Reciprocal distance energy of complete multipartite graphs

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Abstract

In this paper, first we compute the energy of a special partitioned matrix under some cases. As a consequence, we obtain the reciprocal distance energy of the complete multipartite graph and also we give various other energies of complete multipartite graphs. Next, we show that among all complete k-partite graphs on n vertices, the complete split graph $CS(n, k-1)$ has minimum reciprocal distance energy and the reciprocal distance energy is maximum for the Turan graph $T(n, k)$. At last, it is shown that the reciprocal distance energy of the complete bipartite graph $K_{m,m}$ decreases under deletion of an edge if $2 \leq m \leq 7$, whereas the reciprocal distance energy increases if $8 \leq m$. Also, we show that the reciprocal distance energy of the complete tripartite graph does not increase under edge deletion.

Keywords: Complete multipartite graphs, reciprocal distance matrix, reciprocal distance energy.

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1 Introduction

Graphs considered in this paper are simple, connected and undirected. We denote the eigenvalues of a Hermitian matrix H of order n by $\lambda_1(H) \geq \lambda_2(H) \geq \ldots \geq \lambda_n(H)$. For a graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, the distance between two distinct vertices v_i and v_j is denoted by $d(v_i, v_j)$ and is equal to the length of a shortest path connecting the vertices v_i and v_j. The reciprocal distance matrix of G, well-known as Harary matrix, is a symmetric matrix of order n, denoted by $RD(G)$ and its ijth entry is equal to $\dfrac{1}{d(v_i, v_j)}$ if $i \neq j$, 0 otherwise. This matrix was introduced by Ivanciuc et al. for the design of topological indices in the year 1993, see [9]. The well-known topological index derived from the reciprocal distance matrix is the Harary index, see [23]. In [11], Ivanciuc et al. used the largest eigenvalue of the $RD(G)$ as one of the structural descriptor to develop structure–property models for the normal boiling temperature, molar heat capacity, standard Gibbs energy of formation, vaporization enthalpy, refractive index, and density of 134 alkanes C_6–C_{10}. In [3], Das obtained a lower and upper bound for the largest eigenvalue of the reciprocal distance matrix of a graph. Also, Nordhaus-Gaddum-type bounds for the largest eigenvalue of the reciprocal distance matrix were obtained therein. Graphs with maximum spectral radius of the reciprocal distance matrix in the classes of graphs (bipartite graphs) with fixed matching number and graphs with given number of cut edges were determined in [7].

Energy of a graph is a well-known graph invariant derived from the adjacency spectrum of a graph. This graph invariant nowadays known as ordinary energy of a graph was introduced by Gutman in connection with Hückel theory [4]. In analogous to the definition of graph energy, Güngör and Cevik in [6] introduced Harary energy of a graph, also called reciprocal distance energy of a graph. It is denoted by $\mathcal{E}_{RD}(G)$ and is defined as $\mathcal{E}_{RD}(G) = \sum_{i=1}^{n} |\lambda_i(RD(G))|$. Several lower and upper bounds for the reciprocal distance energy in terms of graph parameters are given in [1, 2, 12, 14]. In [19], Ramane et al. constructed pairs of reciprocal distance equienergetic graphs by determining the reciprocal distance energy of line graph of certain regular graph, and its complement. In [20], reciprocal distance equienergetic graphs are presented using the reciprocal distance spectrum of some generalized composition of graphs. Recent studies on the reciprocal distance matrix can be found in [15, 24].

The energy of a complex matrix M is the sum of all singular values of the matrix M and is denoted by $\mathcal{E}(M)$. The definition of energy of a complex matrix was put forward by Nikiforov as an extension of the concept of graph energy, see [16] for more details. We denote a complete k-partite graph by $K_{n_1,n_2,\ldots,n_k}$. The complete split graph $CS(n,k)$ is a graph on n vertices obtained by taking one copy of the complete graph K_k and joining each of its vertices with $n-k$ isolated vertices, i.e., $CS(n,k) \cong K_{n-k,1,1,\ldots,1}$. The Turan graph $T(n,k)$ is the complete k-partite graph on n vertices given by $T(n,k) \cong K_{q+1,q+1,\ldots,q+1,q,q\ldots,q}$, where $n = kq + r$ and $r \geq 0$. We denote the adjacency matrix of G by $A(G)$. For terminologies not defined in the paper, we refer to [5, 13].

In Section 2 of the paper, we compute the energy of a special partitioned matrix under some cases. As a consequence, we obtain the reciprocal distance energy of the complete multipartite graph and also we give various other energies of complete multipartite graphs. In Section 3, we show that among all complete k-partite graph on n vertices, the complete split graph $CS(n,k-1)$ has minimum reciprocal distance energy and the reciprocal distance energy is maximum for the Turan graph $T(n,k)$. In Section 4, it is shown that the reciprocal distance energy of the complete bipartite graph $K_{m,m}$ decreases under deletion of an edge if $2 \leq m \leq 7$, whereas the reciprocal distance energy increases if $8 \leq m$. Also, we show that the reciprocal distance energy of the complete tripartite graph does not increase under edge deletion.

2 Energy of a special partitioned matrix

Let $M_1, M_2, \ldots, M_k$ $(k \geq 2)$ be real symmetric matrices of order $n_1, n_2, \ldots, n_k$ such that $M_i \mathbf{1}_{n_i} = r_i \mathbf{1}_{n_i}$, where $r_i \geq 0$, and $trace(M_i) = 0$. Denote by $M = M[M_1, M_2, \ldots, M_k, a]$, a square matrix of order $n = n_1 + n_2 + \ldots + n_k$, defined as

$$M = \begin{bmatrix} M_1 & aJ_{n_1\times n_2} & aJ_{n_1\times n_3} & \cdots & aJ_{n_1\times n_k} \\ aJ_{n_2\times n_1} & M_2 & aJ_{n_2\times n_3} & \cdots & aJ_{n_2\times n_k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ aJ_{n_{k-1}\times n_1} & aJ_{n_{k-1}\times n_2} & \cdots & M_{k-1} & aJ_{n_{k-1}\times n_k} \\ aJ_{n_k\times n_1} & aJ_{n_k\times n_2} & \cdots & aJ_{n_k\times n_{k-1}} & M_k \end{bmatrix},$$

where a $(\neq 0)$ is a real constant and $J_{n_i\times n_j}$ is a rectangular matrix of order $n_i \times n_j$ with all its entries equal to 1.

