Dual r-Rank Decomposition and Its Applications

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Abstract

In this paper, we introduce the dual r-rank decomposition of dual matrix, get its existence conditions and equivalent forms of the decomposition. Then we derive some characterizations of dual Moore-Penrose generalized inverse(DMPGI). Based on DMPGI, we introduce one special dual matrix(dual EP matrix). By applying the dual r-rank decomposition, we derive several characterizations of dual EP matrix, dual idempotent matrix, dual generalized inverses, and relationships among dual Penrose equations.

Keywords: Dual matrix; dual EP matrix; dual r-rank decomposition; dual Moore-Penrose generalized inverse; dual Penrose equations

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1. Introduction

Clifford firstly proposed the dual number [4] in 1873, then Study [12] gave its specific form. Subsequently, the dual algebra has developed rapidly and been widely applied to dynamic analysis of spatial mechanisms, sensor calibration, robotics and other fields (see [6, 9, 13, 17]). In recent years, some researches of dual matrix, dual generalized inverse, dual equation and their applications have further promoted the development of dual algebra theory and its applications (see [1, 3, 5, 6, 7, 10]).

In this paper, we adopt the following notations: $\mathbb{R}_{m\times n}$ stands for the set of all $m\times n$ real matrices; $\mathrm{rk}(A)$ for the rank of A; $Q^S_{A,B}$ for A^TB+B^TA. Let the dual number be $\widehat{a}$ and have the following form:

$$\widehat{a}=a+\epsilon a^{\circ}$$

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in which a and a° are real numbers, and ϵ is the dual unit subjected to the rules

$$\epsilon \neq 0,\ 0\epsilon = \epsilon 0 = 0,\ 1\epsilon = \epsilon 1 = \epsilon \ \text{ and } \ \epsilon^2 = 0.$$

If a matrix has the form of $A_0 + \epsilon A_1$ and $A_i \in \mathbb{R}_{m\times n}(i = 0, 1)$, it can be called the dual matrix and denoted as $\widehat{A}$. Furthermore, denote the set of all $m \times n$ dual matrices as $\mathbb{D}_{m\times n}$.

The dual Moore-Penrose generalized inverse(DMPGI for short) of $\widehat{A}$ is the unique dual matrix $\widehat{X}$, which satisfies the following four dual Penrose equations [10]:

$$\left(\widehat{1}\right)\ \widehat{A}\widehat{X}\widehat{A} = \widehat{A}, \left(\widehat{2}\right)\ \widehat{X}\widehat{A}\widehat{X} = \widehat{A}, \left(\widehat{3}\right)\ \left(\widehat{A}\widehat{X}\right)^T = \widehat{A}\widehat{X}, \left(\widehat{4}\right)\ \left(\widehat{X}\widehat{A}\right)^T = \widehat{X}\widehat{A}, \tag{1.1}$$

and the unique dual matrix $\widehat{X}$ is denoted by $\widehat{X} = \widehat{A}^{\dagger}$. Especially, the DMPGI has expanded the application range of the generalized inverse theory. It is worth noting that, unlike real matrix, dual matrix may not have DMPGI. When $A_1 = 0$, $\widehat{A} = A_0$ is a real matrix, then the Moore-Penrose generalized inverse of A_0 is the unique matrix X satisfying the following four Penrose equations:

$$(1)\ A_0XA_0 = A_0,\ (2)\ XA_0X = X,\ (3)\ (A_0X)^T = A_0X,\ (4)\ (XA_0)^T = XA_0$$

and the unique matrix X is denoted by $X = A_0^{\dagger}$. Let $A_0\{i, ..., k\}$ denote the set of solutions which satisfy equations $(i), ..., (k)$ from the above four Penrose equations (1)-(4). Therefore X can be called $\{i, ..., k\}$-inverse of A_0, and denoted by $A_0^{(i,...,k)}$(see [2]). It is well known that a variety of generalized inverses, such as Drazin inverse, group inverse, core inverse and core-EP inverse, have been established successively. The achievements of generalized inverse theory have been greatly enriched, and the scope of their applications has been expanded to physics, statistics, etc. For more information about generalized inverse theory and its applications, please refer to [2, 11, 16].

Full-rank decomposition is one of the basic decompositions in matrix theory. It has the following definitions [2, 18]: Let $A \in \mathbb{R}_{m\times n}$ and $\mathrm{rk}(A) = r$, then there exist full column rank matrix $F \in \mathbb{R}_{m\times r}$ and full row rank matrix $G \in \mathbb{R}_{r\times n}$ such that $A = FG$. Not only does full rank decomposition play an important role in solving generalized inverse matrix, but also has a wide range of applications in many fields such as mathematical statistics, systems theory, optimization and cybernetics. For example, the full rank decomposition can be used to represent the $\{i, ..., k\}$-inverse of matrix A [2]: Let $A \in \mathbb{R}_{m\times n}$, $\mathrm{rk}(A) = r$, and its full rank decomposition is $A = FG$, in which $\mathrm{rk}(F) = \mathrm{rk}(G) = r$, then

$$A^{\dagger} = G^{\dagger}F^{\dagger},\ G^{\dagger} = G^T\left(GG^T\right)^{-1},\ F^{\dagger} = \left(F^TF\right)^{-1}F^T, \tag{1.2}$$

$$G^{(i)}F^{(1)} \in A\{i\}, i = 1, 2, 4 \text{ and } \ G^{(1)}F^{(j)} \in A\{j\}, j = 1, 2, 3. \tag{1.3}$$

For more details, please refer to literatures [2, 11].

In this paper, we extend the full rank decomposition from real matrix to dual matrix, introduce the dual r-rank decomposition, get some equivalent characterizations of the existence of dual r-rank decomposition, and give a method of calculating dual r-rank decomposition. By applying the decomposition, we get characterizations of DMPGI and relationships among dual Penrose equations. Furthermore, we give a method of calculating DMPGI and some examples. In addition, we consider two special dual matrices: dual EP matrix and dual idempotent matrix. We give the definition of dual EP matrix, and get characterizations and dual r-rank decompositions of both dual EP matrix and dual idempotent matrix. At last, by applying the dual r-rank decomposition and definitions of these special dual matrix, we get characterizations of both DMPGIs of dual EP matrix and dual idempotent matrix.

2. Preliminaries

This section provides several results that will be used in the following sections.

Lemma 2.1 ([15]). *Let $\widehat{A} \in \mathbb{D}_{m\times n}$ and $\widehat{A} = A_0 + \epsilon A_1$. Then the DMPGI of $\widehat{A}$ exists if and only if*

$$\left(I_m - A_0A_0^{\dagger}\right) A_1 \left(I_n - A_0^{\dagger}A_0\right) = 0. \tag{2.1}$$

Furthermore,

$$\begin{aligned} \widehat{A}^{\dagger} = A_0^{\dagger} - \varepsilon \Big(& A_0^{\dagger}A_1A_0^{\dagger} - \left(A_0^TA_0\right)^{\dagger} A_1^T \left(I_m - A_0A_0^{\dagger}\right) \\ & - \left(I_n - A_0^{\dagger}A_0\right) A_1^T \left(A_0A_0^T\right)^{\dagger}\Big). \end{aligned} \tag{2.2}$$

Lemma 2.2 ([14]). *Let $\widehat{A_1} \in \mathbb{D}_{m\times r}$, $\widehat{A_2} \in \mathbb{D}_{r\times n}$, $\widehat{A_1} = A_2 + \epsilon A_3$, $\widehat{A_2} = A_4 + \epsilon A_5$,* $\mathrm{rk}(A_2) = r$ *and* $\mathrm{rk}(A_4) = r$. *Then*

$$\widehat{A_1}^{\dagger} = \left(\widehat{A_1}^T\widehat{A_1}\right)^{-1} \widehat{A_1}^T \tag{2.3}$$

$$= \left(A_2^TA_2\right)^{-1} A_2^T + \epsilon \left(\left(A_2^TA_2\right)^{-1} A_3^T - \left(A_2^TA_2\right)^{-1} Q^S_{A_2,A_3} \left(A_2^TA_2\right)^{-1} A_2^T\right) \tag{2.4}$$

and

$$\widehat{A_2}^{\dagger} = \widehat{A_2}^T \left(\widehat{A_2}\widehat{A_2}^T\right)^{-1} \tag{2.5}$$

$$= A_4^T \left(A_4A_4^T\right)^{-1} + \epsilon \left(A_5^T \left(A_4A_4^T\right)^{-1} - A_4^T \left(A_4A_4^T\right)^{-1} Q^S_{A_4^T,A_5^T} \left(A_4A_4^T\right)^{-1}\right), \tag{2.6}$$

where $Q^S_{A_2,A_3} = A_2^TA_3 + A_3^TA_2$ and $Q^S_{A_4^T,A_5^T} = A_4A_5^T + A_5A_4^T$.

