Minimal models of field theories: SDYM and SDGR

Evgeny Skvortsov^{*a,b} & Richard Van Dongen^a

^a Service de Physique de l'Univers, Champs et Gravitation, Université de Mons, 20 place du Parc, 7000 Mons, Belgium

> ^b Lebedev Institute of Physics, Leninsky ave. 53, 119991 Moscow, Russia

Abstract

There exists a natural L_{∞} -algebra or Q-manifold that can be associated to any (gauge) field theory. Perturbatively, it can be obtained by reducing the L_{∞} -algebra behind the jet space BV-BRST formulation to its minimal model. We explicitly construct the minimal models of self-dual Yang-Mills and self-dual gravity theories, which also represents their equations of motion as Free Differential Algebras. The minimal model regains all relevant information about the field theory, e.g. actions, charges, anomalies, can be understood in terms of the corresponding Q-cohomology.

^{*}Research Associate of the Fund for Scientific Research – FNRS, Belgium

Contents

1	Introduction	2
2	Minimal models	3
3	SDYM	4
	3.1 Action, initial data	4
	3.2 FDA, flat space	8
	3.3 FDA, constant curvature space	13
4	SDGR	16
	4.1 Action, initial data	16
	4.2 FDA	18
5	Conclusions and Discussion	22
\mathbf{A}	Notation	24
в	Technicalities: SDYM	26
	B.1 Ψ -sector	26
	B.2 Absence of higher order corrections	28
	B.3 Higher gravitational corrections	30
С	Technicalities: SDGR	32
	C.1 Ψ -sector	33
	C.2 Absense of higher order corrections	35
Bi	Bibliography	

1 Introduction

Self-dual theories have a number of remarkable properties that make them very useful toy models in general and first order approximations to more complicated theories: (a) self-dual theories are closed subsectors of the corresponding complete theories; (b) as a result, all solutions of self-dual theories are solutions of the full ones; (c) all amplitudes of self-dual theories are also amplitudes of the full ones; (d) self-dual theories are integrable; (e) self-dual theories are finite and one-loop exact; (f) existence of a self-dual truncation allows one to rearrange the perturbation theory in a nontrivial way, e.g. to represent Yang-Mills theory as expansion over self-dual rather than flat backgrounds; (g) tools from twistor theory are very-well adapted to self-dual theories, see e.g. [1–8]. In this letter we are interested in constructing L_{∞} -algebras of the simplest self-dual theories: SDYM and SDGR, to uncover their algebraic structure.

There is a hierarchy of L_{∞} -algebras that originate from (quantum) field theories and string field theory, see e.g. [9–16]. The simplest L_{∞} -algebras emerge from a re-interpretation of the BV-BRST formalism: upon expanding the master action in ghosts and anti-fields one finds multilinear maps that obey L_{∞} -relations. Another L_{∞} -algebra emerges from the jet space version of the BV-BRST formulation of a given gauge theory [17–21]. Such L_{∞} is especially useful when investigating various properties of this gauge theory systematically, e.g. classification of deformations of the action, or the question of possible anomalies [17, 18]. Given an L_{∞} -algebra one can consider various equivalent reductions. The smallest possible quasi-isomorphic algebra is the minimal model, which still captures all the relevant properties of the field theory. Another closely related algebraic structure is Free Differential Algebra [22], which emerges as the sigma-model based on the minimal model.

In this letter we construct the minimal models for self-dual Yang-Mills and self-dual gravity theories. As a starting point we take the Chalmers-Siegel action [4] for SDYM and the recently constructed action for SDGR with vanishing cosmological constant [23], which is equivalent to other actions in the literature [24, 25].

Our general motivation stems from several possible applications, where we hope to understand from the algebraic, L_{∞} , point of view: (i) integrability of self-dual theories; (ii) the double-copy relations, see [26, 27] and [28, 29] for the recent results in this direction. Also, the results serve as a starting point for covariantization [30] of Chiral Higher Spin Gravity [31–37].

We begin with a short review of relation between L_{∞} and field theory and then proceed to SDYM and SDGR, respectively, with some technicalities left to appendices.

2 Minimal models

As it was already sketched in the introduction, given any (gauge) field theory in the BV-BRST language it is natural to consider its jet space extension [19–21, 38, 39], which is what is done when investigating the local BRST-cohomology [17, 18]. The jet space extension leads to a rather big L_{∞} -algebra, better say to a Q-manifold provided global issues are taken into account. Various Q-cohomology groups correspond to all physically relevant quantities, e.g. deformations/interactions, anomalies, charges, etc., see e.g. [17, 18]. For every L_{∞} -algebra there always exists a (usually much smaller) L_{∞} -algebra, known as the minimal model, see e.g. [21, 40, 41], that contains the same information — it is said to be quasi-isomorphic.² Some care is needed to prove the same statement for field theories [21, 43], where relevant L_{∞} -algebras are necessarily infinite-dimensional. Minimal models were first introduced by Sullivan [22] in the context of differential graded algebras to study rational homotopy theory. We construct such minimal models for SDYM and SDGR.

Given any non-negatively graded supermanifold \mathcal{N} equipped with a homological vector field Q, QQ = 0, e.g. given by the minimal model, one can write down a sigma-model [19]:

$$d\Phi = Q(\Phi)\,,$$

where $\Phi \equiv \Phi^{\mathcal{A}}$ are maps $\Pi T \mathcal{M} \to \mathcal{N}$ from the exterior algebra of differential forms on a spacetime manifold \mathcal{M} to \mathcal{N} . Together with natural gauge symmetries the sigma-model is equivalent to the classical equations of motion of the initial field theory [19–21], thereby having the form of a Free Differential Algebra, see [22] for exact definitions.³ In the paper we adopt a more pragmatic point of view on minimal models: we seek for the classical equations of motion as an FDA [46]. If $\Phi = {\Phi^{\mathcal{A}}}$ are coordinates on \mathcal{N} , then $Q = Q^{\mathcal{A}} \partial / \partial \Phi^{\mathcal{A}}$ and

$$Q^2 = 0 \qquad \Longleftrightarrow \qquad Q^{\mathcal{B}} \frac{\partial}{\partial \Phi^{\mathcal{B}}} Q^{\mathcal{A}} = 0.$$

This condition is equivalent to the Frobenius integrability of the field equations, i.e. the equations are formally consistent. The L_{∞} -relations emerge by Taylor expanding QQ = 0 at a stationary point of Q [13]. By abuse of notation we always denote coordinates on \mathcal{N} and the corresponding fields by the same symbols. For a large class of field theories \mathcal{N} has coordinates

²There is also another, 'quantum', minimal model [42] — the L_{∞} -algebra given by 1PI correlation functions.

³FDA was introduced by Sullivan and applied to problems in topology. Later, FDA's sneaked into physics in the context of supersymmetry and supergravity [44, 45] and, even later, applied to construct formally consistent deformations of the FDA for free higher spin fields [46].

of degree-one and degree-zero to be associated with gauge connection(s) A and with some matter-like zero-forms L. The simplest FDA with this data reads

$$dA = \frac{1}{2}[A, A], \qquad \qquad dL = \rho(A)L$$

and is equivalent to A taking values in some Lie algebra and to L taking values in its module ρ . We consider these equations free. In particular, it is easy to solve them locally in the pure gauge form, e.g. $A = g^{-1}dg$. The most general deformation of the free equations here-above that is consistent with the form-degree counting reads

$$dA = l_2(A, A) + l_3(A, A, L) + l_4(A, A, L, L) + \ldots = F_A(A; L),$$

$$dL = l_2(A, L) + l_3(A, L, L) + \ldots = F_L(A; L).$$

Our strategy for each of the cases, SDYM and SDGR, is to start off with an action, rewrite the variational equations of motion in the 'almost' FDA form, where 'almost' means that at each step the equations/Q-structure will require new fields/coordinates on \mathcal{N} be introduced. At the end of the day we find the complete Q. Interacting field theories are defined modulo admissible field redefinitions (those that do not change the S-matrix). We found a field frame where no structure maps higher than $l_3(\bullet, \bullet, \bullet)$ are needed for SDYM and SDGR, which also fixes all field redefinitions.⁴

3 SDYM

3.1 Action, initial data

The theory can be formulated [4] with two fields:⁵ the usual one-form gauge potential $A \equiv A_{\mu} dx^{\mu} \equiv A_{\mu}^{a} dx^{\mu} T_{a}$ and a zero-form $\Psi^{AB} \equiv \Psi^{BA}$, $\Psi^{AB} \equiv \Psi^{AB;a} T_{a}$. Here T_{a} are generators of some Lie algebra with a non-degenerate invariant bilinear form. We usually suppress form indices and dx's, as well as the Lie algebra indices. In practice, it is convenient to think of generators T_{a} as of taking values in some matrix algebra and assume A and Ψ^{AB} to take values

⁴This is a key difference with respect to [46], where locality and field redefinitions are not taken into account [47, 48], which results in a general ansatz for interactions rather than a concrete theory.

⁵We use almost exclusively the two-component spinor language, which is well-suited for 4*d*-theories. A short compendium can be found in Appendix A. A classical source is [49]. The most important fact about our notation is that symmetric or to be symmetrized indices can be denoted by the same letter. Also, $A(k) \equiv A_1 \dots A_k$.

in Mat_N , with matrix indices again suppressed. The action reads⁶

$$S_{SDYM} = \operatorname{tr} \int \Psi^{A'B'} \wedge H_{A'B'} \wedge F$$
,

where $F = dA - A \wedge A$ and we prefer to omit \wedge -symbol. The equations of motion imply

$$F_{A'B'} = 0, \qquad D^{A}{}_{B'} \Psi^{A'B'} = 0, \qquad (3.1)$$

where $D \equiv dx^{AA'} D_{AA'} \equiv \nabla - [A, \bullet]$ is the gauge and Lorentz covariant derivative. We also used the decomposition of F into (anti)self-dual parts

$$F = H^{BB}F_{BB} + H^{B'B'}F_{B'B'}$$

We can rewrite the variational equations as

$$dA - AA = H^{BB}F_{BB}, \qquad D\Psi^{A'B'} = e_{CC'}\Psi^{C,A'B'C'}, \qquad (3.2)$$

which is the starting point for constructing the corresponding L_{∞} -algebra. The first equation simply states that $F_{A'B'} = 0$ and, hence, connection A is self-dual. Therefore, only the self-dual part may not be trivial and it is parameterized by F^{AB} . A simple consequence is the Bianchi identity for F^{AB} . In the second equation we introduced a field $\Psi^{A,A'B'C'}$ that parameterizes the first derivative of Ψ that is consistent with (3.1), i.e. it corresponds to a coordinate on the on-shell jet of $\Psi^{A'B'}$.⁷

The problem is, therefore, to find a completion of (3.2), which requires an infinite set of coordinates on \mathcal{N} and Q defined on them in such a way that QQ = 0. The first few terms of Q and \mathcal{N} are already clear from (3.2). The on-shell jet space is also well-known [49]. It is the same as for the free theory where we turned off non-Abelian Yang-Mills groups that result in non-linearities. That the coordinates on \mathcal{N} are the same for the free and interacting theories is due to the requirement for them to have the same number of local degrees of freedom.

Coordinates, on-shell jet. The coordinates on \mathcal{N} are: degree-one A; degree-zero $F^{A(k+2),A'(k)}$ and $\Psi^{A(k),A'(k+2)}$, $k = 0, 1, 2, \dots$ The free equations, i.e. (self-dual) Maxwell equations on

⁶Here, see also appendix A, $H^{AB} \equiv H^{BA}$, $H^{A'B'} \equiv H^{B'A'}$ is the basis of self-dual two-forms, $H^{AB} \equiv e^{A}{}_{C'} \wedge e^{BC'}$, *idem.* for $H^{A'B'}$. Vierbein one-form is $e^{AA'}$.

⁷Equations of motion for free fields of arbitrary spin can be recast into the FDA form [50]. The on-shell jet is very easy to describe in spinorial language [49].

Minkowski space, can be written as [50]

$$dA = H^{BB}F_{BB} + \epsilon H^{B'B'}\Psi_{B'B'}, \qquad (3.3)$$

which just defines F^{AB} and $\Psi^{A'B'}$ as (anti)-self-dual components of dA. The Bianchi identities imply

$$dF^{A(k+2),A'(k)} = e_{BB'}F^{A(k+2)B,A'(k)B'}, \qquad (3.4)$$

and a similar chain of equations for the field Ψ

$$d\Psi^{A(k),A'(k+2)} = e_{CC'}\Psi^{A(k)C,A'(k+2)C'}.$$
(3.5)

The system (3.3), (3.4), (3.5) is equivalent to Maxwell equations, i.e. no self-dual truncation has yet been taken. The free SDYM equations are obtained by setting $\epsilon = 0$ in (3.3), while no other modifications are needed. What erasing $\Psi^{A'B'}$ from (3.3) does is that it makes the anti-selfdual part of dA vanish. The Ψ -subsystem (3.5) decouples and describes the second degree of freedom (say, helicity -1). The first equations in (3.4) and (3.5) are equivalent to the well-known [51]

$$D_A{}^{B'}F^{AB} = 0, \qquad \qquad D^A{}_{B'}\Psi^{A'B'} = 0,$$

and describe helicity +1 and -1 degrees of freedom. Subsystems (3.4) and (3.5) are closed and identical to each other (upon swapping primed and unprimed indices). What makes them different is that only the physical degree of freedom carried by F gets embedded into A once we set $\epsilon = 0$. There is no change in the number of physical degrees of freedom in the $\epsilon = 0$ limit.