Let $\dfrac{r_1}{n_1} = \max\limits_{1\leq i\leq k}\left\{\dfrac{r_i}{n_i}\right\}$ and $\dfrac{r_k}{n_k} = \min\limits_{1\leq i\leq k}\left\{\dfrac{r_i}{n_i}\right\}$. In this section, we compute the energy of the matrix $M = M[M_1, M_2, \ldots, M_k, a]$ under some cases. As a consequence, we determine the reciprocal distance energy of complete multipartite graph and also other energies of complete multipartite graph are obtained. For a Hermitian matrix H, $S^{-}(H)$ denote the sum of all negative eigenvalues of H.

The following lemma is a quantitative formulation of Sylvester's law of inertia due to Ostrowski.

Lemma 2.1 (Ostrowski [18])**.** *Let A be a Hermitian matrix of order n and S be a real non-singular matrix of order n. Then $\lambda_i(S^T AS) = \theta_i \lambda_i(A)$, where $\lambda_n(S^T S) \leq \theta_i \leq \lambda_1(S^T S)$.*

The following result is widely used in the study of graph eigenvalues.

Lemma 2.2. [8] *Let $M = N + P$, where N and P are Hermitian matrices of order n. Then for $1 \leq i, j \leq n$, we have (i) $\lambda_i(N) + \lambda_j(P) \leq \lambda_{i+j-n}(M)$ $(i + j > n)$ and (ii) $\lambda_{i+j-1}(M) \leq \lambda_i(N) + \lambda_j(P)$ $(i + j - 1 \leq n)$.*

Theorem 2.3. *For the matrix $M = M[M_1, M_2, \ldots, M_k, a]$ as defined above. We have*

i. $\mathcal{E}(M) = \sum_{i=1}^{k} \mathcal{E}(M_i)$ *if* $a > 0$ *and* $-a + \frac{r_k}{n_k} \geq 0$.

ii. $\mathcal{E}(M) = 2\lambda_1(M)$ *if* M_i *has at most one positive eigenvalue, namely* r_i, *for* $1 \leq i \leq k$, *and* $-a + \frac{r_1}{n_1} \leq 0$.

iii. $\mathcal{E}(M) = \sum_{i=1}^{k} \mathcal{E}(M_i) - 2\lambda_n(M)$ *if* $a < 0$ *and* $a(k-1) + \frac{r_k}{n_k} \leq \lambda_{n_i} \leq 0$ *for all* $1 \leq i \leq k$.

Proof. Let s_i denote the number of positive eigenvalues of the matrix M_i and let $\lambda_{i1} \geq \lambda_{i2} \geq \ldots \geq \lambda_{is_i} \geq \lambda_{i(s_i+1)} \geq \ldots \geq \lambda_{in_i}$ be the eigenvalues of the matrix M_i. Then

$$\begin{aligned}
\mathcal{E}(M_i) &= \lambda_{i1} + \lambda_{i2} + \ldots + \lambda_{i(s_i-1)} + \lambda_{is_i} - \lambda_{i(s_i+1)} - \ldots - \lambda_{in_i} \\
&= -2(\lambda_{i(s_i+1)} + \lambda_{i((s_i+2))} + \ldots + \lambda_{in_i}) \quad (\text{ Because}, trace(M_i) = 0) \\
&= -2S^-(M_i) \qquad (2.1)
\end{aligned}$$

Since M_i is a real symmetric matrix of n_i, there exists an orthogonal matrix P_i such that $P_i^T M_i P_i = D_i$, where D_i is a diagonal matrix of order n_i having $\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in_i}$ as its diagonal entries. Further, since $M_i 1_{n_i} = r_i 1_{n_i}$, we have $\lambda_{i\ell} = r_i$ for some $1 \leq \ell \leq n_i$, and without loss of generality, we can assume that the first column of P_i is equal to $\frac{1_{n_i}}{\sqrt{n_i}}$. Therefore, $P_i^T J_{n_i \times n_j} P_j = \sqrt{n_i n_j} e_{n_i \times n_j}$, where $e_{n_i \times n_j}$ is a rectangular matrix of order $n_i \times n_j$ whose all entries are 0, except the first diagonal entry which is equal to 1. Consider

$$M = \begin{bmatrix}
M_1 & aJ_{n_1 \times n_2} & aJ_{n_1 \times n_3} & \cdots & aJ_{n_1 \times n_k} \\
aJ_{n_2 \times n_1} & M_2 & aJ_{n_2 \times n_3} & \cdots & aJ_{n_2 \times n_k} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
aJ_{n_{k-1} \times n_1} & aJ_{n_{k-1} \times n_2} & \cdots & M_{k-1} & aJ_{n_{k-1} \times n_k} \\
aJ_{n_k \times n_1} & aJ_{n_k \times n_2} & \cdots & aJ_{n_k \times n_{k-1}} & M_k
\end{bmatrix}$$

$$= \begin{bmatrix} P_1D_1P_1^T & aJ_{n_1\times n_2} & aJ_{n_1\times n_3} & \dots & aJ_{n_1\times n_k} \\ aJ_{n_2\times n_1} & P_2D_2P_2^T & aJ_{n_2\times n_3} & \dots & aJ_{n_2\times n_k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ aJ_{n_{k-1}\times n_1} & aJ_{n_{k-1}\times n_2} & \dots & P_{k-1}D_{k-1}P_{k-1}^T & aJ_{n_{k-1}\times n_k} \\ aJ_{n_k\times n_1} & aJ_{n_k\times n_2} & \dots & aJ_{n_k\times n_{k-1}} & P_kD_kP_k^T \end{bmatrix}$$

$$= \begin{bmatrix} P_1 & 0 & 0 & \dots & 0 \\ 0 & P_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_{k-1} & 0 \\ 0 & 0 & \dots & 0 & P_k \end{bmatrix} \begin{bmatrix} D_1 & aP_1^TJ_{n_1\times n_2}P_2 & aP_1^TJ_{n_1\times n_3}P_3 \\ aP_2^TJ_{n_2\times n_1}P_1 & D_2 & aP_2^TJ_{n_2\times n_3}P_3 \\ \vdots & \vdots & \ddots \\ aP_{k-1}^TJ_{n_{k-1}\times n_1}P_1 & aP_{k-1}^TJ_{n_{k-1}\times n_2}P_2 & \dots \\ aP_k^TJ_{n_k\times n_1}P_1 & aP_k^TJ_{n_k\times n_2}P_2 & \dots \end{bmatrix}$$