Lemma 2.3 ([8]). *Let $A \in \mathbb{R}_{m\times p}$, $B \in \mathbb{R}_{q\times n}$ and $C \in \mathbb{R}_{m\times n}$. Then the matrix equation*

$$AX + YB = C \tag{2.7}$$

is consistent if and only if

$$\left(I_m - AA^{\dagger}\right) C \left(I_n - B^{\dagger}B\right) = 0. \tag{2.8}$$

then the solution of this equation is

$$\begin{cases} X = A^{\dagger}C + UB + \left(I_p - A^{\dagger}A\right) V, & \text{(2.9a)} \\ Y = \left(I_m - AA^{\dagger}\right) CB^{\dagger} - AU + W \left(I_q - BB^{\dagger}\right), & \text{(2.9b)} \end{cases}$$

where $U \in \mathbb{R}_{p\times q}$, $V \in \mathbb{R}_{p\times n}$ and $W \in \mathbb{R}_{m\times q}$ are arbitrary.

3. Dual r-rank Decomposition

In this section we extend the full rank decomposition of real matrix to dual matrix. We also give the definitions of r-row full rank dual matrix, r-column full rank dual matrix, and dual r-rank decomposition. Furthermore, we give characterizations of the existence of the dual r-rank decomposition, a method of calculating the decomposition, and two examples.

Definition 3.1. *Let $\widehat{A_1} \in \mathbb{D}_{m\times r}$, $\widehat{A_2} \in \mathbb{D}_{r\times n}$, $\widehat{A_1} = A_2 + \epsilon A_3$ and $\widehat{A_2} = A_4 + \epsilon A_5$. If the real part matrix A_2 of $\widehat{A_1}$ is a column full rank matrix, then we call $\widehat{A_1}$ r-column full rank dual matrix; if the real part matrix A_4 of $\widehat{A_2}$ is a row full rank matrix, then we call $\widehat{A_2}$ r-row full rank dual matrix.*

Definition 3.2. (Dual r-rank Decomposition) *Let $\widehat{A} \in \mathbb{D}_{m\times n}$, $\widehat{A} = A_0 + \epsilon A_1$, $\mathrm{rk}(A_0) = r$, and $A_0 = A_2A_4$ be a full rank decomposition of A_0. If there exist an r-column full rank dual matrix $\widehat{A_1} = A_2 + \epsilon A_3$ and an r-row full rank dual matrix $\widehat{A_2} = A_4 + \epsilon A_5$, such that*

$$\widehat{A} = \widehat{A_1}\widehat{A_2},$$

which we call a dual r-rank decomposition of $\widehat{A}$.

From Definition 3.2, the following results can be inferred.

Theorem 3.1. *Let $\widehat{A} \in \mathbb{D}_{m\times n}$, $\widehat{A} = A_0 + \epsilon A_1$, $\mathrm{rk}(A_0) = r$, and $A_0 = A_2A_4$ be a full rank decomposition of A_0. Then the dual r-rank decomposition of $\widehat{A}$ exists if and only if*

$$\left(I_m - A_2A_2^{\dagger}\right) A_1 \left(I_n - A_4^{\dagger}A_4\right) = 0. \tag{3.1}$$

Furthermore, if $\widehat{A}$ has a dual r-rank decomposition $\widehat{A} = \widehat{A_1}\widehat{A_2}$, in which $\widehat{A_1} = A_2 + \epsilon A_3$ and $\widehat{A_2} = A_4 + \epsilon A_5$, then

$$\begin{cases} A_3 = \left(I_m - A_2A_2^{\dagger}\right) A_1A_4^{\dagger} - A_2P, \\ A_5 = A_2^{\dagger}A_1 + PA_4, \end{cases} \tag{3.2}$$

for arbitrary $P \in \mathbb{R}_{r\times r}$.

Proof. " $\Rightarrow$ ": Suppose the dual r-rank decomposition of the dual matrix $\widehat{A}$ exists. Let $\widehat{A} = \widehat{A_1}\widehat{A_2}$ be a dual r-rank decomposition of $\widehat{A}$, where

$$\widehat{A_1} = A_2 + \epsilon Y \quad \text{and} \quad \widehat{A_2} = A_4 + \epsilon X.$$

Then $A_0 + \epsilon A_1 = (A_2 + \epsilon Y)(A_4 + \epsilon X)$. By expanding this equation, we have

$$A_2X + YA_4 = A_1. \tag{3.3}$$

By applying Lemma 2.3 to the equation(3.3), we get (3.1).

" $\Leftarrow$ ": Let $A_0 = A_2A_4$ be a full rank decomposition of A_0. Because (3.1) holds, by applying Lemma 2.3, we get that the equation $A_2X + YA_4 = A_1$ is consistent, and the solution to this equation is

$$\begin{cases} X = A_2^{\dagger}A_1 + PA_4, \\ Y = \left(I_m - A_2A_2^{\dagger}\right) A_1A_4^{\dagger} - A_2P, \end{cases} \tag{3.4}$$

for arbitrary $P \in \mathbb{R}_{r\times r}$. Let $\widehat{A_1} = A_2 + \epsilon Y$ and $\widehat{A_2} = A_4 + \epsilon X$. Then $\widehat{A_1} = A_2 + \epsilon Y$ is an r-column full rank dual matrix; $\widehat{A_2} = A_4 + \epsilon X$ is an r-row full rank dual matrix;

$$\widehat{A_1}\widehat{A_2} = (A_2 + \epsilon Y)(A_4 + \epsilon X) = A_2A_4 + \epsilon(A_2X + YA_4) = A_0 + \epsilon A_1 = \widehat{A}.$$

Therefore, the dual r-rank decomposition of $\widehat{A}$ exists.

In summary, the dual r-rank decomposition of $\widehat{A}$ exists if and only if the equation (3.1) is consistent. Furthermore, by applying (3.4), we get (3.2). □

Based on Theorem 3.1, the detailed calculation process of dual r-rank decomposition is given as follows, and corresponding examples are also given to verify this process.

(1). Input matrix A_0 and A_1, and the form of dual matrix $\widehat{A}$ is $\widehat{A} = A_0 + \epsilon A_1,\ A_i \in \mathbb{R}_{m\times n},\ \mathrm{rk}(A_0) = r$;

(2). Perform full rank decomposition on A_0: $A_0 = A_2 A_4$, in which A_2 is a column full rank matrix and A_4 is a row full rank matrix;

(3). Calculate the Moore-Penrose inverses of A_2 and A_4: $A_2^\dagger$ and $A_4^\dagger$;

(4). Check whether the matrix equation $A_2 X + Y A_4 = A_1$ is consistent:

$$\left(I_m - A_2 A_2^\dagger\right) A_1 \left(I_n - A_4^\dagger A_4\right) = 0.$$

If the matrix equation holds, then proceed to step (5);

(5). Calculate the solution to matrix equation $A_2 X + Y A_4 = A_1$:

$$\begin{cases} X = A_2^\dagger A_1 + P A_4, \\ Y = \left(I_m - A_2 A_2^\dagger\right) A_1 A_4^\dagger - A_2 P, \end{cases}$$

where P is arbitrary;

(6). Get one dual r-rank decomposition of the dual matrix $\widehat{A}$: $\widehat{A} = \widehat{A_1}\widehat{A_2} = (A_2 + \epsilon A_3)(A_4 + \epsilon A_5)$.

Example 3.1. *Let*

$$\widehat{A} = A_0 + \epsilon A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

By performing full rank decomposition of $A_0 = A_2 A_4$ where

$$A_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \textit{and} \quad A_4 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

we have

$$A_2^\dagger = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \textit{and} \quad A_4^\dagger = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and by calculating $\left(I_2 - A_2 A_2^\dagger\right) A_1 \left(I_2 - A_4^\dagger A_4\right)$, we can get

$$\begin{aligned} \left(I_2 - A_2 A_2^\dagger\right) A_1 \left(I_2 - A_4^\dagger A_4\right) &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}\right) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq 0. \end{aligned}$$

By applying Theorem 3.1, we know that $\widehat{A}$ does not have the dual r-rank decomposition.

