Stäckel representations of stationary KdV systems

Maciej Błaszak[†], Błażej M. Szablikowski[†] and Krzysztof Marciniak[‡]

[†]Department of Mathematical Physics and Computer Modelling
Faculty of Physics, Adam Mickiewicz University
Uniwersytetu Poznańskiego 2, 61-614 Poznań, Poland
blaszakm@amu.edu.pl, bszablik@amu.edu.pl

[‡]Department of Science and Technology
Linköping University, Campus Norrköping
601 74 Norrköping, Sweden
krzma@itn.liu.se

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Abstract

In this article we study Stäckel representations of stationary KdV systems. Using Lax formalism we prove that these systems have two different representations as separable Stäckel systems of Benenti type, related with different foliations of the stationary manifold. We do it by constructing an explicit transformation between the jet coordinates of stationary KdV systems and separation variables of the corresponding Benenti systems for arbitrary number of degrees of freedom. Moreover, on the stationary manifold, we present the explicit form of Miura map between both representations of stationary KdV systems, which also yields their bi-Hamiltonian formulation.

1 Introduction

It is well known that various reductions of soliton hierarchies lead to Liouville integrable finite-dimensional systems. Stationary flows, restricted flows, Lax constrained flows are examples of such reductions (see survey [7] and the literature therein). The KdV hierarchy is by far the most studied of soliton hierarchies, also from the point of view of its reductions. Theory of its stationary flows was studied since the early 70's. Its finite gap solutions were found by Dubrovin and Novikov [18, 16, 17] and its Riemann theta function representation was presented by Its and Matveev [21, 22] (see the comprehensive survey [20] and the literature therein). Bogoyavlenskii and Novikov proved [14] that these flows have the structure of finite-dimensional Hamiltonian systems, a result generalized to stationary flows of other evolution equations by Mokhov in [24]. Al'Ber [2] constructed canonical coordinates for the stationary KdV flows based on the algebraic recursion for conserved 1-forms (co-symmetries) established in [1] (or, equivalently, on the integro-differential recursion for the KdV hierarchy [23]). Consequently, the bi-Hamiltonian formulation for the KdV stationary flows was presented by means of the degenerate Poisson tensors [3] and as result Liouville integrability of the stationary KdV flows was fully proved. It was also observed that, in fact, in the case of the KdV hierarchy there are two Hamiltonian finite-dimensional representations of the stationary flows, connected by the Miura map [4, 28, 26]. Further, the bi-Hamiltonian structure of the stationary KdV flows and their separability was comprehensively studied in [19].

In this article we revisit these ideas in a novel, systematic way. We prove that each stationary KdV system (by which we mean a stationary flow of KdV hierarchy together with all lower flows) has two different Stäckel representations from the Benenti class [5, 6, 10]. We prove this by stitching together Lax representations of (an integrated form of) a given stationary KdV system and the corresponding Stäckel separable system, which allows us to construct the explicit transformation between jet coordinates of KdV stationary system and separation variables (via Viète's coordinates) of the corresponding Stäckel system. We also present an explicit formula for a Miura map between both Stäckel representations of the same KdV stationary system which leads to its bi-Hamiltonian representation.

Let us also mention that the inverse construction is also possible. Starting from a carefully chosen family of Stäckel systems one can reconstruct the related hierarchies of stationary systems and hence reconstruct the whole KdV hierarchy. This idea was explored for the first time in [10, 11].

Denote the infinite KdV hierarchy by $u_{t_k} = \mathcal{K}_k(u)$, $k = 1, 2, \ldots$. The main result of this article is contained in the following theorem.

Theorem 1 *The n-th stationary KdV system, which consists of first n flows from the KdV hierarchy* (2.1) *and the $(n+1)$-st stationary flow of KdV, i.e.*

$$u_{t_1} = \mathcal{K}_1, \qquad u_{t_2} = \mathcal{K}_2, \qquad \ldots, \qquad u_{t_n} = \mathcal{K}_n, \qquad \mathcal{K}_{n+1} = 0, \tag{1.1}$$

can be integrated once either to the form (2.30) *or to the form* (2.33)*. Each form is equivalent, through the appropriately defined map* (4.7)*, to a Stäckel system from Benenti class. In the first representation the corresponding Stäckel system is defined by the spectral (separation) curve*

