On Nash-Solvability of Finite Two-Person Tight Vector Game Forms

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Abstract

We consider finite two-person normal form games. The following four properties of their game forms are equivalent: (i) Nash-solvability, (ii) zero-sum-solvability, (iii) win-lose-solvability, and (iv) tightness. For (ii, iii, iv) this was shown by Edmonds and Fulkerson in 1970. Then, in 1975, (i) was added to this list and it was also shown that these results cannot be generalized for n-person case with $n > 2$.

In 1990, tightness was extended to vector game forms (v-forms) and it was shown that such v-tightness and zero-sum-solvability are still equivalent, yet, do not imply Nash-solvability. These results are applicable to several classes of stochastic games with perfect information.

Here we suggest one more extension of tightness introducing v^+-tight vector game forms (v^+-forms). We show that such v^+-tightness and Nash-solvability are equivalent in case of weakly rectangular game forms and positive cost functions. This result allows us to reduce the so-called bi-shortest path conjecture to v^+-tightness of v^+-forms. However, both (equivalent) statements remain open.

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1 Nash-solvability of tight game forms

1.1 Game forms and cost functions

We consider finite normal form games of two players, Alice and Bob.

A *game form* is a mapping $g : X \times Y \to O$, where O is a set of possible outcomes, while X and Y are the sets of strategies of Alice and of Bob, respectively. These three sets are finite.

Remark 1. *In this paper we restrict ourselves and the players by their pure strategies. The mixed ones are not mentioned.*

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Furthermore, let $r^A : O \to \mathbb{R}$ and $r^B : O \to \mathbb{R}$ be *rewards* or *payoffs* of the players, that is, if outcome $o \in O$ appears, Alice and Bob get $r^A(o)$ and $r^B(o)$, respectively. Both players are maximizers. The triplet (g, r^A, r^B) is called a *finite two-person normal form game*, or just a game, for short.

Remark 2. *"Separating" rewards from game forms allows us to make the latter responsible for the structural properties of games, which hold for arbitrary rewards.*

1.2 Nash equilibria and Nash-solvability

A pair of strategies $(x, y) \in X \times Y$ is called a *strategy profile* or a *situation*.

A situation (x, y) is called a *Nash equilibrium* (NE) if

$$r^A(g(x,y)) \geq r^A(g(x',y)) \text{ for any } x' \in X \text{ and } r^B(g(x,y)) \geq r^B(g(x,y')) \text{ for any } y' \in Y;$$

in other words, if no player can profit by replacing her/his strategy, provided the opponent keeps his/her strategy unchanged. Another equivalent reformulation is as follows:

(x, y) is a NE if and only if y is a best response for x and x is a best response for y.

This concept was introduced by John Nash in [36, 37].

A game form g is called *Nash-solvable* (NS) if the corresponding game (g, r^A, r^B) has a NE for any rewards r^A and r^B. In particular, g is called *zero-sum-solvable* (respectively, *win-lose-solvable*) if game (g, r^A, r^B) has a saddle point for any r^A and r^B such that $r^A + r^B \equiv 0$ (respectively, if $r^A + r^B \equiv 0$ and r^A, r^B take only values ± 1). In the zero-sum case we assume that Alice is a maximizer, while Bob is the minimizer.

Obviously, NS implies zero-sum-solvability, which in its turn implies win-lose-solvability.

1.3 Examples of game forms

Several examples are given in Figure 1, where game forms are represented by tables with rows, columns, and entries labelled by $x \in X$, $y \in Y$, and $o \in O$. Mapping g is assumed to be surjective, but not necessarily injective, that is, an outcome $o \in O$ may occupy an arbitrary array in the table of g.

1.4 Basic strategies and simple situations

Sets $g(x) = \{g(x,y) \mid y \in Y\}$ and $g(y) = \{g(x,y) \mid x \in X\}$ are called the *supports* of strategies $x \in X$ and $y \in Y$, respectively.

A strategy is called *basic* or *minimal* if its support is not a proper subset of the support of any other strategy. For example, in g_6 the first strategies of Alice and Bob are basic, while the second are not; in the remaining eight game forms all strategies are basic. Moreover, any two strategies of a player, Alice or Bob, have distinct supports.

A situation (x, y) is called *simple* if $g(x) \cap g(y) = \{g(x,y)\}$. For example, all situations of game forms g_1, g_2, g_8, g_9 are simple; in contrast, no situation is simple in g_7; in g_3 all are simple, except three on the main diagonal; in g_4 all are simple, except the central one; in g_6 all are simple, except one with the outcome o_2.

Game form g is called *rectangular* if all its situations are simple, or other words, each outcome $o \in O$ fills a box, that is, $g^{-1}(o) = X' \times Y' \subseteq X \times Y$ for some subsets $X' \subseteq X$ and $Y' \subseteq Y$. In Figure 1, game forms g_1, g_2, g_8, g_9 are rectangular.

g_1

o_1	o_1
o_2	o_3

g_2

o_1	o_1	o_2	o_2
o_3	o_4	o_3	o_4

g_3

o_1	o_1	o_3
o_1	o_2	o_2
o_3	o_2	o_3

g_4

o_1	o_1	o_3
o_1	o_1	o_2
o_4	o_2	o_2

g_5

o_1	o_2	o_1	o_2
o_3	o_4	o_4	o_3
o_1	o_4	o_1	o_5
o_3	o_2	o_6	o_2

g_6

o_1	o_1
o_1	o_2

g_7

o_1	o_2
o_2	o_1

g_8

o_1	o_1	o_2
o_3	o_4	o_3

g_9

o_1	o_1	o_2
o_4	o_5	o_2
o_4	o_3	o_3

Fig. 1. Nine game forms. Alice and Bob choose rows and columns, respectively. Forms g_1 - g_6 are tight, forms g_7 - g_9 are not; see Section 1.5) for the definitions.