Example 3.2. *Calculate the dual r-rank decomposition of*

$$\widehat{A} = A_0 + \epsilon A_1 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix} + \epsilon \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 14 \end{bmatrix}.$$

The rank of matrix A_0 is $\mathrm{rk}(A_0) = 2$. By performing full rank decomposition of $A_0 = A_2 A_4$ where

$$A_2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{bmatrix} \quad \text{and} \quad A_4 = \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix},$$

we have

$$A_2^{\dagger} = \begin{bmatrix} -\frac{4}{9} & \frac{5}{9} & \frac{1}{9} \\ \frac{5}{9} & -\frac{4}{9} & \frac{1}{9} \end{bmatrix} \quad \text{and} \quad A_4^{\dagger} = \begin{bmatrix} \frac{10}{11} & -\frac{1}{11} \\ -\frac{1}{11} & \frac{10}{11} \\ \frac{3}{11} & \frac{3}{11} \end{bmatrix}.$$

It is easy to check that $\left(I_3 - A_2 A_2^{\dagger}\right) A_1 \left(I_3 - A_4^{\dagger} A_4\right) = 0$. Therefore, the matrix equation (3.3) *is consistent. Let*

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}.$$

Then the solution to (3.3) *is*

$$\begin{cases} X = A_2^{\dagger} A_1 + P A_4 = \begin{bmatrix} \frac{3}{2} & \frac{13}{6} & \frac{29}{9} \\ -1 & \frac{7}{6} & \frac{31}{18} \end{bmatrix}, \\ Y = \left(I_3 - A_2 A_2^{\dagger}\right) A_1 A_4^{\dagger} - A_2 P = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} \\ \frac{3}{2} & -4 \end{bmatrix}, \end{cases}$$

Let $X = A_5$ and $Y = A_3$, then we can get

$$\begin{cases} \widehat{A_1} = A_2 + \epsilon A_3 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{bmatrix} + \epsilon \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} \\ \frac{3}{2} & -4 \end{bmatrix}, \\ \widehat{A_2} = A_4 + \epsilon A_5 = \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix} + \epsilon \begin{bmatrix} \frac{3}{2} & \frac{13}{6} & \frac{29}{9} \\ -1 & \frac{7}{6} & \frac{31}{18} \end{bmatrix}. \end{cases}$$

Next we verify that $\widehat{A} = \widehat{A_1}\widehat{A_2}$ is a dual r-rank decomposition of $\widehat{A}$. Multiplying $\widehat{A_1}$ by $\widehat{A_2}$ gives

$$\begin{aligned}\widehat{A_1}\widehat{A_2} &= (A_2 + \epsilon A_3)(A_4 + \epsilon A_5) = A_2A_4 + \epsilon A_2A_5 + \epsilon A_3A_4 = A_0 + \epsilon A_1\\ &= \begin{bmatrix}1 & 2 & 1\\ 2 & 1 & 1\\ 3 & 3 & 2\end{bmatrix} + \epsilon \begin{bmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 14\end{bmatrix}.\end{aligned}$$

Hence, $\widehat{A} = \widehat{A_1}\widehat{A_2}$ is a dual r-rank decomposition of $\widehat{A}$.

Remark 3.1. *Since the full rank decomposition of the real part matrix A_0 of $\widehat{A}$ is not unique, the solutions X and Y to the matrix equation(3.3) are not unique. Let P is a zero matrix. By applying Theorem 3.1, it is obvious that $A_2 + \epsilon\left(I_m - A_2A_2^{\dagger}\right)A_1A_4^{\dagger}$ is an r-column full rank dual matrix; $A_4 + \epsilon A_2^{\dagger}A_1$ is an r-row full rank dual matrix;*

$$\widehat{A} = \left(A_2 + \epsilon\left(I_m - A_2A_2^{\dagger}\right)A_1A_4^{\dagger}\right)\left(A_4 + \epsilon A_2^{\dagger}A_1\right). \tag{3.5}$$

Therefore, (3.5) is one dual r-rank decomposition of $\widehat{A}$.

4. Applications of Dual r-rank Decomposition

In this section, we apply dual r-rank decomposition to studying several related problems, including characterization and calculation of DMPGI, special dual matrices and their properties, and dual Penrose equations.

4.1. Dual Moore-Penrose Generalized Inverse

Let $A_0 \in \mathbb{R}_{m\times n},\ \mathrm{rk}(A_0) = r$, and $A_0 = A_2A_4$ be a full rank decomposition of A_0. It is well known that

$$A_0A_0^{\dagger} = A_2A_2^{\dagger} \text{ and } A_0^{\dagger}A_0 = A_4^{\dagger}A_4. \tag{4.1}$$

By using (4.1), we can get the following Theorems.

Theorem 4.1. *Let $\widehat{A} \in \mathbb{D}_{m\times n}$, $\widehat{A} = A_0 + \epsilon A_1$ and $\mathrm{rk}(A_0) = r$. Then the following conditions are equivalent:*

(a). *the dual r-rank decomposition of $\widehat{A}$ exists;*

(b). $\left(I_m - A_0A_0^{\dagger}\right)A_1\left(I_n - A_0^{\dagger}A_0\right) = 0$;

(c). *the DMPGI of $\widehat{A}$ exists.*

Proof. (a)⇒(b): If the dual r-rank decomposition of $\widehat{A}$ exists, according to Theorem 3.1, we can get (3.1). It follows from (4.1) that $\left(I_m - A_0A_0^\dagger\right) A_1 \left(I_n - A_0^\dagger A_0\right) = 0$ holds.

(b)⇐(a): When $\left(I_m - A_0A_0^\dagger\right) A_1 \left(I_n - A_0^\dagger A_0\right) = 0$ holds, by applying (4.1) we get $\left(I_m - A_2A_2^\dagger\right) A_1 \left(I_n - A_4^\dagger A_4\right) = 0$. It follows from Theorem 3.1, that the dual r-rank decomposition of $\widehat{A}$ exists.

Since DMPGI of $\widehat{A}$ exists if and only if $\left(I_m - A_0A_0^\dagger\right) A_1 \left(I_n - A_0^\dagger A_0\right) = 0$, then (b)⇔(c). □

Theorem 4.2. *Let $\widehat{A} \in \mathbb{D}_{m\times n}$, $\widehat{A} = A_0 + \epsilon A_1$, $\mathrm{rk}(A_0) = r$, the dual r-rank decomposition of $\widehat{A}$ exist, and the dual r-rank decomposition of $\widehat{A}$ be $\widehat{A} = \widehat{A_1}\widehat{A_2}$. Then*

$$\widehat{A}^\dagger = \widehat{A_2}^\dagger \widehat{A_1}^\dagger \tag{4.2}$$

$$= \widehat{A_2}^T \left(\widehat{A_2}\widehat{A_2}^T\right)^{-1} \left(\widehat{A_1}^T\widehat{A_1}\right)^{-1} \widehat{A_1}^T. \tag{4.3}$$

Proof. Since the dual r-rank decomposition of $\widehat{A}$ exists, from Theorem 4.1, we see that the DMPGI of $\widehat{A}$ exists. Let $\widehat{A} = \widehat{A_1}\widehat{A_2}$ be a dual r-rank decomposition of $\widehat{A}$, and denote

$$\widehat{X} = \widehat{A_2}^T \left(\widehat{A_2}\widehat{A_2}^T\right)^{-1} \left(\widehat{A_1}^T\widehat{A_1}\right)^{-1} \widehat{A_1}^T.$$

We verify that $\widehat{X}$ satisfies the four dual Penrose equations(1.1):

(1) $\widehat{A}\widehat{X}\widehat{A} = \widehat{A_1}\widehat{A_2}\widehat{A_2}^T \left(\widehat{A_2}\widehat{A_2}^T\right)^{-1} \left(\widehat{A_1}^T\widehat{A_1}\right)^{-1} \widehat{A_1}^T\widehat{A_1}\widehat{A_2} = \widehat{A}$;

(2) $\widehat{X}\widehat{A}\widehat{X} = \widehat{A_2}^T \left(\widehat{A_2}\widehat{A_2}^T\right)^{-1} \left(\widehat{A_1}^T\widehat{A_1}\right)^{-1} \widehat{A_1}^T\widehat{A_1}\widehat{A_2}\widehat{A_2}^T \left(\widehat{A_2}\widehat{A_2}^T\right)^{-1} \left(\widehat{A_1}^T\widehat{A_1}\right)^{-1} \widehat{A_1}^T = \widehat{X}$;

(3) $\left(\widehat{A}\widehat{X}\right)^T = \left(\widehat{A_1}\widehat{A_2}\widehat{A_2}^T \left(\widehat{A_2}\widehat{A_2}^T\right)^{-1} \left(\widehat{A_1}^T\widehat{A_1}\right)^{-1} \widehat{A_1}^T\right)^T = \widehat{A_1} \left(\widehat{A_1}^T\widehat{A_1}\right)^{-1} \widehat{A_1}^T = \widehat{A}\widehat{X}$;

(4) $\left(\widehat{X}\widehat{A}\right)^T = \left(\widehat{A_2}^T \left(\widehat{A_2}\widehat{A_2}^T\right)^{-1} \left(\widehat{A_1}^T\widehat{A_1}\right)^{-1} \widehat{A_1}^T\widehat{A_1}\widehat{A_2}\right)^T = \widehat{A_2}^T \left(\widehat{A_2}\widehat{A_2}^T\right)^{-1} \widehat{A_2} = \widehat{X}\widehat{A}$.

Since $\widehat{A}^\dagger$ satisfying the four equations is unique, then $\widehat{X} = \widehat{A}^\dagger$.