General form. In order to have a genuine FDA we should incorporate the background gravitational fields: vierbein $e^{AA'}$ and the (anti)-self-dual components ω^{AB} , $\omega^{A'B'}$ of spin-connection. Finally, we have

$$\begin{split} \mathcal{N}: & \qquad 1:e^{AA'}, \omega^{AB}, \omega^{A'B'}, A, \\ & 0:F^{A(k+2),A'(k)}, \Psi^{A(k),A'(k+2)}, k=0,1,2, \ldots \end{split}$$

We will prove below that the complete L_{∞} -algebra of SDYM can be cast into the following simple form:

$$\begin{split} de^{AA'} &= \omega^A{}_B \wedge e^{BA'} + \omega^{A'}{}_{B'} \wedge e^{AB'} \,, \\ d\omega^{AB} &= \omega^A{}_C \wedge \omega^{BC} \,, \\ d\omega^{A'B'} &= \omega^{A'}{}_{C'} \wedge \omega^{B'C'} \,, \\ dA &= AA + H_{BB}F^{BB} \,, \\ dF &= l_2(\omega, F) + l_2(A, F) + l_2(e, F) + l_3(e, F, F) \,, \\ d\Psi &= l_2(\omega, \Psi) + l_2(A, \Psi) + l_2(e, \Psi) + l_3(e, F, \Psi) \,. \end{split}$$

Some of the maps above are self-evident, e.g. $l_2(\omega, \bullet)$ and $l_2(A, \bullet)$ are parts of the usual Lorentz and gauge covariant derivatives. $H_{BB}F^{BB}$ is a specific tri-linear map $l_3(e, e, F)$. Introducing the standard Lorentz covariant derivative ∇ and appending it with the gauge part $[A, \bullet]$ we define $D = \nabla - [A, \bullet]$. The equations reduce to

$$\begin{split} \nabla e^{AA'} &= 0 \,, & \nabla^2 = 0 \,, \\ dA &= AA + H_{BB} F^{BB} \,, \\ DF &= l_2(e,F) + l_3(e,F,F) \,, \\ D\Psi &= l_2(e,\Psi) + l_3(e,F,\Psi) \,. \end{split}$$

The first line is equivalent to living in Minkowski space. Covariant derivative D allows us to absorb $l_2(A, F) = [A, F]$ and $l_2(A, \Psi) = [A, \Psi]$. The L_{∞} -structure relations are equivalent to (i) e, ω being a flat connection of Poincare algebra; (ii) a bit more nontrivial L_{∞} -relations that follow from

$$D^{2}F + l_{2}(e, DF) + l_{3}(e, DF, F) + l_{3}(e, F, DF) \equiv 0,$$

$$D^{2}\Psi + l_{2}(e, D\Psi) + l_{3}(e, DF, \Psi) + l_{3}(e, F, D\Psi) \equiv 0$$

and decompose into

$$l_2(e, l_2(e, F)) \equiv 0$$
, (3.8a)

$$-[H_{BB}F^{BB}, F] + l_2(e, l_3(e, F, F)) + l_3(e, l_2(e, F), F) + l_3(e, F, l_2(e, F)) \equiv 0, \qquad (3.8b)$$

$$l_3(e, l_3(e, F, F), F) + l_3(e, F, l_3(e, F, F)) \equiv 0, \qquad (3.8c)$$

$$l_2(e, l_2(e, \Psi)) \equiv 0$$
, (3.8d)

$$-[H_{BB}F^{BB},\Psi] + l_2(e,l_3(e,F,\Psi)) + l_3(e,l_2(e,F),\Psi) + l_3(e,F,l_2(e,\Psi)) \equiv 0, \qquad (3.8e)$$

$$l_3(e, l_3(e, F, F), \Psi) + l_3(e, F, l_3(e, F, \Psi)) \equiv 0.$$
 (3.8f)

The first and the fourth relations are guaranteed by the free equations of motion.

3.2 FDA, flat space

Appetizer. Firstly, let us explain why a non-linear completion of (3.3), (3.4), (3.5) is necessary. The root of the nonlinear completion is in the fact that $D^2 \neq 0$ and, for a field χ in representation ρ of the Yang-Mills algebra we find $D^2\chi = -\rho(F)\chi$. In the adjoint representation, one gets (matrix/Lie algebra indices are implicit)

$$D^2 \chi = -[F, \chi] = -H_{BB}[F^{BB}, \chi].$$

Therefore, the Bianchi identity for the first equation in the F-subsystem

$$DF^{AA} = e_{BB'} \wedge F^{AAB,B'}$$

leads to

$$D^{2}F^{AA} = -H_{BB}[F^{BB}, F^{AA}] = -e_{BB'} \wedge DF^{AAB,B'}.$$
(3.9)

The aim is to use the above equation to obtain $DF^{AAA,A'}$. Matching the indices and imposing that $DF^{AAA,A'}$ is a one-form, one may take the ansatz

$$\begin{split} DF^{AAA,A'} &= e_{CC'}F^{AAAC,A'C'} + \alpha e_C{}^{A'}\left[F^{AC},F^{AA}\right] + \beta e_C{}^{A'}\left[F^{AA},F^{AC}\right] \\ &+ \gamma e^{AA'}[F^A{}_C,F^{AC}]\,. \end{split}$$

The Fierz identity (A.3) and the anti-symmetry of the commutator reduce this to

$$DF^{AAA,A'} = e_{CC'}F^{AAAC,A'C'} + \alpha_{00}e_{C}{}^{A'}[F^{AC},F^{AA}],$$

where the label on α_{00} was added for future convenience. Upon contraction with $e_{BB'}$ this yields⁸

$$e_{BB'} \wedge DF^{AAB,B'} = 0 + \frac{1}{3}\alpha_{00}e_{BB'} \wedge e_C^{B'}[F^{BC}, F^{AA}] + \frac{2}{3}\alpha_{00}e_{BB'} \wedge e_C^{B'}[F^{AC}, F^{AB}] = \frac{1}{3}\alpha_{00}H_{BB}[F^{BB}, F^{AA}],$$
(3.10)

where (A.2) was used. Comparing this to (3.9), one obtains the solution

$$DF^{AAA,A'} = e_{CC'}F^{AAAC,A'C'} + 3e_C{}^{A'}[F^{AC},F^{AA}], \qquad (3.11)$$

where the first term on the r.h.s. is there since the free equations. Similarly, taking the covariant derivative of the above result yields another consistency equation. Following the same steps as before, one finds

$$D^{2}F^{AAA,A'} = -H_{BB}[F^{BB}, F^{AAA,A'}] = -e_{BB'} \wedge DF^{AAAB,A'B'} - 3e_{C}{}^{A'} \wedge [DF^{AC}, F^{AA}] - 3e_{C}{}^{A'}[F^{AC}, DF^{AA}],$$

which results in

$$e_{BB'} \wedge DF^{AAAB,A'B'} = H_{BB}[F^{BB}, F^{AAA,A'}] - \frac{3}{2}H_{BB}[F^{AA}, F^{ABB,A'}] + \frac{3}{2}H_{BB}[F^{AB}, F^{AAB,B'}] + \frac{3}{2}H_{B'}{}^{A'}[F^{AB}, F^{AA}{}_{B}{}^{,B'}].$$
(3.12)

The minimal ansatz for $DF^{AAAA,A'A'}$ reads

$$DF^{AAAA,A'A'} = e_{CC'}F^{AAAAC,A'A'C'} + \alpha_{02}e_{C}{}^{A'}[F^{AC}, F^{AAA,A'}] + \alpha_{12}e_{C}{}^{A'}[F^{AAC,A'}, F^{AA}].$$

We contract this with $e_{BB'}$ to find

$$e_{BB'} \wedge DF^{AAAB,A'B'} = \frac{3\alpha_{02}}{16} H_{BB}[F^{BB}, F^{AAA,A'}] + \left(\frac{9\alpha_{02}}{16} + \frac{3\alpha_{12}}{8}\right) H_{BB}[F^{AB}, F^{AAB,A'}] + \frac{3\alpha_{12}}{8} H_{BB}[F^{AA}, F^{ABB,A'}] + \left(\frac{3\alpha_{02}}{18} - \frac{\alpha_{12}}{8}\right) H_{B'}{}^{A'}[F^{AB}, F^{AA}{}_{B}{}^{B'}].$$
(3.13)

⁸The first term can be rewritten as $e_{BB'} \wedge e_{CC'} F^{AABC,B'C'} = \frac{1}{2} (H_{BC} \epsilon_{B'C'} + \epsilon_{BC} H_{B'C'}) F^{AABC,B'C'}$ and vanishes as the contracted indices are symmetrized in $F^{AABC,B'C'}$ and anti-symmetrized in the ϵ 's. The last term must be zero, because $e_{BB'} \wedge e_C^{B'} = \frac{1}{2} H_{BC}$ is symmetric in B, C, whereas the commutator is anti-symmetric.

We compare this to (3.12) to obtain the result

$$DF^{AAAA,A'A'} = e_{CC'}F^{AAAAC,A'A'C'} + \frac{16}{3}e_C{}^{A'}[F^{AC}, F^{AAA,A'}] + 4e_C{}^{A'}[F^{AAC,A'}, F^{AA}].$$

The procedure presented above is nothing more than the practical realisation of solving the L_{∞} -relation (3.8b). This procedure will be generalized next.

Main course, *F*-sector. By looking at the first few equations in the system it is easy to come up with an ansatz:

$$DF_{A(k+2),A'(k)} = e^{BB'} F_{A(k+2)B,A'(k)B'} + \sum_{n=0}^{k-1} \alpha_{nk} e^{B}{}_{A'} \left[F_{A(n+1)B,A'(n)}, F_{A(k-n+1),A'(k-n-1)} \right],$$
(3.14)

for any $k \ge 0$. This ansatz makes use of the fact that $DF_{A(k+2),A'(k)}$ should be a one-form, which requires the presence of $e^{BB'}$ and it matches the number of (un)-primed indices. In any non-linear theory there is always a freedom to perform field redefinitions. We have also fixed the redefinitions by requiring that there are no index contractions between F in [F, F]. Terms with contracted indices can easily be introduced by field-redefinitions. Our ansatz contains only the terms that are necessary to ensure consistency and, thereby, is the minimal one.

Taking the covariant derivative of the ansatz yields

$$D^{2}F_{A(k+2),A'(k)} = -H^{BB}[F_{BB}, F_{A(k+2),A'(k)}] = -e^{BB'} \wedge DF_{A(k+2)B,A'(k)B'} - e^{B}{}_{A'} \wedge \sum_{n=0}^{k-1} \alpha_{nk}[DF_{A(n+1)B,A'(n)}, F_{A(k-n+1),A'(k-n-1)}] - e^{B}{}_{A'} \wedge \sum_{n=0}^{k-1} \alpha_{nk}[F_{A(n+1)B,A'(n)}, DF_{A(k-n+1),A'(k-n-1)}].$$
(3.15)

and considering only terms quadratic in F gives⁹

$$e^{BB'} \wedge DF_{A(k+2)B,A'(k)B'} = H^{BB}[F_{BB}, F_{A(k+2),A'(k)}] - \frac{1}{2}H^{BB}\sum_{n=0}^{k-1} \alpha_{nk}[F_{A(n+1)BB,A'(n+1)}, F_{A(k-n+1),A'(k-n-1)}] - \frac{1}{4}H^{BB}\sum_{n=0}^{k} (\alpha_{nk} - \alpha_{(k-n)k})[F_{A(n+1)B,A'(n)}, F_{A(k-n+1)B,A'(k-n)}] + \frac{1}{2}H^{B'}{}_{A'}\sum_{n=0}^{k-1} \alpha_{nk}[F_{A(n+1)}{}^{B}{}_{,A'(n)}, F_{A(k-n+1)B,A'(k-n-1)B'}],$$
(3.16)

where terms cubic in F are ignored for now. Alternatively, we contract $e^{BB'}$ with $DF_{A(k+3),A'(k+1)}$ to obtain

$$e^{BB'} \wedge DF_{A(k+2)B,A'(k)B'} = -\frac{1}{2}H^{BB}\alpha_{0(k+1)}\frac{k+2}{(k+3)(k+1)}[F_{BB}, F_{A(k+2),A'(k)}] \\ -\frac{1}{2}H^{BB}\sum_{n=0}^{k-1}\alpha_{(n+1)(k+1)}\frac{(n+2)(k+2)}{(k+3)(k+1)}[F_{A(n+1)BB,A'(n+1)}, F_{A(k-n+1),A'(k-n-1)}] \\ -\frac{1}{4}H^{BB}\sum_{n=0}^{k}(\alpha_{n(k+1)}\frac{(k-n+2)(k+2)}{(k+3)(k+1)} - \alpha_{(k-n)(k+1)})\frac{(n+2)(k+2)}{(k+3)(k+1)})[F_{A(n+1)B,A'(n)}, F_{A(k-n+1)B,A'(k-n)}] \\ +\frac{1}{2}H^{B'}{}_{A'}\sum_{n=0}^{k}(\alpha_{(k-n)(k+1)}\frac{(n+2)(k-n)}{(k+3)(k+1)} + \alpha_{n(k+1)}\frac{(k-n+2)(k-n)}{(k+3)(k+1)})[F_{A(n+1)}{}^{B}_{,A'(n)}, F_{A(k-n+1)B,A'(k-n-1)B'}].$$

Comparing this with (3.16) results in the following system of recurrence relations:

$$\begin{split} 0 &= \alpha_{0k} + \frac{2k(k+2)}{k+1} ,\\ 0 &= \alpha_{(n+1)(k+1)} \frac{(n+2)(k+2)}{(k+3)(k+1)} - \alpha_{nk} ,\\ 0 &= \alpha_{n(k+1)} \frac{(k-n+2)(k+2)}{(k+3)(k+1)} - \alpha_{(k-n)(k+1)} \frac{(n+2)(k+2)}{(k+3)(k+1)} - \alpha_{nk} + \alpha_{(k-n)k} ,\\ 0 &= \alpha_{(k-n)(k+1)} \frac{(n+2)(k-n)}{(k+3)(k+1)} + \alpha_{n(k+1)} \frac{(k-n+2)(k-n)}{(k+3)(k+1)} - \alpha_{nk} .\end{split}$$

This system is over-determined, but, nevertheless, is solved by

$$\alpha_{nk} = -\frac{2}{(n+1)!} \frac{(k+2)!}{(k-n-1)!(k-n+1)(k+1)}$$

⁹In the third term we have made the anti-symmetry of the commutator explicit by writing $[X, Y] = \frac{1}{2}([X, Y] - [Y, X])$ and renaming the dummy indices accordingly. This automatically gets rid of terms that vanish because of symmetry reasons, like the last term in the middle expression of (3.10). As the summation now runs up to n = k, the coefficient α_{kk} shows up, so we set $\alpha_{kk} = 0$ by hand since it was not present in the ansatz.

The full solution reads

$$DF_{A(k+2),A'(k)} = e^{BB'} F_{A(k+2)B,A'(k)B'} - e^{B}_{A'} \sum_{n=0}^{k-1} \frac{2}{(n+1)!} \frac{(k+2)!}{(k-n-1)!(k-n+1)(k+1)} [F_{A(n+1)B,A'(n)}, F_{A(k-n+1),A'(k-n-1)}].$$
(3.18)

It was assumed that the ansatz only contains linear and quadratic terms in F. The fact that terms cubic in F vanish in (3.16) is proved in Appendix B.2. This confirms the L_{∞} -relation in (3.8c) and it implies that $DF_{A(k+2),A'(k)}$ indeed truncates at quadratic order.