$$\lambda^{2n+1} + c\lambda^n + \sum_{k=1}^{n} H_k \lambda^{n-k} = \mu^2,$$

and in the second representation by the spectral curve

$$\lambda^{2n} + \bar{c}\lambda^{-1} + \sum_{k=1}^{n} \bar{H}_k \lambda^{n-k} = \lambda\mu^2,$$

where c and $\bar{c}$ are respective integration constants of stationary flow $K_{n+1} = 0$. In both Stäckel representations the evolution equations generated by Hamiltonians $(H_1, \ldots, H_n)$ and $(\bar{H}_1, \ldots, \bar{H}_n)$ are mapped by the corresponding map (4.7) *to the evolution equations of integrated stationary systems* (2.30) *and* (2.33)*, respectively. Moreover, the two Stäckel representations,* (2.30) *and* (2.33) *are related by the finite-dimensional Miura map* (5.3) *on the extended $(2n+1)$-dimensional phase space on which the stationary system* (1.1) *is defined.*

Further, in the article we carefully explain all the ingredients of this theorem, including the formulas this theorem refers to.

Recently, the (isospectral) Lax representations for the whole Benenti class of Stäckel systems were constructed [9]. These matrix Lax equations belong to the class of the so-called Mumford systems that are associated with the separable systems with separation curves of hyperelliptic type [25, 29]. Using the Lax formalism developed in [9] we are able in this article to construct, in an explicit form and for arbitrary number of degrees of freedom, transformations between jet coordinates of a given KdV stationary system and separation variables of the associated Stäckel systems, which is a new result. Let us emphasize that comparing to the results for instance from [2, 28, 19], we not only consider here the second representation of stationary KdV systems, but also show the one-to-one equivalence between evolution equations from the stationary systems and the Hamiltonian evolution equations from the corresponding Stäckel systems.

This article is organized as follows. In Section 2 we remind some important facts about the KdV hierarchy, its Lax- and zero-curvature representations. Next, we define the notion of KdV stationary systems and obtain their two representations together with their respective Lax equations. In Section 3 we present basic facts about a particular class of Stäckel systems of Benenti type as well as information about its Lax representation. Finally, in Section 4, we prove Theorem 1 formulated above and use the Miura map (5.3) to construct a bi-hamiltonian formulation for the stationary KdV system.

2 KdV hierarchy

2.1 Bi-Hamiltonian structure

Let us collect some, important for further considerations, facts about the KdV hierarchy. The KdV equation

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x$$

is a member of the bi-Hamiltonian chain of nonlinear PDE's

$$u_{t_n} \equiv \mathcal{K}_n = \pi_0 d\mathcal{H}_n = \pi_1 d\mathcal{H}_{n-1}, \qquad n = 1, 2, ... \tag{2.1}$$

where the two Poisson operators are

$$\pi_0 = \partial_x, \qquad \pi_1 = \frac{1}{4}\partial_x^3 + \frac{1}{2}u\partial_x + \frac{1}{2}\partial_x u.$$

The hierarchy (2.1) can be generated by the recursion operator and its adjoint

$$N \equiv \pi_1\pi_0^{-1} = \frac{1}{4}\partial_x^2 + u + \frac{1}{2}u_x\partial_x^{-1}, \qquad N^\dagger = \frac{1}{4}\partial_x^2 + u - \frac{1}{2}\partial_x^{-1}u_x,$$

in the sense that

$$\mathcal{K}_{n+1} = N^n\mathcal{K}_1, \qquad \gamma_n = d\mathcal{H}_n = \left(N^\dagger\right)^n\gamma_0, \qquad n = 1, 2, \dots . \tag{2.2}$$

In particular, we find that that the first vector fields (symmetries) $\mathcal{K}_n$ are:

$$\begin{aligned}
\mathcal{K}_1 &= u_x, \\
\mathcal{K}_2 &= \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x, \\
\mathcal{K}_3 &= \frac{1}{16}u_{5x} + \frac{5}{8}uu_{3x} + \frac{5}{4}u_xu_{xx} + \frac{15}{8}u^2u_x, \\
\mathcal{K}_4 &= \frac{1}{64}u_{7x} + \frac{7}{32}uu_{5x} + \frac{21}{32}u_xu_{4x} + \frac{35}{32}u_{xx}u_{3x} + \frac{35}{32}u_x^3 + \frac{35}{8}uu_xu_{xx} + \frac{35}{32}u^2u_{3x} + \frac{35}{16}u^3u_x, \\
&\vdots
\end{aligned}$$