$$\begin{matrix} \dots & aP_1^TJ_{n_1\times n_k}P_k \\ \dots & aP_2^TJ_{n_2\times n_k}P_k \\ \vdots & \vdots \\ D_{k-1} & aP_{k-1}^TJ_{n_{k-1}\times n_k}P_k \\ aP_k^TJ_{n_k\times n_{k-1}}P_{k-1} & D_k \end{matrix} \Bigg] \begin{bmatrix} P_1^T & 0 & 0 & \dots & 0 \\ 0 & P_2^T & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_{k-1}^T & 0 \\ 0 & 0 & \dots & 0 & P_k^T \end{bmatrix}$$

$$= \begin{bmatrix} P_1 & 0 & 0 & \dots & 0 \\ 0 & P_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_{k-1} & 0 \\ 0 & 0 & \dots & 0 & P_k \end{bmatrix} \begin{bmatrix} D_1 & a\sqrt{n_1n_2}e_{n_1\times n_2} & a\sqrt{n_1n_3}e_{n_1\times n_3} \\ a\sqrt{n_2n_1}e_{n_2\times n_1} & D_2 & a\sqrt{n_2n_3}e_{n_2\times n_3} \\ \vdots & \vdots & \ddots \\ a\sqrt{n_{k-1}n_1}e_{n_{k-1}\times n_1} & a\sqrt{n_{k-1}n_2}e_{n_{k-1}\times n_2} & \dots \\ a\sqrt{n_kn_1}e_{n_k\times n_1} & a\sqrt{n_kn_2}e_{n_k\times n_2} & \dots \end{bmatrix}$$

$$
\begin{matrix} \cdots & a\sqrt{n_1n_k}e_{n_1\times n_k} \\ \cdots & a\sqrt{n_2n_k}e_{n_2\times n_k} \\ \vdots & \vdots \\ D_{k-1} & a\sqrt{n_{k-1}n_k}e_{n_{k-1}\times n_k} \\ a\sqrt{n_kn_{k-1}}e_{n_k\times n_{k-1}} & D_k \end{matrix}\Bigg]
\begin{bmatrix} P_1^T & 0 & 0 & \dots & 0 \\ 0 & P_2^T & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_{k-1}^T & 0 \\ 0 & 0 & \dots & 0 & P_k^T \end{bmatrix}.
$$

Let

$$
M' = \Bigg[\begin{matrix} D_1 & a\sqrt{n_1n_2}e_{n_1\times n_2} & a\sqrt{n_1n_3}e_{n_1\times n_3} \\ a\sqrt{n_2n_1}e_{n_2\times n_1} & D_2 & a\sqrt{n_2n_3}e_{n_2\times n_3} \\ \vdots & \vdots & \ddots \\ a\sqrt{n_{k-1}n_1}e_{n_{k-1}\times n_1} & a\sqrt{n_{k-1}n_2}e_{n_{k-1}\times n_2} & \dots \\ a\sqrt{n_kn_1}e_{n_k\times n_1} & a\sqrt{n_kn_2}e_{n_k\times n_2} & \dots \end{matrix}
$$

$$
\begin{matrix} \dots & a\sqrt{n_1n_k}e_{n_1\times n_k} \\ \dots & a\sqrt{n_2n_k}e_{n_2\times n_k} \\ \vdots & \vdots \\ D_{k-1} & a\sqrt{n_{k-1}n_k}e_{n_{k-1}\times n_k} \\ a\sqrt{n_kn_{k-1}}e_{n_k\times n_{k-1}} & D_k \end{matrix} \Bigg].
$$

Then from equation (2.2) the matrices M and M' are similar. Thus $Spec(M) = Spec(M')$. Now, consider $det(xI - M')$. Expanding $det(xI - M')$ by Laplace's method along all the columns except 1^{st}, $(n_1 + 1)^{\text{th}}$, $(n_1 + n_2 + 1)^{\text{th}}, \dots, (n_1 + n_2 + \ldots + n_{k-1} + 1)^{\text{th}}$ columns, we get

$$
det(xI - M') = det(xI - M'') \prod_{i=1}^{k} \left[det(xI - D_i)(x - r_i)^{-1} \right],
$$

where

$$M'' = \begin{bmatrix} r_1 & a\sqrt{n_1n_2} & a\sqrt{n_1n_3} & \dots & a\sqrt{n_1n_k} \\ a\sqrt{n_2n_1} & r_2 & a\sqrt{n_2n_3} & \dots & a\sqrt{n_2n_k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a\sqrt{n_{k-1}n_1} & a\sqrt{n_{k-1}n_2} & \dots & r_{k-1} & a\sqrt{n_{k-1}n_k} \\ a\sqrt{n_kn_1} & a\sqrt{n_kn_2} & \dots & a\sqrt{n_kn_{k-1}} & r_k \end{bmatrix}.$$

Thus,

$$Spec(M) = \bigcup_{i=1}^{k}(Spec(M_i)\backslash\{r_i\}) \bigcup Spec(M''). \tag{2.2}$$

Since $trace(M) = 0$ and $r_i \geq 0$, we get

$$\begin{aligned} \mathcal{E}(M) &= -2\sum_{i=1}^{k} S^-(M_i) - 2S^-(M'') \\ &= \sum_{i=1}^{k} \mathcal{E}(M_i) - 2S^-(M''), \text{by equation (2.1).} \end{aligned} \tag{2.3}$$

Note that $M'' = C(aJ_{k\times k} - aI_k + D)C$, where $C = diag(\sqrt{n_1}, \sqrt{n_2}, \dots, \sqrt{n_k})$ and $D = diag\left(\frac{r_1}{n_1}, \frac{r_2}{n_2}, \dots, \frac{r_k}{n_k}\right)$. Thus the matrices M'' and $(aJ_{k\times k} - aI_k) + D$ are congruent to each other. Thus by Sylvester's law of inertia the matrices M'' and $(aJ_{k\times k} - aI_k) + D$ have same rank, inertia and signature.

