

## 2-ROOTS FOR SIMPLY LACED WEYL GROUPS

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**ABSTRACT.** We introduce and study “2-roots”, which are symmetrized tensor products of orthogonal roots of Kac–Moody algebras. We concentrate on the case where  $W$  is the Weyl group of a simply laced Y-shaped Dynkin diagram  $Y_{a,b,c}$  having  $n$  vertices and with three branches of arbitrary finite lengths  $a$ ,  $b$  and  $c$ ; special cases of this include types  $D_n$ ,  $E_n$  (for arbitrary  $n \geq 6$ ), and affine  $E_6$ ,  $E_7$  and  $E_8$ . We show that a natural codimension-1 submodule  $M$  of the symmetric square of the reflection representation of  $W$  has a remarkable canonical basis  $\mathcal{B}$  that consists of 2-roots. We prove that, with respect to  $\mathcal{B}$ , every element of  $W$  is represented by a column sign-coherent matrix in the sense of cluster algebras. If  $W$  is a finite simply laced Weyl group, each  $W$ -orbit of 2-roots has a highest element, analogous to the highest root, and we calculate these elements explicitly. We prove that if  $W$  is not of affine type, the module  $M$  is completely reducible in characteristic zero and each of its nontrivial direct summands is spanned by a  $W$ -orbit of 2-roots.

### INTRODUCTION

A *2-root* is a symmetrized tensor product  $\alpha \vee \beta := \alpha \otimes \beta + \beta \otimes \alpha$ , where  $\alpha$  and  $\beta$  are orthogonal roots for a Kac–Moody algebra. In this paper, we develop the theory of 2-roots, concentrating on the case where the Dynkin diagram  $\Gamma$  is a Y-shaped, simply laced Dynkin diagram of rank  $n = a + b + c = 1$ , with arbitrarily long branches of positive lengths  $a$ ,  $b$ , and  $c$ . The Weyl groups  $W = W(Y_{a,b,c})$  of these types play an important role in group theory even outside the finite and affine types, in part because some of them have very interesting finite quotients. For example, by adding one extra relation to the Coxeter presentation for the Weyl group of type  $Y_{3,4,4}$ , it is possible to obtain the group  $C_2 \times \mathbf{M}$ , where  $C_2$  has order 2 and  $\mathbf{M}$  is the Monster simple group [20]. The special cases of  $Y_{1,2,6}$  and  $Y_{1,2,7}$ , also known respectively as  $E_{10}$  and  $E_{11}$ , appear in the physics literature in M-theory and related contexts.

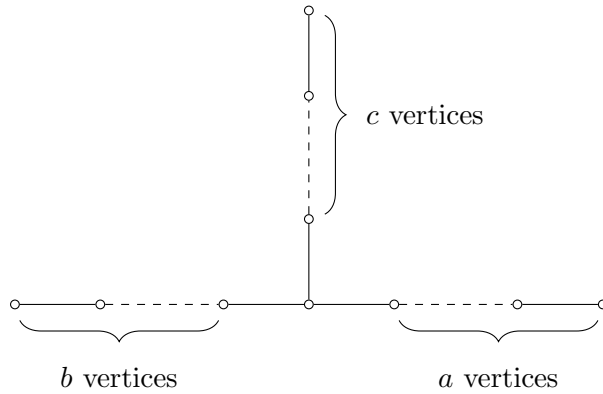


FIGURE 1. The Dynkin diagram of type  $Y_{a,b,c}$

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The reflection representation of the Weyl group  $W$  of  $Y_{a,b,c}$  is an  $n$ -dimensional real representation  $V$  of  $W$  that is equipped with a symmetric  $W$ -invariant bilinear form  $B$ . As we recall in Proposition 1.1, the module  $V$  turns out to be irreducible unless  $Y_{a,b,c}$  is one of the three affine types: affine  $E_6$ ,  $E_7$  and  $E_8$ . The symmetric square  $S^2(V)$  is never an irreducible module (except in trivial cases) because the kernel of  $B$  (regarded as a map from  $S^2(V)$  to  $\mathbb{R}$ ) forms a codimension-1 submodule  $M$  that contains the set  $\Phi^2$  of 2-roots. We call a 2-root *real* if it arises from an orthogonal pair of real roots. The Weyl group  $W$  acts on the set  $\Phi_{\text{re}}^2$  of real 2-roots in a natural way, and it follows from known results that the action has three orbits in type  $D_4$ , two orbits in type  $D_n$  for  $n > 4$ , and one orbit otherwise (Proposition 3.9). If  $Y_{a,b,c}$  is not of affine type, then in characteristic zero, the module  $M$  is a direct sum of irreducible submodules, each of which is the span of one of the  $W$ -orbits of real 2-roots (Theorem 7.8). Furthermore, in the non-affine case, the module  $M$  has a complement in  $S^2(V)$ , spanned by a kind of Virasoro element (Proposition 7.7), so that  $S^2(V)$  is completely reducible.

We show in Theorem 1.8 that there is a canonically defined subset of  $\Phi_{\text{re}}^2$  that forms a basis for  $M$ , which we call the canonical basis of  $M$ . One way to construct this basis is in terms of the stabilizer in  $W$  of a simple root  $\alpha_i$ , which is known by work of Brink [4] and Allcock [1] to be a reflection group with simple system  $\{\beta_{i,1}, \dots, \beta_{i,n-1}\}$ . The canonical basis is then given (Theorem 2.7) by the (redundantly described) set  $\mathcal{B} := \{\alpha_i \vee \beta_{i,j} : 1 \leq i \leq n, 1 \leq j < n\}$ . If  $s_i$  is a simple reflection and  $v$  is a canonical basis element, then  $s_i(v)$  is equal either to  $-v$ , or to  $v$ , or to  $v + v'$  for some other basis element  $v'$  (Theorem 4.7). This is very similar to how a simple reflection acts on a simple root, which is one of the reasons for the name “2-roots”.

On the module  $M$ , the matrices representing group elements  $w \in W$  with respect to the canonical basis have integer entries. We prove (Theorem 5.1) that these matrices are column sign-coherent in the sense of cluster algebras, which means that any two nonzero entries in the same column of a matrix have the same sign. Because every real 2-root is  $W$ -conjugate to a basis element (Proposition 3.3 (iii)), an equivalent way to say this is that each real 2-root is an integer linear combination of canonical basis elements with coefficients of like sign, similar to how every root of  $W$  is an integer linear combination of simple roots with coefficients of like sign. It follows that the elements of  $\mathcal{B}$  have a simple characterization: they are the positive real 2-roots that cannot be expressed as a positive linear combination of two or more positive real 2-roots.

We use the canonical basis  $\mathcal{B}$  to define a partial order  $\leq_2$  on  $\Phi_{\text{re}}^2$  by declaring that  $v_1 \leq_2 v_2$  if  $v_2 - v_1$  is a positive linear combination of elements of  $\mathcal{B}$ . We prove in Proposition 6.4 that  $\leq_2$  is a refinement of the so-called monoidal partial order defined by Cohen, Gijsbers, and Wales on sets of orthogonal positive roots in [7]. In the case where  $W$  is finite, it then follows (Theorem 6.6) that  $\Phi_{\text{re}}^2$  contains a unique maximal 2-root with respect to  $\leq_2$ , which we describe explicitly (Theorem 6.8).

Although we concentrate on the case of type  $Y_{a,b,c}$  in this paper, some of the results hold for type  $A_n$  by restriction. The difference in type  $A$  is that the orthogonal complement of a root is not spanned by the roots it contains. When  $V$  is the reflection representation in type  $A$  (corresponding to the partition  $(n-1, 1)$ ), it is known that  $S^2(V)$  is the direct sum of three representations, corresponding to the partitions  $(n)$ ,  $(n-1, 1)$  and  $(n-2, 2)$ . (This follows from [2, Example 2], using the fact that the exterior square  $\Lambda^2(V)$  corresponds to the partition  $(n-2, 1^2)$ ; see also [12, Proposition 5.4.12].) In this case, the submodule  $(n-2, 2)$  corresponds to the span of the 2-roots, the submodule  $(n-1, 1)$  corresponds to its complement in the module  $M$ , and the submodule  $(n)$  corresponds to the Virasoro element.

We also note that the results of this paper do not seem to generalize readily to all simply laced Weyl groups. For example, let  $n > 6$  and consider the simply laced Weyl group  $W = W(\tilde{D}_{n-1})$  of type affine  $D$  and rank  $n$ . Then by the third example in [1, Section 4], for any simple root  $\alpha_i$  of  $W$  the stabilizer  $W_{\alpha_i}$  of  $\alpha_i$  in  $W$  is a Weyl group of rank  $n$ , not  $n-1$ . It follows that the elements of

the form  $\alpha_i \vee \beta$  where  $\beta$  is a simple root of  $W_{\alpha_i}$  no longer form a linearly independent set, therefore the conclusions in Theorem 2.7 no longer hold.

The paper is organized as follows. Section 1 defines the canonical basis  $\mathcal{B}$  of 2-roots (Theorem 1.8). Section 2 explains how to construct the basis  $\mathcal{B}$  in terms of the stabilizers of real roots (Theorem 2.7). Section 3 describes the  $W$ -orbits of real 2-roots. Section 4 describes the action of reflections on 2-roots, and gives a simple formula (Theorem 4.7) for the action of a simple reflection on a canonical basis element. In Section 5, we prove the sign-coherence properties of the canonical basis (Theorem 5.1 and Theorem 5.2). In Section 6, we prove that a  $W$ -orbit of 2-roots for a finite simply laced Weyl group has a unique maximal element (Theorem 6.6) and we determine this maximal element explicitly (Theorem 6.8). In Section 7, we use  $W$ -orbits of 2-roots to describe the submodules of  $S^2(V)$ , both in general characteristic (Theorem 7.3) and in characteristic zero (Theorem 7.8). In Section 8, we determine when  $W$  acts faithfully on the representations arising from  $W$ -orbits (Theorem 8.6). The results in this paper immediately suggest directions for future research, which we summarize in the conclusion.

## 1. THE CANONICAL BASIS OF 2-ROOTS

Throughout this paper, we will work over a field  $F$  that is of characteristic zero unless otherwise stated. By default, we will assume that  $F = \mathbb{R}$ , but everything will be defined over  $\mathbb{Q}$ , and scalars can be extended if necessary.

Let  $\Gamma = Y_{a,b,c}$  be a simply laced Dynkin diagram with  $n = a + b + c + 1$  vertices, consisting of three paths with  $a$ ,  $b$ , and  $c$  vertices emanating from a trivalent branch vertex. Let  $A$  be the associated Cartan matrix, whose entries  $A_{ij}$  are equal to 2 if  $i = j$ ,  $-1$  if  $i$  and  $j$  are adjacent in  $\Gamma$ , and 0 otherwise.

Let  $W$  be the Weyl group associated to  $\Gamma$ . It is generated by the set  $S = \{s_1, \dots, s_n\}$  indexed by the vertices of  $\Gamma$ , and subject to the defining relations  $s_i^2 = 1$ ,  $s_i s_j = s_j s_i$  if  $A_{ij} = 0$ , and  $s_i s_j s_i = s_j s_i s_j$  if  $A_{ij} = -1$ .

Let  $\Pi = \{\alpha_1, \dots, \alpha_n\}$  be the set of simple roots of  $W$ , and let  $V$  be the  $\mathbb{R}$ -span of  $\Pi$ . Let  $B$  be the Coxeter bilinear form on  $V$ , normalized so that  $B(\alpha_i, \alpha_j) = A_{ij}$ , and let

$$V^\perp = \{v \in V : B(v, v') = 0 \text{ for all } v' \in V\}$$

be the radical of  $B$ . A real root of  $W$  is an element of  $V$  of the form  $w(\alpha_i)$ , where  $w \in W$  and  $\alpha_i$  is a simple root. The real root  $\alpha = w(\alpha_i)$  is associated with the reflection  $s_\alpha = w s_i w^{-1}$ ; in particular, we have  $s_{\alpha_i} = s_i$  for each  $i$ . The reflection  $s_\alpha$  acts on basis elements of  $V$  by the formula

$$s_\alpha(\alpha_j) = \alpha_j - B(\alpha, \alpha_j)\alpha.$$

When  $V$  is endowed with this action, we call  $V$  the *reflection representation* of  $W$ .

It is immediate from the above formula that  $W$  stabilizes the  $\mathbb{Z}$ -span,  $\mathbb{Z}\Pi$ , of the simple roots. The lattice  $\mathbb{Z}\Pi$  is called the *root lattice* and is often denoted by  $Q$ . The form  $B$  is invariant under this action of the Weyl group, meaning that we always have  $B(v, v') = B(w.v, w.v')$ . This implies that  $V^\perp$  is a  $W$ -submodule of  $V$ , and that we have  $B(\alpha, \alpha) = 2$  for every real root  $\alpha$ .

A Kac–Moody algebra may have roots other than real roots; such roots are called *imaginary roots*. We will not give the full definition of imaginary roots, but we will need the result that in the case of affine Kac–Moody algebras, the imaginary roots are precisely the nonzero integer multiples  $n\delta$  of the lowest positive imaginary root,  $\delta$ . The root  $\delta$  satisfies  $B(\delta, v) = 0$  for all  $v \in V$ .

**Proposition 1.1.** *If  $W$  is a Weyl group of type  $Y_{a,b,c}$ , then  $V$  is an irreducible  $W$ -module if and only if  $W$  is not of type affine  $E_6$ , affine  $E_7$  or affine  $E_8$ .*

*Proof.* By [19, Proposition 6.3], it suffices to show that  $V^\perp = 0$ . We omit the rest of the proof, because the result is well known; see for example [9, Example 4.3].  $\square$

We regard the symmetric square,  $S^2(V)$  of  $V$  as a submodule (rather than as a quotient) of  $V \otimes V$ . If  $\alpha, \beta \in V$ , we write  $\alpha \vee \beta$  (or  $\beta \vee \alpha$ ) for the element of  $S^2(V)$  given by  $\alpha \otimes \beta + \beta \otimes \alpha$ . The basis  $\Pi$  of  $V$  gives rise to a basis of  $S^2(V)$  given by

$$\{\alpha_s \vee \alpha_t : s, t \in \Pi\},$$

which we call the *standard basis* of  $S^2(V)$  (with respect to  $\Pi$ ). Restricting the diagonal action of  $W$  on  $V \otimes V$  gives  $S^2(V)$  the structure of a  $W$ -module.

The following result is an immediate consequence of the  $W$ -invariance of  $B$ .

**Lemma 1.2.** *Regard  $B$  as a map  $B : S^2(V) \rightarrow F$ , and let  $M = \ker B$ . Then  $M$  is a  $W$ -submodule of  $S^2(V)$  of dimension*

$$\dim(M) = \dim(S^2(V)) - 1 = \binom{n+1}{2} - 1,$$

and  $S^2(V)/M$  affords the trivial representation of  $W$ . □

Recall that the positive roots of  $W$  are partially ordered in such a way that  $\alpha \leq \beta$  if and only if  $\beta - \alpha$  is a nonnegative linear combination of simple roots, and that if  $W$  is finite, then there exists a highest root with respect to this order. In type  $A_n$ , the highest root is the sum of all the simple roots. In type  $D_n$ , the highest root is  $\sum_{i=1}^n \lambda_i \alpha_i$ , where we have

$$\lambda_i = \begin{cases} 1 & \text{if } i \text{ is an endpoint of } \Gamma; \\ 2 & \text{otherwise.} \end{cases}$$

**Definition 1.3.** We define a positive real root  $\alpha$  of type  $Y_{a,b,c}$  to be *elementary* if  $\alpha$  is a simple root, or the highest root in a type  $A_3$  standard parabolic subsystem, or the highest root in a type  $D_m$  standard parabolic subsystem for  $m \geq 4$ . To each elementary root  $\alpha$ , we associate a nonempty subset  $L(\alpha)$  of the vertices of  $\Gamma$ , defined as follows.

- (1) If  $\alpha_i$  is a simple root, then we define  $L(\alpha_i)$  to be the set of all  $j$  for which  $A_{ij} = 0$ ; in other words, the set of all vertices in  $\Gamma$  that are not equal to or adjacent to  $i$ .
- (2) If  $\alpha$  is the highest root in a standard parabolic subgroup of type  $A_3$ , then  $\alpha = \alpha_i + \alpha_j + \alpha_k$  for some path  $i-j-k$  in  $\Gamma$ , and we define  $L(\alpha) = \{j\}$ . We denote  $\alpha_i + \alpha_j + \alpha_k$  by  $\eta_{i,k}$ . If  $j$  is not the branch point of  $\Gamma$ , then  $i$  and  $k$  can be deduced from a knowledge of  $j$ , and we may write  $\eta_j$  for  $\eta_{i,k}$ .
- (3) Suppose that  $\alpha$  is the highest root of a standard parabolic subgroup of type  $D_m$ . If  $m = 4$ , we define  $L(\alpha)$  to be the three-element set consisting of the neighbours of the branch point in  $\Gamma$ . If  $m > 4$ , we define  $L(\alpha)$  to be the single element  $\{i\}$  indexing the unique simple root in the support of  $\alpha$  that is maximally far from the branch point. In either case, we may denote  $\alpha$  by  $\theta_i$  for any  $i \in L(\alpha)$ .

We say that an elementary root  $\alpha$  is of *type 1, 2, or 3*, depending on which of the three mutually exclusive conditions above applies. If  $\alpha$  is an elementary root and  $i \in L(\alpha)$ , then we say  $\alpha$  is *elementary with respect to  $\alpha_i \in \Pi$* .

*Remark 1.4.* Any simple root  $\alpha_i$  other than the one corresponding to the branch point of  $\Gamma$  lies in a unique standard parabolic subsystem of type  $D_m$  that is of minimal rank. The simple roots involved in this parabolic subsystem are those on the path between  $\alpha_i$  and the branch point, together with all the neighbours of the branch point. In the notation of Definition 1.3, part (3), the highest root of this parabolic subsystem is  $\theta_i$ , and it is elementary with respect to  $i$ .

**Lemma 1.5.** *Let  $\alpha_i$  be a simple root of type  $Y_{a,b,c}$ , and let  $n = a + b + c + 1$ . Then there are precisely  $n - 1$  elementary roots that are elementary with respect to  $\alpha_i$ , and each such elementary root  $\alpha$  satisfies  $B(\alpha_i, \alpha) = 0$ .*

*Proof.* A case-by-case check based on Definition 1.3 shows that whenever  $\alpha$  is a simple root and  $i \in L(\alpha)$ , we have  $B(\alpha_i, \alpha) = 0$ . For the other assertion, we consider three cases, according as  $\alpha_i$  is an endpoint of the Dynkin diagram  $\Gamma$ , or the branch point, or one of the other vertices.

If  $\alpha_i$  is an endpoint of  $\Gamma$ , then the  $n - 1$  elementary roots  $\alpha$  with  $i \in L(\alpha)$  are (a) the  $n - 2$  simple roots that are not equal or adjacent to  $\alpha_i$ , and (b) the root  $\theta_i$  of Definition 1.3, part (3).

If  $\alpha_i$  is the branch point of  $\Gamma$ , then the  $n - 1$  elementary roots  $\alpha$  with  $i \in L(\alpha)$  are (a) the  $n - 4$  simple roots that are not equal or adjacent to  $\alpha_i$ , and (b) the three elementary roots  $\eta_{h,j}$  of type 2 with  $i \in L(\eta_{h,j})$ .

If  $\alpha_i$  is neither an endpoint nor the branch point of  $\Gamma$ , then the  $n - 1$  elementary roots  $\alpha$  with  $i \in L(\alpha)$  are (a) the  $n - 3$  simple roots that are not equal or adjacent to  $\alpha_i$ ; (b) the root  $\eta_i$  of Definition 1.3, and (c) the root  $\theta_i$  of Definition 1.3, part (3).  $\square$

Recall that the roots of  $W$  are partially ordered by stipulating that  $\alpha \leq \beta$  if  $\beta - \alpha$  is a linear combination of simple roots with nonnegative coefficients.