the first conserved one-forms (co-symmetries) γ_n are

$$\begin{aligned}
\gamma_0 &= 2, \\
\gamma_1 &= u, \\
\gamma_2 &= \frac{1}{4}u_{xx} + \frac{3}{4}u^2, \\
\gamma_3 &= \frac{1}{16}u_{4x} + \frac{5}{8}uu_{xx} + \frac{5}{16}u_x^2 + \frac{5}{8}u^3, \\
\gamma_4 &= \frac{1}{64}u_{6x} + \frac{7}{32}uu_{4x} + \frac{7}{16}u_xu_{3x} + \frac{21}{64}u_{xx}^2 + \frac{35}{32}u^2u_{xx} + \frac{35}{32}uu_x^2 + \frac{35}{64}u^4, \\
&\vdots
\end{aligned}$$

while the first Hamiltonian densities $\mathcal{H}_n$ of conserved functionals are

$$\begin{aligned}
\mathcal{H}_0 &= 2u, \\
\mathcal{H}_1 &= \frac{1}{2}u^2, \\
\mathcal{H}_2 &= -\frac{1}{8}u_x^2 + \frac{1}{4}u^3, \\
\mathcal{H}_3 &= \frac{1}{32}u_{xx}^2 + \frac{5}{32}u^2u_{xx} + \frac{5}{32}u^4, \\
\mathcal{H}_4 &= -\frac{1}{128}u_{3x}^2 + \frac{7}{64}uu_{xx}^2 - \frac{35}{64}u^2u_x^2 + \frac{7}{64}u^5, \\
&\vdots
\end{aligned}$$

As u belongs to the whole hierarchy (2.1) we can consider it as depending on infinitely many evolution parameters t_i and one spatial variable x: $u = u(x, t_1, t_2, t_3, ...)$.

2.2 Lax representation

It well know that the hierarchy (2.1) can be reconstructed from an isospectral problem. Consider the following pair of a spectral problem and its auxiliary problem

$$\begin{aligned} L\psi &= \lambda\psi, \qquad \lambda_{t_n} = 0, \\ \psi_{t_n} &= B_n\psi, \qquad n = 1, 2, \ldots, \end{aligned} \tag{2.3}$$

where L and B_n are some differential operators. The compatibility conditions for (2.3) take the form

$$L_{t_n} = [B_n, L], \qquad n = 1, 2, \ldots, \tag{2.4}$$

known as the isospectral deformation equations, since the eigenvalues of the operator L are independent of all times t_i. The equations (2.4) are equivalent with the evolutionary hierarchy of PDE's (2.1). For the KdV hierarchy

$$L = \partial_x^2 + u, \qquad B_n \equiv \left(L^{n-\frac{1}{2}}\right)_{\geq 0} = \sum_{i=0}^{n-1}\Big(-\frac{1}{4}(\gamma_i)_x + \frac{1}{2}\gamma_i\partial_x\Big)L^{n-i-1}, \qquad n = 1, 2, \ldots, \tag{2.5}$$

where in particular

$$\begin{aligned} B_1 &= \partial_x, \\ B_2 &= \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}u_x, \\ B_3 &= \partial_x^5 + \frac{5}{2}u\partial_x^3 + \frac{15}{4}u_x^2\partial_x^2 + \frac{5}{8}(3u^2 + 5u_{xx})\partial_x + \frac{15}{16}(u_{3x} + 2uu_x), \\ &\vdots \end{aligned}$$

As a consequence of (2.5) we can represent the linear problem (2.3) by means of polynomials in the spectral variable λ:

$$\psi_{xx} = \lambda\psi - u\psi, \tag{2.6a}$$

$$\psi_{t_n} = P_n\psi_x - \frac{1}{2}(P_n)_x\,\psi, \qquad n = 1, 2, \ldots, \tag{2.6b}$$

where

$$P_n \equiv \frac{1}{2}\sum_{i=0}^{n-1}\gamma_i\lambda^{n-i-1}. \tag{2.7}$$

Then, the compatibility conditions $(\psi_{xx})_{t_n} = (\psi_{t_n})_{xx}$ of the equations (2.6) provide the hierarchy (2.4) in the form

$$u_{t_n} = 2(P_n)_x(u - \lambda) + u_xP_n + \frac{1}{2}(P_n)_{3x} \equiv \mathcal{K}_n, \qquad n = 1, 2, \ldots. \tag{2.8}$$

The consistency of the KdV hierarchy causes that all the λ terms in (2.8) mutually cancel.