Main course, Ψ -sector. As was clear from the L_{∞} -relations in (3.8), the non-linear extension of the Ψ -sector is different from the *F*-sector. The minimal ansatz for $D\Psi_{A(k),A'(k+2)}$ is slightly more involved as it reads

$$D\Psi_{A(k),A'(k+2)} = e^{CC'}\Psi_{A(k)C,A'(k+2)C'} + \sum_{n=0}^{k-1} \beta_{nk} e^{C}{}_{A'} \left[F_{A(n+1)C,A'(n)}, \Psi_{A(k-n-1),A'(k-n+1)} \right]$$

$$+ \sum_{n=0}^{k-2} \gamma_{nk} e^{C}{}_{A'} \left[F_{A(n+2),A'(n)}, \Psi_{A(k-n-2)C,A'(k-n+1)} \right].$$
(3.19)

We follow the same steps as for the *F*-sector: we write the Bianchi identity for the ansatz above and as a parallel calculation we contract $e^{BB'}$ with $\Psi_{A(k+1),A'(k+3)}$ to obtain two expressions for $e^{BB'} \wedge D\Psi_{A(k)B,A'(k+2)B'}$ and compare them. This provides us with a system of recurrence relations for β_{nk} and γ_{nk} . The details of the calculation are left for Appendix B.1. The system is solved by

$$\beta_{nk} = -\frac{2}{(n+1)!} \frac{k-n+2}{k+3} \frac{k!}{(k-n-1)!} , \qquad \gamma_{nk} = \frac{2}{(n+2)!} \frac{n+1}{k+3} \frac{k!}{(k-n-2)!} .$$

The full solution reads

$$D\Psi_{A(k),A'(k+2)} = e^{CC'}\Psi_{A(k)C,A'(k+2)C'} - e^{C}{}_{A'}\sum_{n=0}^{k-1} \frac{2}{(n+1)!} \frac{k-n+2}{k+3} \frac{k!}{(k-n-1)!} [F_{A(n+1)C,A'(n)}, \Psi_{A(k-n-1),A'(k-n+1)}] + e^{C}{}_{A'}\sum_{n=0}^{k-2} \frac{2}{(n+2)!} \frac{n+1}{k+3} \frac{k!}{(k-n-2)!} [F_{A(n+2),A'(n)}, \Psi_{A(k-n-2)C,A'(k-n+1)}].$$
(3.20)

In Appendix B.2 we show that this solution ensures consistency of the L_{∞} -relation in (3.8f), i.e. the above solution does not require higher order corrections.

Summary. SDYM can be cast in the form of an L_{∞} -algebra. This gives rise to three L_{∞} relations for the *F*-sector and the Ψ -sector of SDYM, see (3.8). The first of each gives rise to
the free equation for $DF_{A(k+2),A'(k)}$ and $D\Psi_{A(k),A'(k+2)}$. The second L_{∞} -relation can be solved
to obtain the quadratic piece of the non-linear extension in both sectors, which are proportional
to [F, F] and $[F, \Psi]$, respectively. In particular, the coefficients can be found by writing down
the minimal ansätze (3.14) and (3.19) and checking their Bianchi identities. This yields two
expressions for $e^{BB'}F_{A(k+2)B,A'(k)B'}$ and $e^{BB'}\Psi_{A(k)B,A'(k+2)B'}$. Comparing them gives rise to a
system of recurrence relations, whose solution gives the final results (3.18) and (3.20), i.e. the
boxed equations above. Furthermore, the third L_{∞} -relation ensures that the system is closed,
i.e. there are no higher order corrections. It is proved that these relation are indeed satisfied
for the obtained solutions and hence the expressions we have found are the complete non-linear
extensions for the two sectors.

An interesting follow up would be to consider the higher-spin extensions of SDYM [34, 52] and the supersymmetric higher spin extensions constructed in [53].

3.3 FDA, constant curvature space

As a simple modification of SDYM on Minkowski background we can consider a constant curvature background, i.e. de Sitter or anti-de Sitter spaces. The action is the same. Let us first recall that the free Maxwell equations on a constant curvature background rewritten as an FDA read [50]

$$dA = H^{BB}F_{BB} + \epsilon H^{B'B'}\Psi_{B'B'}, \qquad (3.21a)$$

$$\nabla F^{A(k+2),A'(k)} = e_{BB'} F^{A(k+2)B,A'(k)B'} + k(k+2)\Lambda e^{AA'} F^{A(k+1),A'(k-1)}, \qquad (3.21b)$$

$$\nabla \Psi^{A(k),A'(k+2)} = e_{CC'} \Psi^{A(k)C,A'(k+2)C'} + k(k+2)\Lambda e^{AA'} \Psi^{A(k-1),A'(k+1)}.$$
(3.21c)

The only difference is the presence of new $e^{AA'}$ -terms that are consistent on their own and do not require any other modifications. It is also convenient to set $\Lambda = 1$ in what follows. The L_{∞} -algebra for SDYM on a constant background is given by

$$\begin{split} de^{AA'} &= \omega^A{}_B \wedge e^{BA'} + \omega^{A'}{}_{B'} \wedge e^{AB'} \,, \\ d\omega^{AB} &= \omega^A{}_C \wedge \omega^{BC} + H^{AB} \,, \end{split}$$

$$\begin{split} d\omega^{A'B'} &= \omega^{A'}{}_{C'} \wedge \omega^{B'C'} + H^{A'B'}, \\ dA &= AA + H_{BB}F^{BB}, \\ dF &= l_2(\omega, F) + l_2(A, F) + l_2(e, F) + \tilde{l}_2(e, F) + l_3(e, F, F), \\ d\Psi &= l_2(\omega, \Psi) + l_2(A, \Psi) + l_2(e, \Psi) + \tilde{l}_2(e, \Psi) + l_3(e, F, \Psi), \end{split}$$

where \tilde{l}_2 encodes the gravitational correction to the free equations (3.21b) and (3.21c). The contributions

$$l_2(\omega, F) = (k+2)\omega^A{}_B F^{A(k+1)B,A'(k)} + k\omega^{A'}{}_{B'} F^{A(k+2),B'A'(k-1)}$$

and $l_2(A, F) = [A, F]$ (and similarly for Ψ) can be absorbed into the covariant derivative $D = \nabla - [A, \bullet]$. As a result, the relations can be rewritten as

$$\begin{split} \nabla e^{AA'} &= 0 \,, \\ dA &= AA + H_{BB}F^{BB} \,, \\ DF &= l_2(e,F) + \tilde{l}_2(e,F) + l_3(e,F,F) \,, \\ D\Psi &= l_2(e,\Psi) + \tilde{l}_2(e,\Psi) + l_3(e,F,\Psi) \,. \end{split}$$

As different from $\nabla^2 = 0$ in flat space, in a constant curvature background we have for any spin-tensor $T^{A(n),A'(m)}$

$$\nabla^2 T^{A(n),A'(m)} = -n H^A{}_B \, T^{A(n-1)B,A'(m)} - m H^{A'}{}_{B'} \, T^{A(n),A'(m-1)B'} \, .$$

The L_{∞} -relations of the sought for L_{∞} -algebra read

$$-[H_{BB}F^{BB}, F] + l_2(e, DF) + \tilde{l}_2(e, DF) + l_3(e, DF, F) + l_3(e, F, DF) \equiv 0, \qquad (3.22a)$$

$$-[H_{BB}F^{BB},\Psi] + l_2(e,D\Psi) + \tilde{l}_2(e,D\Psi) + l_3(e,DF,\Psi) + l_3(e,F,D\Psi) \equiv 0.$$
(3.22b)

Since \tilde{l} can be viewed as a deformation of the previously found FDA, all terms without \tilde{l} vanish already. The remaining nontrivial relations read

$$\tilde{l}_2(e, l_3(e, F, F)) + l_3(e, \tilde{l}_2(e, F), F) + l_3(e, F, \tilde{l}_2(e, F)) = 0, \qquad (3.23a)$$

$$\tilde{l}_2(e, l_3(e, F, \Psi)) + l_3(e, \tilde{l}_2(e, F), \Psi) + l_3(e, F, \tilde{l}_2(e, \Psi)) = 0, \qquad (3.23b)$$

where we ignore terms quadratic in the cosmological constant. These relations are satisfied automatically. A proof of this given in Appendix B.3. Consequently, on a constant curvature gravitational background we obtain

$$DF_{A(k+2),A'(k)} = e^{BB'} F_{A(k+2)B,A'(k)B'} + k(k+2) e_{AA'} F_{A(k+1),A'(k-1)} - e^{B}_{A'} \sum_{n=0}^{k-1} \frac{2}{(n+1)!} \frac{(k+2)!}{(k-n-1)!(k-n+1)(k+1)} [F_{A(n+1)B,A'(n)}, F_{A(k-n+1),A'(k-n-1)}],$$
(3.24a)

$$D\Psi_{A(k),A'(k+2)} = e^{CC'}\Psi_{A(k)C,A'(k+2)C'} + k(k+2)e_{AA'}\Psi_{A(k-1),A'(k+1)} - e^{C}{}_{A'}\sum_{n=0}^{k-1} \frac{2}{(n+1)!} \frac{k-n+2}{k+3} \frac{k!}{(k-n-1)!} [F_{A(n+1)C,A'(n)}, \Psi_{A(k-n-1),A'(k-n+1)}] + e^{C}{}_{A'}\sum_{n=0}^{k-2} \frac{2}{(n+2)!} \frac{n+1}{k+3} \frac{k!}{(k-n-2)!} [F_{A(n+2),A'(n)}, \Psi_{A(k-n-2)C,A'(k-n+1)}].$$
(3.24b)

Summary. We constructed the L_{∞} -algebra of SDYM on a constant curvature background and derived the corresponding L_{∞} -relations. The free Maxwell equations on a constant curvature background in terms of an FDA, (3.21), are well-known in the literature and solve the first L_{∞} -relation of both the *F*-sector and Ψ -sector. In section 3.2 we computed the non-linear extension of $DF_{A(k+2),A'(k)}$ and $D\Psi_{A(k),A'(k+2)}$ on a flat background. In the second L_{∞} -relation of each sector we see an interplay between the gravitational contribution of the free equations and the non-linear extension on flat space. We demonstrated that the second L_{∞} -relation for both sectors decomposes into the flat space L_{∞} -relation and a new relation containing the gravitational contributions in such a way that the latter does not contribute to the quadratic order in $DF_{A(k+2),A'(k)}$ and $D\Psi_{A(k),A'(k+2)}$. The third L_{∞} -relation then contains no gravitational contribution and remains satisfied. The complete non-linear extension of both sectors are only modified in the linear terms according to the free equations and are shown in the boxed equation (3.24a) and (3.24b) above.

4 SDGR

4.1 Action, initial data

Self-dual gravity with vanishing cosmological constant can be formulated with the help of two fields [23]: one-form $\omega^{A'B'}$ and zero-form $\Psi^{A'B'C'D'}$. The action reads

$$\int \Psi^{A'B'C'D'} \wedge d\omega_{A'B'} \wedge d\omega_{C'D'} \,. \tag{4.1}$$

The equations of motion are $(F^{A'B'} = d\omega^{A'B'})$

$$F_{(A'B'} \wedge F_{C'D'}) = 0, \qquad d\Psi^{A'B'C'D'} \wedge F_{A'B'} = 0.$$
(4.2)

One-form $\omega^{A'B'}$ looks like the anti-self-dual part of the Lorentz spin-connection, but it is not. The curvature $F_{A'B'}$ for $\omega^{A'B'}$ lacks the " $\omega\omega$ "-part. Nevertheless, this interpretation is not very far from the reality since action (4.1) can be understood as a limit of that for self-dual gravity with cosmological constant [54]. In the latter $F^{A'B'} = d\omega^{A'B'} - \omega^{A'}_{C'} \wedge \omega^{C'B'}$ is the canonical one and the limit is to drop the $\omega\omega$ -part.

Minkowski space is a special solution of (4.2): $\omega_0^{A'A'} = x_C^{A'} dx^{CA'}$ such that $d\omega_0^{A'B'} = H^{A'B'}$, where $H^{A'B'}$ is built from the Minkowski's space vierbein $e^{AA'} = dx^{AA'}$, $H^{A'B'} \equiv e_C^{A'} \wedge e^{CB'}$ and its conjugate is $H^{AB} \equiv e^A_{C'} \wedge e^{BC'}$. One can easily write down the first few equations of the FDA that corresponds to variational equations (4.2):¹⁰

$$d\omega^{A'A'} = e_B{}^{A'} \wedge e^{BA'},$$

$$de^{AA'} = \omega^A{}_B \wedge e^{BA'},$$

$$d\omega^{AA} = \omega^A{}_C \wedge \omega^{CA} + H_{MM}C^{MMA}{}_B,$$

$$d\Psi^{A'A'A'A'} = e_{BB'}\Psi^{B,A'A'A'A'B'}.$$
(4.4b)

$$\begin{split} d\omega^{AA} &- \omega^{A}{}_{C} \wedge \omega^{CB} - e^{A}{}_{B'} \wedge e^{AB'} = R^{AA}, \\ de^{AA'} &- \omega^{A'}{}_{B'} \wedge e^{AB'} - \omega^{A}{}_{B} \wedge e^{BA'} = T^{AA'}, \\ d\omega^{A'A'} &- \omega^{A'}{}_{C'} \wedge \omega^{C'B'} - e_{B}{}^{A'} \wedge e^{BA'} = R^{A'A'}, \end{split}$$

The gauge algebra for the SDGR with zero scalar curvature can be understood as a limit of so(3, 2)-algebra where $L_{A'A'}$ become abelian [23].

¹⁰As a side remark, let us write the curvature for $so(3,2) \sim sp(4)$, which is relevant for anti-de Sitter space (they correspond to Lorentz generators $L_{A'A'}$, L_{AA} and to translations $P_{AA'}$):

The main idea is to identify the right gauge algebra [23]. This is the starting for constructing the L_{∞} -algebra. The first equation of (4.4a) implies that the gravitational degrees of freedom fully reside in the anti-self-dual part. The last equation of (4.4a) identifies the only nonvanishing part of the curvature with the self-dual Weyl tensor C^{ABCD} , $R^A{}_B = H_{MM}C^{MMA}{}_B$. As a result one obtains a Bianchi identity for $R^A{}_B$. Eq. (4.4b) introduces a new field $\Psi^{A,A'B'C'D'E'}$, which parameterizes the first derivative of Ψ and is contained in the on-shell jet of $\Psi^{A'B'C'D'E'}$. Similarly to SDYM we aim to find a completion of (4.4a) and we need to define an infinite set of coordinates on \mathcal{N} and Q such that QQ = 0.