Case I: Suppose $a > 0$ and $-a + \frac{r_k}{n_k} \geq 0$. By Lemma 2.2, we have

$$\lambda_k(aJ_{k\times k} - aI_k) + \lambda_k(D) \leq \lambda_k(aJ_{k\times k} - aI_k + D).$$

Therefore,

$$0 \leq -a + \frac{r_k}{n_k} \leq \lambda_k(aJ_{k\times k} - aI_k + D).$$

Thus, $aJ_{k\times k} - aI_k + D$ is positive semidefinite. Since M'' and $aJ_{k\times k} - aI_k + D$ are congruent, it follows that M'' is positive semidefinite. Hence $S^-(M'') = 0$. So, from (2.3), we get

$$\mathcal{E}(M) = \sum_{i=1}^{k} \mathcal{E}(M_i).$$

Thus proof of (i) is done.

Case II: Suppose M_i has at most one positive eigenvalue, namely r_i, for $1 \leq i \leq k$, and $-a + \dfrac{r_1}{n_1} \leq 0$. By Lemma 2.2, we have

$$\lambda_2(aJ_{k\times k} - aI_k + D) \leq \lambda_2(aJ_{k\times k} - aI_k) + \lambda_1(D)$$

Therefore,

$$\lambda_2(aJ_{k\times k} - aI_k + D) \leq -a + \frac{r_1}{n_1} \leq 0.$$

Thus, $aJ_{k\times k} - aI_k + D$ has only one positive eigenvalue. Since M'' and $aJ_{k\times k} - aI_k + D$ are congruent matrices, it follows that M'' has only one positive eigenvalue. Also, since M_i has at most one positive eigenvalue, namely r_i, for $1 \leq i \leq k$, M has only one positive eigenvalue by (2.2). Further since $trace(M) = 0$, we must have $\mathcal{E}(M) = 2\lambda_1(M)$. This proves (ii).

Case III: Suppose $a < 0$ and $\dfrac{r_1}{n_1} + a(k-1) \leq \lambda_{in_i} \leq 0$ for all $1 \leq i \leq k$. From Lemma 2.2, we get

$$\lambda_{k-1}(aJ_{k\times k} - aI_k) + \lambda_k(D) \leq \lambda_{k-1}(aJ_{k\times k} - aI_k + D)$$

and

$$\lambda_k(aJ_{k\times k} - aI_k + D) \leq \lambda_k(aJ_{k\times k} - aI_k) + \lambda_1(D).$$

Therefore,

$$0 \leq -a + \frac{r_k}{n_k} \leq \lambda_{k-1}(aJ_{k\times k} - aI_k + D)$$

and

$$\lambda_k(aJ_{k\times k} - aI_k + D) \leq a(k-1) + \frac{r_1}{n_1} \leq 0.$$

Thus $aJ_{k\times k} - aI_k + D$ has at most one negative eigenvalue. Since $aJ_{k\times k} - aI_k + D$ and M'' are congruent matrices, M'' has at most one negative eigenvalue. Moreover, $\lambda_k(M'') \leq \lambda_k(aJ_{k\times k} - aI_k + D)$, by Lemma 2.1. Therefore by (2.2), $\lambda_k(M'') = \lambda_n(M)$ because $\lambda_k(M'') \leq a(k-1) + \dfrac{r_1}{n_1} \leq \lambda_{in_i}$. Hence from equation (2.3), we get $\mathcal{E}(M) = \sum_{i=1}^{k} \mathcal{E}(M_i) - 2\lambda_n(M)$. This completes the proof. □

Corollary 2.4. [21] The distance energy of the complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$ is equal to $4(n-k)$, where $n = n_1+n_2+\ldots+n_k$ and $n_1 \geq n_2 \geq \ldots \geq n_k \geq 2$.

Proof. From the definition of the distance matrix of a connected graph, the distance matrix of the complete multipartite graph $K_{n_1,n_2,\ldots n_k}$ is $M[2(J_{n_1}-I_{n_1}), 2(J_{n_1}-I_{n_1}), \ldots, 2(J_{n_1}-I_{n_1}), 1]$. Since $-1+\dfrac{r_i}{n_i} = -1+\dfrac{2(n_i-1)}{n_i} = 1-\dfrac{2}{n_i} \geq 0$ for all i and $\mathcal{E}(2(J_{n_i}-I_{n_i})) = 4(n_i-1)$, from Theorem 2.3 (i), we get $\mathcal{E}_D(G) = \sum_{i=1}^{n} \mathcal{E}(2(J_{n_i}-I_{n_i})) = \sum_{i=1}^{n} 4(n_i-1) = 4(n-k)$. □

Corollary 2.5. *The reciprocal distance energy of the complete multipartite graph $G = K_{n_1,n_2,\ldots,n_k}$ is equal to $2\lambda_1(RD(G))$.*

Proof. From the definition of the reciprocal distance matrix of a connected graph, the reciprocal distance matrix of the complete multipartite graph G is $M[\frac{1}{2}(J_{n_1}-I_{n_1}), \frac{1}{2}(J_{n_2}-I_{n_2}), \ldots, \frac{1}{2}(J_{n_k}-I_{n_k}), 1]$. Since $-1+\dfrac{r_i}{n_i} = -1+\dfrac{n_i-1}{2n_i} = -\dfrac{1}{2}-\dfrac{1}{2n_i} \leq 0$ for all i, from Theorem 2.3 (ii), we get $\mathcal{E}_{RD}(G) = 2\lambda_1(RD(G))$. □

Corollary 2.6. [17] The Seidel energy of the complete multipartite graph $G = K_{n_1,n_2,\ldots,n_k}$ with $k \geq 3$ is equal to $2(n-k)-2\lambda_n(S(G))$, where $n = n_1+n_2+\ldots+n_k$.

Proof. From the definition of the Seidel matrix of a graph, the Seidel matrix of the complete multipartite graph G is $S(G) = M[J_{n_1}-I_{n_1}, J_{n_2}-I_{n_2}, \ldots, J_{n_k}-I_{n_k}, -1]$.Therefore from Theorem 2.3 (iii) we are done. □

Corollary 2.7. [22] The energy of the complete multipartite graph $G = K_{n_1,n_2,\ldots,n_k}$ is $2\lambda_1(A(G))$.