Remark 3. *This concept can be generalized to the n-person case. In [16], it was shown that an n-person game form is the normal form of a positional structure with perfect information modeled by a tree if and only if this game form is tight and rectangular; see also [15, Remark 2], [17], and [22] for more details.*

1.5   Tight game forms

Mappings $\phi : X \to Y$ and $\psi : Y \to X$ are called *response strategies* of Bob and Alice, respectively. The motivation for this name is clear: a player chooses his/her strategy as a function of a known strategy of the opponent. As usual, $gr(\phi)$ and $gr(\psi)$ denote the graphs of mappings ϕ and ψ in $X \times Y$. Game form g is called *tight* if

(j) $g(gr(\phi)) \cap g(gr(\psi)) \neq \emptyset$ for every pair of mappings ϕ and ψ.

It is not difficult to verify that in Figure 1 the first six game forms $(g_1 - g_6)$ are tight, while the last three $(g_7 - g_9)$ are not.

In [10, 14, 15, 19, 24, 30] the reader can find several equivalent properties characterizing tightness. Here we mention some of them.

(jjA) For every $\phi : X \to Y$ there exists a $y \in Y$ such that $g(y) \subseteq g(gr(\phi))$.

(jjB) For every $\psi : Y \to X$ there exists a $x \in X$ such that $g(x) \subseteq g(gr(\phi))$.

We leave to the careful reader to show that (j) is equivalent to (jjA) and to (jjB) as well.

Properties (jjA) and (jjB) show that playing a game $(g; u, w)$ with a tight game form g, the players, Bob and Alice, do not need non-trivial response strategies but can restrict themselves by the constant ones, that is, by Y and X, respectively, at least in case of the zero-sum games.

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Given a game form $g : X \times Y \to O$, introduce on the ground set O two multi-hypergraphs $A = A(g)$ and $B = B(g)$ whose edges are the supports of strategies of Alice and Bob:

$$A(g) = \{g(x) \mid x \in X\} \text{ and } B(g) = \{g(y) \mid y \in Y\}.$$

By construction, the edges of A and B pairwise intersect, that is, $g(x) \cap g(y) \neq \emptyset$ for all $x \in X$ and $y \in Y$. Furthermore, g is tight if and only if

(jjj) hypergraphs $A(g)$ and $B(g)$ are *dual*, that is, satisfy also the following two properties:

(jjjA) for every $O_A \subseteq O$ such that $O_A \cap g(y) \neq \emptyset$ for all $y \in Y$ there exists an $x \in X$ such that $g(x) \subset O_A$;

(jjjB) for every $O_B \subseteq O$ such that $O_B \cap g(x) \neq \emptyset$ for all $x \in X$ there exists an $y \in Y$ such that $g(y) \subset O_B$.

1.6   Tightness and solvability

Consider the following properties of a game form:

(i) NS, (ii) zero-sum-solvability, (iii) win-lose-solvability, and (iv) tightness.

Implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are immediate from the definitions.

Also (iii) $\Rightarrow$ (iv) is easily seen. Indeed, if g is not tight then there exist response strategies ϕ and ψ such that $g(gr(\phi)) \cap g(gr(\psi)) = \emptyset$. Define the zero-sum reward r such that $r(o) = -1$ for $o \in g(gr(\phi))$, $r(o) = 1$ for $o \in g(gr(\phi))$, and $r(o) = \pm 1$, arbitrarily, for the remaining outcomes, if any. Set $r^A = r$ and $r^B = -r$. Obviously, $-1 = minmax < maxmin = 1$ in the obtained win-lose game. Hence, it has no saddle point.

Implication (iv) $\Rightarrow$ (ii) is implicit in [10] and explicit in [14]. Finally, (iv) $\Rightarrow$ (i) appears in [15]; see also [19]. Thus, all four properties (i-iv) are equivalent.

However, for the three-person case tightness is no longer sufficient [15] nor necessary [15, 19] for NS.

Recently, implication (iv) $\Rightarrow$ (i) was strengthened and its proof simplified in [30, 31].

Given a tight game game form $g : X \times Y \to O$ and reward functions $r^A : O \to \mathbb{R}$ and $r^B : O \to \mathbb{R}$, game (g, r^A, r^B) has a simple NE in basic strategies, that is, situation (x, y) is simple and strategies x, y are basic. Moreover, there exist some special NE (x^0, y^*) defined by a lexicographically safe strategy x^0 of Alice, which depends only on her cost function r^A, while Bob's cost function r^B is irrelevant. This concept is a refinement of the classic safe (max min) strategy. Then, Bob's NE strategy x^* maximizes r^B over set $g(x^0)$, which is the support of x^0. Of course, Alice and Bob can be swapped. Thus, we obtain two sets of NE: NE-A and NE-B. These two sets coincide in the zero-sum case and in some other cases too; for example, when game (g, r^A, r^B) has a unique NE. However, in general, NE-A and NE-B differ. Furthermore, a pair of lexicographically safe strategies (x^0, y^0) is not necessarily a NE. See more details in [30, 31].

These statements are constructive: a polynomial algorithm determining NE-A and NE-B is suggested. This is trivial when a tight game form g is explicit, but the algorithm works in a more general case, when g is given by a polynomial oracle $\mathcal{O}$ such that the size of g is exponential in the size of $\mathcal{O}$. In the next subsection we consider an example of such oracle.