Coordinates, on-shell jet. Coordinates on supermanifold \mathcal{N} coincide with those of the free massless spin-two field, i.e. with [50] and [49]. Indeed, the set of one-forms turned out to be the same, while the zero-forms begin with (anti)-self-dual components of Weyl tensor and are just the on-shell nontrivial derivatives of those. Therefore, the coordinates on \mathcal{N} are: degreeone ω^{AB} , $e^{AA'}$ and $\omega^{A'B'}$; degree-zero $C^{A(k+4),A'(k)}$ and $\Psi^{A(k+4),A'(k)}$, k = 0, 1, 2, ... A similar discussion follows as for SDYM. In particular, the free equations for helicity ±2 fields are [51]

$$\nabla^{A}{}_{B'} \Psi^{A'B'C'D'} = 0, \qquad \qquad \nabla_{A}{}^{B'} C^{ABCD} = 0,$$

and can be rewritten in the FDA form as [50]

$$\nabla C^{A(k+4),A'(k)} = e_{CC'} C^{A(k+4)C,A'(k)C'}, \qquad \nabla \Psi^{A(k),A'(k+4)} = e_{CC'} \Psi^{A(k)C,A'(k+4)C'}.$$
(4.5)

One needs to supplement these equations with the free limit of (4.4). Our problem is to find a nonlinear completion of (4.5) that is consistent with (4.4).

General form. The supermanifold \mathcal{N} has coordinates

$$\mathcal{N}: \qquad \begin{array}{l} 1:\omega^{A'B'},e^{AA'},\omega^{AB},\\ 0:C^{A(k+4),A'(k)},\Psi^{A(k),A'(k+4)},k=0,1,2,... \end{array}$$

Now, we try to reformulate the theory in the L_{∞} -form. Given the data above and our desire to truncate the FDA at $l_3(\bullet, \bullet, \bullet)$, we write

$$\begin{aligned} d\omega^{A'A'} &= e_B{}^{A'} \wedge e^{BA'} ,\\ de^{AA'} &= \omega^A{}_B \wedge e^{BA'} ,\\ d\omega^{AA} &= \omega^A{}_C \wedge \omega^{CA} + H_{MM} C^{MMA}{}_B , \end{aligned}$$

$$dC = l_2(\omega, C) + l_2(e, C) + l_3(e, C, C) ,$$

$$d\Psi = l_2(\omega, \Psi) + l_2(e, \Psi) + l_3(e, C, \Psi) .$$

We define the covariant derivative $\nabla = d - \omega$, which lacks the $\omega^{A'B'}$ -part. For an arbitrary spin-tensor $T^{A(n),A'(m)}$ we get

$$\nabla^2 T^{A(n),A'(m)} = -n H_{MM} C^{MMA}{}_B T^{BA(n-1),A'(m)} .$$
(4.6)

The covariant derivative allows one to absorb the terms $l_2(\omega, C)$ and $l_2(\omega, \Psi)$ and we can write

$$\nabla C = l_2(e, C) + l_3(e, C, C),$$
 $\nabla \Psi = l_2(e, \Psi) + l_3(e, C, \Psi).$

This gives rise to the L_{∞} -relations for SDGR, which read

$$\begin{split} -(k+4)H_{MM}C^{MMA}{}_B \, C^{A(k+3)B,A'(k)} + l_2(e,\nabla C) + l_3(e,\nabla C,C) + l_3(e,C,\nabla C) &= 0 \,, \\ -kH_{MM}C^{MMA}{}_B \, \Psi^{A(k-1)B,A'(k+4)} + l_2(e,\nabla \Psi) + l_3(e,\nabla C,\Psi) + l_3(e,C,\nabla \Psi) &= 0 \,, \end{split}$$

and decompose into

$$l_2(e, l_2(e, C)) = 0,$$
 (4.7a)

$$l_3(e, l_3(e, C, C), C) + l_3(e, C, l_3(e, C, C)) = 0, \qquad (4.7b)$$

$$l_2(e, l_2(e, \Psi)) = 0,$$
 (4.7c)

$$l_3(e, l_3(e, C, C), \Psi) + l_3(e, C, l_3(e, C, \Psi)) = 0, \qquad (4.7d)$$

$$-nH_{MM}C^{MMA}{}_{B}C^{A(k+3)B,A'(k)} + l_{2}(e, l_{3}(e, C, C)) + l_{3}(e, C, l_{2}(e, C)) = 0,$$

$$(4.7e)$$

$$-nH_{MM}C^{MMA}{}_{B}\Psi^{A(k-1)B,A'(k+4)} + l_{2}(e, l_{3}(e, C, \Psi)) + l_{3}(e, C, l_{2}(e, \Psi)) = 0.$$

$$(4.7f)$$

4.2 FDA

Appetizer. Let us first illustrate our approach by presenting the source of the non-linear extension with an explicit example. We follow roughly the same steps as for SDYM, though some subtle differences arise. The most important ones come from the commutativity of the C's and the additional contraction of unprimed indices that we will see shortly.

The Bianchi identity for the curvature, $\nabla R_{AA} = 0$ implies

$$\nabla C_{AAAA} = e^{BB'} C_{AAAB,B'} \,.$$

Its own Bianchi identity via (4.6) imposes

We need to construct an ansatz for $\nabla C_{AAAAA,A'}$. Commutativity of the C's and the Fierz identity allow us to construct the minimal ansatz as

$$\nabla C_{AAAAA,A'} = e^{CC'} C_{AAAAAB,A'C'} + a_{01} e^{C}{}_{A'} C_{AAC}{}^{D} C_{AAAD} .$$
(4.8)

Contracting the ansatz with $e^{BB'}$ yields

$$e^{BB'} \wedge \nabla C_{AAAAB,B'} = -\frac{2a_{01}}{5} H^{BB} C_{ABB}{}^{D} C_{AAAD} - \frac{3a_{01}}{5} H^{BB} C_{AAB}{}^{D} C_{AABD}$$
$$= -\frac{2a_{01}}{5} H^{BB} C_{ABB}{}^{D} C_{AAAD}.$$

One term is dropped, as commuting the two C's and raising/lowering the contracted indices tells us that this term vanishes. Comparing the result with (4.8) yields the solution

$$\nabla C_{AAAAA,A'} = e^{CC'} C_{AAAAAC,A'C'} + 10e^{C}{}_{A'} C_{AAC}{}^D C_{AAAD}.$$

The procedure that we have followed is a practical realisation of solving the L_{∞} -relation (4.7e). This procedure will be generalized next.

Main course, C-sector. Using the same criteria as before we propose the minimal ansatz

$$\nabla C_{A(k+4),A'(k)} = e^{CC'} C_{A(k+4)C,A'(k)C'} + \sum_{n=0}^{k-1} a_{nk} e^{C}{}_{A'} C_{A(n+2)C}{}^{D}{}_{,A'(n)} C_{A(k-n+2)D,A'(k-n-1)}.$$

Taking another derivative leads to

$$\nabla^{2}C_{A(k+4),A'(k)} = (k+4)H^{BB'}C_{ABB}{}^{D}C_{A(k+3)D,A'(k)} = -e^{CC'} \wedge \nabla C_{A(k+4)C,A'(k)C'} - \sum_{n=1}^{k-1} a_{nk}e^{C}{}_{A'} \wedge \nabla C_{A(n+2)C}{}^{D}{}_{,A'(n)}C_{A(k-n+2)D,A'(k-n-1)} - \sum_{n=0}^{k-2} a_{nk}e^{C}{}_{A'} \wedge C_{A(n+2)C}{}^{D}{}_{,A'(n)}\nabla C_{A(k-n+2)D,A'(k-n-1)}.$$

$$(4.9)$$

Considering only terms quadratic in C yields

$$e^{CC'} \wedge \nabla C_{A(k+4)C,A'(k)C'} = -(k+4)H^{BB}C_{ABB}{}^{D}C_{A(k+3)D,A'(k)} - \frac{1}{2}H^{BB}\sum_{n=0}^{k-1}a_{nk}C_{A(n+2)BB}{}^{D}{}_{,A'(n+1)}C_{A(k-n+2)D,A'(k-n-1)} - \frac{1}{2}H^{BB}\sum_{n=0}^{k}(\frac{a_{nk}}{2} - \frac{a_{(k-n)k}}{2})C_{A(n+2)B}{}^{D}{}_{,A'(n)}C_{A(k-n+2)BD,A'(k-n)} - \frac{1}{2}H_{A'}{}^{B'}\sum_{n=0}^{k-1}a_{nk}C_{A(n+2)B}{}^{D}{}_{,A'(n)}C_{A(k-n+2)}{}^{B}{}_{D,A'(k-n-1)B'},$$

$$(4.10)$$

where in the third line we made the anti-commuting property of the C's explicit, together with the anti-symmetry of the spinorial inner product. At the same time we contract $e^{BB'}$ with $\nabla C_{A(k+5),A'(k+1)}$ to obtain

$$e^{BB'} \wedge \nabla C_{A(k+4)B,A'(k)B'} = -H^{BB} \frac{k+2}{(k+5)(k+1)} a_{0(k+1)} C_{ABB}{}^{D} C_{A(k+3)D,A'(k)}$$

$$- \frac{1}{2} H^{BB} \sum_{n=0}^{k-1} \frac{(k+2)(n+3)}{(k+5)(k+1)} a_{(n+1)(k+1)} C_{A(n+2)BB}{}^{D}_{,A'(n+1)} C_{A(k-n+2)D,A'(k-n-1)}$$

$$- \frac{1}{4} H^{BB} \sum_{n}^{k} \left(\frac{(k+2)(k-n+3)}{(k+5)(k+1)} a_{n(k+1)} - \frac{(k+2)(n+3)}{(k+5)(k+1)} a_{(k-n)(k+1)} \right)$$

$$\times C_{A(n+2)B}{}^{D}_{,A'(n)} C_{A(k-n+2)BD,A'(k-n)}$$

$$- \frac{1}{2} H_{A'}{}^{B'} \sum_{n=0}^{k-1} \left(\frac{(k-n)(n+3)}{(k+5)(k+1)} a_{(k-n)(k+1)} + \frac{(k-n)(k-n+3)}{(k+5)(k+1)} a_{n(k+1)} \right)$$

$$\times C_{A(n+2)B}{}^{D}_{,A'(n)} C_{A(k-n+2)}{}^{B}_{D,A'(k-n-1)B'} .$$

Comparing this expression with (4.10) brings about the following system of recurrence relations:

$$\begin{split} 0 &= a_{0k} - \frac{(k+4)(k+3)k}{k+1} ,\\ 0 &= a_{(n+1)(k+1)} - \frac{(k+5)(k+1)}{(k+2)(n+3)} a_{nk} ,\\ 0 &= \frac{(k+2)(k-n+3)}{(k+5)(k+1)} a_{n(k+1)} - \frac{(k+2)(n+3)}{(k+5)(k+1)} a_{(k-n)(k+1)} - a_{nk} + a_{(k-n)n} ,\\ 0 &= a_{nk} - \frac{(k-n)(n+3)}{(k+5)(k+1)} a_{(k-n)(k+1)} - \frac{(k-n)(k-n+3)}{(k+5)(k+1)} a_{n(k+1)} . \end{split}$$

This over-determined system is solved by

$$a_{nk} = \frac{2}{(n+2)!} \frac{(k+4)!(k-n)}{(k-n+2)!(k+1)}$$

and the full solution reads¹¹

$$\nabla C_{A(k+4),A'(k)} = e^{CC'} C_{A(k+4)C,A'(k)C'} + \sum_{n=0}^{k-1} \frac{2}{(n+2)!} \frac{(k+4)!(k-n)}{(k-n+2)!(k+1)} e^{C}{}_{A'} C_{A(n+2)C}{}^{D}{}_{,A'(n)} C_{A(k-n+2)D,A'(k-n-1)}.$$
(4.11)

In appendix C.2 we prove that this solution is complete, i.e. no higher order terms arise.

Main course, \Psi-sector. For the Ψ -sector we follow a similar approach. The minimal ansatz reads

$$\nabla \Psi_{A(k),A'(k+4)} = e^{CC'} \Psi_{A(k)C,A'(k+1)C'} + \sum_{n=0}^{k} b_{nk} e^{C}{}_{A'} C_{A(n+2)C}{}^{D}{}_{,A'(n)} \Psi_{A(k-n-2)D,A'(k-n+3)} + \sum_{n=0}^{k} c_{nk} e^{C}{}_{A'} C_{A(n+3)}{}^{D}{}_{,A'(n)} \Psi_{A(k-n-3)CD,A'(k-n+3)}.$$
(4.12)

The details of the calculations are left to Appendix C.1, but the approach is as follows: we take the covariant derivative of the ansatz above. We also contract $e^{BB'}$ with $\nabla \Psi_{A(k+1),A'(k+5)}$. Both will give us an expression for $e^{BB'} \wedge \nabla \Psi_{A(k)B,A'(k+4)B'}$ and we compare them. This results

¹¹A closely related problem was addressed in [55], which is to find an FDA form of the full gravity to the next to the leading order (the problem to find the complete minimal model for gravity does not seem to admit a solution in a closed form, even though it does always exist as a matter of principle). It would be interesting to understand what [55] describes since it does not coincide with the FDA of SDGR with (non)-vanishing cosmological constant. The physical degrees of freedom are the same though.

in a system of recurrence relations, which is solved by

$$b_{nk} = \frac{2}{(n+2)!} \frac{k!}{(k-n-2)!} \frac{k-n+4}{k+5}, \qquad c_{nk} = -\frac{2}{(n+2)!} \frac{k!}{(k-n-3)!} \frac{n+1}{(k+5)(n+3)}$$

and the solution in the Ψ -sector reads

$$\nabla \Psi_{A(k),A'(k+4)} = e^{CC'} \Psi_{A(k)C,A'(k+1)C'} + \sum_{n=0}^{k} \frac{2}{(n+2)!} \frac{k!}{(k-n-2)!} \frac{k-n+4}{k+5} e^{C}{}_{A'} C_{A(n+2)C}{}^{D}{}_{,A'(n)} \Psi_{A(k-n-2)D,A'(k-n+3)} - \sum_{n=0}^{k} \frac{2}{(n+2)!} \frac{k!}{(k-n-3)!} \frac{n+1}{(k+5)(n+3)} e^{C}{}_{A'} C_{A(n+3)}{}^{D}{}_{,A'(n)} \Psi_{A(k-n-3)CD,A'(k-n+3)}.$$
(4.13)

We prove in Appendix C.2 that $\nabla \Psi_{A(k),A'(k+4)}$ is consistent as it is and does not require higher order terms.