**Lemma 1.6.** *Let  $\alpha$  be a positive root that is elementary with respect to the simple root  $\alpha_i$  in type  $Y_{a,b,c}$ . If  $\alpha$  is a linear combination of positive roots  $\beta_1, \dots, \beta_r$  with positive integer coefficients and with  $r > 1$ , then not all of the  $\beta_k$  can be orthogonal to  $\alpha_i$ .*

*Proof.* Note that the hypotheses imply that we have  $\beta_k < \alpha$  for all  $k$ . If  $\alpha$  is a simple root, then the statement holds vacuously.

If  $\alpha = \eta_{h,j} = \alpha_h + \alpha_i + \alpha_j$ , then the only positive roots  $\beta < \alpha$  are

$$\beta \in \{\alpha_h, \alpha_i, \alpha_j, \alpha_h + \alpha_i, \alpha_i + \alpha_j\},$$

and none of the roots in this list is orthogonal to  $\alpha_i$ .

Finally, suppose that  $\alpha = \theta_i$ , and consider the parabolic subgroup of type  $D_m$  in which  $\theta_i$  is the highest root. Define  $\alpha_j$  to be the simple root adjacent to  $\alpha_i$  in the support of  $\theta$ . It is well known (and mentioned in [1, §4]) that the roots orthogonal to  $\alpha_i$  in  $D_m$  form a root system of type  $D_{m-2} \cup A_1$ , if we interpret  $D_3$  as  $A_3$  and  $D_2$  as  $A_1 \cup A_1$ . The roots in the  $D_{m-2}$  component are those that do not involve  $\alpha_i$  or  $\alpha_j$ , and the roots in the  $A_1$  component are  $\{\pm\theta_i\}$ . It follows that the only positive roots  $\beta < \theta_i$  that are orthogonal to  $\alpha_i$  come from the  $D_{m-2}$  component, so that  $\alpha_i$  and  $\alpha_j$  both appear with zero coefficient in every  $\beta_k$ . This contradicts the fact that  $\alpha_i$  appears with a nonzero coefficient in  $\alpha$ .  $\square$

**Definition 1.7.** Let  $V$  be the reflection representation associated with the Dynkin diagram  $Y_{a,b,c}$ . We define  $\mathcal{B} = \mathcal{B}(a, b, c)$  to be the subset of  $S^2(V)$  consisting of all elements of the form  $\alpha_i \vee \beta$ , where  $\alpha_i \in \Pi$  and where  $\beta$  is elementary with respect to  $\alpha_i$ .

**Theorem 1.8.** *Let  $\Gamma$  be a Dynkin diagram of type  $Y_{a,b,c}$ , let  $n = a + b + c + 1$ , and let  $B$  be the bilinear form on the associated reflection representation  $V$ . The set  $\mathcal{B} = \mathcal{B}(a, b, c)$  is a basis for the submodule  $M = \ker B$  of  $S^2(V)$ .*

*Proof.* Lemma 1.5 implies that every element of  $\mathcal{B}$  lies in  $M$ . The proof now reduces to showing that  $\mathcal{B} = \mathcal{B}(a, b, c)$  is linearly independent and has cardinality  $\binom{n+1}{2} - 1$ , which by Lemma 1.2 is equal to  $\dim(M)$ . We prove these two claims by induction on  $k = \max(a, b, c)$ .

The base case,  $k = 1$ , corresponds to  $\Gamma$  being of type  $D_4$ . We label the vertices of  $\Gamma$  by  $\{1, 2, 3, 4\}$ , where 2 is the branch point. The canonical basis is then given by

$$\{\alpha_1 \vee \theta_1, \alpha_3 \vee \theta_3, \alpha_4 \vee \theta_4, \alpha_2 \vee \eta_{1,3}, \alpha_2 \vee \eta_{1,4}, \alpha_2 \vee \eta_{3,4}, \alpha_1 \vee \alpha_3, \alpha_1 \vee \alpha_4, \alpha_3 \vee \alpha_4\},$$

which has size  $9 = \binom{5}{2} - 1$ , as required.

Suppose for a contradiction that there is a nontrivial dependence relation between these nine elements. We can show that  $\mathcal{B}$  is linearly independent by expanding everything in terms of the standard basis  $\{\alpha_i \vee \alpha_j : 1 \leq i, j \leq n\}$  of  $S^2(V)$ , as follows. For each  $i \in \{1, 3, 4\}$ , the only element

of  $\mathcal{B}$  with  $\alpha_i \vee \alpha_i$  in its support is  $\alpha_i \vee \theta_i$ . This implies that the elements  $\alpha_i \vee \theta_i$  for  $i \in \{1, 3, 4\}$  appear with coefficient zero in the dependence relation. Next, equating coefficients of  $\alpha_1 \vee \alpha_2$  implies that  $\alpha_2 \vee \eta_{1,3}$  and  $\alpha_2 \vee \eta_{1,4}$  occur with equal and opposite coefficients in the dependence relation. Extending this argument to all standard basis elements  $\alpha_i \vee \alpha_2$  for  $i \in \{1, 3, 4\}$  implies that all basis elements  $\alpha_2 \vee \eta_{1,3}$ ,  $\alpha_2 \vee \eta_{1,4}$  and  $\alpha_2 \vee \eta_{3,4}$  occur with coefficient zero in the dependence relation. The remaining elements of  $\mathcal{B}$ ,  $\alpha_1 \vee \alpha_3$ ,  $\alpha_1 \vee \alpha_4$  and  $\alpha_3 \vee \alpha_4$ , are all standard basis elements and are therefore linearly independent, completing the base case.

For the inductive step, we will prove that the statements hold when  $\max(a, b, c) = k+1$ , assuming that they hold when  $\max(a, b, c) = k$ . Since we now have  $\max(a, b, c) > 1$ , it follows that  $n = a + b + c + 1 > 4$ . We assume without loss of generality that  $a \leq b \leq c$ . Denote the vertex of  $\Gamma$  at the end of the  $c$ -branch by 1, and denote the vertex next to it by 2; note that the hypothesis  $n > 4$  guarantees that 2 is not the branch point of  $\Gamma$ . Let  $V'$  be the reflection representation in type  $Y_{a,b,c-1}$ , so that the set  $\Pi \setminus \{\alpha_1\}$  is a basis for  $V'$ , and let  $\mathcal{B}' = \mathcal{B}(a, b, c-1)$ , so that  $\mathcal{B}' \subset \mathcal{B}$ . The elements of  $\mathcal{B} \setminus \mathcal{B}'$  are  $\alpha_1 \vee \theta_1$ ,  $\alpha_2 \vee \eta_2$ , and the  $n-2$  elements  $\alpha_1 \vee \alpha_j$  for  $j \notin \{1, 2\}$ . This implies that  $|\mathcal{B}| = |\mathcal{B}'| + n$ , and therefore by induction that

$$|\mathcal{B}| = |\mathcal{B}'| + n = \binom{n}{2} - 1 + n = \binom{n+1}{2} - 1,$$

as required.

It remains to show that  $\mathcal{B}$  is linearly independent. If not, then the linear independence of  $\mathcal{B}'$  (by induction) means that we must have

$$\sum_{b_i \in \mathcal{B} \setminus \mathcal{B}'} \lambda_i b_i = \sum_{b'_j \in \mathcal{B}'} \mu_j b'_j,$$

for some scalars  $\lambda_i$  and  $\mu_j$ , where some  $\lambda_i$  is nonzero. Now express both sides of this equation with respect to the standard basis of  $S^2(V)$ , so that the right hand side is a linear combination of the standard basis of  $S^2(V')$ . The only element of  $\mathcal{B}$  with a nonzero coefficient of  $\alpha_1 \vee \alpha_1$  is  $\alpha_1 \vee \theta_1$ , so equating coefficients of  $\alpha_1 \vee \alpha_1$  in the above equation implies that  $\alpha_1 \vee \theta_1$  appears with coefficient zero. The only elements of  $\mathcal{B}$  with a nonzero coefficient of  $\alpha_1 \vee \alpha_2$  are  $\alpha_1 \vee \theta_1$  and  $\alpha_2 \vee \eta_2$ , so equating coefficients of  $\alpha_1 \vee \alpha_2$  implies that  $\alpha_2 \vee \eta_2$  appears with coefficient zero. The other elements of  $\mathcal{B} \setminus \mathcal{B}'$  are all standard basis elements that do not lie in  $S^2(V')$ , so they also appear with coefficient zero. This contradiction completes the proof.  $\square$

## 2. STABILIZERS OF REAL ROOTS

Recall that a root of  $W$  is called *real* if it is  $W$ -conjugate to a simple root. We denote the set of real roots of  $W$  by  $\Phi_{\text{re}}$ . In Section 2, we describe the relationship between the basis  $\mathcal{B}$  and the stabilizers of the real roots of  $W$ .

To do so, it is helpful to introduce some graph theoretic terminology. For each integer  $k \geq -1$ , there is a notion of attaching a path of length  $k$  to a graph  $G$  with  $n$  vertices to form a graph with  $n+k$  vertices.

**Definition 2.1.** Let  $k \geq 1$  be an integer, and let  $\alpha$  be a vertex of a graph  $G$ . To *attach a path of length  $k$  to  $G$  at  $\alpha$* , we take the disjoint union of  $G$  and a path  $P$  with  $k$  vertices, and then add an edge between  $\alpha$  and one of the endpoints of  $P$ .

To *attach a path of length 0 to  $G$  at  $\alpha$* , we simply take the graph  $G$  itself. To *attach a path of length  $-1$  to  $G$  at  $\alpha$* , we remove the vertex  $\alpha$  and all edges incident to  $\alpha$ .

**Definition 2.2.** Let  $a, b, c \geq -1$  be integers, and let  $H$  be the 6-cycle  $h_1-h_2-h_3-h_4-h_5-h_6-h_1$ . We define the  $H_{a,b,c}$  to be the graph obtained by attaching paths of lengths  $a$ ,  $b$ , and  $c$  to  $H$  at the vertices  $h_1$ ,  $h_3$ , and  $h_5$ , respectively.

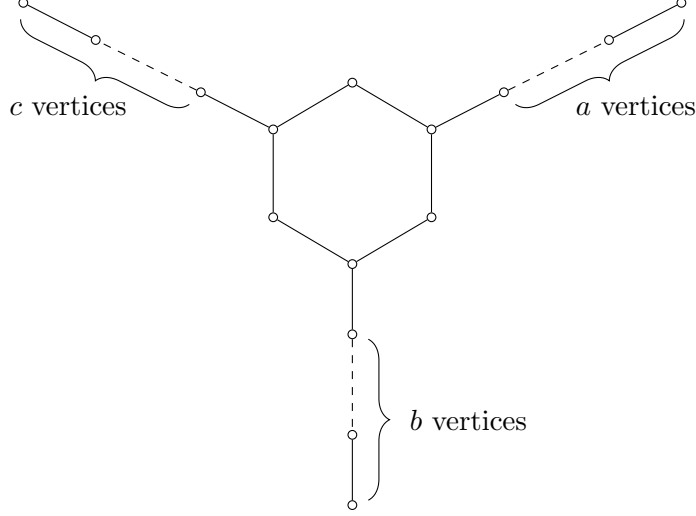


FIGURE 2. The graph  $H_{a,b,c}$

*Remark 2.3.* The graphs  $H_{a,b,c}$  are denoted by  $Q_{a+1,b+1,c+1}$  in the ATLAS of Finite Groups [8, pp 232–233], where they play an important role in the structure of the Monster simple group.

**Lemma 2.4.** *The number of connected components of  $H_{a,b,c}$ , where  $a \leq b \leq c$ , is 3 if  $a = b = c = -1$ , is 2 if  $a = b = -1$  and  $c \geq 0$ , and is 1 otherwise.*

*Proof.* This follows from the definition of  $H_{a,b,c}$ . □

In the case of the Dynkin diagram  $Y_{a,b,c}$ , the stabilizer in  $W$  of a real root has been determined explicitly by Allcock [1], using a result of Brink [4].

**Theorem 2.5** (Allcock, Brink). *Let  $W$  be a Weyl group of type  $Y_{a,b,c}$  with  $a, b, c \geq 1$ , and let  $\alpha$  be a real root of  $W$ . Then the stabilizer  $W_\alpha = \text{Stab}_W(\alpha)$  of  $\alpha$  in  $W$  is generated by the reflections it contains, and  $W_\alpha$  is a simply laced Weyl group of type  $H_{a-2,b-2,c-2}$ .*

*Proof.* Since  $Y_{a,b,c}$  is simply laced, all real roots are  $W$ -conjugate to each other and therefore have conjugate stabilizers. It therefore suffices to prove the theorem in the case where  $\alpha$  is a simple root  $\alpha_i$  associated to a Coxeter generator  $s \in W$ .

It follows from the main result of [4] (see also [1, Corollary 7]) that  $W_{\alpha_i}$  can be expressed as a semidirect product  $W_\Omega \rtimes \Gamma_\Omega$ , where  $W_\Omega$  is the subgroup generated by all the reflections that fix  $\alpha_i$ , and  $\Gamma_\Omega$  is the free group  $\pi_1(\Delta^{\text{odd}}, s)$ . Since every edge in  $Y_{a,b,c}$  has an odd label of 3, the graph  $\Delta^{\text{odd}}$  is simply the Dynkin diagram  $\Gamma$ . The connected component of  $\Gamma$  containing  $s$  has no circuits, which means that the free group in question is trivial, and that  $W_\alpha \cong W_\Omega$ .

The proof is completed from the discussion following [1, Theorem 13], which describes an equivalent construction of the graphs  $H_{a-2,b-2,c-2}$  as the graphs of  $W_\Omega$ . □

**Example 2.6.** Let  $W = W(Y_{a,b,c})$  and let  $\alpha$  be a real root of  $W$ . If  $W$  is a Weyl group of type  $D_4 = Y_{1,1,1}$ ,  $D_5 = Y_{1,1,2}$ , or  $E_8 = Y_{1,2,4}$ , then by Theorem 2.5 the corresponding Dynkin diagrams  $H_{a-2,b-2,c-2}$  for  $W_\alpha$  are as pictured from left to right in Figure 3. If  $W$  is the affine Weyl group of type  $\tilde{E}_6 = Y_{2,2,2}$ , then the corresponding Dynkin diagram  $H_{a-2,b-2,c-2} = H_{0,0,0}$  for  $W_\alpha$  is simply a hexagon, which equals the Dynkin diagram of type  $\tilde{A}_5$ . In other words, the stabilizer of each real root in type affine  $E_6$  is isomorphic (as a reflection group) to the Weyl group of type affine  $A_5$ .

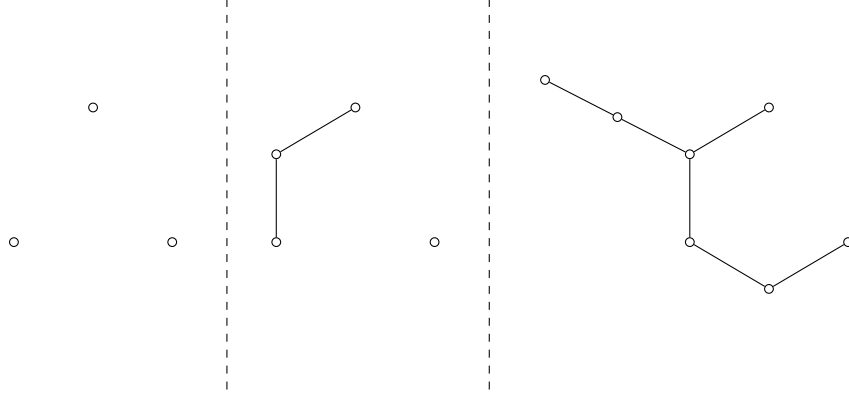


FIGURE 3. The  $H$ -diagrams corresponding to the Weyl groups  $D_4, D_5$  and  $E_8$

**Theorem 2.7.** *Let  $W$  be a Weyl group of type  $Y_{a,b,c}$  with  $a, b, c \geq 1$ , and let  $n = a + b + c + 1$  be the rank of  $W$ .*

- (i) *Let  $\alpha_i$  be a simple root of  $W$ , and regard the stabilizer  $W_{\alpha_i}$  as a Weyl group as in Theorem 2.5. Then the simple roots of  $W_{\alpha_i}$  are precisely the  $(n - 1)$  elementary roots with respect to  $i$  from Definition 1.3.*
- (ii) *The canonical basis  $\mathcal{B}(a, b, c)$  consists of all elements  $\alpha_i \vee \beta$ , where  $\alpha_i$  is a simple root of  $W$ , and  $\beta$  is a simple root of the stabilizer  $W_{\alpha_i}$ .*

*Proof.* By Theorem 2.5, the group  $W_{\alpha_i}$  has  $(a + b + c)$  simple roots. In general, the simple roots can be characterized as the roots that are not expressible as positive integer linear combinations of other positive roots. The elementary roots with respect to  $i$  have this property by Lemma 1.6, and there are  $n - 1 = a + b + c$  of them by Lemma 1.5. The conclusion of (i) follows.

The assertion of (ii) follows from (i) and Theorem 1.8.  $\square$

**Corollary 2.8.** *Let  $i$  and  $j$  be adjacent vertices of  $W$ , and for  $k \in \{i, j\}$ , define  $R_k \subset \mathcal{B}$  be the set of basis elements of the form  $\alpha_k \vee \beta$ . Then the map  $\phi_{ij} : R_i \rightarrow R_j$  defined by  $\phi_{ij}(v) = s_i s_j(v)$  is a well-defined bijection with inverse  $\phi_{ji}$ .*

*Proof.* Note that for any real root  $\beta$  we have

$$\phi_{ij}(\alpha_i \vee \beta) = (s_i s_j(\alpha_i)) \vee (s_i s_j(\beta)) = \alpha_j \vee s_i s_j(\beta).$$

In this way,  $\phi_{ij}$  induces a bijection  $\phi'_{ij}$  between the real roots  $\beta$  orthogonal to  $\alpha_i$  and the real roots  $\beta' = s_i s_j(\beta)$  orthogonal to  $\alpha_j$ . Because the reduced word  $s_i s_j$  has a length of 2, it makes precisely two positive roots negative. These are  $\alpha_j$  and  $s_i(\alpha_j) = \alpha_i + \alpha_j$ , neither of which is orthogonal to  $\alpha_i$ . It follows that  $\phi'_{ij}$  sends positive roots to positive roots, and negative roots to negative roots. In turn, this implies that  $\phi'_{ij}$  sends the simple roots of  $W_{\alpha_i}$  to the simple roots of  $W_{\alpha_j}$ , which proves that  $\phi_{ij}$  has the claimed property by Theorem 2.7 (i). The claim about inverses is immediate from the fact that  $s_i s_j$  is the inverse of  $s_j s_i$ .  $\square$

We record below a technical lemma for future use.

**Lemma 2.9.** *Let  $\alpha_i$  be a simple root, let  $\alpha_i \vee \beta \in \mathcal{B}(a, b, c)$  be a canonical basis element, and let  $\gamma \in \Pi \setminus \{\alpha_i, \beta\}$  be another simple root.*

- (1) *At least one of the following holds:*
  - (i)  $B(\alpha_i, \gamma) = B(\beta, \gamma) = 0$ ;



- (ii)  $B(\alpha_i, \gamma) = -1$ ;
  - (iii)  $B(\alpha_i, \gamma) = 0$  and  $B(\beta, \gamma) = -1$ .
- (2) We have  $B(\beta, \gamma) \in \{-1, 0, 1\}$ .