Furthermore, according to Lemma 2.2, we see $\widehat{A_1}^\dagger = \left(\widehat{A_1}^T\widehat{A_1}\right)^{-1} \widehat{A_1}^T$ and $\widehat{A_2}^\dagger = \widehat{A_2}^T \left(\widehat{A_2}\widehat{A_2}^T\right)^{-1}$. So, $\widehat{A}^\dagger$ can be further expressed as $\widehat{A}^\dagger = \widehat{A_2}^\dagger\widehat{A_1}^\dagger$, that is, (4.2). □

Theorem 4.3. *Let $\widehat{A} \in \mathbb{D}_{m\times n}$, $\widehat{A} = A_0 + \epsilon A_1$ and $\mathrm{rk}(A_0) = r$. Let $A_0 = A_2A_4$ be a full rank decomposition of A_0. Let $\widehat{A} = \widehat{A_1}\widehat{A_2}$ be a dual r-rank decomposition of $\widehat{A}$ where*

$\widehat{A_1} = A_2 + \epsilon A_3$ and $\widehat{A_2} = A_4 + \epsilon A_5$. Then the DMPGI of $\widehat{A}$ exists, and

$$\widehat{A}^{\dagger} = A_4^{\dagger} A_2^{\dagger} + \epsilon \left(A_4^{\dagger} \left(A_2^T A_2 \right)^{-1} \left(A_3^T - Q_{A_2, A_3}^S A_2^{\dagger} \right) + \left(A_5^T - A_4^{\dagger} Q_{A_4^T, A_5^T}^S \right) \left(A_4 A_4^T \right)^{-1} A_2^{\dagger} \right), \tag{4.4}$$

where $Q_{A_4^T, A_5^T}^S = A_4 A_5^T + A_5 A_4^T$ and $Q_{A_2, A_3}^S = A_2^T A_3 + A_3^T A_2$.

Proof. According to Lemma 2.2, by substituting (2.4) and (2.6) into (4.2), we can get

$$\begin{aligned} \widehat{A}^{\dagger} = &\left(A_4^T \left(A_4 A_4^T \right)^{-1} + \epsilon \left(A_5^T \left(A_4 A_4^T \right)^{-1} - A_4^T \left(A_4 A_4^T \right)^{-1} Q_{A_4^T, A_5^T}^S \left(A_4 A_4^T \right)^{-1} \right) \right) \\ &\left(\left(A_2^T A_2 \right)^{-1} A_2^T + \epsilon \left(\left(A_2^T A_2 \right)^{-1} A_3^T - \left(A_2^T A_2 \right)^{-1} Q_{A_2, A_3}^S \left(A_2^T A_2 \right)^{-1} A_2^T \right) \right). \end{aligned}$$

Furthermore, from $A_4^{\dagger} = A_4^T \left(A_4 A_4^T \right)^{-1}$ and $A_2^{\dagger} = \left(A_2^T A_2 \right)^{-1} A_2^T$, we can get the formula for DMPGI $\widehat{A}^{\dagger}$ as shown in (4.4). □

Based on Theorem 4.3, the detailed calculation process of DMPGI is given below, and one corresponding example is also given to verify.

(1). Input matrix A_0, A_1, and the form of the dual matrix $\widehat{A}$ is $\widehat{A} = A_0 + \epsilon A_1,\ A_i \in \mathbb{R}_{m \times n},\ \mathrm{rk}(A_0) = r$;

(2). According to the method of calculating dual r-rank decomposition, we get $\widehat{A} = \widehat{A_1}\widehat{A_2}$ where $\widehat{A_1} = A_2 + \epsilon A_3$ is an r-column full rank dual matrix and $\widehat{A_2} = A_4 + \epsilon A_5$ is an r-row full rank dual matrix;

(3). Calculate $A_4^{\dagger}$, $A_2^{\dagger}$ and $A_4^{\dagger} A_2^{\dagger}$;

(4). Calculate $A_4^{\dagger} \left(A_2^T A_2 \right)^{-1} \left(A_3^T - Q_{A_2, A_3}^S A_2^{\dagger} \right) + \left(A_5^T - A_4^{\dagger} Q_{A_4^T, A_5^T}^S \right) \left(A_4 A_4^T \right)^{-1} A_2^{\dagger}$;

(5). Get the DMPGI $\widehat{A}^{\dagger}$ of $\widehat{A}$.

Example 4.1. *Let $\widehat{A}$, A_2, A_3, A_4 and A_5 be as given in Example 3.2. By applying (4.4), we can get the following result:*

$$\widehat{X} = A_4^{\dagger} A_2^{\dagger} + \epsilon \left(A_4^{\dagger} \left(A_2^T A_2 \right)^{-1} \left(A_3^T - Q_{A_2, A_3}^S A_2^{\dagger} \right) + \left(A_5^T - A_4^{\dagger} Q_{A_4^T, A_5^T}^S \right) \left(A_4 A_4^T \right)^{-1} A_2^{\dagger} \right)$$

$$= \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix}^T \left(\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix}^T \right)^{-1} \left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{bmatrix}^T$$

$$+ \epsilon \left\{ \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix}^T \left(\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix}^T \right)^{-1} \left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{bmatrix} \right)^{-1} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} \\ \frac{3}{2} & -4 \end{bmatrix}^T \right.$$

$$- \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix}^T \left(\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix}^T \right)^{-1} \left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} \\ \frac{3}{2} & -4 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} \\ \frac{3}{2} & -4 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{bmatrix} \right)$$

$$
\times\left(\begin{bmatrix}1&2\\2&1\\3&3\end{bmatrix}^T\begin{bmatrix}1&2\\2&1\\3&3\end{bmatrix}\right)^{-1}\begin{bmatrix}1&2\\2&1\\3&3\end{bmatrix}^T+\begin{bmatrix}\frac{3}{2}&\frac{13}{6}&\frac{29}{9}\\-1&\frac{7}{6}&\frac{31}{18}\end{bmatrix}^T\left(\begin{bmatrix}1&0&\frac{1}{3}\\0&1&\frac{1}{3}\end{bmatrix}\begin{bmatrix}1&0&\frac{1}{3}\\0&1&\frac{1}{3}\end{bmatrix}^T\right)^{-1}
$$

$$
\times\left(\begin{bmatrix}1&2\\2&1\\3&3\end{bmatrix}^T\begin{bmatrix}1&2\\2&1\\3&3\end{bmatrix}\right)^{-1}\begin{bmatrix}1&2\\2&1\\3&3\end{bmatrix}^T+\begin{bmatrix}1&0&\frac{1}{3}\\0&1&\frac{1}{3}\end{bmatrix}^T\left(\begin{bmatrix}1&0&\frac{1}{3}\\0&1&\frac{1}{3}\end{bmatrix}\begin{bmatrix}1&0&\frac{1}{3}\\0&1&\frac{1}{3}\end{bmatrix}^T\right)^{-1}
$$

$$
\times\left(\begin{bmatrix}1&0&\frac{1}{3}\\0&1&\frac{1}{3}\end{bmatrix}\begin{bmatrix}\frac{3}{2}&\frac{13}{6}&\frac{29}{9}\\-1&\frac{7}{6}&\frac{31}{18}\end{bmatrix}^T+\begin{bmatrix}\frac{3}{2}&\frac{13}{6}&\frac{29}{9}\\-1&\frac{7}{6}&\frac{31}{18}\end{bmatrix}\begin{bmatrix}1&0&\frac{1}{3}\\0&1&\frac{1}{3}\end{bmatrix}^T\right)
$$

$$
\left.\times\left(\begin{bmatrix}1&0&\frac{1}{3}\\0&1&\frac{1}{3}\end{bmatrix}\begin{bmatrix}1&0&\frac{1}{3}\\0&1&\frac{1}{3}\end{bmatrix}^T\right)^{-1}\left(\begin{bmatrix}1&2\\2&1\\3&3\end{bmatrix}^T\begin{bmatrix}1&2\\2&1\\3&3\end{bmatrix}\right)^{-1}\begin{bmatrix}1&2\\2&1\\3&3\end{bmatrix}^T\right\}
$$

$$
=\begin{bmatrix}-\frac{5}{11}&\frac{6}{11}&\frac{1}{11}\\\frac{6}{11}&-\frac{5}{11}&\frac{1}{11}\\\frac{1}{33}&\frac{1}{33}&\frac{2}{33}\end{bmatrix}+\epsilon\begin{bmatrix}-\frac{31}{33}&-\frac{16}{33}&\frac{1}{33}\\\frac{2}{11}&\frac{7}{11}&-\frac{8}{11}\\-\frac{25}{99}&\frac{38}{99}&\frac{10}{99}\end{bmatrix}.
$$