Summary. We rewrote SDGR as an L_{∞} -algebra. This gives rise to three L_{∞} -relations for the *C*-sector and the Ψ -sector, see (4.7). Solving the first relation of each sector yields the free equations for $\nabla C^{A(k+4),A'(k)}$ and $\nabla \Psi^{A(k),A'(k+4)}$. We constructed a minimal ansatz for a non-linear extension of the free equations and used the second L_{∞} -relation to determine its structure. The results are shown in the boxes expressions above, (4.11) and (4.13). The third L_{∞} -relation is found to be satisfied for the obtained solutions, which implies that the minimal ansatz is sufficient to solve the whole system.

An interesting followup of this project is construct FDA for SDGR in the constant-curvature background. The action of this theory [54] is even more natural

$$\int \Psi^{A'B'C'D'} \wedge F_{A'B'} \wedge F_{C'D'}$$

where $F^{A'B'} = d\omega^{A'B'} - \omega^{A'}_{C'} \wedge \omega^{C'B'}$. However, it is more nonlinear, featuring quartic terms (the quintic one vanishes). A simpler problem is to consider the higher spin extensions of SDGR [34, 52] with vanishing cosmological constant.

5 Conclusions and Discussion

The present paper is the first in a series of papers where we plan to construct minimal models of various field theories, including some examples of higher spin gravities. Since every (gauge) field

theory defines and is defined by its minimal model, a certain L_{∞} -algebra, our general motivation is to first understand how various properties of field theories, e.g. integrability, asymptotic symmetries, conserved charges, actions, anomalies etc., can be understood in the known cases and derived from this L_{∞} -algebra in the cases where this information is yet unavailable. For example, it would be interesting to understand the Ward construction of Yang-Mills instantons [2] from the L_{∞} point of view.

As we have reviewed in section 2, the minimal model can naturally be associated to any gauge theory and it is the smallest L_{∞} -algebra that captures all local BRST cohomology of this field theory. However, the minimal model is usually difficult to construct explicitly. Apart from this paper, the only available examples where minimal models were explicitly constructed are (a) Chern-Simons theory, which is just dA = AA and, for that reason, is hard to consider this as a genuine example of a minimal model (nevertheless, this toy model was quite useful to prove that all matter-free higher spin gravities in 3d are of Chern-Simons form [41]); (b) another example is discussed in [55] and is closely related to the SDGR FDA of the present paper. It is tempting to argue that minimal models can explicitly be constructed only for theories that feature some kind of (hidden) simplicity, e.g. they are integrable like Chern-Simons theory, SDYM and SDGR.

Some obvious future directions include: (a) self-dual gravity with cosmological constant [54]; (b) higher spin extensions of SDYM and SDGR [34, 52]; (c) the supersymmetric higher spin extensions of [53]; Chiral higher spin gravity [31–37].

All physically relevant local information about a given theory is encoded in its minimal model via the Q-cohomology. For example, conserved charges, actions correspond to H(Q)with values in the trivial module. A more complicated example is the presymplectic structure $\Omega_{AB} \delta \Phi^A \wedge \delta \Phi^B$ which is a two-form on the field space and is a degree (d-1) form from the space-time point of view. It corresponds to $H(Q, \Lambda^2(\mathcal{N}))$, where the action of Q is understood as Lie derivative L_Q along Q that is defined canonically on (p, q)-tensors on \mathcal{N} , see [56–59] for more detail and examples. As the last example, Q-cohomology with values in vector fields, $H(Q, T^{1,0}(\mathcal{N}))$, is responsible for deformations of Q itself, i.e. it classifies possible interactions. It is worth noting that Q-cohomology can often be computed without having to know the minimal model explicitly. The latter is an additional bonus that should be a signal of integrability.

Acknowledgments

We are grateful to Maxim Grigoriev, Yannick Herfray, Kirill Krasnov and Alexey Sharapov for useful discussions. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 101002551). The work was partially supported by the Fonds de la Recherche Scientifique — FNRS under Grant No. F.4544.21.

A Notation

The most important conventions and definitions that were used in the main text are introduced here. A short discussion on the spinor formalism is given, a more elaborate treatment can be found in [49].

It is useful for 4*d* theories to express all space-time indices in terms of spinor indices, using $so(3,1) \sim sl(2,\mathbb{C})$. This isomorphism allows to map 4-dimensional space-time vectors to 2×2 Hermitian matrices, which can be extended to tensors. As basis for the 2×2 Hermitian matrices in flat space-time we choose the Pauli matrices and the unit matrix $\sigma^{\mu}_{AB'} = (\mathbb{1}, \sigma^i)$. The Greek letters run over space-time indices, whereas the lower case Latin letters are space indices and capitals are the matrix indices. The Pauli matrices satisfy $\operatorname{Tr} \sigma^i = 0$ and $\{\sigma^{\mu}, \sigma^{\nu}\}_{AA'} = 2\eta^{\mu\nu}\mathbb{1}_{AA'}$, with $\eta^{\mu\nu} = \operatorname{diag}(-1, 1, 1, 1)$ the Minkowski metric. We define

$$x_{AA'} = x^{\mu}(\sigma_{\mu})_{AA'} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^1 - x^3 \end{pmatrix},$$

which is Hermitian. We also introduce a dual set $(\overline{\sigma}^{\mu})^{AA'} = (\mathbf{1}, -\sigma^i)$, such that

$$v^{\mu} = -\frac{1}{2} x_{AA'} \overline{\sigma}^{\mu A' A}$$

The two sets are related by $\sigma_{\mu}^{AA'} = \overline{\sigma}_{\mu}^{A'A}$. We also introduce raising and lowering rules for the primed (and similarly for unprimed indices):

$$y_A = y^B \epsilon_{BA}, \qquad \qquad y^A = \epsilon^{AB} y_B$$

The inner product in spinor indices is defined as $(xy) = x^A y^B \epsilon_{AB} = x_A y^A = -x^A y_A$. We define

the ϵ 's as

$$\epsilon^{AB} = \epsilon^{A'B'} = i(\sigma^2)^{AB} = \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix}$$

and their inverse $-\epsilon_{AB} = (-\sigma^2)^{-1} = -i\sigma^2$. ϵ_{AB} is anti-symmetric and $\epsilon_{AC}\epsilon^{BC} = \delta_A^{\ B}$. Inner products in spinor indices look slightly different from inner products in space-time indices:

$$x_{AA'}y^{AA'} = -2x_{\mu}y^{\mu}, \qquad \qquad z_A z^A = z^A z^B \epsilon_{AB} = 0$$

Any bi-spinor T_{AB} can be decomposed into symmetric and anti-symmetric parts:

$$T_{AB} = \frac{1}{2}(T_{AB} + T_{BA} + T_{AB} - T_{BA}) = T_{(AB)} + \frac{1}{2}\epsilon_{AB}T_C^{\ C}, \qquad (A.1)$$

where $T_{(AB)} = \frac{1}{2!}(T_{AB} + T_{BA})$ denotes the symmetric part of T_{AB} . From now on, we will use the convention that if a tensor carries identical indices, it is implied that the tensor is symmetric in them, e.g. $T^{AA} = \frac{1}{2!}(T^{A_1A_2} + T^{A_2A_1})$ and in a more condensed notation, $T^{A(n)}$ is symmetrized over the *n* indices in a similar fashion. Tensors can carry two types of unrelated indices, primed and unprimed. In the most general case we write $T^{A(m),A'(n)} = \frac{1}{m!n!} \sum_{permutations} T^{A_1...A_m,A'_1...A'_n}$. An object that we will often use is the vierbein $e^{AA'} \equiv e^{AA'}_{\mu} dx^{\mu}$, which is a one-form. A direct consequence of the decomposition of (A.1) leads to the important identity

$$e_{AA'} \wedge e_{BB'} = \frac{1}{2} (\epsilon_{A'B'} H_{AB} + \epsilon_{AB} H_{A'B'}), \qquad (A.2)$$

where $H_{AB} = e_{AC'} \wedge e_B^{C'}$ and $H_{A'B'} = e_{CA'} \wedge e_B^{C'}$. This identity allows one, for example, to rewrite the Yang-Mills field strength in terms of its (anti)-self-dual parts

$$F = F_{AA'|BB'}e^{AA'} \wedge e^{BB'} = H^{BB}F_{BB} + H^{B'B'}F_{B'B'}$$

with $F_{AB} = \frac{1}{2} F_{AC'|B}{}^{C'}$ and $F_{A'B'} = \frac{1}{2} F_{CA'|}{}^{C}{}_{B'}$.

Another useful feature of the spinor formalism is the Fierz identity. Given three spinors, ϕ_A , χ_B , ψ_C , the anti-symmetrization over their indices equals zero, as their indices only run over two values A, B, C = 0, 1. The Fierz identity is obtained by contracting this anti-symmetrized product with ϵ^{BC} , which leads to

$$3\epsilon^{BC}\phi_{[A}\chi_{B}\psi_{C]} = \phi_{A}(\chi\psi) + \chi_{A}(\psi\phi) + \psi_{A}(\phi\chi) \equiv 0.$$
(A.3)

B Technicalities: SDYM

The calculations in the main text have been highly compacted for the sake of brevity. In this appendix we aim to present some proofs and additional details to the reader.

B.1 Ψ -sector

The calculations of the Ψ -sector have been moved to this appendix as they are very much similar to the *F*-sector. The approach is as follows: we apply a covariant derivative to the ansatz (3.19) and we also contract $e^{BB'}$ with $\Psi_{A(k+1),A'(k+3)}$ as to obtain two expressions for $e^{BB'}D\Psi_{A(k)B,A'(k+2)B'}$, which we then compare. The former yields

$$D^{2}\Psi_{A(k),A'(k+2)} = -H^{BB}[F_{BB}, \Psi_{A(k),A'(k+2)}] = -e^{CC'} \wedge D\Psi_{A(k)C,A'(k+2)C'} - e^{C}{}_{A'} \wedge \sum_{n=0}^{k-1} \beta_{nk}[DF_{A(n+1)C,A'(n)}, \Psi_{A(k-n-1),A'(k-n+1)}] - e^{C}{}_{A'} \wedge \sum_{n=0}^{k-1} \beta_{nk}[F_{A(n+1)C,A'(n)}, D\Psi_{A(k-n-1),A'(k-n+1)}] - e^{C}{}_{A'} \wedge \sum_{n=0}^{k-1} \gamma_{nk}[DF_{A(n+2),A'(n)}, \Psi_{A(k-n-2)C,A'(k-n+1)}] - e^{C}{}_{A'} \sum_{n=0}^{k-1} \gamma_{nk}[F_{A(n+2),A'(n)}, D\Psi_{A(k-n-2),A'(k-n+1)}].$$
(B.1)

Considering only quadratic terms in the fields gives

$$e^{BB'} \wedge D\Psi_{A(k)C,A'(k+2)C'} = H^{BB}[F_{BB}, \Psi_{A(k),A'(k+2)}] - \sum_{n=1}^{k} \frac{\beta_{(n-1)k}}{2} H^{BB}[F_{A(n)BB,A'(n)}, \Psi_{A(k-n),A'(k-n+2)}] - \sum_{n=0}^{k} \frac{\beta_{nk} + \gamma_{(n-1)k}}{2} H^{BB}[F_{A(n+1)B,A'(n)}, \Psi_{A(k-n-1)B,A'(k-n+2)}] - \sum_{n=0}^{n} \frac{\gamma_{nk}}{2} H^{BB}[F_{A(n+2),A'(n)}, \Psi_{A(k-n-2)BB,A'(k-n+2)}] + \sum_{n=0}^{k} \frac{\gamma_{(n-1)k}}{2} H_{A'}^{B'}[F_{A(n+1)}^{B},A'(n-1)B', \Psi_{A(k-n-1)B,A'(k-n+2)}] - \sum_{n=0}^{k} \frac{\beta_{nk}}{2} H_{A'}^{B'}[F_{A(n+1)}^{B},A'(n), \Psi_{A(k-n-1)B,A'(k-n+1)B'}].$$
(B.2)

We have renamed the dummy indices in some terms in order to match the summation limits with the expression for $e^{BB'}\Psi_{A(k)B,A'(k+2)B'}$ that we will derive next. This makes some coefficients show up that were not present in the minimal ansatz, so we have to set them to zero by hand: $\beta_{kk} = 0, \gamma_{(-1)k} = 0$. Contracting $e^{BB'}$ with $D\Psi_{A(k),A'(k+4)}$ gives

$$\begin{split} e^{BB'} \wedge D\Psi_{A(k)B,A'(k+2)B'} &= \\ &- \sum_{n=0}^{k} \frac{(n+1)(k+4)}{(k+1)(k+3)} \frac{\beta_{n(k+1)}}{2} H^{BB} [F_{A(n)BB,A'(n)}, \Psi_{A(k-n),A'(k-n+2)}] \\ &- \sum_{n=0}^{k} \left(\frac{(k-n)(k+4)}{(k+1)(k+3)} \frac{\beta_{n(k+1)}}{2} + \frac{(n+2)(k+4)}{(k+1)(k+3)} \frac{\gamma_{n(k+1)}}{2} \right) H^{BB} [F_{A(n+1)B,A'(n)}, \Psi_{A(k-n-1)B,A'(k-n+2)}] \\ &- \sum_{n=0}^{k} \left(\frac{n(k-n)}{(k+1)(k+3)} \frac{\beta_{n(k+1)}}{2} - \frac{n(n+2)}{(k+1)(k+3)} \frac{\gamma_{n(k+1)}}{2} \right) H_{A'}^{B'} [F_{A(n+1)}^{B}_{,A'(n-1)B'}, \Psi_{A(k-n-1)B,A'(k-n+2)}] \\ &- \sum_{n=0}^{k} \left(\frac{(k-n)(k-n+2)}{(k+1)(k+3)} \frac{\beta_{n(k+1)}}{2} - \frac{(n+2)(k-n+2)}{(k+1)(k+3)} \frac{\gamma_{n(k+1)}}{2} \right) H_{A'}^{B'} [F_{A(n+1)}^{B}_{,A'(n)}, \Psi_{A(k-n-1)B,A'(k-n+1)B'}] \\ &- \sum_{n=0}^{k} \frac{(k-n-1)(k+4)}{(k+1)(k+3)} \frac{\gamma_{n(k+1)}}{2} H^{BB} [F_{A(n+2),A'(n)}, \Psi_{A(k-n-2)BB,A'(k-n+2)}] \,. \end{split}$$

Comparing this expression to (B.2), one obtains the recurrence relations

$$0 = \beta_{0k} + \frac{2k(k+2)}{k+3} \,,$$

$$\begin{split} 0 &= \frac{(n+2)(k+4)}{(k+1)(k+3)} \frac{\beta_{(n+1)(k+1)}}{2} - \frac{\beta_{nk}}{2} ,\\ 0 &= \frac{(k-n)(k+4)}{(k+1)(k+3)} \frac{\beta_{n(k+1)}}{2} + \frac{(n+2)(k+4)}{(k+1)(k+3)} \frac{\gamma_{n(k+1)}}{2} - \frac{\beta_{nk} + \gamma_{(n-1)k}}{2} ,\\ 0 &= \frac{(k-n)(k-n+2)}{(k+1)(k+3)} \frac{\beta_{n(k+1)}}{2} - \frac{(n+2)(k-n+2)}{(k+1)(k+3)} \frac{\gamma_{n(k+1)}}{2} - \frac{\beta_{nk}}{2} ,\\ 0 &= \frac{(k-n-1)(n+1)}{(k+1)(k+3)} \frac{\beta_{(n+1)(k+1)}}{2} - \frac{(n+3)(n+1)}{(k+1)(k+3)} \frac{\gamma_{(n+1)(k+1)}}{2} + \frac{\gamma_{nk}}{2} ,\\ 0 &= \frac{(k-n-1)(k+4)}{(k+1)(k+3)} \frac{\gamma_{n(k+1)}}{2} - \frac{\gamma_{nk}}{2} . \end{split}$$

The system is solved by

$$\beta_{nk} = -\frac{2}{(n+1)!} \frac{k-n+2}{k+3} \frac{k!}{(k-n-1)!} , \qquad \qquad \gamma_{nk} = \frac{2}{(n+2)!} \frac{n+1}{k+3} \frac{k!}{(k-n-2)!}$$

B.2 Absence of higher order corrections

In section 3.2 it was mentioned that the obtained solutions for $DF_{A(k+2),A'(k)}$ and $D\Psi_{A(k),A'(k+2)}$ ensured that no higher order corrections were needed. This result is equivalent to the consistency of the L_{∞} -relation in (3.8c) and (3.8f). Here we shall present the proof.