*Proof.* (1) Since  $\gamma \neq \alpha_i$ , it follows from the definition of the generalized Cartan matrix that we must have  $B(\alpha_i, \gamma) \in \{0, -1\}$ . If  $B(\alpha_i, \gamma) = -1$  then (ii) holds and we are done, so assume that we have  $B(\alpha_i, \gamma) = 0$ .

It now follows from Theorem 2.7 that  $\gamma$  and  $\beta$  are both simple roots of  $W_{\alpha_i}$ . Consideration of the generalized Cartan matrix of  $W_{\alpha_i}$  now shows that either  $B(\beta, \gamma) = 0$  or  $B(\beta, \gamma) = -1$ , which completes the proof.

(2) From the explicit description of  $\mathcal{B}$ , the root  $\beta$  is either a simple root, or is the highest root in a parabolic subsystem of  $Y_{a,b,c}$  of type  $A_3$  or  $D_m$ . Note that if  $\alpha'$  is a simple root that occurs with coefficient  $c \geq 2$  in  $\beta$ , then it must be the case that  $\beta$  is the highest root in a subsystem of type  $D_n$  and  $c = 2$ . In this case, the only simple roots  $\gamma$  in  $Y_{a,b,c}$  that are adjacent to  $\alpha'$  must also be in the support of  $\beta$ .

Suppose first that  $\gamma$  is not in the support of  $\beta$ . If  $\gamma$  is not adjacent to a simple root in the support of  $\beta$ , then we have  $B(\beta, \gamma) = 0$ , which satisfies the conclusion. If, on the other hand,  $\gamma$  is adjacent to a simple root  $\alpha'$  in the support of  $\beta$ , then the previous paragraph shows that  $\gamma$  is adjacent to a simple root  $\alpha'$  in the support of  $\beta$  that occurs with coefficient 1. There is a unique such simple root  $\alpha'$ , because the support of  $\beta$  is a tree and there are no circuits in the subgraph of  $Y_{a,b,c}$  consisting of  $\gamma$  and the support of  $\beta$ . It follows that  $B(\beta, \gamma) = -1$  in this case.

The final possibility is that  $\gamma$  is in the support of  $\beta$ . In this case,  $\gamma$  and  $\beta$  lie in a subsystem of type  $A_3$  or  $D_m$ , and a case-by-case check (depending on whether the subsystem is of type  $A_3$ ,  $D_4$ , or  $D_m$  where  $m > 4$ ) shows that  $B(\beta, \gamma) \in \{0, 1\}$ .  $\square$

### 3. ORBITS OF 2-ROOTS

In Section 3, we investigate the action of  $W$  on pairs of orthogonal roots in more detail (Proposition 3.3), which leads to a detailed description of the  $W$ -orbits of 2-roots (Proposition 3.9). The following result is well known, and follows for example from [3, Lemma 3.6].

**Lemma 3.1.** *Let  $W$  be a simply laced Weyl group, and let  $\alpha_i$  and  $\alpha_j$  be two Coxeter generators of  $W$ . Then  $\alpha_i$  and  $\alpha_j$  are conjugate in  $W$  if and only if they lie in the same connected component of the Dynkin diagram of  $W$ .*  $\square$

The next result will be used in the proof of Proposition 3.3 below.

**Lemma 3.2.** *Let  $\Gamma$  be a Dynkin diagram of type  $Y_{a,b,c}$  and let  $\Gamma'$  be a parabolic subsystem of type  $D_m$  for  $m \geq 4$ . Number the vertices of  $\Gamma'$  such that  $\beta_0$  is the branch vertex, and such that the paths are  $\beta_0 - \beta'$ ,  $\beta_0 - \beta''$ , and  $\beta_0 - \beta_1 - \cdots - \beta_{m-3}$ . Let  $\theta$  be the highest root of  $\Gamma'$ . Then the ordered pairs  $(\beta_{m-3}, \theta)$  and  $(\beta', \beta'')$  are in the same  $W(D_m)$ -orbit.*

*Proof.* Let  $s_i, s', s''$  be the reflections associated to the roots  $\beta_i, \beta'$  and  $\beta''$ , respectively. Direct calculation shows that

$$(s_{m-4}s_{m-3})(s_{m-5}s_{m-4}) \cdots (s_0s_1)(s's_0)((\beta', \beta'')) = (\beta_{m-3}, \theta),$$

which completes the proof.  $\square$

**Proposition 3.3.** *Let  $W$  be a Weyl group of type  $Y_{a,b,c}$ .*

- (i) *The group  $W$  acts transitively on  $\Phi_{\text{re}}$ .*
- (ii) *Every ordered pair of orthogonal real roots of  $W$  is  $W$ -conjugate to a pair of orthogonal simple roots of  $W$ .*
- (iii) *Every real 2-root is  $W$ -conjugate to an element of  $\mathcal{B}$ .*

- (iv) Every ordered pair  $(\alpha, \beta)$  of orthogonal real roots of  $W$  is  $W$ -conjugate to its reversal,  $(\beta, \alpha)$ .
- (v) Two ordered pairs of orthogonal real roots  $(\alpha_1, \beta_1)$  and  $(\alpha_2, \beta_2)$  are  $W$ -conjugate if and only if the corresponding unordered pairs  $\{\alpha_1, \beta_1\}$  and  $\{\alpha_2, \beta_2\}$  are  $W$ -conjugate. The number of  $W$ -orbits in each case is equal to the number connected components of  $H_{a-2, b-2, c-2}$ . This number is 3 if  $W$  is of type  $D_4 = Y_{1,1,1}$ , is 2 if  $W$  is of type  $D_n = Y_{1,1, n-3}$  for  $n > 4$ , and is 1 otherwise.

*Proof.* Any real root is  $W$ -conjugate to a simple root by definition, and the simple roots are in the same  $W$ -orbit by Lemma 3.1 because  $Y_{a,b,c}$  is simply laced and connected. It follows that  $W$  acts transitively on the set  $\Phi_{\text{re}}$  of real roots, proving (i).

Let  $\alpha_1$  be a simple root that maximally far from the branch point of  $\Gamma$ , and let  $W_{\alpha_1}$  be its stabilizer in  $W$ . By the previous paragraph, any ordered pair of orthogonal real roots,  $(\alpha, \beta)$ , is  $W$ -conjugate to one of the form  $(\alpha_1, \gamma)$ . By Theorem 2.5,  $\gamma$  is a real root for the a simply laced Weyl group  $W_{\alpha_1}$ . It follows that there exists  $w \in W_{\alpha_1}$  such that  $w(\gamma)$  is a simple root in the root system of  $W_{\alpha_1}$ . By Theorem 2.7, we have  $w((\alpha_1, \gamma)) = (\alpha_1, \beta')$ , where  $\alpha_1 \vee \beta' \in \mathcal{B}$  is a canonical basis element.

The explicit description of  $\mathcal{B}$  in Definition 1.7 shows that either (a)  $\beta'$  is a simple root of  $W$ , or (b)  $\beta' = \theta_1$ . In the second case, Lemma 3.2 implies that  $(\alpha_1, \beta')$  is  $W$ -conjugate to an ordered pair of simple roots of  $W$ . This completes the proof of (ii).

Part (iii) follows from (ii), because if  $\alpha_i$  and  $\alpha_j$  are orthogonal simple roots, then  $\alpha_i \vee \alpha_j$  is an element of  $\mathcal{B}$ .

To prove (iv), it suffices by (ii) to consider the case where  $\alpha$  and  $\beta$  are both simple roots. By repeatedly using the identity  $s_i s_j(\alpha_i) = \alpha_j$  when  $i$  and  $j$  are adjacent vertices of  $\Gamma$ , we may assume that there is a subgraph  $i-j-k$  of  $\Gamma$  in which  $\alpha = \alpha_i$  and  $\beta = \alpha_k$ . Direct calculation now shows that

$$s_j s_i s_k s_j((\alpha, \beta)) = (\beta, \alpha),$$

from which (iv) follows.

The first assertion of (v) follows from (iv), so it is enough to prove the second assertion for ordered pairs of roots. We claim that there is a bijection between the set of  $W_{\alpha_1}$ -orbits of real roots of  $W_{\alpha_1}$  and the set of  $W$ -orbits of ordered orthogonal pairs of real roots of  $W$ , given by

$$\phi([\gamma]) = [(\alpha_1, \gamma)],$$

where  $[\gamma]$  is the  $W_{\alpha_1}$ -orbit of  $\gamma$ , and  $[(\alpha_1, \gamma)]$  is the  $W$ -orbit of the pair  $(\alpha_1, \gamma)$ . The map  $\phi$  is well-defined and injective because  $W_{\alpha_1}$  is the stabilizer of  $\alpha_1$ , and  $\phi$  is surjective because  $W$  acts transitively on  $\Pi$ .

It follows from Lemma 3.1, applied to the simply laced Weyl group  $W_{\alpha_1}$ , that the orbits of real roots of  $W_{\alpha_1}$  are in bijection with the connected components of  $H_{a-2, b-2, c-2}$ . The number of these connected components is as claimed by Lemma 2.4.  $\square$

The following basic result from linear algebra turns out to be very helpful.

**Lemma 3.4.** *Let  $V$  be a finite dimensional vector space, and let  $\alpha_1$  and  $\alpha_2$  be two linearly independent vectors in  $V$ . If there exist  $\beta_1, \beta_2 \in V$  such that  $\alpha_1 \vee \alpha_2 = \beta_1 \vee \beta_2$ , then the vectors  $\beta_i$  agree with the vectors  $\alpha_i$  up to changing the order and multiplication by nonzero scalars.*

*Proof.* We extend  $\{\alpha_1, \alpha_2\}$  to a basis  $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\}$  of  $V$ . Let  $\{\alpha_i \vee \alpha_j : 1 \leq i < j \leq n\}$  be the associated standard basis of  $S^2(V)$ , and consider the expansion of  $\beta_1 \vee \beta_2 = \alpha_1 \vee \alpha_2$  in terms of this standard basis. Because the coefficient of  $\alpha_k \vee \alpha_k$  in  $\alpha_1 \vee \alpha_2$  is zero for all  $k$ , it follows that the supports of  $\beta_1$  and  $\beta_2$  with respect to  $\mathcal{A}$  are disjoint.

In turn, it follows that if  $\alpha_k \vee \alpha_l$  is in the support of  $\beta_1 \vee \beta_2$ , then either  $\alpha_k$  is in the support of  $\beta_1$  and  $\alpha_l$  is in the support of  $\beta_2$ , or vice versa, but not both. By considering the coefficient of  $\alpha_k \vee \alpha_l$  in  $\alpha_1 \vee \alpha_2$ , this can only be possible if either  $k = 1$  and  $l = 2$ , or  $l = 1$  and  $k = 2$ . This

implies that either  $\beta_1$  is a nonzero scalar multiple of  $\alpha_1$  and  $\beta_2$  is a nonzero scalar multiple of  $\alpha_2$ , or vice versa, which completes the proof.  $\square$

**Definition 3.5.** If  $v$  is an element of  $S^2(V)$  of the form  $\alpha \vee \beta$ , then we call  $\alpha$  and  $\beta$  the *components* of  $v$ . By Lemma 3.4, the components of  $v \in S^2(V)$  are well defined up to order and multiplication by nonzero scalars. We will therefore say “ $\alpha$  is a component of  $v$ ” to mean the same as “some scalar multiple of  $\alpha$  is a component of  $v$ ”. We call a 2-root *real* (respectively, *positive*) if its components can be taken to be real (respectively, positive).

**Proposition 3.6.** Let  $f$  be the function from the set of unordered pairs of orthogonal real roots of  $Y_{a,b,c}$  to  $\Phi_{\text{re}}^2$  defined by

$$f(\{\alpha, \beta\}) = \alpha \vee \beta.$$

- (i) The fibre of each 2-root consists of the two pairs  $\{\alpha, \beta\}$  and  $\{-\alpha, -\beta\}$ , and these two pairs are conjugate to each other under the action of the Weyl group.
- (ii) The function  $f$  induces a bijection between  $W$ -orbits of unordered pairs of orthogonal real roots, and  $W$ -orbits of real 2-roots.

*Proof.* The statement about fibres follows from Lemma 3.4 and the fact ([21, Proposition 5.1 (b)]) that the only scalar multiples of a real root  $\alpha$  are  $\pm\alpha$ . The two pairs listed are conjugate to each other by the Weyl group element  $s_\alpha s_\beta$ , which completes the proof of (i).

For part (ii), Proposition 3.3 (v) gives the equivalence between ordered and unordered pairs of real roots. Part (i) then implies that the function  $f$  gives a well-defined correspondence between ordered pairs of orthogonal real roots and real 2-roots.  $\square$

*Remark 3.7.* Proposition 3.6 allows us to identify the action of  $W$  on pairs of orthogonal real roots with the action of  $W$  on real 2-roots. We will use this implicitly from now on, for example in Proposition 3.9 below.

**Notation 3.8.** In order to give a precise description of the  $W$ -orbits of 2-roots, we recall the standard constructions of root systems of types  $A$  and  $D$  as described in [19, §2]. We endow  $\mathbb{R}^n$  with the usual positive definite inner product and with an orthonormal basis  $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ .

In type  $A_{n-1}$ , the positive roots are  $\{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq n\}$ , the simple roots are  $\{\varepsilon_i - \varepsilon_{i+1} : 1 \leq i < n\}$ , and the highest root is  $\varepsilon_1 - \varepsilon_n$ . The Weyl group is isomorphic to the symmetric group  $S_n$  and it acts on the basis elements  $\varepsilon_i$  by permutations. For  $1 \leq i < n$ , the simple reflection  $s_i$  corresponding to  $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$  acts as the transposition  $(i, i+1)$ .

In type  $D_n$ , the positive roots are  $\{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\}$ , the simple roots are

$$\{\alpha_i := \varepsilon_i - \varepsilon_{i+1} : 1 \leq i < n\} \cup \{\alpha_n := \varepsilon_{n-1} + \varepsilon_n\},$$

and the highest root is  $\varepsilon_1 + \varepsilon_2$ . The numbering scheme for the simple roots is shown in Figure 4. The Weyl group acts on the elements  $\pm\varepsilon_i$  by signed permutations. For  $1 \leq i < n$ , the simple reflection  $s_i$  corresponding to  $\varepsilon_i - \varepsilon_{i+1}$  acts as the transposition  $(i, i+1)$ . The simple reflection  $s_n$  corresponding to  $\varepsilon_{n-1} + \varepsilon_n$  acts as the signed permutation switching  $\varepsilon_{n-1}$  and  $-\varepsilon_n$ , and fixing  $\varepsilon_j$  for  $j \notin \{n-1, n\}$ .

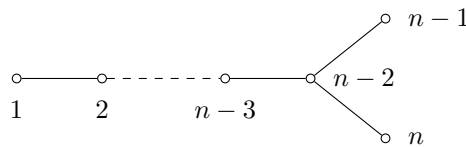


FIGURE 4. The Dynkin diagram of type  $D_n$  ( $n \geq 4$ )

**Proposition 3.9.** *Let  $W$  be a simply laced Weyl group of finite type.*

(i) *If  $W$  is of type  $A_n$  then there is a single orbit of positive 2-roots. The elements of  $\mathcal{B}$  in this orbit are*

$$\{\alpha_i \vee \alpha_j : 1 \leq i < j - 1 \leq n - 1\} \cup \{\alpha_i \vee \eta_{i-1, i+1} : 1 < i < n\}.$$

(ii) *If  $W$  is of type  $D_4$  (where  $\theta_1 = \theta_3 = \theta_4$  is the highest root), then there are three orbits of positive 2-roots. The orbits intersect  $\mathcal{B}$  in the sets*

$$\{\alpha_1 \vee \alpha_3, \alpha_2 \vee \eta_{1,3}, \alpha_4 \vee \theta_4\}, \quad \{\alpha_1 \vee \alpha_4, \alpha_2 \vee \eta_{1,4}, \alpha_3 \vee \theta_3\}, \quad \text{and} \quad \{\alpha_3 \vee \alpha_4, \alpha_2 \vee \eta_{3,4}, \alpha_1 \vee \theta_1\}.$$

(iii) *If  $W$  is of type  $D_n$  for  $n \geq 5$  then there are two orbits of positive 2-roots,  $X_1$  and  $X_2$ , where*

$$X_1 = \{(\varepsilon_i - \varepsilon_j) \vee (\varepsilon_i + \varepsilon_j) : 1 \leq i < j \leq n\}$$

*and  $X_2 = \Phi_+^2 \setminus X_1$  where  $\Phi_+^2$  is the set of positive 2-roots. The elements of  $\mathcal{B}$  in the orbit  $X_1$  are the  $n - 1$  elements*

$$\alpha_1 \vee \theta_1, \alpha_2 \vee \theta_2, \dots, \alpha_{n-3} \vee \theta_{n-3}, \alpha_{n-2} \vee \eta_{n-1, n}, \alpha_{n-1} \vee \alpha_n.$$

(iv) *If  $W$  is of type  $E_6, E_7$ , or  $E_8$ , then there is a single orbit of positive 2-roots.*

*Proof.* In type  $A$ , two positive roots  $\varepsilon_i - \varepsilon_j$  and  $\varepsilon_k - \varepsilon_l$  are orthogonal if and only if their supports,  $\{i, j\}$  and  $\{k, l\}$ , are disjoint. There is therefore a single  $A_n$ -orbit of orthogonal roots, which proves the first assertion for type  $A_n$ . The second assertion follows by restricting Definition 1.7 to a parabolic subgroup of type  $A$ .

Proposition 3.3 (v) implies that there are three orbits in part (ii). The other claims of (ii) follow from computations similar to those in Lemma 3.2.

In type  $D_n$  for  $n \geq 5$ , Proposition 3.3 (v) shows that there are two orbits of positive 2-roots. In this case, two positive roots are orthogonal if and only if their supports are either identical or disjoint. Because the Weyl group acts by signed permutations, these two types of orthogonal roots must form separate orbits. This proves the statement describing  $X_1$  and  $X_2$ .

To prove the last assertion of (iii), we need to identify all the root pairs in  $X_1$  that contain a simple root. Recall that  $\theta_1$  is the highest root in type  $D_n$  and that  $\theta_1 = \varepsilon_1 + \varepsilon_2$ . It follows that  $\{\alpha_1, \theta_1\} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2\}$ , which is an element of  $X_1$ . Similarly, we have  $\theta_k = \varepsilon_k + \varepsilon_{k+1}$  for all  $1 \leq k \leq n - 3$ . Direct calculation shows that  $\eta_{n-1, n} = \varepsilon_{n-2} + \varepsilon_{n-1}$ , and the proof follows.  $\square$

Part (iv) holds by Proposition 3.3 (v).  $\square$

**Definition 3.10.** If  $W$  has type  $D_n$  for  $n \geq 5$ , we will refer to the orbits  $X_1$  and  $X_2$  of Proposition 3.9 (iii) as the *small orbit* and the *large orbit* of  $W$ , respectively.

*Remark 3.11.* When  $W$  has type  $D_n$  for  $n \geq 5$ , the small orbit of 2-roots behaves like the root system of type  $A_{n-1}$ . More precisely, a 2-root of the form  $(\varepsilon_i - \varepsilon_j) \vee (\varepsilon_i + \varepsilon_j)$ , which can be simplified to  $(\varepsilon_i \vee \varepsilon_i) - (\varepsilon_j \vee \varepsilon_j)$ , can be identified with the root  $\varepsilon_i - \varepsilon_j$  of type  $A_{n-1}$ . With this identification, the action of  $W(D_n)$  by signed permutations is equivalent to the action of  $W(A_{n-1})$  by unsigned permutations. This means that the action factors through the surjective homomorphism of groups from  $W(D_n)$  to  $W(A_{n-1})$  that sends the generators  $s_{n-1}$  and  $s_n$  of  $W(D_n)$  to the same generator,  $s_{n-1}$ , of  $W(A_{n-1})$ .