Furthermore, we prove that $\widehat{X}$ *satisfies the following four Penrose equations::*

$$
(1).\quad \widehat{A}\widehat{X}\widehat{A}=\begin{bmatrix}1&2&1\\2&1&1\\3&3&2\end{bmatrix}+\epsilon\begin{bmatrix}1&4&7\\2&5&8\\3&6&14\end{bmatrix}=\widehat{A};
$$

$$
(2).\quad \widehat{X}\widehat{A}\widehat{X}=\begin{bmatrix}-\frac{5}{11}&\frac{6}{11}&\frac{1}{11}\\\frac{6}{11}&-\frac{5}{11}&\frac{1}{11}\\\frac{1}{33}&\frac{1}{33}&\frac{2}{33}\end{bmatrix}+\epsilon\begin{bmatrix}-\frac{31}{33}&-\frac{16}{33}&\frac{1}{33}\\\frac{2}{11}&\frac{7}{11}&-\frac{8}{11}\\-\frac{25}{99}&\frac{38}{99}&\frac{10}{99}\end{bmatrix}=\widehat{X};
$$

$$
(3).\quad \left(\widehat{A}\widehat{X}\right)^T=\left(\begin{bmatrix}\frac{2}{3}&-\frac{1}{3}&\frac{1}{3}\\-\frac{1}{3}&\frac{2}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{2}{3}\end{bmatrix}+\epsilon\begin{bmatrix}\frac{10}{9}&\frac{1}{9}&-\frac{4}{9}\\\frac{1}{9}&-\frac{8}{9}&\frac{5}{9}\\-\frac{4}{9}&\frac{5}{9}&-\frac{2}{9}\end{bmatrix}\right)^T=\widehat{A}\widehat{X};
$$

$$
(4).\quad \left(\widehat{X}\widehat{A}\right)^T=\left(\begin{bmatrix}\frac{10}{11}&-\frac{1}{11}&\frac{3}{11}\\-\frac{1}{11}&\frac{10}{11}&\frac{3}{11}\\\frac{3}{11}&\frac{3}{11}&\frac{2}{11}\end{bmatrix}+\epsilon\begin{bmatrix}-\frac{10}{11}&-\frac{9}{11}&\frac{12}{11}\\-\frac{9}{11}&-\frac{8}{11}&\frac{9}{11}\\\frac{12}{11}&\frac{9}{11}&\frac{18}{11}\end{bmatrix}\right)^T=\widehat{X}\widehat{A}.
$$

Therefore, $\widehat{A}^{\dagger}=\widehat{X}=\begin{bmatrix}-\frac{5}{11}&\frac{6}{11}&\frac{1}{11}\\\frac{6}{11}&-\frac{5}{11}&\frac{1}{11}\\\frac{1}{33}&\frac{1}{33}&\frac{2}{33}\end{bmatrix}+\epsilon\begin{bmatrix}-\frac{31}{33}&-\frac{16}{33}&\frac{1}{33}\\\frac{2}{11}&\frac{7}{11}&-\frac{8}{11}\\-\frac{25}{99}&\frac{38}{99}&\frac{10}{99}\end{bmatrix}.$

4.2. Dual Idempotent Matrix

In [14], Udwadia discussed several types of special dual idempotent matrices, such as $\widehat{A}\widehat{A}^{\dagger}$, $\widehat{A}^{\dagger}\widehat{A}$, $I_m-\widehat{A}\widehat{A}^{\dagger}$ and $I_n-\widehat{A}^{\dagger}\widehat{A}$. In this subsection, we give some characterizations of dual idempotent matrix and its DMPGI by applying the dual r-rank decomposition.

Definition 4.1 ([14])**.** *Let* $\widehat{A}\in\mathbb{D}_{n\times n}$*,* $\widehat{A}=A_0+\epsilon A_1$ *and* $\mathrm{rk}(A_0)=r$*. If* $\widehat{A}$ *satisfies* $\widehat{A}^2=\widehat{A}$*, then* $\widehat{A}$ *is called dual idempotent matrix.*

Theorem 4.4. *Let $\widehat{A} \in \mathbb{D}_{n\times n}$, $\widehat{A} = A_0 + \epsilon A_1$ and $\mathrm{rk}(A_0) = r$. Then $\widehat{A}$ is a dual idempotent matrix if and only if*

$$A_0 = A_0^2 \quad \text{and} \quad A_1 = A_0A_1 + A_1A_0. \tag{4.5}$$

Proof. " $\Rightarrow$ ": If $\widehat{A} = A_0 + \epsilon A_1$ is a dual idempotent matrix, then we have $\widehat{A}^2 = \widehat{A}$ and $A_0^2 + \epsilon\,(A_0A_1 + A_1A_0) = A_0 + \epsilon A_1$. Therefore (4.5) is established.

" $\Leftarrow$ ": Since $\widehat{A} = A_0 + \epsilon A_1$, it is obvious that $\widehat{A}^2 = A_0^2 + \epsilon\,(A_0A_1 + A_1A_0)$. It follows from (4.5) that $\widehat{A}^2 = A_0 + \epsilon A_1 = \widehat{A}$. Therefore, according to Definition 4.1, we see that $\widehat{A}$ is a dual idempotent matrix. □

Corollary 4.5. *Let $\widehat{A} \in \mathbb{D}_{n\times n}$, $\widehat{A} = A_0+\epsilon A_1$ and $\mathrm{rk}(A_0) = r$. If $\widehat{A}$ is a dual idempotent matrix, and the real part matrix A_0 is invertible, then $\widehat{A} = I_n$.*

Proof. According to the Theorem 4.4, if $\widehat{A}$ is a dual idempotent matrix, then the equation (4.5) holds. If the real matrix A_0 is invertible, we can get $A_0 = I_n$. Since $A_0 = I_n$ and $A_1 = A_0A_1 + A_1A_0$, it is easy to check that $A_1 = 0$. Hence, $\widehat{A} = I_n$. □

Theorem 4.6. *Let $\widehat{A} \in \mathbb{D}_{n\times n}$, $\widehat{A} = A_0 + \epsilon A_1$ and $\mathrm{rk}(A_0) = r$. Let $A_0 = A_2A_4$ be a full rank decomposition of A_0. Then the dual r-rank decomposition of $\widehat{A}$ exists, and*

$$\widehat{A} = (A_2 + \epsilon A_1A_2)\,(A_4 + \epsilon A_4A_1) \tag{4.6}$$

which is a dual r-rank decomposition of $\widehat{A}$.

Proof. Let $\widehat{A}$ be a dual idempotent matrix, then the equation (4.5) holds. Let $A_0 = A_2A_4$ be a full rank decomposition of A_0, where A_2 is a column full rank matrix, and A_4 is a row full rank matrix. Write $\widehat{X} = A_2 + \epsilon A_1A_2$ and $\widehat{Y} = A_4 + \epsilon A_4A_1$. It is obvious that $\widehat{X}$ is an r-column full rank dual matrix and $\widehat{Y}$ is an r-row full rank dual matrix. It follows from (4.5) that

$$\begin{aligned}\widehat{X}\widehat{Y} &= (A_2 + \epsilon A_1A_2)\,(A_4 + \epsilon A_4A_1) = A_2A_4 + \epsilon\,(A_2A_4A_1 + \epsilon A_1A_2A_4)\\ &= A_0 + \epsilon\,(A_0A_1 + A_1A_0) = A_0 + \epsilon A_1.\end{aligned}$$

Therefore, the dual r-rank decomposition of $\widehat{A}$ exists and $\widehat{A} = (A_2 + \epsilon A_1A_2)\,(A_4 + \epsilon A_4A_1)$ is a dual r-rank decomposition of $\widehat{A}$. □

Theorem 4.7. *Let $\widehat{A} = A_0 + \epsilon A_1 \in \mathbb{D}_{n\times n}$ be a dual idempotent matrix. Then*

$$\widehat{A}^\dagger = A_0^\dagger + \epsilon\left(A_0^\dagger A_1^T + A_1^TA_0^\dagger - A_0^\dagger\left(A_1 + A_1^T\right)A_0A_0^\dagger - A_0^\dagger A_0\left(A_1^T + A_1\right)A_0^\dagger\right). \tag{4.7}$$

Proof. If $\widehat{A}$ is a dual idempotent matrix, according to Theorem 4.6, the dual r-rank decomposition of $\widehat{A}$ exists. Let $A_0 = A_2A_4$ be a full rank decomposition of A_0 and $\widehat{A} = \widehat{A_1}\widehat{A_2}$ be a dual r-rank decomposition of $\widehat{A}$ where $\widehat{A_1} = A_2 + \epsilon A_1A_2$ and $\widehat{A_2} = A_4 + \epsilon A_4A_1$. Because $\widehat{A_1}$ is an r-column full rank dual matrix and $\widehat{A_2}$ is an r-row full rank dual matrix, then

$$\begin{cases} \left(\widehat{A_1}^T\widehat{A_1}\right)^{-1} = (A_2^TA_2)^{-1} - \epsilon\left(A_2^\dagger(A_1^T + A_1)\left(A_2^\dagger\right)^T\right) \\ \left(\widehat{A_2}\widehat{A_2}^T\right)^{-1} = (A_4A_4^T)^{-1} - \epsilon\left(\left(A_4^\dagger\right)^T(A_1^T + A_1)A_4^\dagger\right) \end{cases}.$$