F-sector. As a starting point we take the solution from (3.18) and plug it into (3.15), from which we only consider only the cubic terms. This gives us the l.h.s. of the L_{∞} -relation (3.8c):

$$\begin{split} &l_{3}(e, l_{3}(e, F, F), F) + l_{3}(e, F, l_{3}(e, F, F)) = \\ &- \frac{1}{2}H_{A'A'} \sum_{n=1}^{k-1} \sum_{m=0}^{n-1} \frac{n-m+1}{n+2} \alpha_{nk} \alpha_{mn} [F_{A(k-n+1),A'(k-n-1)}, [F_{A(m+1)B,A'(m)}, F_{A(n-m)}]_{,A'(n-m-1)}]] \\ &+ \frac{1}{2}H_{A'A'} \sum_{n=0}^{k-2} \sum_{m=0}^{k-n-2} \alpha_{nk} \alpha_{m(k-n-1)} [F_{A(n+1)}]_{,A'(n)}, [F_{A(n+1)B,A'(m)}, F_{A(k-n-m),A'(k-n-m-2)}]] \\ &= -\frac{1}{2}H_{A'A'} \sum_{n=1}^{k-1} \sum_{m=0}^{n-1} \frac{n-m+1}{n+2} \alpha_{nk} \alpha_{mn} [F_{A(m+1)}]_{,A'(m)}, [F_{A(n-m)B,A'(n-m-1)}, F_{A(k-n+1),A'(k-n-1)}]] \\ &- \frac{1}{2}H_{A'A'} \sum_{n=1}^{k-1} \sum_{m=0}^{n-1} \frac{n-m+1}{n+2} \alpha_{nk} \alpha_{mn} [F_{A(n-m)}]_{,A'(n-m-1)}, [F_{A(m+1)B,A'(m)}, F_{A(k-n+1),A'(k-n-1)}]] \\ &+ \frac{1}{2}H_{A'A'} \sum_{n=0}^{k-2} \sum_{m=0}^{k-n-2} \alpha_{nk} \alpha_{mn} [F_{A(n-m)}]_{,A'(n)}, [F_{A(n+1)B,A'(m)}, F_{A(k-n-m),A'(k-n-1)}]] \end{split}$$

where we applied the Jacobi identity on the very first term. In order to compare the three terms on the r.h.s., the nested commutators must be cast into the same form, which can be achieved by renaming the dummy indices. The final result allows all terms to be collected into one and evaluates to

$$\frac{1}{2}H_{A'A'}\sum_{n=0}^{k-2} \left(\sum_{m=0}^{n} \alpha_{mk}\alpha_{(n-m)(k-m-1)} - \frac{m+2}{n+3}\alpha_{(n+1)k}\alpha_{(n-m)(n+1)} - \frac{n-m+2}{n+3}\alpha_{(n+1)k}\alpha_{m(n+1)}\right) \times \left[F_{A(m+1)}\right]_{,A'(m)}^{B}, F_{A(n-m+1)B,A'(n-m)}, F_{A(k-n),A'(k-n-2)}\right] = 0,$$

for which the solution for α_{nk} was used. This proves the L_{∞} -relation (3.8c).

\Psi-sector. We isolate the terms cubic in the fields in (B.1) and we plug in (3.18) and (3.20), which yields the l.h.s. of L_{∞} -relation (3.8f) and reads

$$\begin{split} &l_{3}(e, l_{3}(e, F, F), \Psi) + l_{3}(e, F, l_{3}(e, F, \Psi)) = \\ &H_{A'A'} \sum_{n=1}^{k-1} \sum_{m=0}^{n-1} \frac{\alpha_{mn}\beta_{nk}}{2} \frac{n-m+1}{n+2} [[F_{A(m+1)B,A'(m)}, F_{A(n-m)}{}^{B}_{,A'(n-m-1)}], \Psi_{A(k-n-1),A'(k-n+1)}] \\ &+ H_{A'A'} \sum_{n=0}^{k-2} \sum_{m=0}^{k-n-2} \frac{\beta_{m(k-n-1)}\beta_{nk}}{2} [F_{(n+1)}{}^{B}_{,A'(n)}, [F_{A(m+1)B,A'(m)}, \Psi_{A(k-n-m-2),A'(k-n-m)}]] \\ &+ H_{A'A'} \sum_{n=0}^{k-3} \sum_{m=0}^{k-n-3} \frac{\beta_{nk}\gamma_{m(k-n-1)}}{2} [F_{A(n+1)}{}^{B}_{,A'(n)}, [F_{A(m+2),A'(m)}, \Psi_{A(k-n-m-3)B,A'(k-n-m)}]] \\ &+ H_{A'A'} \sum_{n=1}^{k-2} \sum_{m=0}^{n-1} \frac{\alpha_{mn}\gamma_{nk}}{2} [[F_{A(m+1)B,A'(m)}, F_{A(n-m+1),A'(n-m-1)}], \Psi_{A(k-n-2)}{}^{B}_{,A'(k-n+1)}] \\ &+ H_{A'A'} \sum_{n=0}^{k-2} \sum_{m=0}^{k-n-3} (\frac{\beta_{m(k-n-1)}\gamma_{nk}}{2} \frac{k-n-m-2}{k-n-1} - \frac{\gamma_{m(k-n-1)}\gamma_{nk}}{2} \frac{m+2}{k-n-1}) \\ &\times [F_{A(n+2),A'(n)}, [F_{A(m+1)}{}^{B}_{,A'(m)}, \Psi_{A(k-n-m-3)B,A'(k-n-m)}]]. \end{split}$$

Our approach is similar to the one for the F-sector: we aim to reduce the equations as much as possible by casting the nested commutators into a similar form. A particular technicality in this case is that a contraction can be either between two F's or between F and Ψ . The Fierz identity is used to convert all contractions into the latter type. However, one must be careful, as the Fierz identity requires some free indices on the available spinors, which might not be present in all terms of the summation. Hence we isolate these cases and check that their contribution vanishes.

$$\begin{aligned} H_{A'A'} \sum_{n=0}^{k-2} \frac{\beta_{(k-1)k}\alpha_{n(k-1)}}{2} \frac{k-n}{k+1} [[F_{A(n+1)B,A'(n)}, F_{A(k-n-1)}^{B}, A'(k-n-2)], \Psi_{A'A'}] \\ &+ H_{A'A'} \sum_{n=0}^{k-2} \frac{\beta_{nk}\beta_{(k-n-2)(k-n-1)}}{2} [F_{A(n+1)}^{B}, A'(n), [F_{A(k-n-1)B,A'(k-n-2)}, \Psi_{A'A'}]] \\ &= H_{A'A'} \sum_{n=0}^{k-2} \left(-\frac{\beta_{(k-1)k}\alpha_{n(k-1)}}{2} \frac{k-n}{k+1} + \frac{\beta_{nk}\beta_{(k-n-2)(k-n-1)}}{2} - \frac{\beta_{(k-1)k}\alpha_{(k-n-2)(k-1)}}{2} \frac{n+2}{k+1} \right) \\ &\times [F_{A(n+1)}^{B}, A'(n), [F_{A(k-n-1)B,A'(k-n-2)}, \Psi_{A'A'}]] = 0, \end{aligned}$$

where we used the solution for β_{nk} and α_{nk} . Finally, applying the Fierz identity, Jacobi identity and renaming of dummy indices allows one to cast the remaining terms into a more practical form that reads

$$\begin{split} H_{A'A'} \sum_{n=0}^{k-3} \sum_{m=0}^{n} \left(\frac{\beta_{(n+1)k}\alpha_{m(n+1)}}{2} \frac{n-m+2}{n+3} + \frac{\beta_{(n+1)k}\alpha_{(n-m)(n+1)}}{2} \frac{m+2}{n+3} + \frac{\beta_{mk}\beta_{(n-m)(k-m-1)}}{2} \right) \\ &+ \frac{\beta_{mk}\gamma_{(n-m)(k-m-1)}}{2} - \frac{\gamma_{(n+1)k}\alpha_{m(n+1)}}{2} \right) \left[F_{A(m+1)}{}^{B}_{,A'(m)}, \left[F_{A(n-m+2),A'(n-m)}, \Psi_{A(k-n-3)B,A'(k-n)} \right] \right] \\ &+ H_{A'A'} \sum_{n=0}^{k-3} \sum_{m=0}^{n} \left(\frac{\beta_{(n+1)k}\alpha_{(n-m)(n+1)}}{2} \frac{m+2}{n+3} + \frac{\beta_{(n+1)k}\alpha_{m(n+1)}}{2} \frac{n-m+2}{n+3} - \frac{\beta_{mk}\beta_{(n-m)(k-m-1)}}{2} \right) \\ &+ \frac{\gamma_{(n+1)k}\alpha_{(n-m)(n+1)}}{2} - \frac{\gamma_{mk}\beta_{(n-m)(k-m-1)}}{2} \frac{k-n-2}{k-m-1} + \frac{\gamma_{mk}\gamma_{(n-m)(k-m-1)}}{2} \frac{n-m+2}{k-m-1} \right) \\ &\times \left[F_{A(m+2),A'(m)}, \left[F_{A(n-m+1)} \right]_{,A'(n-m)}^{B}, \Psi_{A(k-n-3)B,A'(k-n)} \right] \right] = 0 \,, \end{split}$$

which is obtained by plugging in the solutions for α_{nk} , β_{nk} and γ_{nk} were used. This implies the consistency of L_{∞} -relation (3.8f).

B.3 Higher gravitational corrections

In section 3.3 we mentioned that the correction due to the constant gravitational background to the linear term in $DF_{A(k+2),A'(k)}$ and $D\Psi_{A(k),A'(k+2)}$ does not propagate to the quadratic term or higher. This appendix is dedicated to prove this.

F-sector. The L_{∞} -relations are modified on a constant curvature background according to (3.23). It was mentioned in (3.23) that the gravitational contribution decouples and vanishes independently. We shall present a proof here.

We are interested in checking consistency of L_{∞} -relation (3.23a). We do so by taking the

covariant derivative of (3.24a), which gives

$$D^{2}F_{A(k+2),A'(k)} = -H^{BB}[F_{BB}, F_{A(k+2),A'(k)}] + (k+2)H_{A}^{B}F_{A(k+1)B,A'(k)} + kH_{A'}^{B'}F_{A(k+2),A'(k-1)B'} = -e^{BB'} \wedge DF_{A(k+2)B,A'(k)B'} - e^{B}_{A'} \wedge \sum_{n=0}^{k-1} \alpha_{nk}[DF_{A(n+1)B,A'(n)}, F_{A(k-n+1),A'(k-n-1)}] - e^{B}_{A'} \wedge \sum_{n=0}^{k-1} \alpha_{nk}[F_{A(n+1)B,A'(n)}, DF_{A(k-n+1),A'(k-n-1)}].$$
(B.4)