The bi-Hamiltonian chain for the KdV hierarchy (2.1), on the level of co-symmetries, takes the form $\pi_\lambda P_\lambda = 0$ or, explicitly

$$(P_\lambda)_x(u - \lambda) + \frac{1}{2}u_xP_\lambda + \frac{1}{4}(P_\lambda)_{3x} = 0, \tag{2.9}$$

where

$$P_\lambda \equiv \sum_{i=0}^{\infty}\gamma_i\lambda^{-i-1}$$

lies in the kernel of the Poisson pencil $\pi_\lambda \equiv \pi_1 - \lambda\pi_0$. In fact, we can integrate (2.9) to the equation

$$-\frac{1}{2}P_\lambda(P_\lambda)_{xx} + \frac{1}{4}(P_\lambda)_x^2 - (u - \lambda)P_\lambda^2 = C(\lambda) \equiv 4\lambda^{-1}, \tag{2.10}$$

where $C(\lambda)$ is an arbitrary function of λ with coefficients being constants of integration appearing in the recursion (2.2). Here we make the simplest possible choice $C(\lambda) \equiv 4\lambda^{-1}$. Solving recursively (2.10) for coefficients of P_λ one finds that $\gamma_0 = 2$, $\gamma_1 = u$ and

$$\gamma_k = \frac{1}{16}\sum_{i=1}^{k-1}\left[2\gamma_{k-i-1}(\gamma_i)_{xx} - (\gamma_{k-i-1})_x(\gamma_i)_x - 4\gamma_{k-i}\gamma_i\right] + \frac{1}{4}\sum_{i=0}^{k-1} u\gamma_{k-i-1}\gamma_i, \qquad k \geq 2. \tag{2.11}$$

Now, the KdV flows can be obtained in the form (2.8) taking $P_n = \frac{1}{2}\left[\lambda^n P_\lambda\right]_+$, where $[\cdot]_+$ means, here, the projection on the polynomial part in λ. The algebraic recursion formula (2.11), for the construction of co-symmetries γ_i, was originally obtained in [1]. Let us note that contrary to the original recursion (2.2) the formula (2.11) does not require integration.

2.3 Zero-curvature representation

The hierarchy (2.4) can also be reconstructed form the so-called zero-curvature equations, which are more suitable for our further considerations. Introducing the vector eigenfunction $\Psi = (\psi, \psi_x)^T$ we can rewrite the linear problem for the KdV hierarchy (2.3), or equivalently (2.6), in the form

$$\Psi_{t_n} = \mathbb{V}_n\Psi, \qquad n = 1, 2, \ldots, \tag{2.12}$$

where

$$\mathbb{V}_n = \begin{pmatrix} -\frac{1}{2}(P_n)_x & P_n \\ P_n(\lambda - u) - \frac{1}{2}(P_n)_{xx} & \frac{1}{2}(P_n)_x \end{pmatrix}, \qquad n = 1, 2, \ldots . \tag{2.13}$$

In particular

$$\mathbb{V}_1 = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, \qquad \mathbb{V}_2 = \begin{pmatrix} -\frac{1}{4}u_x & \lambda + \frac{1}{2}u \\ \lambda^2 - \frac{1}{2}u\lambda - \frac{1}{2}u^2 - \frac{1}{4}u_{xx} & \frac{1}{4}u_x \end{pmatrix} \tag{2.14a}$$

and

$$\mathbb{V}_3 = \begin{pmatrix} -\frac{1}{4}u_x\lambda - \frac{1}{16}(u_{3x} + 6uu_x) & \lambda^2 + \frac{1}{2}u\lambda + \frac{1}{8}(u_{xx} + 3u^2) \\ \lambda^3 - \frac{1}{2}u\lambda^2 - \frac{1}{8}(u_{xx} + u^2)\lambda - (\frac{1}{16}u_{4x} + \frac{1}{2}uu_{xx} + \frac{3}{8}u_x^2 + \frac{3}{8}u^3) & \frac{1}{4}u_x\lambda + \frac{1}{16}(u_{3x} + 6uu_x) \end{pmatrix}. \tag{2.14b}$$