1.7 Deterministic graphical multi-stage game structures

Let $\Gamma = (V, E)$ be a directed graph (digraph) whose vertices and arcs are interpreted as positions and moves, respectively. Furthermore, denote by V_T the set of terminal positions, of out-degree zero, and by V_A, V_B the positions of positive out-degree, controlled by Alice and Bob, respectively. We assume that $V = V_A \cup V_B \cup V_T$ is a partition. A strategy $x \in X$ of Alice (resp., $y \in Y$ of Bob) is a mapping that assigns to each position $v \in V_A$ (resp., $v \in V_B$) an arbitrary move from this position. An initial position $v_0 \in V_A \cup V_B$ is fixed. Each situation (x, y) defines a unique a walk that begins and v_0 and then follows the decisions made by x and y. This walk $P(x, y)$ is called a *play*. Each play either terminates in V_T or is infinite. In the latter case, it forms a "lasso": first, an initial path, which may be empty, and then a directed cycle (dicycle) repeated infinitely (This holds, because we restrict players by their stationary strategies, that is, a move may depend only on the current position but not on previous positions and/or moves).

The positional structure defined above can also be represented in normal form. We introduce a game form $g : X \times Y \to O$, where, as before, O denotes a set of outcomes. Yet, there are several ways to define this set. One is to "merge" all infinite plays (lassos) and consider them as a single outcome c, thus, setting $O = V_T \cup \{c\}$. This model was introduced by Washburn [40] and called *deterministic graphical game structure* (DGGS).

The following generalization was suggested in [24]. Digraph Γ is called *strongly connected* if for any $v, v' \in V$ there is a directed path from v to v' (and, hence, from v' to v, as well). By this definition, the union of two strongly connected digraphs is strongly connected whenever they have a common vertex. A vertex-inclusion-maximal strongly connected induced subgraph of Γ is called its *strongly connected component* (SCC). In particular, each terminal position $v \in V_T$ is an SCC. It is both obvious and well-known that any digraph $\Gamma = (V, E)$ admits a unique decomposition into SCCs: $\Gamma^o = \Gamma[V^o] = (V^o, E^o)$ for $o \in O$, where O is a set of indices. Furthermore, partition $V = \cup_{o \in O} V^o$ can be constructed in time linear in the size of Γ, that is, in $(|V| + |E|)$. It has numerous applications; see [38, 39] for more details. One more was suggested in [24]. For each $o \in O$, contract the SCC Γ^o into a single vertex v^o. Then, all edges of E^o (including loops) disappear and we obtain an acyclic digraph $\Gamma^* = (O, E^*)$. The set O can be treated as the set of outcomes. Each situation (x, y) uniquely defines a play $P = P(x, y)$. This play either comes to a terminal $v \in V_T$ or forms a lasso. The cycle of this lasso is contained in an SCC o of Γ. Each terminal is an SCC as well. In both cases an SCC $o \in O$ is assigned to the play $P(x, y)$. Thus, we obtain a game form $g : X \times Y \to O$, which is the normal form of the *multi-stage DGGS* (MSDGGS) defined by Γ.

An SCC is called *transient* if it is not a terminal and contains no dicycles. No play can result in such SCC; or in other words it does not generate an outcome. For example, $O = V_T$ in an acyclic digraph, while all remaining SCC are transient.

DGGS and MSDGGS can be viewed as polynomial oracles the corresponding game forms g' and g, respectively. Note that the size of game forms may be exponential in the size of these oracles. Note also that g' is obtained from g by merging some outcomes. Namely, all outcomes corresponding to the non-terminal SCCs are replaced by a single outcome c. Obviouslly, merging outcomes respects tightness. Thus, it is enough to verify it for MSDGGSs, As we know, it is sufficient to prove the win-lose solvability. For DGGS it was done in [40]; see also [7, Section 3], [1], [8, Section 12]. The result was extended to MSDGGS in [24]. All proofs were constructive, the corresponding win-lose games were solved in time polynomial in the size of Γ.

For reader's convenience, we briefly sketch here the proof of from [24]. Consider a win-lose game $(g; r^A, r^B)$ with game form $g = g(\mathcal{O})$ generated by a MSDGGS oracle $\mathcal{O}$. Let $O = O_A \cup O_B$ denote the partition of O into two sets of outcomes: winning for Alice and Bob, respectively. We would like to apply the Backward Induction, yet, digraph Γ may have dicycles. So we modify Backward Induction to make it work in presence of dicycles. Recall that O is the set of SCCs of Γ and $\Gamma^* = (O, E^*)$ is acyclic. Consider an SCC $o = \Gamma' = (V', E')$ in Γ that is not terminal, but each move (v', v) from a position $v' \in V'$ either ends in a terminal $v \in V_T$, or stays in Γ', that is, $v' \in V'$. Wlog assume that $o \in O_A$, that is, Alice wins if the play cycles in Γ'. Then Bob wins in a position $v' \in V'$ if and only if he can force the play to come to a terminal $v \in O_B$ and Alice wins in all other positions of V'. Note that for Alice it is not necessary to force the play to come to O_A, it is enough if it cycles in Γ'. Thus, every position of Γ' can be added either to O_A or to O_B. Then we eliminate all edges E' of Γ' and repeat until the initial position v_0 of Γ is evaluated. This procedure proves solvability of game form $g(\mathcal{O})$ and solves a win-lose game $(g; O_A, O_B)$ in time linear in the size of $\mathcal{O} = \Gamma$.

More polynomial oracles for (finite two-person) tight game forms can be found in [29, 30].

2 Zero-sum-solvability of tight v-forms

Here we survey results of [20].

2.1 Main concepts and theorem

A *vector game form of type v* (or *v-form*) is a mapping $g_v : X \times Y \to W$, where $W \subset \mathbb{R}^m$ is a finite set of real m-vectors $\{w = (w_1, \dots, w_m) \mid w \in W\}$.

Given an arbitrary real utility m-vector $u = (u_1, \dots, u_m)$, we define a real-valued reward function $r : W \to \mathbb{R}$, where $r(w) = (u, w) = u_1 w_1 + \dots u_m w_m$ for each $w \in W$.

In the zero-sum case we assume that $r^A = r$ and $r^B = -r$; Alice is the maximizer and Bob is the minimizer. In general case, we introduce $u^A : W \to \mathbb{R}$ and $u^B : W \to \mathbb{R}$ separately, set $r^A(w) = (u^A, w)$ and $r^B(w) = (u^B, w)$ for all vectors $w \in W$, and assume that both players are maximizers.