Considering only the terms coming from gravitational contributions gives the l.h.s of L_{∞} relation (3.23a) and reads after introducing $f_k = k(k+2)$:

$$\begin{split} \tilde{l}_{2}(e, l_{3}(e, F, F)) + l_{3}(e, \tilde{l}_{2}(e, F), F) + l_{3}(e, F, \tilde{l}_{2}(e, F)) &= \\ H_{A'A'} \sum_{n=0}^{k-2} \frac{\frac{1}{2}n+2}{n+3} f_{n+1} \alpha_{(n+1)k} [F_{A(n+2),A'(n)}, F_{A(k-n),A'(k-n-2)}] \\ &+ H_{A'A'} \sum_{n=0}^{k-2} \frac{\alpha_{nk}}{2} f_{k-n-1} [F_{A(n+2),A'(n)}, F_{A(k-n),A'(k-n-2)}] \\ &- H_{A'A'} \sum_{n=0}^{k-2} \frac{\alpha_{n(k-1)}}{2} f_{k} [F_{A(n+2),A'(n)}, F_{A(k-n),A'(k-n-2)}] \\ &= \sum_{n=0}^{k-2} \frac{1}{2} (\frac{\frac{1}{2}n+2}{n+3} f_{n+1} \alpha_{(n+1)k} - \frac{\frac{1}{2}(k-n)+1}{k-n+1} f_{k-n-1} \alpha_{(k-n-1)k} + \frac{\alpha_{nk}}{2} f_{k-n-1} - \frac{\alpha_{(k-n-2)k}}{2} f_{n+1} \\ &- \frac{\alpha_{n(k-1)}}{2} f_{k} + \frac{\alpha_{(k-n-2)(k-1)}}{2} f_{k}) [F_{A(n+2),A'(n)}, F_{A(k-n),A'(k-n-2)}] = 0 \,, \end{split}$$

where the anti-symmetry of the commutator has been made explicit and the solution for α_{nk} was applied. Thus, the modification to the second L_{∞} -relation of the *F*-sector vanishes, which means that the gravitational background only modifies $DF_{A(k+2),A'(k)}$ on the linear level, identically to the free equations. This is equivalent to the consistency of (3.23a). **\Psi-sector.** The second L_{∞} -relation for Ψ on a gravitational background is modified according to (3.23b). This gives

$$D^{2}\Psi_{A(k),A'(k+2)} = -H^{BB}[F_{BB}, \Psi_{A(k),A'(k+2)}] + kH_{A}^{B}\Psi_{A(k-1)B,A'(k+2)} + (k+2)H_{A'}^{B'}\Psi_{A(k),A'(k+1)B'} = -e^{CC'} \wedge D\Psi_{A(k)C,A'(k+2)C'} - e^{C}{}_{A'} \wedge \sum_{n=0}^{k-1} \beta_{nk}[DF_{A(n+1)C,A'(n)}, \Psi_{A(k-n-1),A'(k-n+1)}] - e^{C}{}_{A'} \wedge \sum_{n=0}^{k-1} \beta_{nk}[F_{A(n+1)C,A'(n)}, D\Psi_{A(k-n-1),A'(k-n+1)}] - e^{C}{}_{A'} \wedge \sum_{n=0}^{k-1} \gamma_{nk}[DF_{A(n+2),A'(n)}, \Psi_{A(k-n-2)C,A'(k-n+1)}] - e^{C}{}_{A'} \sum_{n=0}^{k-1} \gamma_{nk}[F_{A(n+2),A'(n)}, D\Psi_{A(k-n-2),A'(k-n+1)}] .$$
(B.5)

Considering only the terms containing a gravitational contribution, one obtains the l.h.s. of the L_{∞} -relation (3.23b):

$$\tilde{l}_{2}(e, l_{3}(e, F, \Psi)) + l_{3}(e, \tilde{l}_{2}(e, F), \Psi) + l_{3}(e, F, \tilde{l}_{2}(e, \Psi))$$

$$= H_{A'A'} \sum_{n=0}^{k-2} (f_{n+1}\beta_{(n+1)k} \frac{\frac{1}{2}n+2}{n+3} + \frac{1}{2}\beta_{nk}f_{k-n-1} + \frac{1}{2}f_{n+1}\gamma_{(n+1)k}$$

$$+ \frac{\frac{1}{2}k - \frac{1}{2}n}{k-n-1}\gamma_{nk}f_{k-n-1} - f_{k}\frac{\beta_{n(k-1)}}{2} - f_{k}\frac{\gamma_{n(k-1)}}{2})[F_{A(n+2),A'(n)}, \Psi_{A(k-n-2),A'(k-n)}] = 0,$$
(B.6)

which we obtain by plugging in the solutions for α_{nk} , β_{nk} and γ_{nk} . This proves the consistency of (3.23b).

The results in this appendix prove that the gravitational contribution to the L_{∞} -relations in both sectors decouples and vanishes independently, which is equivalent to consistency of (3.23a) and (3.23b). Thus, the gravitational background only modifies $DF_{A(k+2),A'(k)}$ and $D\Psi_{A(k),A'(k+2)}$ on the linear level, identically to the free equations.

C Technicalities: SDGR

Several technicalities have been left out from the main text. In this section we aim to present the calculation of the Ψ -sector, as well as the proofs of the truncation of ∇C and $\nabla \Psi$, as promised in section 4.2

C.1 Ψ -sector

We have left the details of the calculation of the Ψ -sector of section 4.2 to this appendix, as it bears a lot of resemblance to the C-sector.

The approach is similar to before: we take the covariant derivative of the ansatz (4.12) and we also contract $e^{BB'}$ with $\nabla \Psi_{A(k+1),A'(k+5)}$ as this will give two expressions for $e^{BB'} \wedge \nabla \Psi_{A(k)B,A'(k+4)B'}$, so we can compare them. This will unveil its structure. The former yields

$$\nabla^{2} \Psi_{A(k),A'(k+4)} = k H^{BB} C_{ABB}{}^{D} \Psi_{A(k-1)D,A'(k+4)} = -e^{CC'} \wedge \nabla \Psi_{A(k)C,A'(k+4)C'} - e^{C}{}_{A'} \wedge \sum_{n=0}^{k} b_{nk} \nabla C_{A(n+2)C}{}^{D}{}_{,A'(n)} \Psi_{A(k-n-2)D,A'(k-n+3)} - e^{C}{}_{A'} \wedge \sum_{n=0}^{k} b_{nk} C_{A(n+2)C}{}^{D}{}_{,A'(n)} \nabla \Psi_{A(k-n-2)D,A'(k-n+3)} - e^{C}{}_{A'} \wedge \sum_{n=0}^{k} c_{nk} \nabla C_{A(n+3)}{}^{D}{}_{,A'(n)} \Psi_{A(k-n-3)CD,A'(k-n+3)} - e^{C}{}_{A'} \wedge \sum_{n=0}^{k} c_{nk} C_{A(n+3)}{}^{D}{}_{,A'(n)} \nabla \Psi_{A(k-n-3)CD,A'(k-n+3)} .$$
(C.1)

Isolating the terms quadratic in the fields gives

$$\begin{split} e^{BB'} \wedge \nabla \Psi_{A(k)C,A'(k+4)C'} &= -kH^{BB}C_{ABB}{}^{D} \Psi_{A(k-1)D,A'(k+4)} \\ &- \frac{1}{2}H^{BB} \sum_{n=0}^{k} b_{nk}C_{A(n+2)BB}{}^{D}_{,A'(n+1)} \Psi_{A(k-n-2)D,A'(k-n+3)} \\ &- \frac{1}{2}H^{BB} \sum_{n=0}^{k} (b_{nk} + c_{(n-1)k})C_{A(n+2)B}{}^{D}_{,A'(n)} \Psi_{A(k-n-2)BD,A'(k-n+4)} \\ &+ \frac{1}{2}H_{A'}{}^{B'} \sum_{n=0}^{k} b_{nk}C_{A(n+2)}{}^{BD}_{,A'(n)} \Psi_{A(k-n-2)BD,A'(k-n+3)B'} \\ &- \frac{1}{2}H_{A'}{}^{B'} \sum_{n=0}^{k} c_{nk}C_{A(n+3)}{}^{BD}_{,A'(n)} \Psi_{A(k-n-3)BD,A'(k-n+3)} \\ &- \frac{1}{2}H^{BB} \sum_{n=0}^{k} c_{nk}C_{A(n+3)}{}^{D}_{,A'(n)} \Psi_{A(k-n-3)BD,A'(k-n+4)} \,, \end{split}$$

whereas the latter gives

$$\begin{split} e^{BB'} \wedge \nabla \Psi_{A(k)B,A'(k+4)B'} &= -H^{BB}b_{0(k+1)}\frac{k+6}{(k+1)(k+5)}C_{ABB}{}^D \Psi_{A(k-1)D,A'(k+4)} \\ &- \frac{1}{2}H^{BB}\sum_{n=0}^{k} b_{(n+1)(k+1)}\frac{(k+6)(n+3)}{(k+1)(k+5)}C_{A(n+2)BB}{}^D_{,A'(n+1)} \Psi_{A(k-n-2)D,A'(k-n+3)} \\ &- \frac{1}{2}H^{BB}\sum_{n=0}^{k} (b_{n(k+1)}\frac{(k+6)(k-n-1)}{(k+1)(k+5)} + c_{n(k+1)}\frac{(k+6)(n+3)}{(k+1)(k+5)}) \\ &\times C_{A(n+2)B}{}^D_{,A'(n)} \Psi_{A(k-n-2)BD,A'(k-n+4)} \\ &+ \frac{1}{2}H_{A'}{}^B' \sum_{n=0}^{k} (b_{(n+1)(k+1)}\frac{(n+1)(k-n-2)}{(k+1)(k+5)} - c_{(n+1)(k+1)}\frac{(n+1)(n+4)}{(k+1)(k+5)}) \\ &\times C_{A(n+3)}{}^{BD}_{,A'(n)B'} \Psi_{A(k-n-3)BD,A'(k-n+3)} \\ &+ \frac{1}{2}H_{A'}{}^B' \sum_{n=0}^{k} (b_{n(k+1)}\frac{(k-n+4)(k-n-1)}{(k+1)(k+5)} - c_{n(k+1)}\frac{(k-n+4)(n+3)}{(k+1)(k+5)}) \\ &\times C_{A(n+2)}{}^{BD}_{,A'(n)} \Psi_{A(k-n-2)BD,A'(k-n+3)B'} \\ &- \frac{1}{2}H^{BB}\sum_{n=0}^{k} c_{n(k+1)}\frac{(k+6)(k-n-2)}{(k+1)(k+5)}C_{A(n+3)}{}^D_{,A'(n)} \Psi_{A(k-n-3)BBD,A'(k-n+4)} \,. \end{split}$$

Comparing them gives the system of recurrence relations

$$\begin{split} 0 &= \frac{k+6}{(k+1)(k+5)} b_{0(k+1)} - k ,\\ 0 &= b_{nk} - b_{(n+1)(k+1)} \frac{(k+6)(n+3)}{(k+1)(k+5)} ,\\ 0 &= b_{nk} + c_{nk} - b_{n(k+1)} \frac{(k+6)(k-n-1)}{(k+1)(k+5)} - c_{n(k+1)} \frac{(k+6)(n+3)}{(k+1)(k+5)} ,\\ 0 &= b_{nk} - b_{n(k+1)} \frac{(k-n+4)(k-n-1)}{(k+1)(k+5)} + c_{n(k+1)} \frac{(k-n+4)(n+3)}{(k+1)(k+5)} ,\\ 0 &= c_{nk} + b_{(n+1)(k+1)} \frac{(n+1)(k-n-2)}{(k+1)(k+5)} - c_{(n+1)(k+1)} \frac{(n+1)(n+4)}{(k+1)(k+5)} ,\\ 0 &= c_{nk} - c_{n(k+1)} \frac{(k+6)(k-n-2)}{(k+1)(k+5)} , \end{split}$$

which is solved by

$$b_{nk} = \frac{2}{(n+2)!} \frac{k!}{(k-n-2)!} \frac{k-n+4}{k+5} \qquad \qquad c_{nk} = -\frac{2}{(n+2)!} \frac{k!}{(k-n-3)!} \frac{n+1}{(k+5)(n+3)} \,.$$

C.2 Absense of higher order corrections

C-sector. We consider (4.9) and isolate the terms cubic in C. Plugging in the solution from (4.11) yields the l.h.s. of the L_{∞} -relation (4.7b) given by

$$\begin{split} l_{3}(e, l_{3}(e, C, C), C) + l_{3}(e, C, l_{3}(e, C, C)) &= \\ \frac{1}{2}H_{A'A'} \sum_{n=1}^{k-1} \sum_{m=0}^{n-1} a_{nk} a_{mn} \frac{n-m+2}{n+4} \frac{m+2}{n+3} C_{A(m+1)B}{}^{DE}_{,A'(m)} C_{A(n-m+1)E}{}^{B}_{,A'(n-m-1)} C_{A(k-n+2)D,A'(k-n-1)} \\ &+ \frac{1}{2}H_{A'A'} \sum_{n=1}^{k-1} \sum_{m=0}^{n-1} a_{nk} a_{mn} \frac{n-m+2}{n+4} \frac{n-m+1}{n+3} \\ &\times C_{A(m+2)B}{}^{E}_{,A'(m)} C_{A(n-m)E}{}^{BD}_{,A'(n-m-1)} C_{A(k-n+2)D,A'(k-n-1)} \\ &+ \frac{1}{2}H_{A'A'} \sum_{n=0}^{k-2} \sum_{m=0}^{k-n-2} a_{nk} a_{m(k-n-1)} \frac{m+2}{k-n+3} \\ C_{A(n+2)}{}^{BD}_{,A'(n)} C_{A(m+1)BD}{}^{E}_{,A'(m)} C_{A(k-n-m+1)E,A'(k-n-m-2)} \\ &+ \frac{1}{2}H_{A'A'} \sum_{n=0}^{k-2} \sum_{m=0}^{k-n-2} a_{nk} a_{m(k-n-1)} \frac{k-n-m+1}{k-n+3} \\ &\times C_{A(n+2)}{}^{BD}_{,A'(n)} C_{A(m+2)B}{}^{E}_{,A'(m)} C_{A(k-n-m)DE,A'(k-n-m-2)} . \end{split}$$
(C.3)

The first three terms can be collected into

$$\frac{1}{2}H_{A'A'}\sum_{n=1}^{k-1}\sum_{m=0}^{n-1} \left(a_{nk}a_{mn}\frac{n-m+2}{n+4}\frac{m+2}{n+3} + a_{nk}a_{(n-m-1)n}\frac{m+3}{n+4}\frac{m+2}{n+3} - a_{(n-m-1)k}a_{m(k-n+m)}\frac{m+2}{k-n+m+4}\right) \times C_{A(m+1)B}{}^{DE}_{,A'(m)}C_{A(n-m+1)}{}^{B}_{,A'(n-m-1)}C_{A(k-n+2)D,A'(k-n-1)} = 0,$$

for which the solution for a_{nk} is applied. The last term in (C.3) may be rewritten as

$$\frac{1}{2}H_{A'A'}\sum_{n=0}^{k-2}\sum_{m=0}^{n}a_{mk}a_{(n-m)(k-m-1)}\frac{k-n+1}{k-m+3}C_{A(m+2)}{}^{BD}_{,A'(m)}C_{A(n-m+2)B}{}^{E}_{,A'(m)}C_{A(k-n)DE,A'(k-n-2)}$$

$$=\frac{1}{4}H_{A'A'}\sum_{n=0}^{k-2}\sum_{m=0}^{n}\left(a_{mk}a_{(n-m)(k-m-1)}\frac{k-n+1}{k-m+3}-a_{(n-m)k}a_{m(k-n+m-1)}\frac{k-n+1}{k-n+m+3}\right)$$

$$\times C_{A(m+2)}{}^{BD}_{,A'(m)}C_{A(n-m+2)B}{}^{E}_{,A'(m)}C_{A(k-n)DE,A'(k-n-2)}=0,$$

where again we used the solution for a_{nk} . This proves that the *C*-sector truncates at quadratic order. This confirms the consistency of (4.7b).