Proof. The adjacency matrix of G is $M = M[\mathbf{0}_{n_1}, \mathbf{0}_{n_2}, \ldots, \mathbf{0}_{n_k}, 1]$, Where $\mathbf{0}_{n_i}$ is a null matrix of order n_i. Therefore by Theorem 2.3 (ii), the corollary follows. □

Corollary 2.8. *The complementary distance energy of $G = K_{n_1,n_2,\ldots,n_k}$ is $2\lambda_1(CD(G))$.*

Proof. The complementary distance matrix [10] of the complete multipartite graph G is $M[J_{n_1}-I_{n_1}, J_{n_2}-I_{n_2}, \ldots, J_{n_k}-I_{n_k}, 2]$. Since $-2+\frac{r_i}{n_i} = -2+\frac{n_i-1}{n_i} = -1+\frac{1}{n_i} \leq 0$ for all i, from Theorem 2.3 (ii), we get $\mathcal{E}_{CD}(G) = 2\lambda_1(CD(G))$. □

Corollary 2.9. *The reciprocal complementary distance energy of* $G = K_{n_1,n_2,\ldots,n_k}$ *is* $2(n-k)$*, where* $n = n_1+n_2+\ldots+n_k$ *and* $n_1 \geq n_2 \geq \ldots \geq n_k \geq 2$.

Proof. The reciprocal complementary distance matrix [10] of the complete multipartite graph G is $M[J_{n_1}-I_{n_1}, J_{n_2}-I_{n_2}, \ldots, J_{n_k}-I_{n_k}, 1/2]$. Since $-\frac{1}{2}+\frac{r_i}{n_i} = -\frac{1}{2}+\frac{n_i-1}{n_i} = \frac{1}{2}-\frac{1}{n_i} \geq 0$ for all i and $\mathcal{E}(J_{n_i}-I_{n_i}) = 2(n_i-1)$, from Theorem 2.3 (i), we get $\mathcal{E}_{RCD}(G) = 2(n-k)$. □

3 Extremal complete multipartite graphs with respect to reciprocal distance energy

In this section, we show that among all complete k-partite graph on n vertices, the complete split graph $CS(n, k-1)$ has minimum reciprocal distance energy and the reciprocal distance energy is maximum for the Turan graph $T(n, k)$.

We need the following Cauchy's interlace theorem.

Lemma 3.1. [8] Let M be a Hermitian matrix of order n and let N be a principal submatrix of M of order k. Then $\lambda_{n-k+i}(M) \leq \lambda_i(N) \leq \lambda_i(M)$ for $i = 1, 2, \ldots, k$.

Lemma 3.2. *Let* $G_1 = K_{n_1,n_2,\ldots,n_s,n_{s+1},\ldots,n_k}$ *and* $G_2 = K_{n_1,n_2,\ldots,n_s-1,n_{s+1}+1,\ldots,n_k}$*, where* $n_s - n_{s+1} \geq 2$*. Then* $\lambda_1(RD(G_2)) > \lambda_1(RD(G_1))$.

Proof. Let $V_1, V_2, V_3, \ldots, V_k$ denote the vertex partition sets of the complete multipartite graph G_1. Let $V_i = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$. Then form the definition of the complete multipartite graph, the vertices $v_{i1}, v_{i2}, \ldots, v_{in_i}$ are symmetric. Since the reciprocal distance matrix of G_1 is non-negative, it follows from Perron-Frobenius theory that $\lambda_1(RD(G_1))$ is simple and there exists an unit eigenvector X (say) corresponding to the eigenvalue $\lambda_1(RD(G_1))$ with all its entries being positive. Let $x_{i1}, x_{i2}, \ldots x_{in_i}$ be the components of X corresponding to the vertices $v_{i1}, v_{i2}, \ldots, v_{in_i}$. Suppose $x_{ij} \neq x_{ik}$ for $1 \leq j, k \leq n_i$. Let X' be the vector obtained from X by interchanging the components x_{ij} and x_{ik}. Then the vectors X and X' are linearly independent and $RD(G_1)X' = \lambda_1(RD(G_1))X'$, because $x_{ij} \neq x_{ik}$ and the vertices $v_{i1}, v_{i2}, \ldots, v_{in_i}$ are symmetric. Therefore multiplicity of $\lambda_1(RD(G_1))$ is at least 2, a contradiction. Hence $x_{i1} = x_{i2} = \ldots = x_{in_i}$ for all $1 \leq i \leq k$. Let $n = n_1 + n_2 + \ldots + n_k$. From Rayleigh quotient inequality, we have

$$\begin{aligned}
\lambda_1(RD(G_2)) &\geq X^T RD(G_2) X \\
&= X^T \big[\frac{1}{2}(J_{n\times n} - I_n) + \frac{1}{2}A(G_2)\big] X \\
&= X^T \frac{1}{2}(J_{n\times n} - I_n)X + \frac{1}{2}X^T A(G_2) X \\
&= X^T \frac{1}{2}(J_{n\times n} - I_n)X + \sum_{v_{ip}v_{jq} \in E(G_2)} x_{ip}x_{jq} \\
&= X^T \frac{1}{2}(J_{n\times n} - I_n)X + \sum_{v_{ip}v_{jq} \in E(G_1)} x_{ip}x_{jq} - \sum_{j=1}^{n_{s+1}} x_{sn_s}x_{(s+1)j} + \sum_{j=1}^{n_s-1} x_{sn_s}x_{sj} \\
&= X^T \frac{1}{2}(J_{n\times n} - I_n)X + \frac{1}{2}X^T A(G_1) X - n_{s+1}x_{sn_s}x_{(s+1)1} + (n_s - 1)x_{sn_s}x_{s1} \\
&= X^T RD(G_1) X - n_{s+1}x_{sn_s}x_{(s+1)1} + (n_s - 1)x_{sn_s}x_{s1} \\
&= \lambda_1(RD(G_1)) - n_{s+1}x_{sn_s}x_{(s+1)1} + (n_s - 1)x_{sn_s}x_{s1}. \qquad (3.1)
\end{aligned}$$