In type  $D_4$ , the argument of the previous paragraph applies verbatim to the orbit

$$\{\alpha_3 \vee \alpha_4, \alpha_2 \vee \eta_{3,4}, \alpha_1 \vee \theta_1\} = \{(\varepsilon_3 - \varepsilon_4) \vee (\varepsilon_3 + \varepsilon_4), (\varepsilon_2 - \varepsilon_3) \vee (\varepsilon_2 + \varepsilon_3), (\varepsilon_1 - \varepsilon_2) \vee (\varepsilon_1 + \varepsilon_2)\},$$

and it applies to the other two orbits of 2-roots by applying graph automorphisms. The action of  $W$  on each of the three orbits of 2-roots factors through a surjective homomorphism from  $W(D_4)$  to  $W(A_3) \cong S_4$  that identifies two of the three branch nodes.

#### 4. REFLECTIONS ACTING ON 2-ROOTS

The goal of Section 4 is to prove Theorem 4.7, which gives a formula for the action of a simple reflection on a canonical basis element.

**Definition 4.1.** For each real root  $\alpha$ , we define the element  $C_\alpha$  of the group algebra  $FW$  to be  $s_\alpha - 1$ .

**Lemma 4.2.** Let  $\alpha$  be a real root of type  $Y_{a,b,c}$ , let  $s_\alpha$  be the associated reflection, and let  $V$  be the reflection representation.

(i) If  $v \in V$ , then we have  $C_\alpha(v) = -B(\alpha, v)\alpha$ , and  $C_\alpha$  acts on  $V \otimes V$  as

$$C_\alpha \otimes C_\alpha + C_\alpha \otimes 1 + 1 \otimes C_\alpha.$$

(ii) If  $v \in S^2(V)$ , then  $C_\alpha(v)$  is of the form  $\alpha \vee v'$  for some  $v' \in V$ .

(iii) If  $\beta$  is a real root orthogonal to  $\alpha$ , then we have  $C_\alpha C_\beta = C_\beta C_\alpha$ . The product  $C_\alpha C_\beta$  acts as zero on  $V$ , and acts on  $V \otimes V$  as

$$C_\alpha \otimes C_\beta + C_\beta \otimes C_\alpha.$$

*Proof.* The formula for  $C_\alpha(v)$  follows from the formula for the action of  $s_\alpha$  on  $V$ . Since  $s_\alpha$  acts diagonally as  $s_\alpha \otimes s_\alpha$  on  $V \otimes V$ , it follows that  $C_\alpha = s_\alpha - 1$  acts on  $V \otimes V$  as

$$s_\alpha \otimes s_\alpha - 1 \otimes 1 = C_\alpha \otimes C_\alpha + C_\alpha \otimes 1 + 1 \otimes C_\alpha,$$

which completes the proof of (i).

By part (i), it follows that if  $v_1 \in V \otimes V$ , then we have

$$C_\alpha(v_1) = \lambda_1 \alpha \otimes \alpha + \alpha \otimes v_2 + v_3 \otimes \alpha$$

for some  $v_2, v_3 \in V$ . In particular, if  $v_1 \in S^2(V)$  then we have  $v_2 = v_3$  and

$$C_\alpha(v_1) = \alpha \vee (\lambda_1 \alpha + v_2),$$

which proves part (ii).

To prove (iii), note that  $s_\alpha$  and  $s_\beta$  commute with each other because  $\alpha$  and  $\beta$  are orthogonal. It follows that  $C_\alpha = s_\alpha - 1$  and  $C_\beta = s_\beta - 1$  also commute with each other. If we take  $v \in S^2(V)$ , part (i) implies that  $C_\alpha C_\beta(v)$  is a scalar multiple of  $\alpha$ , and that  $C_\beta C_\alpha(v)$  is a scalar multiple of  $\beta$ . It follows that  $C_\alpha C_\beta(v) = 0$ . The formula for the action on  $V \otimes V$  follows from this by composing the actions of  $C_\alpha$  and  $C_\beta$  on  $V \otimes V$  as given in (i).  $\square$

**Lemma 4.3.** Let  $\alpha$  and  $\beta$  be real roots of type  $Y_{a,b,c}$  such that  $B(\alpha, \beta) = \pm 1$ , and define  $t = s_\alpha$  and  $u = s_\beta$ . Then we have  $tut = utu$ , and the element of  $FW$  given by

$$C = 1 - t - u + tu + ut - tut$$

acts as zero on the submodule  $M \leq S^2(V)$ .

*Proof.* Since  $\alpha$  and  $\beta$  are real roots, we have  $B(\alpha, \alpha) = 2 = B(\beta, \beta)$ , so the condition  $B(\alpha, \beta) = \pm 1$  implies that  $\alpha$  and  $\beta$  are linearly independent by linear algebra.

When computing products of  $t$  and  $u$ , we note that since  $s_\gamma = s_{-\gamma}$  for any real root  $\gamma$  of  $W$ , by replacing  $\beta$  with  $-\beta$  if necessary we may assume that  $B(\alpha, \beta) = -1$ . It then follows from the formula for a reflection that  $tut$  and  $utu$  both negate the vector  $\alpha + \beta$  and fix its  $B$ -orthogonal complement, therefore  $tut = utu = s_{\alpha+\beta}$ .

Note that we have

$$C = (t - 1)(-1 + u - ut) = C_\alpha(-1 + u - ut).$$

It follows from Lemma 4.2 (i) that if  $v \in S^2(V)$  then  $C(v) = \alpha \vee v'$  for some  $v' \in V$ , so that  $\alpha$  is a component of  $C(v)$ .

Similarly, we have

$$C = (u - 1)(-1 + t - tu) = C_\beta(-1 + t - tu),$$

which implies that  $\beta$  is a component of  $C(v)$ .

Lemma 3.4 now implies that  $C(v)$  is a scalar multiple of  $\alpha \vee \beta$ . However, the hypothesis that  $B(\alpha, \beta) \neq 0$  implies that  $\alpha \vee \beta$  does not lie in  $M$ . We conclude that  $C(v) = 0$ .  $\square$

*Remark 4.4.* In the setting of Lemma 4.3, the element  $C$  in fact annihilates all of the symmetric square  $S^2(V)$  rather than just the module  $M$ . To see this, it suffices to show that  $C \cdot (\alpha \vee \beta) = 0$  because  $M$  has codimension 1 in  $S^2(V)$  and  $\alpha \vee \beta \in S^2(V) \setminus M$ . The fact that  $C \cdot (\alpha \vee \beta) = 0$  can be checked by a straightforward computation: assuming that  $B(\alpha, \beta) = -1$  without loss of generality as in the proof of Lemma 4.3, we have

$$\begin{aligned} C \cdot (\alpha \vee \beta) &= \alpha \vee \beta - t \cdot (\alpha \vee \beta) - u \cdot (\alpha \vee \beta) + tu \cdot (\alpha \vee \beta) + ut \cdot (\alpha \vee \beta) - tut \cdot (\alpha \vee \beta) \\ &= \alpha \vee \beta + \alpha \vee (\alpha + \beta) + (\alpha + \beta) \vee \beta - \beta \vee (\alpha + \beta) - (\alpha + \beta) \vee \alpha - \beta \vee \alpha \\ &= 0. \end{aligned}$$

On the other hand, we note that  $C$  does not annihilate all of the tensor square  $V \otimes V$ : a similar computation to the one shown above proves that if  $\alpha, \beta$  and another root  $\gamma$  are the simple roots of type  $A_3$  subsystem of  $Y_{a,b,c}$  with  $B(\alpha, \gamma) = 0$  and  $B(\beta, \gamma) = -1$ , then  $C \cdot (\beta \otimes \gamma) = \alpha \otimes \beta - \beta \otimes \alpha \neq 0$ .

The next result describes a situation where applying a reflection to a 2-root is analogous to applying a reflection to a root. In each case, one obtains a sum of two 2-roots: the original one, and a different 2-root in the same  $W$ -orbit.

**Proposition 4.5.** *Let  $\alpha \vee \beta$  be an arbitrary real 2-root of type  $Y_{a,b,c}$ , and let  $\gamma$  be a simple root.*

(i) *We have  $s_\gamma(\alpha \vee \beta) = (\alpha \vee \beta) + (\gamma \vee v)$ , where  $v$  is given by*

$$v = B(\alpha, \gamma)B(\beta, \gamma)\gamma - B(\alpha, \gamma)\beta - B(\beta, \gamma)\alpha.$$

(ii) *Let  $\alpha \vee \beta$  be a real 2-root of type  $Y_{a,b,c}$ , and let  $\gamma$  be a real root for which  $B(\alpha, \gamma) = \pm 1$ . Then we have*

$$s_\gamma(\alpha \vee \beta) = (\alpha \vee \beta) + s_\alpha s_\gamma(\alpha \vee \beta) = (\alpha \vee \beta) \mp (\gamma \vee s_\alpha s_\gamma(\beta)).$$

(iii) *The following are equivalent:*

- (1) *the element  $v \in V$  from (i) is a real root;*
- (2) *either  $B(\alpha, \gamma) = \pm 1$ , or  $B(\beta, \gamma) = \pm 1$ , or both.*

*Proof.* To prove (i), we use the formula for a reflection acting on  $V$ :

$$\begin{aligned} s_\gamma(\alpha \vee \beta) &= (s_\gamma(\alpha) \vee s_\gamma(\beta)) \\ &= (\alpha - B(\gamma, \alpha)\gamma) \vee (\beta - B(\gamma, \beta)\gamma) \\ &= (\alpha \vee \beta) - (B(\gamma, \alpha)\gamma \vee \beta) - (\alpha \vee B(\gamma, \beta)\gamma) + (B(\gamma, \alpha)\gamma \vee B(\gamma, \beta)\gamma) \\ &= (\alpha \vee \beta) + (\gamma \vee (-B(\gamma, \alpha)\beta - B(\gamma, \beta)\alpha + B(\gamma, \alpha)B(\gamma, \beta)\gamma)), \end{aligned}$$

and the stated formula follows.

To prove (ii), let  $t$  and  $u$  be the reflections associated to  $\alpha$  and  $\gamma$ , respectively. Since  $\alpha$  and  $\beta$  are orthogonal, it follows that  $t$  fixes  $\beta$ , so that we have

$$(t - 1)(\alpha \vee \beta) = (t(\alpha) \vee t(\beta)) - (\alpha \vee \beta) = -2(\alpha \vee \beta).$$

Let  $C$  be the element defined from  $t, u$  in Lemma 4.3, and note that we have

$$C = (-1 + u - tu)(t - 1).$$

By Lemma 4.3, we have  $C(\alpha \vee \beta) = 0$ . Since  $(t-1)(\alpha \vee \beta)$  is a nonzero multiple of  $\alpha \vee \beta$ , it follows that we have

$$(-1 + u - tu)(\alpha \vee \beta) = 0,$$

which implies the first equation in the statement of (ii). The second equation follows because  $s_\alpha s_\gamma(\alpha)$  is equal to  $\gamma$  if  $B(\alpha, \gamma) = -1$  and to  $-\gamma$  if  $B(\alpha, \gamma) = +1$ . This completes the proof of (ii).

To show (1) implies (2) in part (iii), assume that the element  $v$  is a real root. It follows that we have  $B(v, v) = 2$ . For brevity, let us define  $x = B(\alpha, \gamma)$  and  $y = B(\beta, \gamma)$ , so that  $v = xy\gamma - x\beta - y\alpha$ . We then have

$$\begin{aligned} B(v, v) &= B(xy\gamma - x\beta - y\alpha, xy\gamma - x\beta - y\alpha) \\ &= x^2 y^2 B(\gamma, \gamma) + x^2 B(\beta, \beta) + y^2 B(\alpha, \alpha) - 2x^2 y B(\gamma, \beta) - 2xy^2 B(\gamma, \alpha) + 2xy B(\beta, \alpha) \\ &= 2x^2 y^2 + 2y^2 + 2x^2 - 2x^2 y^2 - 2x^2 y^2 + 0 \\ &= 2x^2 + 2y^2 - 2x^2 y^2. \end{aligned}$$

Since we also know that  $B(v, v) = 2$ , we have  $2x^2 + 2y^2 - 2x^2 y^2 = 2$ . This is equivalent to the condition

$$(x^2 - 1)(y^2 - 1) = 0,$$

so that either  $x = \pm 1$  or  $y = \pm 1$ , as required.

Now assume that (2) holds. If  $B(\alpha, \gamma) = \pm 1$ , it follows from (ii) that  $v = \mp s_\alpha s_\gamma(\beta)$ , which is a real root. The case  $B(\beta, \gamma) = \pm 1$  follows by a symmetrical argument, proving (1).  $\square$

*Remark 4.6.* If  $\gamma$  is a simple root, then the root  $s_\alpha s_\gamma(\beta)$  in the statement of Proposition 4.5 (ii) agrees up to sign with the positive root  $\delta_r$  appearing in [7, Lemma 2.2].

The next result gives a short formula for the action of a simple reflection on a canonical basis element in terms of the element  $v$  of Proposition 4.5 (i).

**Theorem 4.7.** *Let  $\mathcal{B}$  be the canonical basis of 2-roots of type  $Y_{a,b,c}$ , let  $\alpha \vee \beta \in \mathcal{B}$ , and let  $\gamma$  be a simple root of  $W$ . Then we have*

$$s_\gamma(\alpha \vee \beta) = \begin{cases} \alpha \vee \beta & \text{if } B(\alpha, \gamma) = B(\beta, \gamma) = 0; \\ -\alpha \vee \beta & \text{if } \gamma \in \{\alpha, \beta\}; \\ (\alpha \vee \beta) + (\gamma \vee v) & \text{otherwise, for some } (\gamma \vee v) \in \mathcal{B}. \end{cases}$$

Furthermore, the basis element  $\gamma \vee v$  appearing above satisfies  $w(\alpha \vee \beta) = \gamma \vee v$  for some  $w \in \langle s_\alpha, s_\beta, s_\gamma \rangle$ .

*Proof.* By Theorem 2.7 (ii), we may assume without loss of generality that  $\alpha$  is a simple root. If  $\gamma \in \{\alpha, \beta\}$  or  $B(\alpha, \gamma) = B(\beta, \gamma) = 0$ , then  $s_\gamma(\alpha \vee \beta)$  equals  $-\alpha \vee \beta$  or  $\alpha \vee \beta$  by direct computation. Otherwise, we must either have  $B(\alpha, \gamma) = -1$  or simultaneously have  $B(\alpha, \gamma) = 0$  and  $B(\beta, \gamma) = -1$  by Lemma 2.9. It remains to show that in both these cases, we have  $s_\gamma(\alpha \vee \beta) = (\alpha \vee \beta) + (\gamma \vee v)$  for a canonical basis element  $\gamma \vee v$  with the claimed properties.

If we have  $B(\alpha, \gamma) = -1$ , then  $\alpha$  and  $\gamma$  correspond to adjacent vertices of  $\Gamma$ , and Proposition 4.5 (ii) implies that

$$s_\gamma(\alpha \vee \beta) = (\alpha \vee \beta) + s_\alpha s_\gamma(\alpha \vee \beta).$$

Corollary 2.8 now implies that  $s_\alpha s_\gamma(\alpha \vee \beta)$  is a canonical basis element of the form  $\gamma \vee v$ .

The other possibility is that  $B(\alpha, \gamma) = 0$  and  $B(\beta, \gamma) = -1$ . In this case, Proposition 4.5 (ii) implies that

$$s_\gamma(\alpha \vee \beta) = (\alpha \vee \beta) + s_\beta s_\gamma(\alpha \vee \beta).$$

Since both  $\beta$  and  $\gamma$  are orthogonal to  $\alpha$ , we have  $s_\beta s_\gamma(\alpha) = \alpha$ . The hypothesis  $B(\beta, \gamma) = -1$  implies that  $s_\beta s_\gamma(\beta) = \gamma$ . We conclude that

$$s_\gamma(\alpha \vee \beta) = (\alpha \vee \beta) + (\alpha \vee \gamma),$$

which completes the proof because we have  $(\alpha \vee \gamma) \in \mathcal{B}$ .  $\square$

We define the *2-root lattice* to be the  $\mathbb{Z}$ -span of the canonical basis  $\mathcal{B}$ .

**Corollary 4.8.** *Let  $W$  be the Weyl group of type  $Y_{a,b,c}$  and let  $\mathcal{B}$  be the canonical basis of 2-roots of  $M$ .*

- (i) *The action of  $W$  on the module  $M$  leaves invariant the 2-root lattice  $\mathbb{Z}\mathcal{B}$ .*
- (ii) *Let  $W_I$  be a parabolic subgroup of  $W$ , let  $\Phi_I$  be the root system of  $W_I$ , and define*

$$\mathcal{B}_I = \{\alpha \vee \beta \in \mathcal{B} : \alpha, \beta \in \Phi_I\}.$$

*Then the action of  $W_I$  leaves invariant the lattice  $\mathbb{Z}\mathcal{B}_I$ .*

*Proof.* The formula in Theorem 4.7 shows that a generator of  $W$  sends a canonical basis element to an integral linear combination of canonical basis elements, which proves (i).

To prove (ii), it is enough to show that if  $s_i$  is a generator of  $W_I$  and we have  $\alpha \vee \beta \in \mathcal{B}_I$ , then  $s_i(\alpha \vee \beta) \in \mathbb{Z}\mathcal{B}_I$ . This is immediate from Theorem 4.7, because the 2-roots  $\alpha \vee \beta$  and  $\gamma \vee v$  in that result are conjugate in  $W_I$ .  $\square$

*Remark 4.9.* Note that if the parabolic subgroup  $W_I$  is also of type  $Y_{a',b',c'}$  (for some values of  $a'$ ,  $b'$ , and  $c'$ ) then the set  $\mathcal{B}_I$  coincides with the canonical basis  $\mathcal{B}(a', b', c')$ .

## 5. SIGN-COHERENCE

Following the theory of cluster algebras ([6, Definition 2.2 (i)], [11, Definition 6.12]), we say that a matrix  $A$  is *column sign-coherent* (or “sign-coherent” for short) if any two nonzero entries in the same column of  $A$  have the same sign. We extend this terminology to say that a basis of a finite dimensional group representation  $V$  is a *sign-coherent basis* of  $V$  if every element of the group acts on  $V$  by a sign-coherent matrix with respect to the basis, and we say  $V$  is a *sign-coherent representation* if it admits a sign-coherent basis.

Sign-coherent representations exist in abundance. Some (trivial) examples of this phenomenon are representations arising from permutations or signed permutations. An interesting and well-known example of a sign-coherent basis is the basis of simple roots for the reflection representation of a Weyl group. It also follows quickly from the definitions that a direct sum or tensor product of sign-coherent representations is sign-coherent, as is the symmetric square of a sign-coherent representation. In particular, the standard basis of  $S^2(V)$  is a sign-coherent basis.

It is more difficult to find sign-coherent bases for irreducible modules, such as the direct summands of the module  $M$  in Theorem 7.8 below. In this section, we will establish the following sign-coherence property of the canonical basis  $\mathcal{B}$ :

**Theorem 5.1.** *Let  $W$  be a Weyl group of type  $Y_{a,b,c}$ . The canonical basis  $\mathcal{B}$  is a sign-coherent basis for the module  $M$ . With respect to this basis, every element  $w \in W$  is represented by a sign-coherent matrix of integers.*

Because each real 2-root is  $W$ -conjugate to an element of  $\mathcal{B}$  (see Proposition 3.3 (iii)), the 2-roots of  $W$  are precisely the set of possible columns of matrices representing the action of elements  $w \in W$  with respect to the basis  $\mathcal{B}$ . We can therefore restate Theorem 5.1 as follows:

**Theorem 5.2.** *Let  $W$  be a Weyl group of type  $Y_{a,b,c}$ . Then any real 2-root  $\alpha \vee \beta$  of  $W$  is an integral linear combination of elements of  $\mathcal{B}$  with coefficients of like sign.*



Here, the fact that any real 2-root of  $W$  is an integral linear combination of elements of  $\mathcal{B}(a, b, c)$  with coefficients of like sign is similar to the fact that any root of  $W$  is an integral linear combination of simple roots of  $W$  with coefficients of like sign. Note also that Theorem 5.2 implies that one can characterize the basis  $\mathcal{B}$  as the set of positive 2-roots that cannot be expressed as a positive linear combination of other positive 2-roots.