By applying (2.3), (2.5), (4.1) and the above equations to (4.3)

$$\begin{aligned} \widehat{A}^\dagger &= A_0^\dagger + \epsilon\left(A_0^\dagger A_1^T - A_0^\dagger\left(A_1A_2 + A_1^TA_2\right)A_2^\dagger + A_1^TA_0^\dagger - A_4^\dagger\left(A_4A_1^T + A_4A_1\right)A_0^\dagger\right) \\ &= A_0^\dagger + \epsilon\left(A_0^\dagger A_1^T - A_0^\dagger\left(A_1 + A_1^T\right)A_2A_2^\dagger + A_1^TA_0^\dagger - A_4^\dagger A_4\left(A_1^T + A_1\right)A_0^\dagger\right) \\ &= A_0^\dagger + \epsilon\left(A_0^\dagger A_1^T + A_1^TA_0^\dagger - A_0^\dagger\left(A_1 + A_1^T\right)A_0A_0^\dagger - A_0^\dagger A_0\left(A_1^T + A_1\right)A_0^\dagger\right). \end{aligned}$$

Therefore, we get (4.7). □

Theorem 4.8. *Let $\widehat{A} \in \mathbb{D}_{n\times n}$, $\widehat{A} = A_0 + \epsilon A_1$ and $\mathrm{rk}(A_0) = r$. Let $\widehat{A} = \widehat{A_1}\widehat{A_2}$ be a dual r-rank decomposition of $\widehat{A}$. Then $\widehat{A}$ is a dual idempotent matrix if and only if $\widehat{A_2}\widehat{A_1} = I_r$.*

Proof. " $\Rightarrow$ ": Let $\widehat{A}$ be a dual idempotent matrix, then the dual r-rank decomposition of $\widehat{A}$ exists. Let $A_0 = A_2A_4$ be a full rank decomposition of A_0, and $\widehat{A} = \widehat{A_1}\widehat{A_2} = (A_2 + \epsilon Y)(A_4 + \epsilon X)$ be a dual r-rank decomposition of $\widehat{A}$. Since $\widehat{A}$ is a dual idempotent matrix, by the first equation in (4.5), we see that A_0 is an idempotent matrix, and $A_4A_2 = I_r$. Therefore,

$$\widehat{A_2}\widehat{A_1} = I_r + \epsilon Z. \tag{4.8}$$

Because $\widehat{A}$ is a dual idempotent matrix, we have $\widehat{A_1}\widehat{A_2}\widehat{A_1}\widehat{A_2} = \widehat{A_1}\widehat{A_2}$,

$$\widehat{A_1}\widehat{A_2} = (A_2 + \epsilon Y)(A_4 + \epsilon X) = A_2A_4 + \epsilon(A_2X + YA_4)$$

and

$$\widehat{A_1}\widehat{A_2}\widehat{A_1}\widehat{A_2} = (A_2 + \epsilon Y)(I_r + \epsilon Z)(A_4 + \epsilon X) = A_2A_4 + \epsilon(A_2X + A_2ZA_4 + YA_4).$$

Therefore, $A_2ZA_4 = 0$. Since A_2 is a column full rank matrix and A_4 is a row full rank matrix, $Z = 0$. It follows from (4.8) that $\widehat{A_2}\widehat{A_1} = I_r$.

" $\Leftarrow$ ": Let $\widehat{A_2}\widehat{A_1} = I_r$. Then $\widehat{A}^2 = \widehat{A_1}\widehat{A_2}\widehat{A_1}\widehat{A_2} = \widehat{A_1}I_r\widehat{A_2} = \widehat{A_1}\widehat{A_2} = \widehat{A}$, that is, $\widehat{A}$ is a dual idempotent matrix. □

4.3. Dual EP Matrix

This subsection introduces one special dual matrix: dual EP matrix, and considers characterizations, dual r-rank decomposition and DMPGI of the special matrix.

Definition 4.2. *Let $\widehat{A} \in \mathbb{D}_{n\times n}$, and $\widehat{A}^{\dagger}$ exist. If*

$$\widehat{A}\widehat{A}^{\dagger} = \widehat{A}^{\dagger}\widehat{A}, \tag{4.9}$$

then $\widehat{A}$ is called a dual EP matrix.

Theorem 4.9. *Let $\widehat{A} \in \mathbb{D}_{n\times n}$, $\widehat{A} = A_0 + \epsilon A_1$ and $\operatorname{rk}(A_0) = r$. Let $\widehat{A} = \widehat{A_1}\widehat{A_2}$ be a dual r-rank decomposition of $\widehat{A}$. Then $\widehat{A}$ is a dual EP matrix if and only if*

$$\widehat{A_1}\widehat{A_1}^{\dagger} = \widehat{A_2}^{\dagger}\widehat{A_2}. \tag{4.10}$$

Proof. " $\Rightarrow$ ": Since the dual r-rank decomposition of $\widehat{A}$ exists, the DMPGI of $\widehat{A}$ exists. Let $\widehat{A} = \widehat{A_1}\widehat{A_2}$ be the dual r-rank decomposition of $\widehat{A}$, and $\widehat{A}$ be a dual EP matrix. According to Definition 4.2, we can get the equation (4.9). Then by applying (4.3) to (4.9), we get

$$\widehat{A_1}\widehat{A_2}\widehat{A_2}^T\left(\widehat{A_2}\widehat{A_2}^T\right)^{-1}\left(\widehat{A_1}^T\widehat{A_1}\right)^{-1}\widehat{A_1}^T = \widehat{A_2}^T\left(\widehat{A_2}\widehat{A_2}^T\right)^{-1}\left(\widehat{A_1}^T\widehat{A_1}\right)^{-1}\widehat{A_1}^T\widehat{A_1}\widehat{A_2},$$

that is, $\widehat{A_1}\left(\widehat{A_1}^T\widehat{A_1}\right)^{-1}\widehat{A_1}^T = \widehat{A_2}^T\left(\widehat{A_2}\widehat{A_2}^T\right)^{-1}\widehat{A_2}$. It follows from (2.3) and (2.5) that we obtain (4.10).

" $\Leftarrow$ ": Conversely, with the precondition that $\widehat{A_1}$ is an r-column full rank dual matrix and $\widehat{A_2}$ is an r-row full rank dual matrix, if the equation (4.10) holds, according to Lemma 2.2, we have $\widehat{A_1}^{\dagger} = \left(\widehat{A_1}^T\widehat{A_1}\right)^{-1}\widehat{A_1}^T$ and $\widehat{A_2}^{\dagger} = \widehat{A_2}^T\left(\widehat{A_2}\widehat{A_2}^T\right)^{-1}$. Then applying these two equations to the equation (4.10), we get $\widehat{A_1}\left(\widehat{A_1}^T\widehat{A_1}\right)^{-1}\widehat{A_1}^T = \widehat{A_2}^T\left(\widehat{A_2}\widehat{A_2}^T\right)^{-1}\widehat{A_2}$. Therefore,

$$\widehat{A_1}\widehat{A_2}\widehat{A_2}^T\left(\widehat{A_2}\widehat{A_2}^T\right)^{-1}\left(\widehat{A_1}^T\widehat{A_1}\right)^{-1}\widehat{A_1}^T = \widehat{A_2}^T\left(\widehat{A_2}\widehat{A_2}^T\right)^{-1}\left(\widehat{A_1}^T\widehat{A_1}\right)^{-1}\widehat{A_1}^T\widehat{A_1}\widehat{A_2}.$$

Hence, the equation (4.9) holds, that is, $\widehat{A}$ is a dual EP matrix. $\square$

Theorem 4.10. *Let $\widehat{A} \in \mathbb{D}_{n\times n}$, $\widehat{A} = A_0 + \epsilon A_1$, and the DMPGI of $\widehat{A}$ exist. Then $\widehat{A}$ is a dual EP matrix if and only if*

$$\begin{cases} A_0A_0^{\dagger} = A_0^{\dagger}A_0, & (4.11a) \\ \left(I_n - A_0^{\dagger}A_0\right)A_1A_0^{\dagger} = \left(A_0^{\dagger}A_1\left(I_n - A_0^{\dagger}A_0\right)\right)^T. & (4.11b) \end{cases}$$

Proof. By applying (2.2) and Definition 4.2, we can get that $\widehat{A}$ is a dual EP matrix if and only if

$$(A_0+\epsilon A_1)\left(A_0^{\dagger}-\epsilon R\right)=\left(A_0^{\dagger}-\epsilon R\right)(A_0+\epsilon A_1), \tag{4.12}$$

in which $R=A_0^{\dagger}A_1A_0^{\dagger}-\left(A_0^TA_0\right)^{\dagger}A_1^T\left(I_n-A_0A_0^{\dagger}\right)-\left(I_n-A_0^{\dagger}A_0\right)A_1^T\left(A_0A_0^T\right)^{\dagger}$.