Now, consider

$$\begin{aligned}
\lambda_1(RD(G_1))x_{i1} &= \sum_{\substack{p=1 \\ p\neq i}}^{k} \sum_{q=1}^{n_p} x_{pq} + \frac{1}{2}\sum_{q=2}^{n_i} x_{iq} \\
&= \sum_{\substack{p=1 \\ p\neq i}}^{k} n_p x_{p1} + \frac{1}{2}(n_i - 1)x_{i1} \\
&= \sum_{p=1}^{k} n_p x_{p1} - \frac{(n_i+1)}{2}x_{i1}.
\end{aligned}$$

Therefore,

$$x_{i1} = \frac{2X}{2\lambda_1(RD(G_1)) + n_i + 1}, \text{ where } X = \sum_{p=1}^{k} n_p x_{p1}. \tag{3.2}$$

Using equation (3.2) in (3.1), we get $\lambda_1(RD(G_2)) - \lambda_1(RD(G_1))$

$$\begin{aligned}
&\geq 2Xx_{sn_s}\left(\frac{-n_{s+1}}{2\lambda_1(RD(G_1)) + n_{s+1} + 1} + \frac{n_s - 1}{2\lambda_1(RD(G_1)) + n_s + 1}\right) \\
&= 2Xx_{sn_s}\left(\frac{2\lambda_1(RD(G_1))(n_s - n_{s+1}) - 2\lambda_1(RD(G_1)) + n_s - 2n_{s+1} - 1}{(2\lambda_1(RD(G_1)) + n_{s+1} + 1)(2\lambda_1(RD(G_1)) + n_s + 1)}\right) \\
&\geq 2Xx_{sn_s}\left(\frac{2\lambda_1(RD(G_1)) + n_s - 2n_{s+1} - 1}{(2\lambda_1(RD(G_1)) + n_{s+1} + 1)(2\lambda_1(RD(G_1)) + n_s + 1)}\right) \qquad (3.3)
\end{aligned}$$

Since $RD(K_{n_s,n_{s+1}})$ is the principal submatrix of $RD(G_1)$, by Lemma 3.1, we have

$$\begin{aligned}
\lambda_1(RD(G_1)) &\geq \lambda_1(RD(K_{n_s,n_{s+1}})) \\
&= \frac{1}{4}\left(n_s + n_{s+1} - 2 + \sqrt{n_s^2 + 14n_s n_{s+1} + n_{s+1}^2}\right) \\
&> \frac{1}{2}(n_s + n_{s+1} - 1) \qquad (3.4)
\end{aligned}$$

Using equation (3.4) in (3.3), we get $\lambda_1(RD(G_2)) - \lambda_1(RD(G_1))$

$$\begin{aligned}
&> 2Xx_{sn_s}\left(\frac{2n_s - n_{s+1} - 2}{(2\lambda_1(RD(G_1)) + n_{s+1} + 1)(2\lambda_1(RD(G_1)) + n_s + 1)}\right) \\
&> 0.
\end{aligned}$$

Thus $\lambda_1(RD(G_2)) > \lambda_1(RD(G_1))$. Hence the proof of the theorem. □

Theorem 3.3. *Let G be a complete k-partite graph on n vertices. Then*

$$\mathcal{E}_{RD}(CS(n, k-1)) \leq \mathcal{E}_{RD}(G) \leq \mathcal{E}_{RD}(T(n,k)).$$

Moreover, the left equality holds if and only if $G \cong CS(n, k-1)$ and the right equality holds if and only if $G \cong T(n,k)$.

Proof. Let $G \cong K_{n_1,n_2,\ldots,n_k}$. Then $n = n_1 + n_2 + \ldots + n_k$.

Left inequality: Suppose $n_2 \geq 2$. Then consider the graph $G_1 \cong K_{n_1+1,n_2-1,\ldots,n_k}$. From Lemma 3.2, we get $\lambda_1(RD(G)) > \lambda_1(RD(G_1))$. Thus by Corollary 2.5, we have $\mathcal{E}_{RD}(G) > \mathcal{E}_{RD}(G_1)$. Now, if $n_2 - 1 \geq 2$. Then by a similar argument, $\mathcal{E}_{RD}(G_1) > \mathcal{E}_{RD}(G_2)$, where $G_2 \cong K_{n_1+2,n_2-2,\ldots,n_k}$. Thus $\mathcal{E}_{RD}(G) > \mathcal{E}_{RD}(G_2)$. Repeating, n_2-1 times, we get $\mathcal{E}_{RD}(G) > \mathcal{E}_{RD}(K_{n_1+n_2-1,n_3,\ldots,n_k,1})$. Hence $\mathcal{E}_{RD}(G) > \mathcal{E}_{RD}(K_{n_1+n_2-1,n_3,\ldots,n_k,1}) > \mathcal{E}_{RD}(K_{n_1+n_2+n_3-2,n_4,\ldots,n_k,1,1}) > \ldots > \mathcal{E}_{RD}(K_{n-k-1,1,\ldots,1,1})$. Therefore, $\mathcal{E}_{RD}(G) > CS(n, k-1)$.

Right inequality: If $n_s - n_{s+1} \geq 2$ for some $1 \leq s < k$. Then by Lemma 3.2, $\lambda_1(RD(G)) < \lambda_1(RD(G_1))$, where $G_1 \cong K_{n'_1,n'_2,\ldots,n'_s,n'_{s+1},\ldots,n'_k}$, $n'_i = n_i$ for $i \neq s, s+1$, $n'_s = n_s - 1$ and $n'_{s+1} = n_{s+1} + 1$. Thus $\mathcal{E}_{RD}(G) < \mathcal{E}_{RD}(G_1)$. Now, if $n'_t - n'_{t+1} \geq 2$ for some $1 \leq t < k$. Then by a similar argument, we get $\mathcal{E}_{RD}(G_1) < \mathcal{E}_{RD}(G_2)$, where $G_2 \cong K_{n''_1,n''_2,\ldots,n''_k}$, $n''_i = n'_i$ for $i \neq t, t+1$, $n''_t = n'_t - 1$ and $n''_{t+1} = n'_{t+1} + 1$. So, $\mathcal{E}_{RD}(G) < \mathcal{E}_{RD}(G_2)$. Repeating the process until the difference of size of any two partitions is less than 2, we get $\mathcal{E}_{RD}(G) < \mathcal{E}_{RD}(T(n,k))$. □

4 Reciprocal distance energy change of some complete multipartite graph due to edge deletion

In this section, we study the change in reciprocal distance energy of complete bipartite graph $K_{q,q}$ and the complete tripartite graph $K_{p,q,r}$ due to edge deletion.

