an independent set, and vertices $z \in A$ and $p \in B$ adjacent iff $z \in p$. Note that distinct vertices of B are at distance two in G_n by (i), and that for each $p \in B$ and $z \in p$, $|R_2(p, z) \cap B| = n$ by (ii). Therefore, defining $y_{p_1 p_2} = 1$ for any distinct $p_1, p_2 \in B$ and $y_{uv} = 0$ for any other pair u, v of vertices of G_n , we have

$$x_p = \max_{z: z \in p} \sum_{p' \in R_2(p, z)} y_{pp'} = n$$

for $p \in B$ and $x_z = 0$ for $z \in A$. Therefore,

$$\frac{\sum_{uv: d_{G_n}(u, v) = 2} y_{uv}}{\sum_{u \in V(G_n)} x_u} = \frac{\binom{|B|}{2}}{|B|n} = \frac{|B| - 1}{2n} = \frac{n + 1}{2}.$$

The claim now follows from Lemma 28. \square

Let us remark that split graphs are a special case of *chordal graphs* (graphs with no induced cycle of length at least four), and thus chordal graphs do not in general admit weak guidance systems of bounded maximum outdegree.

Finally, let us consider the graphs of bounded *clique-width*. A *k-labeled graph* is a graph where each vertex is assigned a label from $[k]$ (several vertices can have the same label, and not all labels must be used). A *k-labeled graph* G is *constructible* if it is obtained by a finite number of applications of the following rules:

- $|V(G)| = 1$, or
- G is the disjoint union of at least two constructible *k-labeled graphs*, or
- G is obtained from a constructible *k-labeled graph* G' by, for some $i, j \in [k]$, changing all labels i to j , or
- G is obtained from a constructible *k-labeled graph* G' by, for some $i, j \in [k]$, adding all edges between vertices with labels i and j .

We say a graph has *clique-width* at most k if we can assign labels to its vertices so that the resulting *k-labeled graph* is constructible. Graphs with bounded shrub-depth also have bounded clique-width (or equivalently, bounded rank-width); indeed, they can be viewed as graphs of bounded clique-width where the corresponding operation tree has bounded depth. It is natural to ask whether Lemma 8 extends to graphs of bounded clique-width. We show that this is not the case, even for weak 2-guidance systems.

Lemma 32. *For every $d \geq 0$ and $a \geq \max(2, 2d - 1)$, there exists a constructible 6-labeled graph $H_{d,a}$ with half its vertices labeled 1 and half its vertices labeled 2, such that*

(i) $|V(H_{d,a})| \leq 8a^d - 6$ and

(ii) *for every partial orientation \vec{G} of $H_{d,a}$ of maximum outdegree less than d , there exist vertices u and v of labels 1 and 2, respectively, at distance exactly two, such that for every common neighbor x of u and v , we have $(u, x), (v, x) \notin E(\vec{G})$.*

Proof. For $d = 0$, we can let $H_{0,a} = K_2$ with one vertex labeled 1 and the other vertex labeled 2. Suppose we already constructed $H_{d-1,a}$, and let us show how to inductively obtain $H_{d,a}$. First, let $H'_{d-1,a}$ be the graph obtained from $H_{d-1,a}$ by adding vertices v_3 and v_4 with labels 3 and 4 and adding all edges between vertices with labels 1 and 4 and between vertices with labels 2 and 3. Next, we form the disjoint union of a copies of $H'_{d-1,a}$. Then we add two vertices v_5 and v_6 with labels 5 and 6, and all edges between

vertices with labels i and $i + 2$ for $i \in \{3, 4\}$. Finally, we relabel vertices with labels 3 and 5 to label 1 and vertices with labels 4 and 6 to label 2.