*Remark 5.3.* In an earlier version of this paper, we conjectured that theorems 5.1 and 5.2 hold for all Coxeter groups of type  $Y_{a,b,c}$  but proved the theorems only in the finite and affine cases (our conjecture beyond these types was based on extensive computer calculations). The proof for the general case that we will give below is based on a proof that was communicated to us by Robert B. Howlett.

To prove Theorem 5.2, we note that Proposition 3.3 (iii) and Corollary 4.8 (i) imply that any real 2-root  $\alpha \vee \beta$  is a linear combination of elements of  $\mathcal{B}$  with integer coefficients, so it remains to prove that these integers are positive. We do so below. In the proof, we will freely use the result [19, Proposition 5.7] that if  $w \in W$  and  $\alpha$  is a positive real root, then either  $w(\alpha) > 0$  and  $\ell(ws_\alpha) > \ell(w)$ , or  $w(\alpha) < 0$  and  $\ell(ws_\alpha) < \ell(w)$ . We will also make use of the following remark in the proof.

*Remark 5.4.* Because the components of any element of  $\mathcal{B}$  can be taken to be positive roots, it follows that a 2-root is positive (respectively, negative) if and only if it is a positive (respectively, negative) linear combination of the standard basis of  $S^2(V)$ . In turn, this implies that if  $w(\alpha \vee \beta)$  is a sign-coherent linear combination of elements of  $\mathcal{B}$ , then  $w(\alpha \vee \beta)$  is a positive linear combination if  $w(\alpha)$  and  $w(\beta)$  are both positive or both negative roots, and  $w(\alpha \vee \beta)$  is a negative linear combination if one of  $w(\alpha)$  and  $w(\beta)$  is a positive root and the other is a negative root.

*Proof of Theorem 5.2.* By the discussions following Theorem 5.1, to prove Theorem 5.2 it suffices to show that for any  $w \in W$  and  $\alpha \vee \beta \in \mathcal{B}$ , the 2-root  $w(\alpha \vee \beta)$  is a linear combination of  $\mathcal{B}$  with coefficients of like sign. We prove this fact by induction on the length,  $\ell(w)$ , of  $w \in W$ . The case  $\ell(w) = 0$  is trivial, and the case  $\ell(w) = 1$  follows from Theorem 4.7. Suppose then that we have  $\ell(w) > 1$ .

If we have  $w(\alpha) < 0$ , then we have  $\ell(ws_\alpha) < \ell(w)$  and  $w(\alpha \vee \beta) = -ws_\alpha(\alpha \vee \beta)$ , and the proof is completed by applying the inductive hypothesis to  $ws_\alpha$ . A similar argument applies if  $w(\beta) < 0$ , so we may assume from now on that both  $w(\alpha) > 0$  and  $w(\beta) > 0$ .

Fix a simple root  $\gamma$  with the property that  $\ell(ws_\gamma) < \ell(w)$ , which implies that  $ws_\gamma(\gamma) > 0$ . If we have  $s_\gamma(\alpha \vee \beta) = \pm(\alpha \vee \beta)$ , then the proof follows by applying the inductive hypothesis to  $ws_\gamma$  as in the previous paragraph. We may therefore assume that we are in the third case of the statement of Theorem 4.7, so that  $\gamma \notin \{\alpha, \beta\}$ , and  $\gamma$  is not orthogonal to both  $\alpha$  and  $\beta$ . Since  $\gamma$  and  $\alpha$  are distinct simple roots, we must have  $B(\alpha, \gamma) \in \{0, -1\}$ .

Suppose that  $ws_\gamma(\alpha) < 0$ , which implies that  $\ell(ws_\gamma s_\alpha) < \ell(ws_\gamma)$ . The assumption that  $w(\alpha) > 0$  implies that  $B(\alpha, \gamma) \neq 0$ , and it follows from the previous paragraph that  $B(\alpha, \gamma) = -1$ . Corollary 2.8 implies that  $s_\alpha s_\gamma(\alpha \vee \beta) = (\gamma \vee s_\alpha s_\gamma(\beta))$  is an element of  $\mathcal{B}$ . We then have

$$w(\alpha \vee \beta) = ws_\gamma s_\alpha(\gamma \vee s_\alpha s_\gamma(\beta)),$$

and the proof follows by applying the inductive hypothesis to  $ws_\gamma s_\alpha$ .

Suppose that  $ws_\gamma(\beta) < 0$ , which implies that  $\ell(ws_\gamma s_\beta) < \ell(ws_\gamma)$ . Let  $c := B(\beta, \gamma)$ . The assumption that  $w(\beta) > 0$  implies that  $B(\beta, \gamma) \neq 0$ , and Lemma 2.9 then implies that  $c = \pm 1$ ; in particular, we have  $c^2 = 1$ . It follows that

$$s_\beta s_\gamma(\beta) = s_\beta(\beta - c\gamma) = -\beta - c(\gamma - c\beta) = -c\gamma.$$

We claim that one of  $\pm s_\beta s_\gamma(\alpha \vee \beta)$  is an element of  $\mathcal{B}$ . If  $B(\alpha, \gamma) = 0$ , then we have  $s_\beta s_\gamma(\alpha) = \alpha$  and  $s_\beta s_\gamma(\alpha \vee \beta) = \alpha \vee (-c\gamma) = -c(\alpha \vee \gamma)$ , which proves the claim in this case because  $\alpha$  and  $\gamma$  are

orthogonal simple roots and  $c = \pm 1$ . The other possibility is that  $B(\alpha, \gamma) = -1$ , in which case we have

$$s_\alpha s_\gamma(\alpha) = \gamma$$

and

$$s_\beta s_\gamma(\alpha) = s_\beta(\alpha + \gamma) = \alpha + \gamma - c\beta = s_\alpha(\gamma - c\beta) = -cs_\alpha s_\gamma(\beta).$$

Combining the three equations displayed above, we find that

$$s_\beta s_\gamma(\alpha \vee \beta) = (-cs_\alpha s_\gamma(\beta)) \vee (-c\gamma) = s_\alpha s_\gamma(\beta) \vee s_\alpha s_\gamma(\alpha) = s_\alpha s_\gamma(\beta \vee \alpha)$$

which completes the proof of the claim by Corollary 2.8. The claim then implies that

$$w(\alpha \vee \beta) = ws_\gamma s_\beta(s_\beta s_\gamma(\alpha \vee \beta)),$$

and the proof follows by applying the inductive hypothesis to  $ws_\gamma s_\beta$ .

By the previous three paragraphs, we may assume from now on that  $ws_\gamma(\alpha)$ ,  $ws_\gamma(\beta)$ , and  $ws_\gamma(\gamma)$  are all positive roots.

If we have  $B(\alpha, \gamma) = 0$  then, since we are assuming that we are not in the first two cases of the statement of Theorem 4.7, we have  $B(\beta, \gamma) = -1$  by Lemma 2.9. This implies that  $s_\gamma(\alpha \vee \beta) = (\alpha \vee \beta) + (\alpha \vee \gamma)$ , and we therefore have

$$w(\alpha \vee \beta) = ws_\gamma(s_\gamma(\alpha \vee \beta)) = ws_\gamma(\alpha \vee \beta) + ws_\gamma(\alpha \vee \gamma).$$

It follows from the previous paragraph that  $ws_\gamma(\alpha \vee \beta)$  and  $ws_\gamma(\alpha \vee \gamma)$  are positive 2-roots. Remark 5.4 and the inductive hypothesis applied to  $ws_\gamma$  then imply that each of  $ws_\gamma(\alpha \vee \beta)$  and  $ws_\gamma(\alpha \vee \gamma)$  is a nonnegative integral linear combination of elements of  $\mathcal{B}$ . It follows that  $w(\alpha \vee \beta)$  is also a nonnegative integral linear combination of elements of  $\mathcal{B}$ , which completes the proof in this case.

We may suppose from now on that  $B(\alpha, \gamma) = -1$ . Let  $c = B(\beta, \gamma)$ , so that  $c \in \{-1, 0, 1\}$  by Lemma 2.9. Define  $\beta' = s_\alpha s_\gamma(\beta) = \beta - c(\alpha + \gamma)$ , and note that  $s_\alpha(\beta') = \beta - c\gamma$  is also a root. Suppose that  $ws_\gamma(\beta') < 0$ , which implies that  $\ell(ws_\gamma s_{\beta'}) < \ell(ws_\gamma)$ . The assumption that  $ws_\gamma(\beta) > 0$  implies that  $\beta \neq \beta'$ , which rules out the case  $c = 0$ . We now have  $c^2 = 1$ ,  $s_{\beta'} s_\gamma(\alpha) = \pm\beta$  and  $s_{\beta'} s_\gamma(\beta) = \pm\alpha$ . This implies that

$$w(\alpha \vee \beta) = ws_\gamma s_{\beta'}(s_{\beta'} s_\gamma(\alpha \vee \beta)) = \pm ws_\gamma s_{\beta'}(\alpha \vee \beta),$$

and the proof follows by applying the inductive hypothesis to  $ws_\gamma s_{\beta'}$ .

We have reduced to the case where  $B(\alpha, \gamma) = -1$  and  $ws_\gamma(\beta') > 0$ . We have  $s_\alpha s_\gamma(\alpha) = \gamma$ , and  $s_\alpha s_\gamma(\beta) = \beta'$ . The case of  $B(\alpha, \gamma) = -1$  in the proof of Theorem 4.7 now implies that

$$w(\alpha \vee \beta) = ws_\gamma(s_\gamma(\alpha \vee \beta)) = ws_\gamma(\alpha \vee \beta) + ws_\gamma(s_\alpha s_\gamma(\alpha \vee \beta)) = ws_\gamma(\alpha \vee \beta) + ws_\gamma(\gamma \vee \beta'),$$

and Corollary 2.8 implies that  $(\gamma \vee \beta') = s_\alpha s_\gamma(\alpha \vee \beta)$  is an element of  $\mathcal{B}$ . We have shown that all of  $ws_\gamma(\alpha)$ ,  $ws_\gamma(\beta)$ ,  $ws_\gamma(\gamma)$ , and  $ws_\gamma(\beta')$  are positive roots, which implies that  $ws_\gamma(\alpha \vee \beta)$  and  $ws_\gamma(\gamma \vee \beta')$  are positive 2-roots. Remark 5.4 and the inductive hypothesis applied to  $ws_\gamma$  then imply that each of  $ws_\gamma(\alpha \vee \beta)$  and  $ws_\gamma(\gamma \vee \beta')$  is a nonnegative integral linear combination of elements of  $\mathcal{B}$ . It follows that  $w(\alpha \vee \beta)$  is also a nonnegative integral linear combination of elements of  $\mathcal{B}$ , which completes the proof.  $\square$

In finite types types  $A$  and  $D$ , Theorem 5.2 can be interpreted diagrammatically using the conventions of Notation 3.8.

We can depict positive roots of types  $A$  and  $D$  as arcs connecting rows of dots labelled  $1, 2, \dots, n$ . A positive root of the form  $\varepsilon_i - \varepsilon_j$  (respectively,  $\varepsilon_i + \varepsilon_j$ ) is depicted as an undecorated (respectively, decorated) arc joining point  $i$  to point  $j$ . We can then depict positive 2-roots as pairs of (possibly decorated) arcs connecting points  $i$  and  $j$ . In type  $D$ , this may result in two arcs connecting the same two points, one of which is decorated and one of which is not.

In this context, the linear relations between 2-roots that one obtains from Theorem 5.2 can be interpreted as a type of skein relation with positive coefficients. For example, in type  $A$  the positive 2-root  $(\alpha_1 + \alpha_2) \vee (\alpha_2 + \alpha_3)$  decomposes into a positive linear combination of canonical basis elements by Theorem 5.2:

$$(\alpha_1 + \alpha_2) \vee (\alpha_2 + \alpha_3) = (\alpha_1 \vee \alpha_3) + (\alpha_2 \vee (\alpha_1 + \alpha_2 + \alpha_3)).$$

Writing this in terms of coordinates, we have

$$(5.1) \quad (\varepsilon_1 - \varepsilon_3) \vee (\varepsilon_2 - \varepsilon_4) = ((\varepsilon_1 - \varepsilon_2) \vee (\varepsilon_3 - \varepsilon_4)) + ((\varepsilon_2 - \varepsilon_3) \vee (\varepsilon_1 - \varepsilon_4)).$$

Pictorially, this shows how to express a diagram with a crossing as a positive linear combination of diagrams with fewer crossings. The canonical basis elements in this case correspond to the legal configurations of arcs in the top half of diagrams for the Temperley–Lieb algebra.

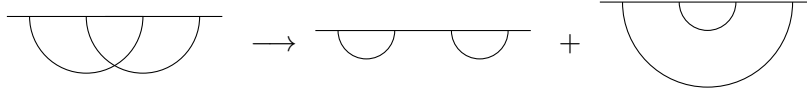


FIGURE 5. Equation 5.1 interpreted as a skein relation

Something similar happens in type  $D$ . The positive 2-root  $(\varepsilon_1 + \varepsilon_4) \vee (\varepsilon_2 + \varepsilon_3)$  in type  $D_4$  decomposes into the following positive linear combination of canonical basis elements:

$$(5.2) \quad (\varepsilon_1 + \varepsilon_4) \vee (\varepsilon_2 + \varepsilon_3) = (\alpha_1 \vee \alpha_3) + (\alpha_2 \vee \eta_{1,3}) + (\alpha_4 \vee \theta_4) \\ = ((\varepsilon_1 - \varepsilon_2) \vee (\varepsilon_3 - \varepsilon_4)) + ((\varepsilon_2 - \varepsilon_3) \vee (\varepsilon_1 - \varepsilon_4)) + ((\varepsilon_3 + \varepsilon_4) \vee (\varepsilon_1 + \varepsilon_2)).$$

Pictorially, this shows how to express a diagram with a non-exposed decorated arc (in this case, the one between 2 and 3) as a linear combination of diagrams that have fewer such features. After performing a left-right reflection, the canonical basis elements in this case correspond to the legal configurations of arcs in the top half of diagrams for the Temperley–Lieb algebra of type  $D$ , as described by the first author in [13, Theorem 4.2]. The positive 2-roots of the form  $(\varepsilon_i - \varepsilon_j) \vee (\varepsilon_i + \varepsilon_j)$  correspond to the “diagrams of type 1” of [13], and the other positive 2-roots correspond to the “diagrams of type 2”.

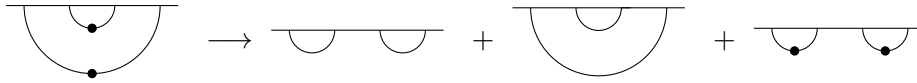


FIGURE 6. Equation 5.2 interpreted as a skein relation

From this point of view, the decoration rules for arcs in these algebras are canonically determined by the basis  $\mathcal{B}$ .

## 6. THE HIGHEST 2-ROOT

In Section 6, we assume that  $\Gamma$  is a Dynkin diagram of finite type  $A_n$ ,  $D_n$ ,  $E_6$ ,  $E_7$ , or  $E_8$  unless otherwise stated, and we continue to work over a subfield of  $\mathbb{R}$ .

Recall that the root lattice  $Q = \mathbb{Z}\Pi$  of arbitrary type is equipped with a standard partial order: if  $q_1, q_2 \in \mathbb{Z}\Pi$ , we say that  $q_1 < q_2$  if  $q_2 - q_1$  is a positive linear combination of simple roots  $\Pi$ . Also recall that the *height* of a positive root  $\alpha = \sum_{b \in \Pi} \lambda_b b$  is defined to be the number  $\text{ht}(\alpha) := \sum_{b \in \Pi} \lambda_b$ .

The basis  $\mathcal{B}$  allows us to define natural analogues of  $\leq$  and  $\text{ht}$  for 2-roots: for 2-roots  $q_1, q_2 \in \mathbb{Z}\mathcal{B}$ , we say that  $q_1 \leq_2 q_2$  if  $q_2 - q_1$  is a positive linear combination of elements of  $\mathcal{B}$ . If  $\alpha \vee \beta$  is a positive 2-root satisfying  $\alpha \vee \beta = \sum_{b \in \mathcal{B}} \lambda_b b$ , then we may define the *height* of  $\alpha \vee \beta$  to be the number  $\text{ht}_2(\alpha \vee \beta) := \sum_{b \in \mathcal{B}} \lambda_b$ . Note that in this context, Theorem 5.2 implies that every real 2-root is

comparable to the zero vector in the order  $\leq_2$ . The same theorem also implies that the height of a positive root must be a positive integer. Theorem 4.7 shows that if  $s_i$  is a generator for  $W$ ,  $b \in \mathcal{B}$  is a canonical basis element, and  $s_i(b)$  is not a scalar multiple of  $b$ , then  $b <_2 s_i(b)$  is a covering pair.

The main purpose of this section is to show that each  $W$ -orbit of 2-roots in  $\mathbb{Z}\mathcal{B}$  has a unique maximal element with respect to the order  $\leq_2$ . To this end, we first use  $\leq_2$  to induce a partial order on the set of pairs of orthogonal positive roots of  $W$ , also denoted by  $\leq_2$ , defined by

$$\{\alpha_1, \beta_1\} \leq_2 \{\alpha_2, \beta_2\} \text{ if } (\alpha_1 \vee \beta_1) \leq_2 (\alpha_2 \vee \beta_2).$$

The new order is well-defined since  $\alpha \vee \beta = \beta \vee \alpha$  for all roots  $\alpha, \beta$  of  $W$ . Similarly, we may define the *height* of each pair of positive orthogonal roots  $\{\alpha, \beta\}$  to be  $\text{ht}_2(\alpha \vee \beta)$ .

To establish the existence of maximal 2-roots in  $\mathbb{Z}\mathcal{B}$ , we shall compare the order  $\leq_2$  on root pairs with two other partial orders on sets of orthogonal roots introduced in [7]. Motivated by the Lawrence–Krammer representation of the Artin group, Cohen, Gijsbers, and Wales defined combinatorially in [7] a partial order  $\leq'$  on each  $W$ -orbit of  $k$ -tuples of mutually orthogonal positive roots, where  $W$  is a simply laced finite Weyl group and  $k \in \mathbb{Z}_{>0}$ . Here, the action of each element  $w \in W$  sends every  $k$ -tuple  $\rho = \{\alpha_1, \dots, \alpha_k\}$  to the set

$$w(\rho) = \Phi_+ \cap \{\pm\alpha_1, \dots, \pm\alpha_k\}$$

where  $\Phi_+$  is the set of positive roots of  $W$ . The definition of the order  $\leq'$  also requires the  $k$ -tuples to be “admissible”, a technical combinatorial property that is always satisfied if  $k = 2$  (see [7, Proposition 2.3]). As a consequence, the order  $\leq'$  restricts to pairs of orthogonal roots as follows.