" $\Rightarrow$ ": Let $\widehat{A}$ be a dual EP matrix. By applying (4.12), we see that

$$A_0A_0^{\dagger}+\epsilon\left(A_1A_0^{\dagger}-A_0R\right)=A_0^{\dagger}A_0+\epsilon\left(A_0^{\dagger}A_1-RA_0\right). \tag{4.13}$$

Therefore, we get (4.11a) and

$$A_1A_0^{\dagger}-A_0R=A_0^{\dagger}A_1-RA_0. \tag{4.14}$$

Since $A_0A_0^{\dagger}=A_0^{\dagger}A_0$, A_0 is EP. Then there exists an orthogonal matrix U such that

$$A_0=U\begin{bmatrix}T & 0\\ 0 & 0\end{bmatrix}U^T, \tag{4.15}$$

where $T\in\mathbb{R}_{r\times r}$ is a nonsingular matrix. It is easy to check that

$$\left(A_0A_0^T\right)^{\dagger}A_0=\left(A_0^T\right)^{\dagger}. \tag{4.16}$$

By applying (4.16) and $A_0A_0^{\dagger}=A_0^{\dagger}A_0$, we see that

$$\begin{aligned}A_1A_0^{\dagger}-A_0R&=A_1A_0^{\dagger}-A_0A_0^{\dagger}A_1A_0^{\dagger}+A_0\left(A_0^TA_0\right)^{\dagger}A_1^T\left(I_n-A_0A_0^{\dagger}\right),\\ &=\left(I_n-A_0A_0^{\dagger}\right)A_1A_0^{\dagger}+\left(A_0^T\right)^{\dagger}A_1^T\left(I_n-A_0A_0^{\dagger}\right),\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}A_0^{\dagger}A_1-RA_0&=A_0^{\dagger}A_1-A_0^{\dagger}A_1A_0^{\dagger}A_0+\left(I_n-A_0^{\dagger}A_0\right)A_1^T\left(A_0A_0^T\right)^{\dagger}A_0\\ &=A_0^{\dagger}A_1\left(I_n-A_0A_0^{\dagger}\right)+\left(I_n-A_0A_0^{\dagger}\right)A_1^T\left(A_0^T\right)^{\dagger}.\end{aligned} \tag{4.18}$$

By substituting (4.17) and (4.18) into (4.14) we get

$$\left(I_n-A_0A_0^{\dagger}\right)\left(A_1A_0^{\dagger}-A_1^T\left(A_0^T\right)^{\dagger}\right)=\left(A_0^{\dagger}A_1-\left(A_0^T\right)^{\dagger}A_1^T\right)\left(I_n-A_0A_0^{\dagger}\right). \tag{4.19}$$

It is obvious that $\left(I_n-A_0A_0^{\dagger}\right)\left(A_1A_0^{\dagger}-A_1^T\left(A_0^T\right)^{\dagger}\right)$ is an antisymmetric matrix.

Furthermore, write

$$A_1=U\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}U^T, \tag{4.20}$$

where $A_{11} \in \mathbb{R}_{r\times r}$. By applying (4.15) and (4.20), we get

$$
\begin{aligned}
&\left(I_n - A_0A_0^{\dagger}\right)\left(A_1A_0^{\dagger} - A_1^T\left(A_0^T\right)^{\dagger}\right)\\
&= U\begin{bmatrix} 0 & 0 \\ 0 & I_{n-r}\end{bmatrix} U^T \left(A_1 U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0\end{bmatrix} U^T - A_1^T U \begin{bmatrix} \left(T^T\right)^{-1} & 0 \\ 0 & 0\end{bmatrix} U^T\right)\\
&= U\begin{bmatrix} 0 & 0 \\ A_{21}T^{-1} - A_{21}^T\left(T^T\right)^{-1} & 0\end{bmatrix} U^T.
\end{aligned}
$$

Since it is an antisymmetric matrix and $A_0A_0^{\dagger} = A_0^{\dagger}A_0$, it is obvious that

$$
\left(I_n - A_0A_0^{\dagger}\right)\left(A_1A_0^{\dagger} - A_1^T\left(A_0^T\right)^{\dagger}\right) = 0. \tag{4.21}
$$

Therefore, we get (4.11b).

" $\Leftarrow$ ": Conversely, from (4.11a), we get that $\left(A_0A_0^T\right)^{\dagger}A_0 = \left(A_0^T\right)^{\dagger}$, A_0 is EP and A_0 has the decomposition (4.15). From (4.11b), we have (4.21). Therefore, we get (4.19).

By applying (4.11a), (4.19) and $\left(A_0A_0^T\right)^{\dagger}A_0 = \left(A_0^T\right)^{\dagger}$, we have (4.13) and (4.14). Therefore, we get (4.12), that is, $\widehat{A}$ is a dual EP matrix. □

Theorem 4.11. *Let $\widehat{A} \in \mathbb{D}_{n\times n}$, $\widehat{A} = A_0 + \epsilon A_1$ and $\mathrm{rk}(A_0) = r$. Let $A_0 = A_2A_4$ be a full rank decomposition of A_0. If the dual r-rank decomposition of $\widehat{A}$ exists, let $\widehat{A} = \widehat{A_1}\widehat{A_2}$ be a dual r-rank decomposition of $\widehat{A}$ where $\widehat{A_1} = A_2 + \epsilon A_3 \in \mathbb{D}_{n\times r}$ and $\widehat{A_2} = A_4 + \epsilon A_5 \in \mathbb{D}_{r\times n}$. then $\widehat{A}$ is a dual EP matrix if and only if*

$$
\begin{cases}
A_2\left(A_2^TA_2\right)^{-1}A_2^T = A_4^T\left(A_4A_4^T\right)^{-1}A_4 & (4.22\text{a})\\
\left(I_n - A_4^T\left(A_4A_4^T\right)^{-1}A_4\right)A_3A_2^{\dagger} = \left(A_4^{\dagger}A_5\left(I_n - A_4^T\left(A_4A_4^T\right)^{-1}A_4\right)\right)^T. & (4.22\text{b})
\end{cases}
$$

Proof. Let the dual r-rank decomposition of $\widehat{A}$ exist, then the DMPGI of $\widehat{A}$ exists. Let $\widehat{A} = \widehat{A_1}\widehat{A_2}$ be a dual r-rank decomposition of $\widehat{A}$ where $\widehat{A_1} = A_2 + \epsilon A_3$, $A_i(i = 2, 3) \in \mathbb{R}_{n\times r}$, $\widehat{A_2} = A_4 + \epsilon A_5$ and $A_i(i = 4, 5) \in \mathbb{R}_{r\times n}$.

" $\Rightarrow$ ": By applying (1.2) and the full rank decomposition of A_0 to (4.11a), we have (4.22a).

By applying (1.2) and $A_1 = A_2A_5 + A_3A_4$, we get

$$
\left(I_n - A_4^T\left(A_4A_4^T\right)^{-1}A_4\right)A_2A_5A_0^{\dagger} = \left(I_n - A_2\left(A_2^TA_2\right)^{-1}A_2^T\right)A_2A_5A_0^{\dagger} = 0, \tag{4.23}
$$

$$
\begin{aligned}
\left(I_n - A_4^T\left(A_4A_4^T\right)^{-1}A_4\right)A_3A_4A_0^{\dagger} &= \left(I_n - A_4^T\left(A_4A_4^T\right)^{-1}A_4\right)A_3A_4A_4^{\dagger}A_2^{\dagger}\\
&= \left(I_n - A_4^T\left(A_4A_4^T\right)^{-1}A_4\right)A_3A_2^{\dagger},
\end{aligned} \tag{4.24}
$$

and

$$
\begin{aligned}
\left(I_n - A_0^{\dagger}A_0\right) A_1 A_0^{\dagger} &= \left(I_n - A_4^T\left(A_4A_4^T\right)^{-1}A_4\right) A_1 A_0^{\dagger} \\
&= \left(I_n - A_4^T\left(A_4A_4^T\right)^{-1}A_4\right)\left(A_2A_5 + A_3A_4\right) A_0^{\dagger} \\
&= \left(I_n - A_4^T\left(A_4A_4^T\right)^{-1}A_4\right) A_3 A_2^{\dagger}.
\end{aligned} \tag{4.25}
$$

In the same way, we have

$$
A_0^{\dagger}A_1\left(I_n - A_0^{\dagger}A_0\right) = A_4^{\dagger}A_5\left(I_n - A_4^T\left(A_4A_4^T\right)^{-1}A_4\right). \tag{4.26}
$$

From (4.25), (4.26) and (4.11b), it follows that we get (4.22b).