The construction uses only 6 labels, and thus $H_{d,a}$ is a constructible 6-labeled graph. Moreover,

$$|V(H_{d,a})| = a(|V(H_{d-1,a})| + 2) + 2 \leq a(8a^{d-1} - 4) + 2 \leq 8a^d - 6,$$

where the last inequality holds since $a \geq 2$. Consider any partial orientation \vec{G} of $H_{d,a}$ of maximum outdegree less than d . Since v_5 and v_6 have outdegree less than d , for one of the $a \geq 2d - 1$ copies of $H'_{d-1,a}$ in $H_{d,a}$, denoted by F' , we have $(v_i, v) \notin \vec{G}$ for every $i \in \{5, 6\}$ and $v \in V(F')$. Let F be the copy of $H_{d-1,a}$ in F' . Suppose that for any two vertices u and v of F of labels 1 and 2, respectively, at distance exactly two in $H_{d,a}$, there exists a common neighbor x of u and v in $H_{d,a}$ such that $(u, x) \in E(\vec{G})$ or $(v, x) \in E(\vec{G})$. The construction of $H'_{d-1,a}$ and $H_{d,a}$ ensures that such a common neighbor x necessarily belongs to F , as we did not add any vertex adjacent both to vertices with label 1 and with label 2. Hence, by the induction hypothesis, the restriction of \vec{G} to F has maximum outdegree at least $d - 1$. Let u be a vertex of F with at least $d - 1$ outneighbors in \vec{G} belonging to F . By symmetry, we can assume u has label 1. Since \vec{G} has maximum outdegree less than d , we have $(u, v_4) \notin E(\vec{G})$. Moreover, by the choice of F' , we have $(v_6, v_4) \notin E(\vec{G})$. Note that v_6 has label 2 in $H_{d,a}$ and the copy of v_4 in F is the only common neighbor of u and v_6 in $H_{d,a}$. This shows that $H_{d,a}$ satisfies the property (ii). \square

By Lemma 32, letting $n = |V(H_{d,2d-1})|$, we conclude that any weak 2-guidance system in $H_{d,2d-1}$, a graph of clique-width at most 6, has maximum outdegree at least $d = \Omega(\log n / \log \log n)$. As we will see in Lemma 33, this is nearly tight. Before that, let us remark that a similar bound also applies to fractional 2-guidance systems, which follows from Lemma 28: For the purpose of the analysis, let us define both vertices of $H_{0,a}$ to be *foundational*, and when constructing $H_{d,a}$, we let the foundational vertices be exactly the foundational vertices in the copies of $H_{d-1,a}$; then, we consider the y -weights defined inductively for each copy of $H_{d-1,a}$, and additionally set $y_{v_i z} = 1$ for each $i \in \{5, 6\}$ and each foundational vertex z at distance two from v_i . Letting $n_d = 2a^d$ be the number of foundational vertices of $H_{d,a}$, this results in $x_{v_i} = |n_{d-1}|/2$; and moreover, $x_z = 1$ for every foundational vertex z . Hence, the lower bound we obtain by Lemma 28 is at least

$$\frac{an_{d-1} + a^2n_{d-2} + \dots + a^dn_0}{(n_{d-1} + an_{d-2} + \dots + a^dn_0) + n_d} = \frac{2da^d}{2da^{d-1} + 2a^d} = \frac{da}{d+a} = \frac{2}{3}d$$

for $a = 2d$.

On the positive side, we show that graphs of bounded clique-width admit weak guidance systems of logarithmic outdegree. Let us start by a useful observation. Suppose (A, B) is a partition of the vertex set of a graph G . For $u, v \in V(G)$, we write $u \equiv_{(A,B)} v$ if either $u, v \in A$ and u and v have the same neighbors in B , or $u, v \in B$ and u and v have the same neighbors in A .

Lemma 33. *Let r be a positive integer or ∞ . Suppose (A, B) is a partition of the vertex set of a graph G and $\equiv_{(A,B)}$ has k equivalence classes. If $G[A]$ and $G[B]$ have a weak r -guidance system of maximum outdegree at most c , then G has a weak r -guidance system of maximum outdegree at most $c + k$.*

Proof. Let \vec{H}_A and \vec{H}_B be weak r -guidance systems of maximum outdegree at most c in $G[A]$ and $G[B]$, respectively. Let \vec{H} consist of $\vec{H}_A \cup \vec{H}_B$ and the following edges: For each $u \in V(G)$ and each equivalence class C of $\equiv_{(A,B)}$ intersecting the component of G containing u , choose a vertex u'_C in C nearest to u in G and a vertex $u_C \in G(u \rightarrow u'_C)$ arbitrarily, and add the edge (u, u_C) . Clearly, \vec{H} has maximum outdegree at most $c + k$.