Lemma 4.1. [8] *Let A and B be two real symmetric matrices of same order such that $0 \leq A \leq B$. Then $\lambda_1(A) \leq \lambda_1(B)$.*

Let $E = \begin{bmatrix} 0 & a & bJ_{1\times p} & cJ_{1\times q} \\ a & 0 & cJ_{1\times p} & bJ_{1\times q} \\ b1_p & c1_n & b(J_{p\times p} - I_p) & cJ_{p\times q} \\ c1_q & b1_q & cJ_{q\times p} & b(J_{q\times q} - I_q) \end{bmatrix}$, where a, b and c are real constants. In the following lemma, we give the spectrum of the matrix E.

Lemma 4.2. *The spectrum of the matrix E consists of $-b$ with multiplicity $p+q-2$ and the four roots of the polynomial $t^4+(-bp-bq+2b)t^3+(b^2pq-c^2pq-2b^2p-2b^2q-c^2p-c^2q-a^2+b^2)t^2+(2b^3pq-2bc^2pq+a^2bp+a^2bq-2abcp-2abcq-b^3p-b^3q-bc^2p-bc^2q-2a^2b)t-a^2b^2pq+a^2c^2pq+2pqab^2c-2pqac^3+b^4pq-2pqb^2c^2+c^4pq+a^2b^2p+qa^2b^2-2ab^2cp-2ab^2cq-a^2b^2$.*

Proof. Let $e_{i,j}$ be a column vector of size $p+q+2$ with its ith and jth entries equal to 1 and -1, respectively, and the remaining entries are 0. Then $Ee_{3,j}=-be_{3,j}$ for $j=4,5,\ldots,p+2$ and $Ee_{p+3,j}=-be_{p+3,j}$ for $j=p+4,p+5,\ldots,p+q+2$. Thus $-b$ is an eigenvalue of E corresponding to the $p+q-2$ linearly independent eigenvectors $e_{3,j}$ $(j=4,5,\ldots,p+2)$ and $e_{p+3,j}$ $(j=p+4,p+5,\ldots,p+q+2)$. Thus we have listed $p+q-2$ eigenvalues of E. Let t_1,t_2,t_3 and t_4 be the remaining eigenvalues of E corresponding to the eigenvectors X_1, X_2, X_3 and X_4, respectively. Let e_i be the column vector with its ith entry equal to 1 and the rest of the entries equal to 0. Also, let e_i^j $(i\le j)$ be the column vector with its kth entry equal to 1 if $i\le k\le j$, and 0 otherwise. Then the vectors e_1, e_2, e_3^{p+2}, e_{p+3}^{p+q+2}, $e_{3,j}$ $(j=4,5,\ldots,p+2)$ and $e_{p+3,j}$ $(j=p+4,p+5,\ldots,p+q+2)$ form a linearly independent set with $p+q+2$ elements. Since the matrix E is real and symmetric, it has $p+q+2$ linearly independent eigenvectors. Thus the vectors X_1, X_2, X_3 and X_4 are in the linear span of the vectors e_1, e_2, e_3^{p+2} and e_{p+3}^{p+q+2}. Let $X_i=a_ie_1+b_ie_2+c_ie_3^{p+2}+d_ie_{p+3}^{p+q+2}$ for $i=1,2,3,4$. Then $EX_i=t_iX_i$ implies $b_ia+c_ibp+d_icq=a_it_i$; $a_ia+cc_ip+bd_iq=b_it_i$; $ba_i+cb_i+b(p-1)c_i+cd_iq=c_it_i$; $ca_i+bb_i+cc_ip+d_ib(q-1)=d_it_i$. Thus $EX_i=t_iX_i$ if and only if

$$det\begin{pmatrix}-t_i & a & bp & cq\\ a & -t_i & cp & bq\\ b & c & b(p-1)-t_i & cq\\ c & b & cp & b(q-1)-t_i\end{pmatrix}=0.$$

Therefore the remaining four eigenvalues of the matrix E are the roots of the equation $t^4+(-bp-bq+2b)t^3+(b^2pq-c^2pq-2b^2p-2b^2q-c^2p-c^2q-a^2+b^2)t^2+(2b^3pq-2bc^2pq+a^2bp+a^2bq-2abcp-2abcq-b^3p-b^3q-bc^2p-bc^2q-$

$2a^2b)t - a^2b^2pq + a^2c^2pq + 2pqab^2c - 2pqac^3 + b^4pq - 2pqb^2c^2 + c^4pq + a^2b^2p + qa^2b^2 - 2ab^2cp - 2ab^2cq - a^2b^2$. ◻

Lemma 4.3. *The reciprocal distance spectrum of $K_{m,n}\backslash\{e\}$ consists of $-1/2$ with multiplicity $m+n-4$ and the four roots of the polynomial $144\,t^4+(-72\,m-72\,n+288)\,\lambda^3+((-108\,n-108)\,m-108\,q+344)\,\lambda^2+((-108\,n-22)\,m-22\,n+136)\,\lambda+(21\,n-41)\,m-41\,n+57$.*

Proof. We have the reciprocal distance matrix of $K_{m,n}\backslash\{e\}$ as

$$\begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{2}J_{1\times p} & J_{1\times q} \\ \frac{1}{3} & 0 & J_{1\times p} & \frac{1}{2}J_{1\times q} \\ \frac{1}{2}\mathbf{1}_p & \mathbf{1}_n & \frac{1}{2}(J_{p\times p}-I_p) & J_{p\times q} \\ \mathbf{1}_q & \frac{1}{2}\mathbf{1}_q & J_{q\times p} & \frac{1}{2}(J_{q\times q}-I_q) \end{bmatrix},$$

where $p = m-1$ and $q = n-1$. Letting $a = 1/3$, $b = 1/2$ and $c = 1$ in Lemma 4.2, we get the reciprocal distance spectrum of $K_{m,n}\backslash\{e\}$. ◻