**Definition 6.1.** Let  $W$  be a simply laced Weyl group of finite type. Let  $\rho_1$  and  $\rho_2$  be two (unordered) pairs of orthogonal positive roots of  $W$  such that  $w(\rho_1) = \rho_2$  for some  $w \in W$ . We say that  $\rho_1 <' \rho_2$  if there exist  $\gamma_1 \in \rho_1 \setminus \rho_2$  and  $\gamma_2 \in \rho_2 \setminus \rho_1$ , of minimal height in  $\rho_1 \setminus \rho_2$  and  $\rho_2 \setminus \rho_1$  respectively, such that  $\text{ht}(\gamma_1) < \text{ht}(\gamma_2)$ .

**Proposition 6.2** (Cohen, Gijsbers, Wales). *Let  $W$  be a simply laced Weyl group of finite type, and let  $X$  be a  $W$ -orbit of pairs of orthogonal positive roots. The relation  $\leq'$  of Definition 6.1 is a partial order on  $X$ .*

*Proof.* This follows from the proof of [7, Proposition 3.1]. (We note that the proof from [7] asserts that “it is readily verified” that  $\leq'$  is an ordering. However, to our knowledge it is not completely trivial to prove the fact that  $\leq'$  is transitive.)  $\square$

The second partial order we need from [7] is defined using  $\leq'$  as follows.

**Definition 6.3.** Let  $W$  be a simply laced Weyl group of finite type, and let  $X$  be a  $W$ -orbit of pairs of orthogonal positive roots. Let  $\leq_m$  be the partial order on  $X$  whose covering relations are those of the form  $\rho <_m s_i(\rho)$  where  $s_i$  is a Coxeter generator such that  $s_i(\rho) \in X \setminus \{\rho\}$  and  $\rho <' s_i(\rho)$ . We call  $\leq_m$  the *monoidal order* on  $X$ .

Note that because  $\leq'$  is a partial order, it follows that  $\leq_m$  is antisymmetric, and thus that the reflexive, transitive extension of the relation in Definition 6.3 is a partial order. It is immediate from the definitions that  $\leq'$  is a refinement of  $\leq_m$ .

The next result also applies in type  $A$ , by using the identifications of Corollary 4.8 (ii).

**Proposition 6.4.** *Let  $W$  be a simply laced Weyl group of finite type, and let  $X$  be a  $W$ -orbit of pairs of orthogonal positive roots.*

- (i) *If  $\rho_1 = \{\alpha, \beta\} \in X$  and  $\rho_1 <_m \rho_2$  is a covering pair in  $X$ , then we have  $\rho_2 = \{s_i(\alpha), s_i(\beta)\}$  for some simple reflection  $s_i$ . Furthermore, if  $x = B(\alpha_i, \alpha)$  and  $y = B(\alpha_i, \beta)$ , then we have  $x, y \in \{-1, 0, 1\}$ , and we do not have  $x = y = 0$ .*

- (ii) The partial order  $\leq_2$  refines the monoidal order  $\leq_m$ ; in other words, if  $\rho_1, \rho_2 \in X$  satisfy  $\rho_1 \leq_m \rho_2$ , then we have  $\rho_1 \leq_2 \rho_2$ .

*Proof.* To prove (i), we note that  $\rho_1 = \{\alpha, \beta\} \subseteq \Phi_+$ . Then by Definition 6.3 we must have

$$\rho_1 <' \rho_2 = s_i(\rho_1) = \Phi_+ \cap \{\pm s_i(\alpha), \pm s_i(\beta)\}$$

for some Coxeter generator  $s_i$ . Let  $\alpha_i \in \Pi$  be the simple root corresponding to  $s_i$ , and let  $x = B(\alpha_i, \alpha)$ , and let  $y = B(\alpha_i, \beta)$  as in the statement. Note that if  $\alpha_i = \alpha$  then we have  $s_i(\alpha) = -\alpha$ ,  $s_i(\beta) = \beta$  and  $\rho_2 = \{\alpha, \beta\}$ . This contradicts the fact that  $\rho_1 \neq \rho_2$ , which proves that  $\alpha_i$  and  $\alpha$  are distinct real roots. Since  $s_i$  permutes the set  $\Phi_+ \setminus \{\alpha_i\}$ , it follows that  $s_i(\alpha) \in \Phi_+$ . A similar argument shows that  $\beta \neq \alpha_i$  and  $s_i(\beta) \in \Phi_+$ , so it follows that  $\rho_2 = \{s_2(\alpha), s_i(\beta)\}$ .

To prove the claims about  $x$  and  $y$ , note first  $\alpha \neq -\alpha_i$  since both  $\alpha$  and  $\alpha_i$  are positive roots. Also note that since  $W$  is simply laced, the roots  $\alpha_i$  and  $\alpha$  have the same norm in the sense that we must have  $B(\alpha_i, \alpha_i) = B(\alpha, \alpha) = 2$ . Since  $\alpha_i$  and  $\alpha$  are distinct, are not opposite, and have the same norm, it then follows from [18, §9.4] that  $x = B(\alpha_i, \alpha) \in \{-1, 0, 1\}$ . Similarly, we have  $y \in \{-1, 0, 1\}$ . Moreover, since  $\rho_2 \neq \rho_1$ , the root  $\alpha_i$  cannot be orthogonal to both  $\alpha$  and  $\beta$ , so we cannot have  $x = y = 0$ . This completes the proof of (i).

It is enough to prove (ii) in the case where  $\rho_1 <_m \rho_2$  forms a covering pair. By the previous paragraph, we may assume without loss of generality that  $x \in \{-1, 1\}$ , and that the 2-root corresponding to  $\rho_2$  is

$$s_i(\alpha) \vee s_i(\beta) = s_i(\alpha \vee \beta) = (\alpha \vee \beta) + s_\alpha s_i(\alpha \vee \beta) = \alpha \vee \beta + s_\alpha s_i(\alpha) \vee s_\alpha s_i(\beta),$$

where the second equality holds by Proposition 4.5 (ii) since  $x \in \{-1, 1\}$ . Every positive 2-root is a linear combination of elements of  $\mathcal{B}$  with positive integral coefficients by Theorem 5.2, so to prove  $\rho_1 \leq_2 \rho_2$  it now suffices to show that  $s_\alpha s_i(\alpha) \vee s_\alpha s_i(\beta)$  equals a positive root. We do so by showing that  $s_\alpha s_i(\alpha)$  and  $s_\alpha s_i(\beta)$  are either both positive or both negative roots, depending on the values of  $x$  and  $y$ .

Suppose first that  $y = 0$ . In this case, both  $\alpha_i$  and  $\alpha$  are orthogonal to  $\beta$ , which implies that  $s_\alpha s_i(\beta) = \beta$  is positive. By assumption, we have

$$\{\alpha, \beta\} <_m \{s_i(\alpha), s_i(\beta)\} = \{\alpha - x\alpha_i, \beta\},$$

which implies (using Definition 6.1) that we have  $x = -1$ . In turn, this implies that  $s_\alpha s_i(\alpha) = \alpha_i$  and  $s_\alpha s_i(\beta) = \beta$  are both positive roots, which completes the proof in this case.

Next, suppose that  $y = x = \pm 1$ . In this case, the condition that  $\{\alpha, \beta\} <_m \{s_i(\alpha), s_i(\beta)\}$  implies that  $y = x = -1$ . This implies that  $s_\alpha s_i(\alpha) = \alpha_i$  and  $s_\alpha s_i(\beta) = \alpha + \beta + \alpha_i$  are both positive, as required.

Finally, suppose that  $y = -x = \pm 1$ . We cannot have  $\text{ht}(\alpha) = \text{ht}(\beta)$ , because one of  $s_i(\alpha)$  and  $s_i(\beta)$  would have a lower height than both of  $\alpha$  and  $\beta$ , which is incompatible with the condition that  $\{\alpha, \beta\} <_m \{s_i(\alpha), s_i(\beta)\}$ . We may therefore assume without loss of generality that  $\text{ht}(\alpha) < \text{ht}(\beta)$ . The condition  $\{\alpha, \beta\} <_m \{s_i(\alpha), s_i(\beta)\}$  then implies that  $x = -1$  and  $y = 1$ . This implies that  $s_\alpha s_i(\alpha) = \alpha_i$ , a positive root, and  $s_\alpha s_i(\beta) = \beta - \alpha - \alpha_i$ . Because  $\beta - \alpha - \alpha_i$  is a root, it cannot have height zero, so we must have  $\text{ht}(\beta) \geq \text{ht}(\alpha) + 2$  and thus that  $\text{ht}(\beta - \alpha - \alpha_i) > 0$ . It follows that  $\beta - \alpha - \alpha_i$  is a positive root, as required, which completes the proof of (ii).  $\square$

The next example shows that the partial order  $\leq_2$  strictly refines the order  $\leq_m$ .

**Example 6.5.** Let  $\Gamma$  be a Dynkin diagram of type  $D_5$ , with vertices numbered 1–2–3–4 and 3–5, so that 3 is the branch point. The pair of positive orthogonal roots  $\{\alpha_3, \theta_1\}$  is not minimal in  $\leq_2$ , because we have

$$\alpha_3 \vee \theta_1 = (\alpha_3 \vee \alpha_1) + (\alpha_3 \vee \eta_{2,4}) + (\alpha_3 \vee \eta_{2,5}).$$

However,  $\{\alpha_3, \theta_1\}$  is minimal in the order  $\leq_m$ , because there is no simple reflection  $s_i$  for which  $s_i(\{\alpha_3, \theta_1\}) <_m \{\alpha_3, \theta_1\}$ .

**Theorem 6.6.** *Let  $W$  be a simply laced Weyl group of finite type, and let  $X$  be a  $W$ -orbit of pairs of orthogonal positive roots. The orbit contains a maximum element  $\{\alpha, \beta\}$  with respect to  $\leq_2$ . In particular, if  $\alpha \vee \beta = \sum_{b \in \mathcal{B}} \mu_b b$  and  $\alpha' \vee \beta' = \sum_{b \in \mathcal{B}} \lambda_b b$  for some other element  $\{\alpha', \beta'\}$  in the orbit, then we have  $\lambda_b \leq \mu_b$  for all  $b \in \mathcal{B}$ .*

*Proof.* The orbit  $X$  has a unique maximal element with respect to  $\leq_m$  by [7, Corollary 3.6], and this is equivalent to having a maximum element because  $X$  is finite. The result now follows from Proposition 6.4 (ii).  $\square$

Recall from Proposition 3.6 and Remark 3.7 that the  $W$ -orbits of pairs of orthogonal real roots and the  $W$ -orbits of real 2-roots can be identified under the correspondence  $\{\alpha', \beta'\} \leftrightarrow \alpha' \vee \beta'$ . Theorem 6.6 may be interpreted as saying that the identification matches the maximum elements of these orbits.

In the sequel, we will refer to the 2-root  $\alpha \vee \beta$  from Theorem 6.6 the *highest 2-root* in its  $W$ -orbit. Our next goal is to give an explicit description of the highest 2-root in each orbit, and the next result will be helpful for this purpose.

**Lemma 6.7.** *Let  $W$  be a simply laced Weyl group of finite type, and let  $\{\alpha, \beta\}$  be a pair of orthogonal positive roots of  $W$  satisfying  $\text{ht}(\alpha) = \text{ht}(\beta)$ . Suppose that every simple root  $\alpha_i$  satisfies the following two conditions:*

- (i) *if  $B(\alpha_i, \alpha) = -1$  then  $B(\alpha_i, \beta) = +1$ ;*
- (ii) *if  $B(\alpha_i, \beta) = -1$  then  $B(\alpha_i, \alpha) = +1$ .*

*Then  $\alpha \vee \beta$  is the highest 2-root in its orbit.*

*Proof.* Suppose that the conditions are satisfied, but that  $\{\alpha, \beta\}$  is not the highest 2-root with respect to  $\leq_m$ . By Proposition 6.4 (i), there must be a simple reflection  $s_i$  such that  $\alpha \vee \beta \leq_m s_i(\alpha \vee \beta)$  is a covering relation. Proposition 6.4 (i) also implies that  $\{B(\alpha_i, \alpha), B(\alpha_i, \beta)\} \subseteq \{-1, 0, 1\}$ , and that  $\alpha_i$  cannot be orthogonal to both  $\alpha$  and  $\beta$ .

Suppose for a contradiction that we have such an  $s_i$ . We claim that in fact neither of  $B(\alpha_i, \alpha)$  and  $B(\alpha_i, \beta)$  can be zero, because in the case that (say)  $B(\alpha_i, \beta) = 0$ , Proposition 4.5 (i) implies that

$$s_i(\alpha \vee \beta) = (\alpha \vee \beta) - B(\alpha_i, \alpha)(\alpha_i \vee \beta).$$

Since  $s_i(\alpha \vee \beta)$  is assumed to be a higher root than  $\alpha \vee \beta$ , we must have  $B(\alpha_i, \alpha) = -1$ . However, we also have  $B(\alpha_i, \beta) = 0$ , which contradicts condition (i) of the statement.

We may now assume that  $B(\alpha_i, \alpha) = \pm 1$  and that  $B(\alpha_i, \beta) = \pm 1$ , with the signs chosen independently. By conditions (i) and (ii), the case  $B(\alpha_i, \alpha) = B(\alpha_i, \beta) = -1$  never occurs, so there are three other cases to consider.

The first case is  $B(\alpha_i, \alpha) = -1$  and  $B(\alpha_i, \beta) = +1$ . In this case, Proposition 4.5 (i) implies that

$$s_i(\alpha \vee \beta) = (\alpha \vee \beta) + (\alpha_i \vee (\beta - \alpha - \alpha_i)) = (\alpha + \alpha_i) \vee (\beta - \alpha_i),$$

and Proposition 4.5 (iii) implies that  $\beta - \alpha - \alpha_i$  is a root. The assumption that  $\text{ht}(\alpha) = \text{ht}(\beta)$  shows that  $\beta - \alpha - \alpha_i$  is a negative simple root, so that  $s_i(\alpha \vee \beta) <_2 (\alpha \vee \beta)$ , a contradiction.

The second case, where  $B(\alpha_i, \alpha) = +1$  and  $B(\alpha_i, \beta) = -1$ , follows by a symmetrical argument exchanging the roles of  $\alpha$  and  $\beta$ .

Finally, suppose that we have  $B(\alpha_i, \alpha) = B(\alpha_i, \beta) = +1$ . In this case, Proposition 4.5 (i) implies that

$$s_i(\alpha \vee \beta) = (\alpha \vee \beta) + (\alpha_i \vee (\alpha_i - \alpha - \beta)) = (\alpha - \alpha_i) \vee (\beta - \alpha_i).$$

Proposition 4.5 (iii) then implies that  $\alpha_i - \alpha - \beta$  is a root, and this root must be negative because  $\text{ht}(\alpha_i) = 1$ , so that  $s_i(\alpha \vee \beta) <_2 (\alpha \vee \beta)$ . This contradiction completes the proof.  $\square$

In order to state the main result about highest 2-roots, it is convenient to fix some notation for the simple reflections in type  $E_n$ . (We maintain the conventions of Notation 3.8 for types  $A_n$  and  $D_n$ .) We number the nodes of the Dynkin diagram of type  $E_n$  so that 3 is the branch node,  $1-2-3-\dots-(n-1)$  is a path, and the last node,  $x$  is adjacent to 3.

With these conventions, the highest root  $\theta$  in type  $E_6$ ,  $E_7$ , and  $E_8$  is given by

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_x,$$

$$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_x,$$

and

$$2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + 3\alpha_x,$$

respectively.

In type  $D_n$  ( $n \geq 4$ ),  $E_6$ ,  $E_7$  and  $E_8$ , there is a unique simple root,  $\alpha_y$ , that is not orthogonal to the highest root. We have  $\alpha_y = \alpha_2$  in type  $D_n$  for all  $n \geq 4$ , and  $\alpha_y = \alpha_x$ ,  $\alpha_1$ , and  $\alpha_7$  in types  $E_6$ ,  $E_7$ , and  $E_8$  respectively.

If  $b$  and  $c$  are nodes of the Dynkin diagram, we write  $\alpha_{b,c}$  to mean  $\sum_{i \in P} \alpha_i$ , where  $P$  is the set of vertices on the unique path between  $b$  and  $c$ , counting both endpoints.

**Theorem 6.8.** *Let  $W$  be a simply laced Weyl group of finite type, let  $\theta$  be the highest root of  $W$ , and maintain the above notation and Notation 3.8. The highest 2-root in each  $W$ -orbit is given as follows.*

(i) *If  $W$  has type  $A_n$  where  $n > 2$ , then the highest 2-root is*

$$\alpha_{1,n-1} \vee \alpha_{2,n} = (\varepsilon_1 - \varepsilon_n) \vee (\varepsilon_2 - \varepsilon_{n+1}).$$

(ii) *If  $W$  has type  $D_4$ , then the highest 2-roots in each of the three orbits are*

$$(\theta - \alpha_{2,4}) \vee (\theta - \alpha_{2,3}) = \alpha_{1,3} \vee \alpha_{1,4} = (\varepsilon_1 - \varepsilon_4) \vee (\varepsilon_1 + \varepsilon_4) = (\varepsilon_1 \vee \varepsilon_1) - (\varepsilon_4 \vee \varepsilon_4),$$

$$(\theta - \alpha_{2,4}) \vee (\theta - \alpha_{1,2}) = \alpha_{1,3} \vee \alpha_{3,4} = (\varepsilon_1 - \varepsilon_4) \vee (\varepsilon_2 + \varepsilon_3), \text{ and}$$

$$(\theta - \alpha_{2,3}) \vee (\theta - \alpha_{1,2}) = \alpha_{1,4} \vee \alpha_{3,4} = (\varepsilon_1 + \varepsilon_4) \vee (\varepsilon_2 + \varepsilon_3).$$

(iii) *If  $W$  has type  $D_n$  for  $n \geq 5$ , then the highest 2-root in the small orbit is*

$$\alpha_{1,n-1} \vee \alpha_{1,n} = (\varepsilon_1 - \varepsilon_n) \vee (\varepsilon_1 + \varepsilon_n) = (\varepsilon_1 \vee \varepsilon_1) - (\varepsilon_n \vee \varepsilon_n).$$

(iv) *If  $W$  has type  $D_n$  for  $n \geq 5$ , then the highest 2-root in the large orbit is*

$$(\theta - \alpha_{1,2}) \vee (\theta - \alpha_{2,3}) = (\theta - \alpha_1 - \alpha_2) \vee (\theta - \alpha_2 - \alpha_3) = (\varepsilon_2 + \varepsilon_3) \vee (\varepsilon_1 + \varepsilon_4).$$

(v) *If  $W$  has type  $E_6$ , then the highest 2-root is  $(\theta - \alpha_{2,x}) \vee (\theta - \alpha_{4,x}) = (\theta - \alpha_{2,y}) \vee (\theta - \alpha_{4,y})$ .*

(vi) *If  $W$  has type  $E_7$ , then the highest 2-root is  $(\theta - \alpha_{x,1}) \vee (\theta - \alpha_{4,1}) = (\theta - \alpha_{x,y}) \vee (\theta - \alpha_{4,y})$ .*

(vii) *If  $W$  has type  $E_8$ , then the highest 2-root is  $(\theta - \alpha_{2,7}) \vee (\theta - \alpha_{x,7}) = (\theta - \alpha_{2,y}) \vee (\theta - \alpha_{x,y})$ .*

*Proof.* The proof is by Lemma 6.7 in each case. Recall from Proposition 3.9 that there are three orbits in type  $D_4$ , two orbits in type  $D_n$  for  $n \geq 5$ , and one orbit otherwise. The three 2-roots appearing in the statement of (ii) can be distinguished by comparing components (see Definition 3.5).

In type  $A_n$  where  $n > 2$ , let  $\alpha = \alpha_{1,n-1}$ , and let  $\beta = \alpha_{2,n}$ . The roots  $\alpha$  and  $\beta$  are orthogonal roots of height  $n-1$ . The only simple root  $\alpha_i$  for which  $B(\alpha_i, \alpha) = -1$  is  $\alpha_n$ , and we have  $B(\alpha_n, \beta) = +1$ . Conversely, the only simple root for which  $B(\alpha_i, \beta) = -1$  is  $\alpha_1$ , and we have  $B(\alpha_i, \alpha) = +1$ . Lemma 6.7 implies that  $\alpha \vee \beta$  is the highest 2-root in its orbit, proving (i).