A v-form g_v is called *tight* if

$$ConvHull\{g_v(x, \phi(x)) \mid x \in X\} \cap ConvHull\{g_v(\psi(y), y) \mid y \in Y\} \neq \emptyset \quad \forall \, \phi : X \to Y, \ \psi : Y \to X;$$

in (other) words, for an arbitrary pair of response strategies $\phi : X \to Y$ of Alice and $\psi : Y \to X$ of Bob, convex hulls of the corresponding two sets of vectors from their graphs, $\{g_v(\phi) = \{g_v(x, \phi(x)) \mid x \in X\}, \{g_v(\psi) = \{g_v(\psi(y), y) \mid y \in Y\} \subseteq W$, intersect in $\mathbb{R}^m$.

A game form g can be viewed as a v-form g_v with only unit vectors (one entry 1 and all others 0). For any two sets of such vectors, their convex hulls are disjoint in $\mathbb{R}^m$ if and only if the sets are disjoint. Thus, definitions of tightness for game forms and for vector game forms agree, that is, g and g_v are tight simultaneously.

The next theorem extends the criterion of zero-sum-solvability of Section 1.

Theorem 1. *([20]) Tightness and zero-sum-solvability of v-forms are equivalent.*

Proof. Suppose that g_v is not tight. Then there exist ϕ and ψ such that $g_v(\phi) \cap g_v(\psi) = \emptyset$. Two disjoint convex sets in $\mathbb{R}^m$ can be separated by a hyperplane; in other words, there exists a vector $u \in \mathbb{R}^m$ such that $(u, g_v(x, \phi(x)) < (u, g_v(\phi(y), y)$ for every $x \in X$ and $y \in Y$. Then, $maxmin < minmax$ in the zero-sum game (g, u) and, hence, it has no saddle point. Thus, g_v is not zero-sum-solvable.

Conversely, suppose that g_v is not zero-sum-solvable, that is, for some $u \in \mathbb{R}^m$, the zero-sum game (g, u) has no saddle point and. Then, $maxmin < minmax$ in this game. Consider arbitrary best response strategies $\phi : X \to Y$ and $\psi : Y \to X$ of Bob and Alice, respectively. By definition, these two strategies guarantee $maxmin$ and $minmax$, respectively. Hence, $g_v(\phi) \cap g_v(\psi) = \emptyset$. Thus, g_v is not tight. $\square$

Remark 4. *A tight game form $g : X \times Y \to O$ is injective if and only if $|X| = 1$ or $|Y| = 1$. In contrast, a tight v-form g_v may be injective for any sizes of X and Y.*

As we know, NS and zero-sum-solvability are equivalent for game forms. In contrast, NS of a v-form does not follow from its zero-sum-solvability. For example, mean payoff games are zero-sum-solvabile but not NS [18]; see next subsection for more details.

Verifying tightness of an explicitly given game form g is an important open problem. A quasi-polynomial algorithm was suggested by Fredman and Khachiyan [12]; see also [28]. "Almost obviously" verifying tightness of an explicitly given v-form is NP-complete. Yet, a proof is required.

2.2 Mean payoff games

Consider the model of Section 1.7. Add a loop to every terminal position $v \in V_T$, if any. Then, the obtained graph $G = (V, E)$ has no terminal positions.

Set O of outcomes consists of all dicycles (dicycles) of G. To each dicycle C assign an m-vector of weights $w(C) = (w_e \mid e \in E)$ such that $w_e = 1/k$ for $e \in C$ and $w(e) = 0$ for $e \notin C$. Here $m = |E|$ is the number of directed edges of G and $k = k(C) = |C|$ is the length of dicycle C. Note that sum of entries of any vector $w(C)$ equals $k(1/k) = 1$. Note also that k may take any positive integer value: $k = 1$ if C is a loop and $k = 2$ if C is formed by a pair of oppositely directed edges.

Given a utility vector $u = (u(e) \mid e \in E) = (u_1, \ldots, u_m)$, a real-valued reward function $r : W \to \mathbb{R}$ is the scalar product $r(w) = (u, w) = u_1 w_1 + \ldots u_m w_m$ for each $w \in W$.

As before, in the zero-sum case we assume that $r^A = r, r^B = -r$; Alice is the maximizer and Bob is the minimizer. In general case we introduce $u^A : W \to \mathbb{R}$ and $u^B : W \to \mathbb{R}$ separately, set $r^A(w) = (u^A, w)$, $r^B(w) = (u^B, w)$ for all vectors $w \in W$, and assume that both players are maximizers.

Since $V_T = \emptyset$, each play $P = P(x, y)$ is infinite; it forms a "lasso" consisting of an initial path, which may be empty, and a dicycle $C = C(x, y)$ repeated infinitely. Thus, the effective payoff of a player in situation (x, y) is the average local payoff along $C(x, y)$, that is, $r(C) = (u, w) = |C|^{-1} \sum_{e \in C} u_e$, where u is u^A or u^B for Alice and Bob, respectively. These definitions justify the name "mean payoff games".

Zero-sum-solvability of these games was proven by Moulin [35] for complete bipartite digraphs, by Ehrenfeucht and Mycielski [11] for any bipartite digraphs, and by Gurvich, Karzanov, and Khachiyan [27] for arbitrary digraphs.

Thus, by Theorem 1, the corresponding v-forms are tight. However, no direct proof of tightness is known and it is a challenge to obtain one.

Furthermore, NS does not hold. An example of NE-free mean payoff game was constructed in [18] for the complete bipartite digraph 3×3. In a way, this example is minimal: NS holds for the complete bipartite digraphs $a \times b$ with $a \leq 2$ or $b \leq 2$ [21].

Thus, unlike game forms, for v-forms NS and zero-sum-solvability are not equivalent.