" $\Leftarrow$ ": Conversely, if the equation (4.22a) holds, by applying the full rank decomposition of A_0, it is easy to check that $A_0A_0^{\dagger} = A_0^{\dagger}A_0$, that is (4.11a). Furthermore, let (4.22a) and (4.22b) hold simultaneously. Because A_0 is EP, $\left(I_n - A_4^T\left(A_4A_4^T\right)^{-1}A_4\right)A_2A_5A_0^{\dagger} = 0$ and $\left(I_n - A_4^T\left(A_4A_4^T\right)^{-1}A_4\right)A_3A_4A_0^{\dagger} = \left(I_n - A_4^T\left(A_4A_4^T\right)^{-1}A_4\right)A_3A_2^{\dagger}$. Therefore, we get that

$$
\left(I_n - A_4^T\left(A_4A_4^T\right)^{-1}A_4\right)A_3A_2^{\dagger} = \left(I_n - A_0^{\dagger}A_0\right)A_1A_0^{\dagger}.
$$

In the same way, we have $A_4^{\dagger}A_5\left(I_n - A_4^T\left(A_4A_4^T\right)^{-1}A_4\right) = A_0^{\dagger}A_1\left(I_n - A_0^{\dagger}A_0\right)$. It follows from applying both (4.22b) and Theorem 4.10 that $\widehat{A}$ is a dual EP matrix. □

4.4. Dual Penrose Equations

This subsection considers dual Penrose equations by applying dual r-rank decomposition.

Theorem 4.12. *Let $\widehat{A} \in \mathbb{D}_{m\times n}$, $\widehat{A} = A_0 + \epsilon A_1$ and $\mathrm{rk}(A_0) = r$. If the dual r-rank decomposition of $\widehat{A}$ exists and $\widehat{A_1}\widehat{A_2}$ is a dual r-rank decomposition of $\widehat{A}$, then*

$$
(a)\ \widehat{A_2}^{(i)}\widehat{A_1}^{(1)} \in \widehat{A}\{i\}(i = 1, 2, 4),\ (b)\ \widehat{A_2}^{\{1\}}\widehat{A_1}^{(j)} \in \widehat{A}\{j\}(i = 1, 2, 3).
$$

Proof. (a). When $i = 1$, both $\widehat{A_1}\widehat{A_1}^{(1)}$ and $\widehat{A_2}^{(1)}\widehat{A_2}$ are dual idempotent matrices, then $\widehat{A_1}^{(1)}\widehat{A_1} = I_r$, and $\widehat{A_2}\widehat{A_2}^{(1)} = I_r$, we get

$$
\widehat{A_1}\widehat{A_2}\widehat{A_2}^{(1)}\widehat{A_1}^{(1)}\widehat{A_1}\widehat{A_2} = \widehat{A_1}\widehat{A_2},
$$

that is, $\widehat{A_2}^{(1)}\widehat{A_1}^{(1)} \in \widehat{A}\{1\}$.

When $i = 2$, $\widehat{A_1}\widehat{A_1}^{(1)}$ is a dual idempotent matrix, then $\widehat{A_1}^{(1)}\widehat{A_1} = I_r$. Since $\widehat{A_2}^{(2)}\widehat{A_2}\widehat{A_2}^{(2)} = \widehat{A_2}^{(2)}$, we get

$$\widehat{A_2}^{(2)}\widehat{A_1}^{(1)}\widehat{A_1}\widehat{A_2}\widehat{A_2}^{(2)}\widehat{A_1}^{(1)} = \widehat{A_2}^{(2)}\widehat{A_1}^{(1)},$$

that is, $\widehat{A_2}^{(2)}\widehat{A_1}^{(1)} \in \widehat{A}\{2\}$.

When $i = 4$, $\widehat{A_1}\widehat{A_1}^{(1)}$ is a dual idempotent matrix, then $\widehat{A_1}^{(1)}\widehat{A_1} = I_r$, we get

$$\widehat{A_2}^{(4)}\widehat{A_1}^{(1)}\widehat{A_1}\widehat{A_2} = \widehat{A_2}^{(4)}\widehat{A_2} = \left(\widehat{A_2}^{(4)}\widehat{A_2}\right)^T = \left(\widehat{A_2}^{(4)}\widehat{A_1}^{(1)}\widehat{A_1}\widehat{A_2}\right)^T,$$

that is, $\widehat{A_2}^{(4)}\widehat{A_1}^{(1)} \in \widehat{A}\{4\}$.

(b) When $i = 1$, both $\widehat{A_1}\widehat{A_1}^{(1)}$ and $\widehat{A_2}^{(1)}\widehat{A_2}$ are dual idempotent matrices, then $\widehat{A_1}^{(1)}\widehat{A_1} = I_r$, $\widehat{A_2}\widehat{A_2}^{(1)} = I_r$, we get

$$\widehat{A_1}\widehat{A_2}\widehat{A_2}^{(1)}\widehat{A_1}^{(1)}\widehat{A_1}\widehat{A_2} = \widehat{A_1}\widehat{A_2},$$

that is, $\widehat{A_2}^{(1)}\widehat{A_1}^{(1)} \in \widehat{A}\{1\}$.

When $i = 2$, $\widehat{A_2}^{(1)}\widehat{A_2}$ is a dual idempotent matrix, then $\widehat{A_2}\widehat{A_2}^{(1)} = I_r$, and since $\widehat{A_1}^{(2)}\widehat{A_1}\widehat{A_1}^{(2)} = \widehat{A_1}^{(2)}$, we get

$$\widehat{A_2}^{(1)}\widehat{A_1}^{(2)}\widehat{A_1}\widehat{A_2}\widehat{A_2}^{(1)}\widehat{A_1}^{(2)} = \widehat{A_2}^{(1)}\widehat{A_1}^{(2)},$$

that is, $\widehat{A_2}^{\{1\}}\widehat{A_1}^{\{2\}} \in \widehat{A}\{2\}$.

When $i = 3$, $\widehat{A_2}^{(1)}\widehat{A_2}$ is a dual idempotent matrix, then $\widehat{A_2}\widehat{A_2}^{(1)} = I_r$, we get

$$\widehat{A_1}\widehat{A_2}\widehat{A_2}^{(1)}\widehat{A_1}^{(3)} = \widehat{A_1}\widehat{A_1}^{(3)} = \left(\widehat{A_1}\widehat{A_1}^{(3)}\right)^T = \left(\widehat{A_1}\widehat{A_2}\widehat{A_2}^{(1)}\widehat{A_1}^{(3)}\right)^T,$$

that is, $\widehat{A_2}^{\{1\}}\widehat{A_1}^{\{3\}} \in \widehat{A}\{3\}$. □

References

[1] Angeles J., The dual generalized inverses and their applications in kinematic synthesis[M]. Springer, Dordrecht, 2012.

[2] Ben-Israel A., Greville T. N. E., Generalized Inverses: Theory and Applications[M]. Springer, New York, 2003.

[3] Belzile B, Angeles J., Reflections over the dual ring-applications to kinematic synthesis[J]. Journal of Mechanical Design, 2019, 141(7): 1-9.

[4] Clifford W. K., Preliminary Sketch of Biquaternions[J]. Proceedings of the London Mathematical Society, 1873, 4(1): 381-395.

[5] Condurache D, Burlacu A., Orthogonal dual tensor method for solving the $AX = XB$ sensor calibration problem[J]. Mechanism and Machine Theory, 2016, 104: 382-404.

[6] Falco D. De., Pennestrì E., Udwadia F. E., On generalized inverses of dual matrices[J]. Mechanism and Machine Theory, 2018, 123: 89-106.

[7] Gutin R., Generalizations of singular value decomposition to dual-numbered matrices[J]. Linear and Multilinear Algebra, 2021: DOI: 10.1080/03081087.2021.1903830.

[8] Liu Y. H., Ranks of solutions of the linear matrix equation $AX + YB = C$[J]. Computers and Mathematics with Applications, 2006, 52(6-7): 861-872.

[9] Pennestrì E., Valentini P. P., Falco D De., The Moore-Penrose dual generalized inverse matrix with application to kinematic synthesis of spatial linkages[J]. Journal of Mechanical Design, 2018, 140(10): 102303.

[10] Pennestrì E., Valentini P. P., Linear dual algebra algorithms and their application to kinematics[M]. Springer, Dordrecht, 2009.

[11] Rao C. R., Linear Statistical Inference and its Applications[M]. Wiley Series in Probability and Statistics, New York, 1973.

[12] Study E., Von den Bewegungen and Umlegungen[J]. Mathematische Annalen, 1891, 39(4): 441-565.

[13] Udwadia F. E., Pennestrì E., Falco D. De., Do all dual matrices have dual Moore-Penrose generalized inverses?[J]. Mechanism and Machine Theory, 2020, 151: 103878.

[14] Udwadia F. E., Dual generalized inverses and their use in solving systems of linear dual equations[J]. Mechanism and Machine Theory, 2021, 156: 104158.

[15] Wang H. X., Characterizations and properties of the MPDGI and DMPGI[J]. Mechanism and Machine Theory. 2021, 158(7): 104212.

[16] Wang G. R., Wei Y. M., Qiao S. Z., Generalized Inverses: Theory and Computations[M]. Springer, Singapore, 2018.

[17] Yang J., Wang X. R., The application of the dual number methods to scara kinematics[C]. in: International Conference on Mechanic Automation and Control Engineering, Wuhan, China, 2010, 3871-3874.

[18] Zhang F. Z., Matrix Theory Basic Results and Techniques[M]. Springer, New York, 1999.