The proof of (iii) is similar to that of (i). Suppose we are in the situation of (iii), and let  $\alpha = \alpha_{1,n-1}$  and  $\beta = \alpha_{1,n}$ . The roots  $\alpha$  and  $\beta$  are orthogonal roots of height  $n-1$ . The only simple root  $\alpha_i$  for which  $B(\alpha_i, \alpha) = -1$  is  $\alpha_n$ , and we have  $B(\alpha_n, \beta) = +1$ . Conversely, the only simple

root for which  $B(\alpha_i, \beta) = -1$  is  $\alpha_{n-1}$ , and we have  $B(\alpha_{n-1}, \alpha) = +1$ . Lemma 6.7 implies that  $\alpha \vee \beta$  is the highest 2-root in its orbit, proving (iii).

Suppose that we are in the situation of (vii), and let  $\alpha = \theta - \alpha_{2,7}$  and  $\beta = \theta - \alpha_{x,y}$ . Since  $\alpha_7$  is the only simple root not orthogonal to  $\theta$  and  $B(\theta, \alpha_7) = 1$ , it follows that for any root  $\gamma = \sum_j \lambda_j \alpha_j$  we have  $B(\theta, \gamma) = \lambda_7$ , so that  $B(\theta, \alpha_{2,7}) = 1$ . It follows that

$$B(\alpha, \alpha_7) = B(\theta, \alpha_7) - B(\alpha_{2,7}, \alpha_7) = 1 - 1 = 0$$

and that for every generator  $s \neq 7$  we have

$$B(\alpha, \alpha_s) = B(\theta, \alpha_s) - B(\alpha_{2,7}, \alpha_s) = -B(\alpha_{2,7}, \alpha_s) = \begin{cases} -1 & \text{if } s = 2, \\ 1 & \text{if } s = 1 \text{ or } s = x, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the only simple root  $\alpha_i$  with  $B(\alpha_i, \alpha) = -1$  is  $\alpha_2$ , and we note that  $B(\alpha_2, \beta) = B(\alpha_2, \theta - \alpha_{x,7}) = 0 - B(\alpha_2, \alpha_{x,7}) = 1$ . Similarly, we can check that only simple root  $\alpha_j$  with  $B(\alpha_j, \beta) = -1$  is  $\alpha_x$  and  $B(\alpha_x, \alpha) = 1$ . By Lemma 6.7, to prove  $\alpha \vee \beta$  is the highest root it now suffices to show that  $\alpha$  and  $\beta$  are positive real roots of the same height. To do so, recall from [21, Proposition 5.10 (i)] that an element  $\gamma$  in the  $\mathbb{Z}$ -span of the simple roots is a real root if and only if  $B(\gamma, \gamma) = 2$ . It follows that

$$B(\alpha, \alpha) = B(\theta, \theta) - 2B(\theta, \alpha_{2,7}) + B(\alpha_{2,7}, \alpha_{2,7}) = 2 - 2 \cdot 1 + 2 = 2,$$

which in turn implies that  $\alpha$  is a real root. A similar argument shows that  $\beta$  is also a real root. Finally, since every simple root appears with positive integer coefficient in  $\theta$  and  $\text{ht}(\alpha_{2,7}) = \text{ht}(\alpha_{x,7})$ , it follows that  $\alpha$  and  $\beta$  are positive roots with the same height.

The proofs of (ii), (iv), (v) and (vi) are the same as the proof of (vii), *mutatis mutandis*.  $\square$

Parts (i) and (iv) of Theorem 6.8 also follow from [7, Example 4.4], which the authors state without proof.

We record without proof the heights of the highest 2-root in each orbit of  $W$ , with respect to the canonical basis  $\mathcal{B}$ . The explicit decomposition of the highest 2-root in terms of the canonical basis  $\mathcal{B}$  is also known in each case, and we remark that in type  $E_8$ ,  $\alpha_7 \vee \theta_7$  is the unique element of  $\mathcal{B}$  that occurs with coefficient 1 in the highest 2-root.

Orbit type	Height of highest 2-root
$A_n$	$(n-2)^2 + 1$
$D_4$ , three orbits	3, 3, 3
$D_n, n \geq 5$ , small orbit	$n-1$
$D_n, n \geq 5$ , large orbit	$4(n-4)(n-3) + 3$
$E_6$	28
$E_7$	85
$E_8$	295

The sequence for  $A_n$  appears as [23, A002522], and the sequence for the large orbit of  $D_n$  appears as [23, A164897]. The reason that the highest 2-root in the small orbit of  $D_n$  has height  $n-1$  is that the highest 2-root is the sum of all  $(n-1)$  basis elements in the small orbit, each with coefficient 1. In turn, this is due to the phenomenon described in Remark 3.11, combined with the fact that the highest root in type  $A$  is the sum of all the simple roots, each with coefficient 1.

## 7. IRREDUCIBILITY

The main results of Section 7 are Theorem 7.3 and Theorem 7.8, which describe the indecomposable summands of the module  $M$  in terms of 2-roots.



Recall from Corollary 4.8 (i) that the lattice of 2-roots,  $\mathbb{Z}\mathcal{B}$ , has the structure of a  $\mathbb{Z}W$ -module. The following result shows that any real 2-root is an integral linear combination of basis 2-roots from the same  $W$ -orbit, and that the 2-roots from a given  $W$ -orbit span a submodule of  $\mathbb{Z}\mathcal{B}$ .

**Proposition 7.1.** *Let  $\mathcal{B}$  be the canonical basis of 2-roots in type  $Y_{a,b,c}$ , and let  $X_1, X_2, \dots, X_r$  be the orbits of the action of  $W$  on the set of real 2-roots,  $\Phi_{\text{re}}^2$ .*

(i) *The  $\mathbb{Z}W$ -module  $\mathbb{Z}\mathcal{B}$  decomposes as a direct sum of  $\mathbb{Z}W$ -modules*

$$\mathbb{Z}\mathcal{B} \cong \bigoplus_{i=1}^r \mathbb{Z}(\mathcal{B} \cap X_i).$$

(ii) *Every 2-root in the orbit  $X_i$  lies in the submodule  $\mathbb{Z}(\mathcal{B} \cap X_i)$ .*

*Proof.* Since the sets  $\mathcal{B} \cap X_i$  partition  $\mathcal{B}$ , it follows that we have a direct sum decomposition of  $\mathbb{Z}\mathcal{B}$  as  $\mathbb{Z}$ -modules of the form given in the statement. Theorem 4.7 implies that the  $\mathbb{Z}$ -modules  $\mathbb{Z}(\mathcal{B} \cap X_i)$  are  $\mathbb{Z}W$ -modules, and this completes the proof of (i).

Proposition 3.3 (iii) shows that for any real 2-root  $\alpha \vee \beta \in X_i$ , there exists  $w \in W$  with

$$w(\alpha \vee \beta) \in \mathcal{B} \cap X_i$$

for some  $i$ . By part (i), the result of applying  $w^{-1}$  to  $w(\alpha \vee \beta)$  also lies in  $\mathcal{B} \cap X_i$ , which proves (ii).  $\square$

The Coxeter bilinear form  $B$  naturally gives a  $W$ -invariant bilinear form on  $S^2(V)$ , which we denote by  $B'$ . It is defined as the unique linear map satisfying

$$\begin{aligned} B'(\alpha_i \vee \alpha_j, \alpha_k \vee \alpha_l) &= \text{perm} \begin{pmatrix} B(\alpha_i, \alpha_k) & B(\alpha_i, \alpha_l) \\ B(\alpha_j, \alpha_k) & B(\alpha_j, \alpha_l) \end{pmatrix} \\ &= B(\alpha_i, \alpha_k)B(\alpha_j, \alpha_l) + B(\alpha_i, \alpha_l)B(\alpha_j, \alpha_k), \end{aligned}$$

where  $\text{perm}$  denotes the permanent of a matrix. The form  $B'$  restricts to an integer-valued form on the lattice  $\mathbb{Z}\mathcal{B}$ . We can also make  $B'$  into a nonzero  $F$ -valued form on the  $FW$ -module  $F\mathcal{B} = F \otimes_{\mathbb{Z}} \mathbb{Z}\mathcal{B}$ .

**Lemma 7.2.** *Let  $F$  be an arbitrary field and let  $B'$  be the bilinear form on  $F\mathcal{B}$  defined above. If  $\alpha$  and  $\beta$  are orthogonal real roots and  $v \in F\mathcal{B}$ , then we have*

$$C_\alpha C_\beta(v) = B'(\alpha \vee \beta, v)(\alpha \vee \beta).$$

*Proof.* It is enough to consider the case where  $v = v_1 \vee v_2$  is a symmetrized simple tensor, because the general case follows by linearity. By Lemma 4.2 (iii), we have

$$C_\alpha C_\beta(v_1 \vee v_2) = (C_\alpha \otimes C_\beta + C_\beta \otimes C_\alpha)(v_1 \vee v_2).$$

Lemma 4.2 (i) implies that

$$(C_\alpha \otimes C_\beta)(v_1 \vee v_2) = B(\alpha, v_1)B(\beta, v_2)(\alpha \otimes \beta) + B(\alpha, v_2)B(\beta, v_1)(\alpha \otimes \beta).$$

Adding this to the analogous expression for  $(C_\beta \otimes C_\alpha)(v_1 \vee v_2)$  gives

$$\begin{aligned} (C_\alpha \otimes C_\beta + C_\beta \otimes C_\alpha)(v_1 \vee v_2) &= (B(\alpha, v_1)B(\beta, v_2) + B(\alpha, v_2)B(\beta, v_1))(\alpha \vee \beta) \\ &= B'(\alpha \vee \beta, v_1 \vee v_2)(\alpha \vee \beta), \end{aligned}$$

as required.  $\square$

If  $N$  is an  $FW$ -submodule of  $F\mathcal{B}$ , we define the *radical*,  $\text{rad}(N)$  of  $N$ , to be

$$\text{rad}(N) = \{v \in N : B'(v, v') = 0 \text{ for all } v' \in N\}.$$

It is immediate from the  $W$ -invariance of  $B'$  that  $\text{rad}(N)$  is a submodule of  $F\mathcal{B}$ .

In the next result, we define the  $FW$ -module  $F(\mathcal{B} \cap X)$  to be  $F \otimes_{\mathbb{Z}} \mathbb{Z}(\mathcal{B} \cap X)$ .

**Theorem 7.3.** *Let  $\mathcal{B}$  be the canonical basis of 2-roots in type  $Y_{a,b,c}$ , let  $X$  be a  $W$ -orbit of 2-roots, let  $F$  be an arbitrary field, let  $N = F(\mathcal{B} \cap X)$ , and assume the restriction of the bilinear form  $B'$  to  $N$  is nonzero.*

- (i) *The radical  $\text{rad}(N)$  of  $N$  (with respect to  $B'$ ) is the unique maximal FW-submodule of  $N$ .*
- (ii) *The module  $N/\text{rad}(N)$  is an irreducible FW-module.*
- (iii) *The module  $N$  is indecomposable, and is irreducible if and only if  $\text{rad}(N) = 0$ .*

*Proof.* Since  $B'$  is assumed not to be zero, it follows that  $\text{rad}(N)$  is a proper submodule of  $N$ . To prove (i), it remains to show that every proper submodule of  $N$  is contained in  $\text{rad}(N)$ . The proof reduces to showing that if  $N' \leq N$  is a submodule containing an element  $v \in N \setminus \text{rad}(N)$  then the submodule  $\langle v \rangle$  generated by  $v$  is equal to  $N$ .

Fix an element  $v \in N \setminus \text{rad}(N)$ . Since  $v \notin \text{rad}(N)$ , there must be a 2-root  $\alpha \vee \beta$  in the  $W$ -orbit  $X$  such that  $B'(\alpha \vee \beta, v) \neq 0$ . Lemma 7.2 then implies that  $\langle v \rangle$  contains  $C_\alpha C_\beta(v)$ , which is a nonzero multiple of  $\alpha \vee \beta$ . It follows that  $\langle v \rangle$  contains  $\alpha \vee \beta$ , which means that  $\langle v \rangle$  contains the whole orbit  $X$ , and thus the whole of  $N$ . This completes the proof of (i).

Part (ii) follows from part (i), and the second assertion of (iii) follows from (ii). If  $N$  could be expressed as a nontrivial direct sum of modules  $N \cong N_1 \oplus N_2$ , then (i) would imply that both  $N_1$  and  $N_2$  were contained in the proper submodule  $\text{rad}(N)$ , which is a contradiction. Part (iii) now follows.  $\square$

*Remark 7.4.* The requirement in Theorem 7.3 that  $B'$  should not vanish on  $N$  is a mild assumption. This condition is always satisfied when the field does not have characteristic 2, because any 2-root  $\alpha \vee \beta \in X$  satisfies

$$B'(\alpha_i \vee \alpha_j, \alpha_i \vee \alpha_j) = 4 \neq 0.$$

Even in characteristic 2, the bilinear form  $B'$  will not be zero provided that  $W_I$  contains a parabolic subgroup of type  $A_4$ , because in type  $A_4$  we have

$$B'(\alpha_1 \vee \alpha_3, \alpha_2 \vee \alpha_4) = 1.$$

However, the form  $B'$  is zero in type  $A_3$  in characteristic 2.

For the rest of Section 7, we will assume that  $F$  is a field of characteristic 0.

Let  $\tilde{B}$  be the bilinear form on  $V \otimes V$  satisfying

$$\tilde{B}(b_i \otimes b_j, b_k \otimes b_l) = B(b_i, b_k)B(b_j, b_l),$$

and let  $\tau : V \otimes V \rightarrow V \otimes V$  be the linear map satisfying  $\tau(b_i \otimes b_j) = b_j \otimes b_i$ . We identify the symmetric square  $S^2(V)$  and the exterior square  $\Lambda^2(V)$  of  $V$  with the eigenspaces of  $\tau$  for the eigenvalues 1 and  $-1$ , respectively.

*Remark 7.5.* The forms  $\tilde{B}$  and  $B'$  are closely related. It follows from the definitions that the restriction of  $\tilde{B}$  to the module  $M \leq S^2(V)$  satisfies

$$\begin{aligned} \tilde{B}(\alpha_i \vee \alpha_j, \alpha_k \vee \alpha_l) &= 2(B(\alpha_i, \alpha_k)B(\alpha_j, \alpha_l) + B(\alpha_i, \alpha_l)B(\alpha_j, \alpha_k)) \\ &= 2B'(\alpha_i \vee \alpha_j, \alpha_k \vee \alpha_l). \end{aligned}$$

In particular, if the characteristic of  $F$  is not 2, the form  $B'$  is nondegenerate on  $M$  if and only if  $\tilde{B}$  is.

**Lemma 7.6.** *The following are equivalent:*

- (i)  *$B$  is nondegenerate on  $V$ ;*
- (ii)  *$\tilde{B}$  is nondegenerate on  $V \otimes V$ ;*
- (iii) *the restrictions of  $\tilde{B}$  to  $S^2(V)$  and to  $\Lambda^2(V)$  are both nondegenerate.*

*Proof.* To prove the equivalence of (i) and (ii), let  $G \in M_n(k)$  be the Gram matrix of  $B$  with  $G_{ij} = B(b_i, b_j)$ . The Gram matrix of  $\tilde{B}$  is the Kronecker product  $G \otimes G$ , whose determinant is given by  $(\det(G))^{2n}$ . It follows that  $G \otimes G$  is invertible if and only if  $G$  is invertible, and therefore that  $\tilde{B}$  is nondegenerate if and only if  $B$  is nondegenerate.

To prove the equivalence of (ii) and (iii), note that we have

$$V \otimes V \cong S^2(V) \oplus \Lambda^2(V)$$

as  $F$ -vector spaces, because the characteristic of  $F$  is zero. Setting  $\alpha_k \wedge \alpha_l := \alpha_k \otimes \alpha_l - \alpha_l \otimes \alpha_k$ , we have

$$\tilde{B}(\alpha_i \vee \alpha_j, \alpha_k \wedge \alpha_l) = \tilde{B}(\alpha_i \otimes \alpha_j + \alpha_j \otimes \alpha_i, \alpha_k \otimes \alpha_l - \alpha_l \otimes \alpha_k),$$

where all the terms cancel in pairs to give zero. It follows that  $S^2(V)$  and  $\Lambda^2(V)$  are orthogonal to each other with respect to the form  $\tilde{B}$ . By computing the Gram matrix  $G$  of  $\tilde{B}$  using a basis compatible with this decomposition, we obtain a block diagonal matrix whose two blocks are the Gram matrix of  $\tilde{B}$  restricted to  $S^2(V)$  and to  $\Lambda^2(V)$ . It follows that  $G$  is invertible if and only if both these blocks are invertible.  $\square$

Now assume that the form  $B$  is nondegenerate, or equivalently by Proposition 1.1 that we are not in any of the three affine types. Let  $\Pi^* = \{\alpha_1^*, \dots, \alpha_n^*\}$  be the dual basis of  $\Pi$ , which we identify with a subset of  $V$  in the usual way, via

$$B(\alpha_i^*, \alpha_j) = \delta_{ij},$$

where  $\delta$  is the Kronecker delta.

Following the theory of vertex operator algebras [17], we define the *Virasoro element* of  $B$  (with respect to the basis  $\Pi$ ) to be the element

$$\omega = \sum_{i=1}^n \alpha_i^* \otimes \alpha_i.$$

In [17, §2], the element  $\omega$  appears in the context of an algebra with identity  $\mathbb{I} = \omega/2$ , and the bilinear form  $B'$  is denoted by  $\langle \cdot, \cdot \rangle$ . We will show that  $\omega$  spans a complement in  $S^2(V)$  to the submodule  $M$ . Although most of the next result is known from the vertex operator algebras literature, we will give a self-contained proof for the convenience of the reader and in order to fix notation.

**Proposition 7.7.** *Maintain the above notation, and assume that  $B$  is nondegenerate.*

- (i) *For any  $v \in V \otimes V$ , we have  $\tilde{B}(\omega, v) = B(v)$ .*
- (ii) *The Virasoro element  $\omega$  is independent of the choice of basis  $\Pi$ .*
- (iii) *The Virasoro element  $\omega$  is symmetric, meaning that  $\tau(\omega) = \omega$ , and*

$$\omega = \frac{1}{2} \sum_{i=1}^n \alpha_i^* \vee \alpha_i.$$

- (iv) *The Virasoro element  $\omega$  is fixed by the action of any  $w \in W$ .*
- (v) *We have  $\tilde{B}(\omega, \omega) = n$ , and  $\omega \in S^2(V) \setminus M$ .*

*Proof.* It follows from the definitions that for all  $1 \leq i, j, k \leq n$ , we have

$$\tilde{B}(\alpha_i^* \otimes \alpha_i, \alpha_j \otimes \alpha_k) = B(\alpha_i^*, \alpha_j)B(\alpha_i, \alpha_k) = \delta_{ij}B(\alpha_j, \alpha_k).$$

Part (i) follows after summing over  $i$ .

The form  $\tilde{B}$  is nondegenerate by Lemma 7.6, and this implies that there is a unique element  $R \in V \otimes V$  with the property that  $\tilde{B}(R, v) = B(v)$  for all  $v \in V \otimes V$ . This implies that  $\omega$  is characterized by the property in (i). This characterization is basis-free, proving (ii).