2.3   Mean payoff games with positions of chance

Replace partition $V = V_A \cup V_B$ by $V = V_A \cup V_B \cup V_R$ by allowing positions V_R with random moves, with a given probabilistic distribution on the edges going from each $v \in V_R$. Introduce also a probabilistic distribution over V, where $p(v)$ is the probability that the game begins in v. In particular, an initial position v_0 may be fixed, which means that $p(v_0) = 1$ and $p(v) = 0$ for all $v \in V \setminus \{v_0\}$.

Then, each situation (x, y) determines a Markov's chain $M(x, y)$ rather than a unique play $P(x, y)$ in the obtained digraph $G = (V, E)$. In accordance with Markov's theory, this chain has a limit probabilistic distribution $p : E \to \mathbb{R}$, with $p(e) = p_e \geq 0$ for all $e \in E$ and $\sum_{e \in E} p_e = 1$. Setting $w_e(x, y) = p_e(x, y)$ for all (x, y) and $e \in E$ we obtain a v-form.

This model was suggested in [27], where BW-games and BWR-games were introduced. (Bob and Alice's positions were called Black (B) and White (W), respectively.) In fact, BWR-games are computationally equivalent [2, 3] to classic stochastic games with perfect information and zero stop probability, which were introduced much earlier, in 1957, by Gillette [13], who proved their zero-sum-solvability. This proof is not simple; it is based on the famous Hardy and Littlewood Tauberian theorem; its conditions were not accurately verified in [13] yet; so the proof was completed only in 12 years by Liggett and Lipman [34].

Thus, by theorem 1, v-forms are tight, yet, no direct proof of their tightness is known.

Remark 5. *In v-forms considered in this and in the previous subsections, all vectors $w \in W$ are non-negative, $w_e \geq 0$ for all $e \in E$. Yet, Theorem 1 holds for arbitrary real vectors.*

In addition to mean payoff games, some other zero-sum-solvabile cases are known in stochastic game theory. An infinite family of effective payoffs, called k-total, was considered in [4] for any integer $k \geq 0$. For example, $k = 0$ and $k = 1$ are associated with mean and *total* [5] effective payoffs, respectively. For each k, the corresponding v-forms are NS [4] and, hence, tight, by Theorem 1. Yet, no direct proof of tightness is known.

3   Nash-solvability of tight v^+-forms

3.1   Main concepts

Vector game forms g_v^+. We modify the concept of v-forms by requiring some extra properties. A *vector game forms of type v^+* (or v^+-*form*) is a mapping $g_v^+ : X \times Y \to \{W \cup \{w^c\}\}$, where $W \subset \mathbb{R}^m$ is a finite set of real *non-negative* and *non-zero* m-vectors, $\{w = (w_1, \ldots, w_m) \mid w \in W\}$ with $w_i \geq 0$ for all $i = 1, \ldots, m$ and $w_i > 0$ for at least one $i \in \{1, \ldots, m\}$; while w^c is as a special m-vector all m coordinates of which are $+\infty$.

Weak rectangularity. We will also assume that mapping g_v^+ is *weakly rectangular*, that is, $(g_v^+)^{-1}(w) = X' \times Y'$, where $X' \subseteq X$ and $Y' \subseteq Y$, or in other words, each vector $w \in W$ fills a box in $X \times Y$. In contrast, w^c fills the rest of $X \times Y$ and may be not a box.

Local and effective costs. Let $u^A = (u_1^A, \ldots, u_m^A)$ and $u^B = (u_1^B, \ldots, u_m^B)$ be strictly positive cost vectors of Alice and Bob, respectively; $u_i^A > 0$ and $u_i^B > 0$ for all $i = 1, \ldots m$.

Now we assume that both players are minimizers, rather than maximizers. Respectively, we replace rewards by costs. Define real-valued cost functions $r^A : W \to \mathbb{R}$ and $r^B : W \to \mathbb{R}$ as scalar products $r^A(w) = (u^A, w)$ and $r^B(w) = (u^B, w)$, respectively, for each $w \in W$.

Note that both functions are strictly positive. Furthermore, set $r^A(w^c) = r^B(w^c) = +\infty$.

Then, effective costs $r^A : X \times Y \to \mathbb{R}$ and $r^B : X \times Y \to \mathbb{R}$ for Alice and Bob are defined as $r^A(x, y) = r^A(g_v^+(x, y))$ and $r^B(x, y) = r^B(g_v^+(x, y))$ for each situation (x, y), including the case when $w(x, y) = w^c$.

Degenerate and non-degenerate situations, NE, strategies, and game forms. Given a v^+-form g_v^+,

- situation (x, y) is called *degenerate* if $g_v^+(x, y) = w^c$;

- strategy x of Alice (respectively, y of Bob) is called *degenerate* if $g_v^+(x, y) = w^c$ for each $y \in Y$ (respectively, for each $x \in X$);

- v^+-form g_v^+ itself is called *degenerate* if $g_v^+(x, y) = w^c$ for each $x \in X$ and $y \in Y$.

The following statements are obvious:
Degenerate situation (x, y) is a NE if and only if both strategies x and y are degenerate.
Each situation of a degenerate v^+-form is a degenerate NE. This case is trivial.

Possibly best response (PBR) strategies. A response strategy $\phi : X \to Y$ of Bob is called a *PBR* if there exists a cost vector u^B such that $y = \phi(x)$ is a best response to x for each Alice's strategy x.

Without any loss of generality (wlog), we can additionally require uniqueness of the best response vector $w(x) = g_v^+(x, y(x))$ in the support $g(x) = \{g(x, y) \mid y \in Y\}$ for all $x \in X$. Indeed, if uniqueness does not hold, it can be achieved by a perturbation of u^B, sufficiently small to respect the best response strategy $y = \phi(x)$.

Similarly, we define PBRs $\psi : Y \to X$ of Alice.