 Ψ -sector. We consider the cubic terms in (C.1) and we assume the solutions (4.11) and (4.13). This gives the l.h.s. of the L_{∞} -relation in (4.7d) and reads

$$\begin{split} &l_{3}(e, l_{3}(e, C, C), \Psi) + l_{3}(e, C, l_{3}(e, C, \Psi)) = \\ &\frac{1}{2}H_{A'A'} \sum_{n=0}^{k=3} \sum_{m=0}^{n} (a_{m(n+1)}b_{(n+1)k} \frac{(n-m+3)(m+2)}{(n+5)(n+4)} + a_{(n-m)(n+1)}b_{(n+1)k} \frac{(m+3)(m+2)}{(n+5)(n+4)} \\ &- b_{m(k-n+m-1)}b_{(n-m)k} \frac{m+2}{k-n+m-1}) \\ &\times C_{A(m+1)B} D^{E}, A'(m) C_{A(n-m+2)E} B^{B}, A'(n-m) \Psi_{A(k-n-3)D,A'(k-n+2)} \\ &+ \frac{1}{4}H_{A'A'} \sum_{n=0}^{k=4} \sum_{m=0}^{n} (b_{(n-m)(k-m-1)}b_{mk} \frac{k-n-3}{k-m-1} + c_{(n-m)(k-m-1)}b_{mk} \frac{n-m+3}{k-m-1} \\ &- a_{m(n+1)}c_{(n+1)k} \frac{n-m+3}{n+5} - b_{m(k-n+m-1)}b_{(n-m)k} \frac{k-n-3}{k-n+m-1} \\ &- c_{m(k-n+m-1)}b_{(n-m)k} \frac{m+3}{k-n+m-1} + a_{(n-m)(n+1)}c_{(n+1)k} \frac{m+3}{n+5}) \\ &\times C_{A(m+2)} D^{B}, A'(m) C_{A(n-m+2)B} E^{E}, A'(n-m) \Psi_{A(k-n-4)DE,A'(k-n+2)} \\ &+ \frac{1}{2}H_{A'A'} \sum_{n=0}^{n} \sum_{m=n}^{n} (b_{(n-m)(k-m-1)}c_{mk} \frac{(k-n-3)(n-m+2)}{(k-m-1)(k-m-2)} - a_{(n-m)(n+1)}c_{(n+1)k} \frac{n-m+2}{n+5}) \\ &\times C_{A(m+3)} D^{A,(m)} C_{A(n-m+1)BD} E^{E}, A'(n-m) \Psi_{A(k-n-4)E} B^{B}, A'(k-n+2) \\ &+ \frac{1}{2}H_{A'A'} \sum_{n=0}^{n} \sum_{m=0}^{n} (c_{m(k-n+m-1)}b_{(n-m)k} \frac{k-n-4}{k-n+m-1} \\ &- b_{(n-m)(k-m-1)}c_{mk} \frac{(k-n-3)(k-n-4)}{(k-m-1)(k-m-2)} + c_{(n-m)(k-m-1)}c_{mk} \frac{(n-m+3)(k-n-4)}{(k-m-1)(k-m-2)}) \\ C_{A(m+3)} C^{A,(m)} C_{A(n-m+2)} D^{BD}, A'(n-m) \Psi_{A(k-n-5)BDE,A'(k-n+2)} = 0 , \end{split}$$

which is obtained by plugging in the results for a_{nk} , b_{nk} and c_{nk} . This proves consistency of (4.7d).

Bibliography

- [1] R. Penrose, "Nonlinear Gravitons and Curved Twistor Theory," Gen. Rel. Grav. 7 (1976) 31–52.
- [2] R. S. Ward, "On Selfdual gauge fields," Phys. Lett. A 61 (1977) 81–82.

- [3] M. F. Atiyah, N. J. Hitchin, and I. M. Singer, "Selfduality in Four-Dimensional Riemannian Geometry," Proc. Roy. Soc. Lond. A 362 (1978) 425–461.
- G. Chalmers and W. Siegel, "The Selfdual sector of QCD amplitudes," *Phys. Rev.* D54 (1996) 7628-7633, arXiv:hep-th/9606061 [hep-th].
- [5] L. J. Mason and N. M. J. Woodhouse, Integrability, selfduality, and twistor theory. 1991.
- [6] E. Witten, "Perturbative gauge theory as a string theory in twistor space," Commun. Math. Phys. 252 (2004) 189-258, arXiv:hep-th/0312171.
- [7] N. Berkovits and E. Witten, "Conformal supergravity in twistor-string theory," JHEP 08 (2004) 009, arXiv:hep-th/0406051.
- [8] M. Atiyah, M. Dunajski, and L. Mason, "Twistor theory at fifty: from contour integrals to twistor strings," Proc. Roy. Soc. Lond. A 473 no. 2206, (2017) 20170530, arXiv:1704.07464 [hep-th].
- B. Zwiebach, "Closed string field theory: Quantum action and the B-V master equation," Nucl. Phys. B390 (1993) 33-152, arXiv:hep-th/9206084 [hep-th].
- M. R. Gaberdiel and B. Zwiebach, "Tensor constructions of open string theories. 1: Foundations," Nucl. Phys. B505 (1997) 569-624, arXiv:hep-th/9705038 [hep-th].
- [11] H. Kajiura, "Noncommutative homotopy algebras associated with open strings," *Rev. Math. Phys.* 19 (2007) 1–99, arXiv:math/0306332 [math-qa].
- T. Lada and J. Stasheff, "Introduction to SH Lie algebras for physicists," Int. J. Theor. Phys. 32 (1993) 1087-1104, arXiv:hep-th/9209099 [hep-th].
- [13] M. Alexandrov, M. Kontsevich, A. Schwarz, and O. Zaboronsky, "The Geometry of the Master Equation and Topological Quantum Field Theory," Int. J. Mod. Phys. A12 (1997) 1405–1429, arXiv:hep-th/9502010 [hep-th].
- [14] G. Barnich, M. Grigoriev, A. Semikhatov, and I. Tipunin, "Parent field theory and unfolding in BRST first-quantized terms," Commun. Math. Phys. 260 (2005) 147–181, arXiv:hep-th/0406192 [hep-th].
- [15] O. Hohm and B. Zwiebach, " L_{∞} Algebras and Field Theory," Fortsch. Phys. 65 no. 3-4, (2017) 1700014, arXiv:1701.08824 [hep-th].
- [16] B. Jurčo, L. Raspollini, C. Sämann, and M. Wolf, " L_{∞} -Algebras of Classical Field Theories and the Batalin-Vilkovisky Formalism," Fortsch. Phys. 67 no. 7, (2019) 1900025, arXiv:1809.09899 [hep-th].
- G. Barnich, F. Brandt, and M. Henneaux, "Local BRST cohomology in the antifield formalism. 1. General theorems," Commun. Math. Phys. 174 (1995) 57-92, arXiv:hep-th/9405109.
- [18] G. Barnich, F. Brandt, and M. Henneaux, "Local BRST cohomology in the antifield formalism. II. Application to Yang-Mills theory," Commun. Math. Phys. 174 (1995) 93-116, arXiv:hep-th/9405194.
- [19] G. Barnich and M. Grigoriev, "First order parent formulation for generic gauge field theories," JHEP 1101 (2011) 122, arXiv:1009.0190 [hep-th].
- [20] M. Grigoriev, "Parent formulations, frame-like Lagrangians, and generalized auxiliary fields," JHEP 12 (2012) 048, arXiv:1204.1793 [hep-th].
- [21] M. Grigoriev and A. Kotov, "Gauge PDE and AKSZ-type Sigma Models," Fortsch. Phys. 67 no. 8-9, (2019) 1910007, arXiv:1903.02820 [hep-th].
- [22] D. Sullivan, "Infinitesimal computations in topology," Publ. Math. IHES 47 (1977) 269–331.

- [23] K. Krasnov and E. Skvortsov, "Flat self-dual gravity," JHEP 08 (2021) 082, arXiv:2106.01397 [hep-th].
- [24] W. Siegel, "Selfdual N=8 supergravity as closed N=2 (N=4) strings," Phys. Rev. D47 (1993) 2504-2511, arXiv:hep-th/9207043 [hep-th].
- [25] M. Abou-Zeid and C. M. Hull, "A Chiral perturbation expansion for gravity," JHEP 02 (2006) 057, arXiv:hep-th/0511189.
- [26] Z. Bern, J. J. M. Carrasco, and H. Johansson, "New Relations for Gauge-Theory Amplitudes," *Phys. Rev. D* 78 (2008) 085011, arXiv:0805.3993 [hep-ph].
- [27] Z. Bern, J. J. M. Carrasco, and H. Johansson, "Perturbative Quantum Gravity as a Double Copy of Gauge Theory," Phys. Rev. Lett. 105 (2010) 061602, arXiv:1004.0476 [hep-th].
- [28] M. Campiglia and S. Nagy, "A double copy for asymptotic symmetries in the self-dual sector," JHEP 03 (2021) 262, arXiv:2102.01680 [hep-th].
- [29] L. Borsten, H. Kim, B. Jurčo, T. Macrelli, C. Saemann, and M. Wolf, "Double Copy from Homotopy Algebras," Fortsch. Phys. 69 no. 8-9, (2021) 2100075, arXiv:2102.11390 [hep-th].
- [30] E. Skvortsov and R. Van Dongen, "Minimal models of field theories: Chiral Higher Spin Gravity," to appear (2022).
- [31] R. R. Metsaev, "Poincare invariant dynamics of massless higher spins: Fourth order analysis on mass shell," Mod. Phys. Lett. A6 (1991) 359–367.
- [32] R. R. Metsaev, "S matrix approach to massless higher spins theory. 2: The Case of internal symmetry," Mod. Phys. Lett. A6 (1991) 2411–2421.
- [33] D. Ponomarev and E. D. Skvortsov, "Light-Front Higher-Spin Theories in Flat Space," J. Phys. A50 no. 9, (2017) 095401, arXiv:1609.04655 [hep-th].
- [34] D. Ponomarev, "Chiral Higher Spin Theories and Self-Duality," JHEP 12 (2017) 141, arXiv:1710.00270 [hep-th].
- [35] E. D. Skvortsov, T. Tran, and M. Tsulaia, "Quantum Chiral Higher Spin Gravity," Phys. Rev. Lett. 121 no. 3, (2018) 031601, arXiv:1805.00048 [hep-th].
- [36] E. Skvortsov, T. Tran, and M. Tsulaia, "More on Quantum Chiral Higher Spin Gravity," *Phys. Rev.* D101 no. 10, (2020) 106001, arXiv:2002.08487 [hep-th].
- [37] E. Skvortsov and T. Tran, "One-loop Finiteness of Chiral Higher Spin Gravity," arXiv:2004.10797 [hep-th].
- [38] F. Brandt, "Gauge covariant algebras and local BRST cohomology," Contemp. Math. 219 (1998) 53-67, arXiv:hep-th/9711171.
- [39] F. Brandt, "Local BRST cohomology and covariance," Commun. Math. Phys. 190 (1997) 459-489, arXiv:hep-th/9604025.
- [40] J. Huebschmann, "The sh-lie algebra perturbation lemma," 23 no. 4, (2011) 669–691.
- [41] M. Grigoriev, K. Mkrtchyan, and E. Skvortsov, "Matter-free higher spin gravities in 3D: Partially-massless fields and general structure," *Phys. Rev. D* 102 no. 6, (2020) 066003, arXiv:2005.05931 [hep-th].

- [42] A. S. Arvanitakis, "The L_{∞} -algebra of the S-matrix," *JHEP* 07 (2019) 115, arXiv:1903.05643 [hep-th].
- [43] G. Barnich and M. Grigoriev, "A Poincare lemma for sigma models of AKSZ type," J. Geom. Phys. 61 (2011) 663-674, arXiv:0905.0547 [math-ph].
- [44] P. van Nieuwenhuizen, "Free graded differential superalgebras," in Group Theoretical Methods in Physics. Proceedings, 11th International Colloquium, Istanbul, Turkey, August 23-28, 1982, pp. 228–247. 1982.
- [45] R. D'Auria, P. Fre, and T. Regge, "Graded Lie Algebra Cohomology and Supergravity," Riv. Nuovo Cim. 3N12 (1980) 1.
- [46] M. A. Vasiliev, "Consistent equations for interacting massless fields of all spins in the first order in curvatures," Annals Phys. 190 (1989) 59–106.
- [47] N. Boulanger, P. Kessel, E. D. Skvortsov, and M. Taronna, "Higher spin interactions in four-dimensions: Vasiliev versus Fronsdal," J. Phys. A49 no. 9, (2016) 095402, arXiv:1508.04139 [hep-th].
- [48] E. D. Skvortsov and M. Taronna, "On Locality, Holography and Unfolding," JHEP 11 (2015) 044, arXiv:1508.04764 [hep-th].
- [49] R. Penrose and W. Rindler, Spinors and Space-Time, vol. 1 of Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1984.
- [50] M. A. Vasiliev, "Free massless fields of arbitrary spin in the de sitter space and initial data for a higher spin superalgebra," Fortsch. Phys. 35 (1987) 741–770.
- [51] R. Penrose, "Zero rest mass fields including gravitation: Asymptotic behavior," Proc. Roy. Soc. Lond. A284 (1965) 159.
- [52] K. Krasnov, E. Skvortsov, and T. Tran, "Actions for Self-dual Higher Spin Gravities," arXiv:2105.12782 [hep-th].
- [53] C. Devchand and V. Ogievetsky, "Interacting fields of arbitrary spin and N > 4 supersymmetric selfdual Yang-Mills equations," Nucl. Phys. B 481 (1996) 188-214, arXiv:hep-th/9606027.
- [54] K. Krasnov, "Self-Dual Gravity," Class. Quant. Grav. 34 no. 9, (2017) 095001, arXiv:1610.01457 [hep-th].
- [55] M. A. Vasiliev, "Triangle identity and free differential algebra of massless higher spins," Nucl. Phys. B324 (1989) 503–522.
- [56] K. B. Alkalaev and M. Grigoriev, "Frame-like Lagrangians and presymplectic AKSZ-type sigma models," *Int. J. Mod. Phys. A* 29 no. 18, (2014) 1450103, arXiv:1312.5296 [hep-th].
- [57] M. Grigoriev, "Presymplectic structures and intrinsic Lagrangians," arXiv:1606.07532 [hep-th].
- [58] A. Sharapov and E. Skvortsov, "Characteristic Cohomology and Observables in Higher Spin Gravity," JHEP 12 (2020) 190, arXiv:2006.13986 [hep-th].
- [59] A. Sharapov and E. Skvortsov, "Higher Spin Gravities and Presymplectic AKSZ Models," arXiv:2102.02253 [hep-th].