Theorem 4.4. *We have $\mathcal{E}_{RD}(K_{q,q}\backslash\{e\}) < \mathcal{E}_{RD}(K_{q,q})$ if $2 \le q \le 7$, and $\mathcal{E}_{RD}(K_{q,q}\backslash\{e\}) > \mathcal{E}_{RD}(K_{q,q})$ if $q \ge 8$.*

Proof. From Lemma 4.3, the reciprocal distance spectrum of $K_{q,q}\backslash\{e\}$ consists of $-1/2$ with multiplicity $2q-4$; $\frac{3}{4}\,q-\frac{5}{6}+\frac{1}{12}\sqrt{81\,q^2+72\,q-128}$; $\frac{3}{4}\,q-\frac{5}{6}-\frac{1}{12}\sqrt{81\,q^2+72\,q-128}$; $-\frac{1}{4}\,q-\frac{1}{6}+\frac{1}{12}\sqrt{9\,q^2+24\,q-32}$; $-\frac{1}{4}\,q-\frac{1}{6}-\frac{1}{12}\sqrt{9\,q^2+24\,q-32}$. Thus for $q = 2$, $\mathcal{E}_{RD}(K_{q,q}\backslash\{e\}) = \dfrac{4}{3}+\dfrac{1}{3}\sqrt{85} < \mathcal{E}_{RD}(K_{q,q}) = 5$, and for $q \ge 3$, we have

$$\mathcal{E}_{RD}(K_{q,q}\backslash\{e\}) = 2\left(\frac{1}{2}\,q-1+\frac{1}{12}\sqrt{81\,q^2+72\,q-128}+\frac{1}{12}\sqrt{9\,q^2+24\,q-32}\right).$$

Let $X = \frac{1}{12}\sqrt{81\,q^2+72\,q-128}$ and $Y = \frac{1}{12}\sqrt{9\,q^2+24\,q-32}$. Then

$$(2XY)^2 - \left[\left(q+\frac{1}{2}\right)^2 - (X^2+Y^2)\right]^2 = \frac{1}{4}q^3 - \frac{73}{48}q^2 - \frac{35}{18}q - \frac{17}{16}.$$

If $2 \le q \le 7$. Then from the above equation, we get

$$(2XY)^2 - \left[\left(q+\frac{1}{2}\right)^2 - (X^2+Y^2)\right]^2 < 0.$$

Thus

$$(X+Y)^2 < \left(q+\frac{1}{2}\right)^2.$$

Therefore, $X+Y < q+\frac{1}{2}$ or $\frac{1}{2}q-1+X+Y < \frac{3}{2}q-\frac{1}{2}$. Hence $\mathcal{E}_{RD}(K_{q,q}\backslash\{e\}) < \mathcal{E}_{RD}(K_{q,q})$

If $q \geq 8$. Then we have

$$\begin{aligned}(2XY)^2 - \left[\left(q+\frac{1}{2}\right)^2 - (X^2+Y^2)\right]^2 &= \frac{1}{4}q^3 - \frac{73}{48}q^2 - \frac{35}{18}q - \frac{17}{16} \\ &\geq 2q^2 - \frac{73}{48}q^2 - \frac{35}{18}q - \frac{17}{16} \\ &= \frac{23}{48}q^2 - \frac{35}{18}q - \frac{17}{16} \\ &\geq \frac{17}{9}q - \frac{17}{16} \\ &> 0.\end{aligned}$$

$$(X+Y)^2 > \left(q+\frac{1}{2}\right)^2.$$

Therefore, $X+Y > q+\frac{1}{2}$ or $\frac{1}{2}q-1+X+Y > \frac{3}{2}q-\frac{1}{2}$. Hence $\mathcal{E}_{RD}(K_{q,q}\backslash\{e\}) > \mathcal{E}_{RD}(K_{q,q})$. This completes the proof. □

Theorem 4.5. *We have* $\mathcal{E}_{RD}(K_{p,q,r}\backslash\{e\}) \leq \mathcal{E}_{RD}(K_{p,q,r})$.

Proof. Case I: If $p,q \geq 2$. Then similar to Lemma 4.2, the reciprocal distance spectrum of $K_{p,q,r}\backslash\{e\}$ consists of $-1/2$ with multiplicity $p+q+r-5$, and the five roots of the polynomial $p(t) = 32\,t^5 + (-16\,p - 16\,q - 16\,r + 80)\,t^4 + ((-24\,q - 24\,r - 32)\,p + (-24\,r - 32)\,q - 32\,r + 104)\,t^3 + [((-20\,r - 36)\,q - 36\,r - 20)\,p + (-36\,r - 20)\,q - 4\,r + 68]\,t^2 + [((-20\,r - 12)\,q - 12\,r - 10)\,p + 26 + (-12\,r - 10)\,q]\,t + (-5\,r - 3)\,p + (-5\,r - 3)\,q + 3\,r + 5$.

Case II: If p=q=1. Then the reciprocal distance spectrum of $K_{p,q,r}\backslash\{e\}$ consists of $-1/2$ with multiplicity $r-1$, and the three roots of the polynomial $p(t) = 8\,\lambda^3 + (4 - 4\,r)\,\lambda^2 + (-16\,r - 2)\,\lambda - 7\,r - 1$.

Case III: If $p \geq 2$ and $q = 1$. Then the reciprocal distance spectrum of $K_{p,q,r} \backslash \{e\}$ consists of $-1/2$ with multiplicity $p+r-3$, and the three roots of the polynomial $p(t) = 16\,t^4+(-8\,p - 8\,r + 24)\,t^3+((-12\,r - 24)\,p - 24\,r + 24)\,t^2+((-22\,r - 16)\,p$ $-8\,r + 12)\,t + (-5\,r - 3)\,p - 2\,r + 2$.

By Descarte's rule of signs, the polynomial $p(t)$ has exactly one positive root. Thus $\mathcal{E}_{RD}(K_{p,q,r} \backslash \{e\}) = 2\lambda_1(RD(K_{p,q,r} \backslash \{e\}))$. Since $\mathcal{E}_{RD}(K_{p,q,r}) = 2\lambda_1(RD(K_{p,q,r}))$ and $\lambda_1(RD(K_{p,q,r} \backslash \{e\})) \leq \lambda_1(RD(K_{p,q,r}))$ by Lemma 4.1, we are done. $\square$

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