By part (ii), we may also define  $\omega$  with respect to the dual basis of  $\Pi$ , proving that

$$\omega = \sum_{i=1}^n \alpha_i \otimes \alpha_i^*.$$

Comparing this with the original definition of  $\omega$  implies that  $\omega = \tau(\omega)$ , proving (iii).

Given  $w \in W$ , part (ii) shows that we can compute the Virasoro element with respect to the basis  $\{w(\alpha_i)\}_{i=1}^n$ . Because  $B$  is  $W$ -invariant, the dual basis in this case is  $\{w(\alpha_i^*)\}_{i=1}^n$ , and we have

$$w(\omega) = \sum_{i=1}^n w(\alpha_i \otimes \alpha_i^*) = \sum_{i=1}^n w(\alpha_i) \otimes w(\alpha_i^*) = \omega,$$

which proves (iv).

By part (iii), we have

$$\tilde{B}(\omega, \omega) = \tilde{B} \left( \sum_{i=1}^n \alpha_i^* \otimes \alpha_i, \sum_{j=1}^n \alpha_j \otimes \alpha_j^* \right) = \sum_{i,j=1}^n B(\alpha_i^*, \alpha_j) B(\alpha_i, \alpha_j^*) = \sum_{i,j=1}^n \delta_{ij}^2 = n,$$

which proves the first assertion of (v). Part (i) implies that an element  $v \in S^2(V)$  lies in  $M$  if and only if  $\tilde{B}(\omega, v) = 0$ , and the second assertion of (v) follows from (iii) and the fact that  $\tilde{B}(\omega, \omega) \neq 0$ .  $\square$

**Theorem 7.8.** *Let  $W$  be a Weyl group of type  $Y_{a,b,c}$  and let  $V$  be the reflection representation of  $W$  over a field  $F$  of characteristic zero. If  $Y_{a,b,c}$  is not of affine type, then the module  $S^2(V)$  decomposes as a direct sum of irreducible modules: the one-dimensional module  $F\omega$ , and the modules  $F(\mathcal{B} \cap X_i)$  corresponding to the orbits  $X_i$  of 2-roots.*

*Proof.* By Proposition 1.1, the form  $B$  is nondegenerate, and it follows from remarks 7.4 and 7.5 that  $B'$  and  $\tilde{B}$  are nonzero when restricted to each summand  $F(\mathcal{B} \cap X_i)$ . Proposition 7.7 (iv) and (v) imply that the module  $S^2(V)$  is isomorphic to  $\text{Span}(\omega) \oplus M$ , where  $\text{Span}(\omega)$  affords the trivial representation of  $W$ . It remains to show that the module  $M$  decomposes as the direct sum of the modules  $F(\mathcal{B} \cap X_i)$ , and that these modules are irreducible.

We first consider the case where  $W$  is finite. It follows from Proposition 7.1 (i), by extending scalars to  $F$ , that we have

$$F\mathcal{B} \cong \bigoplus_{i=1}^r F(\mathcal{B} \cap X_i).$$

The modules in the direct sum are indecomposable by Theorem 7.3 (iii), and irreducible by Maschke's Theorem, which completes the proof in this case.

Assume from now on that  $W$  is infinite, which means by Lemma 2.4 that there is a single orbit of 2-roots, and that the module  $F(\mathcal{B} \cap X)$  is  $M$  itself. For any subspace  $N \leq S^2(V)$ , define  $N^\perp$  to be the subspace

$$N^\perp := \{v \in S^2(V) : \tilde{B}(v, v') = 0 \text{ for all } v' \in N\}.$$

The nondegeneracy of  $\tilde{B}$  on  $S^2(V)$ , proved in Lemma 7.6 (iii), shows that we always have  $\dim(N) + \dim(N^\perp) = \dim(S^2(V))$ .

Assume for a contradiction that there exists a nonzero element  $m \in M \cap M^\perp$ , so that we have  $M \leq \text{Span}(m)^\perp$ . The previous paragraph shows that  $\dim(\text{Span}(m)^\perp) = \dim(M)$ , which implies that  $M = \text{Span}(m)^\perp$ . However, Proposition 7.7 (i) shows that  $M = \text{Span}(\omega)^\perp$ , and the nondegeneracy of  $\tilde{B}$  on  $S^2(V)$  then implies that  $\text{Span}(m) = \text{Span}(\omega)$ , which contradicts Proposition 7.7 (v). It follows that  $M \cap M^\perp = \text{rad}(M)$  is zero. Theorem 7.3 (iii) now implies that  $M$  is irreducible.  $\square$

*Remark 7.9.* In the cases where  $Y_{a,b,c}$  is of type affine  $E_n$  for  $n \in \{6, 7, 8\}$ , the form  $B$  is degenerate and there is no obvious analogue of the Virasoro element. The module  $M$  is indecomposable as in the case for other infinite Weyl groups, but it has an  $n$ -dimensional radical  $\text{rad}(M)$  consisting of the elements  $\delta \vee v$ , where  $v \in V$  and where  $\delta$  is the lowest positive imaginary root. The module  $\text{rad}(M)$  is isomorphic to the reflection representation, and it in turn has a submodule spanned by  $\delta \vee \delta$ .

## 8. FAITHFULNESS

In Section 8, we find the kernels of the action of the Weyl group in its action on the nontrivial summands of  $S^2(V)$ . The main result is Theorem 8.6, which describes when  $W$  acts faithfully on the nontrivial summands and on the associated orbits of 2-roots. This description depends on the centre  $Z(W)$  of  $W$ , which has the following explicit description.

**Lemma 8.1.** *Let  $W$  be a Weyl group of type  $Y_{a,b,c}$ . The centre  $Z(W)$  of  $W$  is trivial unless  $W$  is of type  $E_7$ ,  $E_8$ , or  $D_n$  for  $n$  even; in particular,  $Z(W)$  is trivial if  $W$  is infinite. In the cases where  $Z(W)$  is nontrivial, we have  $Z(W) = \{1, w_0\}$ , where  $w_0$  is the longest element of  $W$  and  $w_0$  acts as the scalar  $-1$  on the reflection representation  $V$ .*

*Proof.* Assume first that  $W$  is finite. It follows from [19, Exercise 6.3.1] that we have  $Z(W) = \{1, w_0\}$  in the case where the longest element  $w_0$  acts as  $-1$  on  $V$ , and  $Z(W) = \{1\}$  otherwise. The assertions about  $Z(W)$  being trivial follow from [19, §3.19], which completes the proof in the finite case.

Now assume that  $W$  is infinite. Qi proves [22, Proposition 2.5] that the center of any irreducible infinite Coxeter group is trivial. In particular, this implies that the center of  $W = W(Y_{a,b,c})$  is trivial if  $W$  is infinite, which completes the proof.  $\square$

**Lemma 8.2.** *Let  $\Gamma$  be a Dynkin diagram of type  $Y_{a,b,c}$  with at least five vertices, and let  $\Gamma_1 \cup \Gamma_2$  be a partition of the vertices into proper nonempty subsets. Then there are vertices  $x_1 \in \Gamma_1$  and  $x_2 \in \Gamma_2$  such that  $x_1$  and  $x_2$  are not adjacent in  $\Gamma$ .*

*Proof.* Without loss of generality, we may assume that  $|\Gamma_1| \geq 3$ .

If  $|\Gamma_1| \geq 4$ , then any  $x_2 \in \Gamma_2$  fails to be adjacent to at least one element of  $\Gamma_1$ , because  $\Gamma$  has no vertex of degree 4.

If  $|\Gamma_1| = 3$  then we must have  $|\Gamma_2| \geq 2$ . Let  $x$  and  $x'$  be distinct elements of  $\Gamma_2$ . There is a unique vertex of degree 3 in  $\Gamma$ , which means that either  $x$  or  $x'$  fails to be adjacent to one of the vertices in  $\Gamma_1$ , completing the proof.  $\square$

**Proposition 8.3.** *Let  $W$  be an infinite group of type  $Y_{a,b,c}$ , and let  $V$  be the reflection representation of  $W$ . Then  $W$  acts faithfully on the irreducible codimension-1 submodule  $M$  of  $S^2(V)$ .*

*Proof.* Since  $W$  is infinite, the rank  $n$  of  $W$  is at least 7, and the module  $M$  is irreducible by Theorem 7.8 because there is a single orbit of real 2-roots. Let  $w$  be a nonidentity element of  $W$ ; we need to show that  $w$  does not act on  $M$  as the identity.

Let  $S_1 = \{\alpha_i \in \Pi : w(\alpha_i) < 0\}$  and let  $S_2 = \Pi \setminus S_1$ . The set  $S_1$  is nonempty because  $w \neq 1$ , and the set  $S_2$  is nonempty because otherwise,  $w$  would be the longest element of  $W$ , which is impossible because  $W$  is infinite. By Lemma 8.2, there exist orthogonal simple roots  $\alpha_i$  and  $\alpha_j$  such that  $w(\alpha_i) < 0$  and  $w(\alpha_j) > 0$ . The element  $w$  sends the 2-root  $\alpha_i \vee \alpha_j$  (which is a standard basis element) to  $w(\alpha_i) \vee w(\alpha_j)$ , which is a negative linear combination of standard basis elements. In particular,  $w$  does not act as the identity on  $M$ , which completes the proof.  $\square$

Note that if  $W$  is a finite group and the longest element  $w_0$  of  $W$  acts on  $V$  as the scalar  $-1$ , then  $w_0$  will act as the identity on  $M$ . In these cases,  $W$  will not act faithfully on  $M$ .

**Proposition 8.4.** *Let  $W$  be a finite Weyl group of type  $D_n$  or  $E_n$  with  $n \geq 4$ , let  $V$  be the reflection representation of  $W$  over a field  $F$  of characteristic zero, and let  $N$  be a nontrivial irreducible direct summand of the module  $S^2(V)$  corresponding to a  $W$ -orbit  $X$  of 2-roots.*

- (i) *If  $X$  is the small orbit in type  $D_n$  (as in Definition 3.10), or any of the three orbits in type  $D_4$ , then the kernel of the action of  $W$  on  $N$  is elementary abelian of order  $2^{n-1}$ .*
- (ii) *In all other cases, the kernel of the action of  $W$  on  $N$  is the centre,  $Z(W)$ .*

*Proof.* Suppose that  $X$  is one of the orbits in the statement of (i). By Remark 3.11, the action of  $W(D_n)$  on  $N$  factors through the action of the symmetric group  $S_n$  on the root system of type  $A_{n-1}$ , and the latter action is faithful. The kernel of the action is the kernel of a homomorphism from  $W(D_n)$  to  $S_n$  that identifies two of the generators on the short branches. The latter is elementary abelian of order  $2^{n-1}$  (see [19, §2.10]), which proves (i).

To prove (ii), we will show that any  $w \notin \{1, w_0\}$  acts nontrivially on  $N$ . Let  $S_1 = \{\alpha_i \in \Pi : w(\alpha_i) < 0\}$  and let  $S_2 = \Pi \setminus S_1$ . The assumptions on  $W$  mean that  $S_1$  and  $S_2$  are both nonempty. By Lemma 8.2, there exist orthogonal simple roots  $\alpha_i \in S_1$  and  $\alpha_j \in S_2$  such that  $w(\alpha_i) > 0$  and  $w(\alpha_j) < 0$ .

The 2-root  $\alpha_i \vee \alpha_j$  will be in the orbit  $X$  as long as  $\alpha_i \vee \alpha_j$  is not in the small orbit in type  $D_n$ . By Proposition 3.9 (iii), this can only happen if we are in type  $D_n$  and  $\{i, j\} = \{n-1, n\}$ . In this case, we can replace the pair  $\{\alpha_{n-1}, \alpha_n\}$  by one of the pairs  $\{\alpha_{n-3}, \alpha_{n-1}\}$  or  $\{\alpha_{n-3}, \alpha_n\}$  to obtain a pair of orthogonal simple roots with one element from each of  $S_1$  and  $S_2$ .

As in the proof of Proposition 8.3, we now have a 2-root  $\alpha_i \vee \alpha_j$  that is a standard basis element in the orbit  $X$  such that  $w(\alpha_i \vee \alpha_j)$  is a negative linear combination of standard basis elements. This shows that  $w$  acts nontrivially on  $N$ .

If  $w_0 \in Z(W)$ , then  $w_0$  acts as  $-1$  on  $V$  and  $w_0$  acts trivially on  $S^2(V)$  and  $N$ . Combined with the fact that  $w$  acts nontrivially on  $N$ , this shows that the kernel of the action is  $Z(W) = \{1, w_0\}$ .

On the other hand, if  $w_0 \notin Z(W)$ , then  $\{1, w_0\}$  is not a normal subgroup of  $W$  and  $Z(W)$  is trivial. In this case, the kernel of the action, which we already know is contained in  $\{1, w_0\}$ , is also trivial, as required.  $\square$

*Remark 8.5.* If  $W$  is of type  $D_4$ , it can be shown that the kernels of the action of  $W$  on each of three nontrivial direct summands of  $S^2(V)$  intersect in the centre,  $Z(W)$ , of order 2.

The results of Section 8 can be summarized as follows.

**Theorem 8.6.** *Let  $W$  be a Weyl group of type  $Y_{a,b,c}$  other than  $W(D_4)$ , let  $V$  be the reflection representation of  $W$  over a field  $F$  of characteristic zero, and let  $N$  be a nontrivial irreducible direct summand of the module  $S^2(V)$  corresponding to a  $W$ -orbit  $X$  of 2-roots. If  $X$  is not the small orbit in type  $D_n$ , then the following hold.*

- (i) *The kernel of the action of  $W$  on  $N$  is the center,  $Z(W)$ , of  $W$ .*
- (ii) *The group  $W$  acts faithfully on  $N$  if and only if one of the following conditions holds:*
  - (1)  *$W$  is infinite;*
  - (2)  *$W$  is of type  $D_n$  and  $n$  is odd;*
  - (3)  *$W$  is of type  $E_6$ .*

*Proof.* Part (i) follows from Lemma 8.1 and Proposition 8.3 if  $W$  is infinite, and from Proposition 8.4 (ii) if  $W$  is finite. Part (ii) follows from (i) and Lemma 8.1.  $\square$

#### CONCLUDING REMARKS

Some natural candidates for generalizing the results of this paper including considering  $k$ -roots for integers  $k > 2$ , meaning symmetrized  $k$ -fold tensor products of mutually orthogonal roots. The most tractable cases may be types  $A$ ,  $B$ , and  $D$ , where the root systems are easy to understand

and there is a diagram calculus [13] to use as a guide. Following the completion of this paper, a notion of  $k$ -roots of type  $D$  has been introduced and studied in the preprint [14] by the first author. These  $k$ -roots have similar properties to the 2-roots of the current paper and have applications to spherical functions of Gelfand pairs  $(S_n, S_k \times S_{n-k})$  arising from maximal Young subgroups of symmetric groups.

In another direction, it would be interesting to know if the relations from Theorem 5.2 expressing 2-roots as positive combinations of other 2-roots may be amenable to an interpretation in terms of categorification.

Although we did not discuss this for reasons of space, the ideas of this paper are motivated by the authors' study of the Kazhdan–Lusztig basis of the Hecke algebra of  $W$ , specifically the elements  $w \in W$  such that  $\mathbf{a}(w) = 2$ , where  $\mathbf{a}$  is Lusztig's  $\mathbf{a}$ -function [15, 16]. Using the Kazhdan–Lusztig basis  $\{C_w : w \in W\}$ , rather than the basis  $\{C'_w : w \in W\}$ , it is possible to construct the canonical basis  $\mathcal{B}$  as follows. When  $q$  is specialized to 1, it can be shown that for each Kazhdan–Lusztig basis element  $C_w$  of  $\mathbf{a}$ -value 2, there are precisely two reflections  $s_\alpha$  and  $s_\beta$  such that  $s_\alpha(C_w)$  and  $s_\beta(C_w)$  are both equal to  $-C_w$  modulo  $I$ , where  $I$  is the ideal spanned by all Kazhdan–Lusztig basis elements of  $\mathbf{a}$ -value at least 3. It then turns out that the function sending  $C_w$  to  $\alpha \vee \beta$  extends to a module homomorphism from each cell module of  $\mathbf{a}$ -value 2 to the module  $M$ . These identifications give rise to  $q$ -analogues of many of the results in this paper.

It can also be shown that for non-affine types in characteristic zero, the irreducible summands of  $S^2(V)$  remain irreducible upon restriction to the derived subgroup  $W'$  of  $W$ . An important special case is the case  $Y_{1,2,6}$ , also known as type  $E_{10}$  or  $E_8^{++}$ . In this case, the derived subgroup  $W'$  can be identified with  $PSL(2, \mathbf{O})$  [10, §6.4]. Here,  $\mathbf{O}$  is the ring of octavians, a discrete subring of the octonions  $\mathbb{O}$ . The module  $M$  in this case is a 54-dimensional irreducible representation of  $PSL(2, \mathbf{O})$  in characteristic zero. There should be some octonionic interpretation of  $M$  in this case, and we note that in this case,  $M$  has twice the dimension of the exceptional Jordan algebra.

It follows from Remark 3.11 that the 2-roots in the small orbit in type  $D_n$  form a scaled copy of the root lattice of type  $A_{n-1}$ . This suggests that the integral lattice of 2-roots  $\mathbb{Z}\mathcal{B}$  may be related in interesting ways to other known integral lattices. Some natural questions to ask are the following.

- (1) Are the 2-roots the only elements  $x \in \mathbb{Z}\mathcal{B}$  for which  $B'(x, x) = 4$ ?
- (2) When does the lattice  $\mathbb{Z}\mathcal{B}$  contain elements  $x \in \mathbb{Z}\mathcal{B}$  such that  $B'(x, x) = 2$ ?
- (3) When does the lattice  $\mathbb{Z}\mathcal{B}$  contain *roots*, meaning elements  $\alpha$  such that the reflection

$$s_\alpha : x \mapsto x - 2 \frac{B'(\alpha, x)}{B'(\alpha, \alpha)} \alpha$$

gives an automorphism of  $\mathbb{Z}\mathcal{B}$ ?

- (4) Does the lattice  $\mathbb{Z}\mathcal{B}$  have any automorphisms other than negation and those induced from automorphisms of the root lattice  $\mathbb{Z}\Pi$ ?

The answer to question (3) above is positive in the case of the small orbit in type  $D_n$ . For question (2), Willson [24] has shown that, outside the finite and affine types, there always exist sign-coherent vectors  $x$  with  $B'(x, x) = 2$ . For example, consider the pairs of Coxeter diagrams of the form  $(\Gamma', \Gamma) \in \{(Y_{2,2,2}, Y_{2,2,3}), (Y_{1,3,3}, Y_{1,3,4}), (Y_{1,2,5}, Y_{1,2,6})\}$  where  $\Gamma'$  is naturally a subdiagram of  $\Gamma$  (and where  $\Gamma'$  is of affine type and  $\Gamma$  is of hyperbolic type). Let  $\alpha_{-1}$  be the unique simple root in  $\Gamma \setminus \Gamma'$ , let  $\alpha$  be a simple root corresponding to one of the other endpoints of  $\Gamma$ , let  $\delta$  be the lowest positive imaginary root for  $\Gamma'$ , and let  $\beta$  be a simple root of  $\Gamma$  that is adjacent to  $\alpha$ . Then the element  $x := ((\alpha + \beta) \vee \alpha_{-1}) + (\alpha \vee \delta)$  is a sign-coherent element with  $B'(x, x) = 2$ .

Finally, we note that in the physics literature, real roots in type  $E_{10}$  correspond to instantons, and two real roots are orthogonal if and only if the corresponding instantons can “bind at threshold” [5, §3.2]. It would be interesting to know if the realization of 2-roots as lattice points in  $\mathbb{Z}\mathcal{B}$  or the linear dependence relations between these lattice points have a physical interpretation.

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## STATEMENTS AND DECLARATIONS

The authors have no conflict of interest.

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