Tightness A v^+-form g_v^+ is called *tight* if supports of any two PBRs ϕ and ψ of Alice and Bob intersect:

$$\{g_v^+(x, \phi(x)) \mid x \in X\} \cap \{g_v^+(\psi(y), y) \mid y \in Y\} \neq \emptyset.$$

3.2 Main theorem

Tightness and NS are equivalent for v^+-forms.

Theorem 2. *A v^+-form g_v^+ is tight if and only if it is NS.*

*Moreover, if g_v^+ is tight and Alice **or** Bob has no degenerate strategy then for any cost vectors u^A and u^B the corresponding game (g_v^+, u^A, u^B) has a non-degenerate NE.*

Proof As we already mentioned, a degenerate v^+-form is tight; furthermore, every its situation is a degenerate NE. This case is trivial.

Obviously, Alice (respectively, Bob) has a degenerate strategy if and only if
$w^c \in \{g_v^+(x, \phi(x)) \mid x \in X\}$ (respectively, $w^c \in \{g_v^+(\psi(y), y) \mid y \in Y\}$.

Here ϕ and ψ are some response strategies of Bob and Alice, respectively.

Suppose that both players have degenerate strategies, Alice x and Bob y. Then, for any u^A and u^B we have:

Both sets $\{g_v^+(x, \phi(x)) \mid x \in X\}$ and $\{g_v^+(\psi(y), y) \mid y \in Y\}$ contain w^c (in fact, only w^c). Hence, their intersection is not empty. Thus, g_v^+ is tight.

Game (g_v^+, u^A, u^B) has a degenerate NE for any cost vectors u^A and u^B.

The statement holds in this case too.

For the rest of the proof, assume that Alice and Bob have no degenerate strategies. Then $w^c \notin \{g_v^+(x, \phi(x)) \mid x \in X\} \cup \{g_v^+(\psi(y), y) \mid y \in Y\}$ for any cost vectors u^A and u^B and corresponding best response strategies ϕ and ψ of Bob and Alice, respectively.

Assume that g_v^+ is tight. Then $\{g_v^+(x, \phi(x)) \mid x \in X\} \cap \{g_v^+(\psi(y), y) \mid y \in Y\} \neq \emptyset$ and, hence, there exists a vector $w^* \in W \cap \{g_v^+(x, \phi(x)) \mid x \in X\} \cap \{g_v^+(\psi(y), y) \mid y \in Y\}$.

Recall that vector game form g_v^+ is weakly rectangular. Hence, for any u^A and u^B, there exist strategies x and y such that $y = \phi(x)$, $x = \psi(y)$, and $g_v^+(x, y) = w^*$, where ϕ and ψ are best response strategies of Bob and Alice, respectively. In other words, situation (x, y) is a non-degenerate NE in game $(g_v^+(x, y), u^A, u^B)$.

Conversely, assume that g_v^+ is not tight. Then there exist PBR ϕ and ψ such that $\{g_v^+(x, \phi(x)) \mid x \in X\} \cap \{g_v^+(\psi(y), y) \mid y \in Y\} = \emptyset$. Obviously, for the corresponding u^A and u^B game $(g_v^+(x, y), u^A, u^B)$ has no NE. Recall that (x, y) is a NE if and only if y is a best response to x and x is a best response to y. □

If a player has no degenerate strategies then delete them from the opponent's set of strategies as well. Obviously, this operation respects tightness and, hence, NS too.

3.3   Asumability conditions

Verifying tightness of a v^+-form looks very difficult, although no accurate results abut its complexity is known. In particular, it is not clear how to check that mapping $\phi : X \to Y$ is a PBR. However, the following, pretty strong, conditions are necessary. Consider a subset $X^* \subseteq X$ and two mappings $\phi' : X^* \to Y$ and $\phi'' : X^* \to Y$. Assume that all vectors of W are pairwise distinct and also that $2|X^*|$ vectors $\{g_v^+(x, \phi'(x)), g_v^+(x, \phi''(x)) \mid x \in X^*\} \subseteq W$ are pairwise distinct, for all $x \in X^*$.

Proposition 1. *If* $|X^*| \geq 1$ *and* $\sum_{x \in X^*} g_v^+(x, \phi'(x)) > \sum_{x \in X^*} g_v^+(x, \phi''(x))$
then mapping $\phi' : X^* \to Y$ *cannot be extended to a PBR* $\phi : X \to Y$.

If $|X^*| \geq 2$ *and* $\sum_{x \in X^*} g_v^+(x, \phi'(x)) = \sum_{x \in X^*} g_v^+(x, \phi''(x))$
then neither $\phi' : X^* \to Y$ *nor* $\phi'' : X^* \to Y$ *can be extended to a PBR* $\phi : X \to Y$.

Proof. The first statement is obvious, since Bob's cost is $r^B(w) = (u^B, w)$ all entries are non-negative, and Bob is the minimizer.

For the second statement we should recall that wlog we cab assume uniqueness of the best response vector $w(x) = g_v^+(x, y(x))$ in the support $g(x) = \{g(x, y) \mid y \in Y\}$ for all $x \in X$; see above.

Furthermore, without any loss of generality (wlog), we can additionally require uniqueness of the best response vector $w(x) = g_v^+(x, y(x))$ in the support $g(x) = \{g(x, y) \mid y \in Y\}$ for all $x \in X$. Indeed, if uniqueness does not hold, it can be achieved by a perturbation of u^B, sufficiently small to respect the best response strategy $y = \phi(x)$. $\square$

Of course, similar necessary conditions hold for Alice's PBRs as well. One should just replace x, X, and ϕ by y, Y, and ψ, respectively.

It is open whether the above asumability conditions only necessary or necessary and sufficient for a response strategy to be a PBR.

4   Shortest path games and bi-shortest path conjecture

We formulate a conjecture from graph theory [25, 6] that is equivalent to NS of the finite two-person vector v^+-forms, which correspond to the so-called shortest path games with positive local costs. For the three-person case this conjecture fails [32].