Consider now any vertices $u, v \in V(G)$ at distance ℓ , where $2 \leq \ell \leq r$, and let P be a shortest path between u and v in G . If an edge of P incident with u or v belongs to $G[A] \cup G[B]$, switch the names of vertices u and v if necessary so that such an edge is incident with u . By symmetry, we can assume $u \in A$. If $P \subseteq G[A]$, then by Observation 7, \vec{H}_A (and thus also \vec{H}) contains an edge directed from u to $G[A](u \rightarrow v) \subseteq G(u \rightarrow v)$ or an edge directed from v to $G[A](v \rightarrow u) \subseteq G(v \rightarrow u)$. Hence, suppose that $P \not\subseteq G[A]$.

If the first edge of P is contained in $G[A]$, then let P' be the longest initial segment of P contained in $G[A]$. If the first edge of P is not contained in $G[A]$, then let P' be the longest initial segment of P contained in $G[B \cup \{u\}]$. Let C be the equivalence class of $\equiv_{(A,B)}$ containing the last vertex z of P' . Note that $z \neq v$: In the first case, this is because P is not contained in $G[A]$. In the second case, this is because $|E(P)| = \ell \geq 2$ and the choice of the names of the vertices u and v implies that the last edge of P is not contained in $G[B]$. Since u'_C is a nearest vertex from u in C , u'_C is at distance at most $|E(P')|$ from u in G . Moreover, u'_C is in the same equivalence class of $\equiv_{(A,B)}$ as z , and thus u'_C is adjacent to the vertex following z in P . Hence, $u_C \in G(u \rightarrow v)$ and \vec{H} contains the edge (u, u_C) .

Observation 7 then implies that \vec{H} is a weak r -guidance system in G . \square

We combine this with the following well-known fact about clique-width.

Observation 34. *If G is a graph with n vertices and clique-width at most k , then there exists a partition (A, B) of vertices of G such that $|A|, |B| \leq \frac{2}{3}n$ and $\equiv_{(A,B)}$ has at most $2k$ equivalence classes.*

Since any induced subgraph of a graph of clique-width at most k also has clique-width at most k , we obtain the following consequence.

Corollary 35. *For every $k \geq 0$, every n -vertex graph of clique-width at most k has a partial orientation \vec{H} of maximum outdegree $O(k \log n)$ such that \vec{H} is a weak ∞ -guidance system.*

7 Conclusions

As we have shown, some interesting graph classes admit weak guidance systems of bounded maximum outdegree, including

- interval graphs,
- classes with structurally bounded expansion, and
- distance powers of graphs with bounded outdegree weak guidance systems.

However, we do not have an exact characterization of the graph classes with this property.

Problem 36. *Characterize hereditary graph classes \mathcal{G} such that for every positive integer r , every graph from \mathcal{G} admits a weak r -guidance system of bounded maximum outdegree.*

We have also exhibited several graph classes that only admit weak guidance systems whose outdegree grows slowly with the number of vertices of the graph, in particular

- structurally nowhere-dense classes, and
- graphs of bounded clique-width.

Again, we do not have a good description of the graph classes with this property.

Problem 37. *Characterize hereditary graph classes \mathcal{G} such that for every positive integer r , every graph $G \in \mathcal{G}$ admits a weak r -guidance system of maximum outdegree at most $|V(G)|^{o(1)}$.*

In sparse graphs, guidance systems and related notions (such as the generalized coloring numbers) have various algorithmic and structural applications. We suspect that similar applications can be found for weak guidance systems as well, generalizing them to dense graphs; we demonstrated this on the example of approximation algorithms for distance domination and independence number.

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