4.1   Definitions and statement of the conjecture

Let $G = (V, E)$ be a finite digraph with two distinct vertices $s, t \in V$. We assume that

(j) every vertex $v \in V \setminus \{t\}$ has an outgoing edge, while t has not;

(jj) G contains a directed path from s to t;

(jjj) every edge $e \in E$ belongs to such a path.

If (j) fails for v we merge v and t; if (jjj) fails for e we delete e from E.

Given a partition $V \setminus \{t\} = V_A \cup V_B$ with non-empty V_A and V_B, assign an ordered pair of positive real numbers $(u^A(e), u^B(e))$ to every $e \in E$.

Fix a mapping x that assigns to each $v \in V_A$ an edge $e \in E$ going from v. Delete all other edges going from v. In the obtained digraph find a directed shortest path (SP) from s to t, assuming that u^B are the lengths (or costs) of the edges $e \in E$. One can use, for example, Dijkstra's SP algorithm. Swapping A and B we obtain two sets of directed (s, t)-paths.

We conjecture that these two sets intersect, that is, have an (s, t)-path in common, and call this *bi-shortest path (Bi-SP) conjecture.*

Wlog, we can assume that all (s, t)-paths have pairwise different lengths, which can be achieved by small perturbations of u^A and u^B. Then, a shortest path is unique.

It may happen that some mappings x or y leave no (s, t)-path. Then, we choose nothing.

Let us slightly modify the procedure choosing in this case some symbolic path c. Then we obtain a weak version of the Bi-SP conjecture. Indeed, if two sets of (s, t)-paths have only c in common then the Bi-SP conjecture fails, but the weak Bi-SP one holds.

Wlog, we can restrict ourselves by bipartite graphs with parts (V_A, V_B). Indeed, if E contains an edge $e = (u, w)$ such that both u and w are in V_A (respectively, in V_B) we subdivide e by a vertex $v \in V_B$ (respectively, by $v \in V_A$) into two edges $e' = (u, v)$ and $e'' = (v, w)$ choosing some lengths $u^D(e') > 0$ and $u^D(e'') > 0$ such that $u^D(e) = u^D(e') + u^D(e'')$, where $D = A$ or $D = B$.

4.2 Finite n-person shortest path games

Players, positions, moves, and local costs. Given a finite digraph $G = (V, E)$ satisfying above assumption (j, jj, jjj), let us generalize case of $n = 2$ players and consider an arbitrary integer $n \geq 2$. Partition vertices into n non-empty subsets $\{V \setminus t\} = V_1 \cup \ldots \cup V_n$ and assign a positive real number $u^i(e)$ to every player $i \in I$ and edge $e \in E$. Consider the following interpretation: $I = \{1, \ldots, n\}$ is a set of *players*, V_i is the set of *positions* controlled by player $i \in I$; furthermore, $s = v_0$ and $t = v_t$ are respectively the *initial* and *terminal* positions; $e \in E$ is a *(legal) move*, $u^i(e)$ is the cost of move $e \in E$ for player $i \in I$ called the *local cost*.

Strategies, plays, and effective costs. A mapping z_i that assigns a move (v, v') to each position $v \in V_i$ is a strategy of player $i \in I$. [1] Each *strategy profile* (also called a *situation*) $z = (z_1, \ldots, z_n) \in Z_1 \times \ldots \times Z_n = Z$ uniquely defines a play $p(z)$, that is, a walk in G that begins in the initial position $s = v_0$ and goes in accordance with the choice of z in every position that appears. Obviously, $p(z)$ either terminates in $t = v_t$ or cycles; respectively, it is called a terminal or a cyclic play. Indeed, after play $p(z)$ revisits a position, this play will repeat its previous moves, thus, making a "lasso", because all strategies in situation z are stationary.

The effective cost of play $p(z)$ for player $i \in I$ is additive, that is,

$$r^i(p(z)) = \sum_{e \in p(z)} u^i(e) \quad \text{if } p(z) \text{ is a terminal play};$$

$$r^i(p(z)) = +\infty \quad \text{if } p(z) \text{ is a cyclic play}.$$

In other word, each player $i \in I$ pays the local cost $u^i(e)$ for every move $e \in p(z)$. Since a cyclic play $p(z)$ never finishes and all local costs are positive, each player pays $+\infty$.

Let $|E| = m$. The effective cost of a terminal play $p = p(z)$ for player $i \in I$ is the scalar product $r^i(p) = (u^i, w_p)$ of two m-vectors: u^i is the player i local cost m-vector $(u^i(e) \mid e \in E)$ and w_p is the support m-vector of (s, t)-path p, that is, $w_p(e) = 1$ if $e \in p$ and $w_p(e) = 0$ if $e \notin p$.

All n players are minimizers. Thus, a finite *n-person shortest path (SP) game* is defined. We study NS of these games, or more precisely, of the corresponding SP game forms.

Shortest path (vector) game forms. Denote by $\mathcal{P} = \mathcal{P}(s, t)$ the set of directed (s, t)-paths of digraph G. The set of outcomes of a shortest path game is $\mathcal{P} \cup \{c\}$ where c is a special outcome merging all infinite plays (lassos). Mapping $g : Z \to \mathcal{P} \cup \{c\}$, defined in the previous subsection, is the SP game form.

Proposition 2. *Game form g is weakly rectangular.*

Proof. We have two show that each (s, t)-path $p \in \mathcal{P}$ fills a box:

$$g^{-1}(p) = Z^* = Z_1^* \times \ldots \times Z_n^* \subseteq Z_1 \times \ldots \times Z_n = Z.$$

Suppose that $g(z') = g(z'') = p$ for two situations $z' = (z_1', \ldots, z_n')$ and $z'' = (z_1'', \ldots, z_n'')$ in Z. It is enough to show that $g(z) = p$ also for each situation $z = (z_1, \ldots, z_n) \in Z$ such that $z_i = z_i'$ or $z_i = z_i''$ for every player $i \in I$. By assumption, all $2n$ strategies

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[1] We restrict ourselves and all players to their pure stationary strategies; no mixed or history dependent ones are considered in this paper.

$z'_1, \ldots, z'_n, z''_1, \ldots, z''_n$ require to follow p in every position $v \in p$. Then obviously, situation z has the same property and, hence, $g(z) = p$ as well. $\square$

An SP game form g is typically not tight, already in case $n = 2$.

Replace in g every (s,t)-path $p \in \mathcal{P}$ by its support m-vector $w_p = (w_p(e) \mid e \in E)$, where $w_p(e) = 1$ for $e \in p$ and $w_p(e) = 0$ for $e \notin p$. Furthermore, replace outcome c in g by m-vector w_c with m entries equal $+\infty$. Then, game form g will be replaced by a (weakly rectangular) v^+-form g_v^+, which will be called, an *SP v^+-form*.

Return to case $n = 2$. The following statement follows immediately from definitions.

Theorem 3. *Every two-person finite SP v^+-form is tight if and only if bi-shortest path conjecture holds.*

Furthermore, by Theorem 2, tightness and NS are equivalent for v^+-forms.

However, it is an open problem whether tightness holds for SP v^+-forms and, thus, bi-shortest path conjecture remains open too.

4.3   NS of n-person SP v^+-forms assigned to symmetric digraphs.

Main result. For $n > 2$, an n-person SP game (with positive local costs) may have no NE. In other words, the corresponding n-person v^+-forms are neither NS nor tight. An example was constructed for $n = 3$ in [32]. However, tightness and NS hold in the following important special case.

Digraph $G = (V, E)$ is called *symmetric* if each non-terminal move in it is reversible, that is, $(u,w) \in E$ if and only if $(w,u) \in E$ unless $u = t$ or $w = t$. It was recently shown that every n-person SP game on a finite symmetric digraph has an NE [6].

Terminal n-person games and costs. A local cost vector $u : I \times E \to \mathbb{R}^m$ is called terminal if $u(i, e) = 0$ for each player $i \in I$ and move $e \in E$ unless e is a terminal move. Note that these terminal costs are arbitrary real numbers: may be positive, negative, or 0. An SP game with terminal costs is called *terminal*.

Remark 6. *In this case, it is convenient to replace the unique terminal $t = v_t$ by a terminal set $V_T \subseteq V$ assuming that each terminal move leads to a separate terminal. Then, costs can be defined in terminals rather than for moves.*

Each (finite) two-person terminal game has a NE, yet, this result does not hold for n-person games with $n > 2$: an NE-free example for $n = 4$ was given in [33] and then, for $n = 3$, in [9]; see Subsection 1.7 for more details. Yet, the problem is open if we assume that the next condition holds:

(C) Any terminal outcome is better than c for each player $i \in I$.

Moreover, every known example of an NE-free terminal game has the following property:

(C22) There are at least 2 players each of whom has at least 2 terminals worse than c.

Is this true for all NE-free terminal games? Such conjecture was called "Catch 22" in [26]. Also. in this preprint, the following strengthening of Catch 22 was suggested:

Partition digraph G into *strongly connected components* (SCCs) and assign an outcome to each. In particular, every terminal vertex is an SCC, which will be called *terminal*, while any other SCC will be called *inner*. Respectively, the corresponding outcomes will be called *terminal* and *inner*, as well. See [24] and subsection 1.7 for more details.

Let us merge all inner outcomes into one special outcome c. It is easily seen that such operation respects NS and tightness. Yet, inverse is not true: these properties may appear after merging even if they did not hold before.

Condition (C) and ($C22$) were generalized in [26] as follows:

(C$'$) Any terminal outcome is better than every inner one for each player $i \in I$.

(C$'22$) There are at least two players each of which has at least two terminal outcomes that are worse than an inner one.

In n-person terminal case, NS remains open if $n > 2$ and a conditions from $\{(C),(C'),(C22),(C'22)\}$ is required.

4.4   Two versions of the bi-shortest path conjecture

In the same way we can strengthen the Bi-SP conjecture assuming that there are several inner outcomes each of which is worse than every terminal outcome for both players, or in other words, that condition (C$'$) holds.

Proposition 3. *Both versions of the Bi-SP conjecture are equivalent.*

Proof. Obviously, the strong one implies the standard one. Let us show that the inverse implication holds too. Consider two cases.

Suppose that digraph G has no (s,t)-path. Then both versions of the Bi-SP conjecture holds. The weak one is trivial, while the strong one holds, due to NS result of [24].

Suppose that digraph G has an (s,t)-path and the Bi-SP conjecture holds. Then, the corresponding v^+-form g_v^+ is NS, that is, for any cost functions u^A and u^B, game (g_v^+, u^A, u^B) has a NE (x,y) such that $g_v + (x,y)$ is a terminal vector. Let us merge all inner outcomes. Then, (x,y) remains a NE, by condition (C$'$). □

If digraph G is symmetric then there exists a unique inner outcome and, hence, both version of the Bi-SP conjecture coincide. Moreover, for symmetric digraphs NS holds even for the n-person SP game forms [6]. In contrast, without assumption of symmetry, already the weak version fails for $n = 3$ [32], and for $n = 2$ both versions of the Bi-SP conjectures are equivalent and open.

For terminal costs, the strong version is proven for $n = 2$ in [24], while for $n > 2$ even the weak version fails; the counterexamples were given for $n = 4$ in [33] and for $n = 3$ in [9]. Yet, the weak and strong versions are open if we assume (C) and (C$'$), respectively, and remain open if we weaken conditions (C) and (C$'$) requiring instead their Catch 22 versions, (C22) and (C$'22$).

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