From the equation (2.6b) it follows that $\psi_{t_1} = \psi_x$ and hence (2.12) for $n = 1$ specifies to

$$\Psi_x = \mathbb{V}_1\Psi. \tag{2.15}$$

The compatibility conditions $(\Psi_x)_{t_n} = (\Psi_{t_n})_x$ between (2.12) and (2.15) take the form of the following zero-curvature equations

$$\frac{d}{dt_n}\mathbb{V}_1 = [\mathbb{V}_n, \mathbb{V}_1] + \frac{d}{dx}\mathbb{V}_n, \qquad n = 1, 2, \ldots , \tag{2.16}$$

which are equivalent to the respective members of the KdV hierarchy (2.8). Here, $\frac{d}{dx}$ and $\frac{d}{dt_n}$ means the total derivatives with respect to spatial x and evolution t_n variables. The remaining zero-curvature equations coming from the conditions $(\Psi_{t_m})_{t_k} = (\Psi_{t_k})_{t_m}$,

$$\frac{d}{dt_k}\mathbb{V}_r - \frac{d}{dt_r}\mathbb{V}_k + [\mathbb{V}_r, \mathbb{V}_k] = 0, \qquad r, k = 1, 2, \ldots, \tag{2.17}$$

are identically satisfied due to the commutativity of all the vector fields of the KdV hierarchy (2.1).

2.4 Stationary systems

The $(n+1)$-st stationary flow is determined by the following restriction on the $(n+1)$-st KdV symmetry:

$$u_{t_{n+1}} = 0 \qquad \text{or equivalently} \qquad \mathcal{K}_{n+1} = 0, \tag{2.18}$$

which can be obtained by imposing on the linear problems (2.12) the constraint

$$\Psi_{t_{n+1}} = \lambda^m\mu\Psi \tag{2.19a}$$

or equivalently

$$\mathbb{V}_{n+1}\Psi = \lambda^m \mu \Psi. \tag{2.19b}$$

The factor λ^m in (2.19) is a matter of later convenience. Indeed, the constraint (2.19a) and the compatibility condition $(\Psi_x)_{t_{n+1}} = (\Psi_{t_{n+1}})_x$ gives

$$\frac{d}{dt_{n+1}}\mathbb{V}_1 = 0, \tag{2.20}$$

which is equivalent to (2.18), or alternatively the compatibility condition between eigenvalue problem (2.19b) and $\Psi_x = \mathbb{V}_1\Psi$ yields the Lax equation

$$\frac{d}{dx}\mathbb{V}_{n+1} = [\mathbb{V}_1, \mathbb{V}_{n+1}],$$

which combined with the zero-curvature equation (2.16) for $k = n+1$ gives again (2.20).

The differential order of $(n+1)$-st vector field $\mathcal{K}_{n+1}$ is equal to $2n+1$, which means that the vector field $\mathcal{K}_{n+1}$ depends on $2n+2$ jet variables: $u, u_x, \ldots, u_{(2n+1)x}$. The stationary restriction (2.18) provides constraint on the infinite-dimensional (functional) manifold, on which the KdV hierarchy is defined, reducing it to the finite-dimensional (stationary) submanifold $\mathcal{M}_n$ of dimension $(2n+1)$. Using (2.18) and its differential consequences we can eliminate all terms of order $2n+1$ and higher. Thus, the coordinates on the stationary manifold $\mathcal{M}_n$ are provided by the jet coordinates: $u, u_x, \ldots, u_{(2n)x}$. Due to the integrability the constraint is invariant with respect to all flows from the KdV hierarchy. As result the infinite hierarchy (2.1) reduces to the finite system:

$$u_{t_1} = \mathcal{K}_1, \qquad u_{t_2} = \mathcal{K}_2, \qquad \ldots, \qquad u_{t_n} = \mathcal{K}_n, \qquad \mathcal{K}_{n+1} = 0 \tag{2.21}$$

which further will be called the *n-th stationary KdV system.*

The finite hierarchy of associated Lax equations is given by equations

$$\frac{d}{dt_k}\mathbb{V}_{n+1} = [\mathbb{V}_k, \mathbb{V}_{n+1}], \qquad k = 1, 2, \ldots, n, \tag{2.22}$$

valid under the constraint (2.18). Notice that from (2.20) or directly (2.18) it follows that $(\mathbb{V}_k)_{t_{n+1}} = 0$. Thus, one obtains the Lax equations (2.22) simply imposing (2.20) on the zero-curvature equations (2.17), with $r = n+1$, or by the compatibility conditions between eigenvalue problem (2.19b) and the respective linear problems (2.12).

After imposing the constraint (2.19b) the existence of nontrivial solutions for the respective linear problems enforces the characteristic equation

$$\det\left(\mathbb{V}_{n+1} - \lambda^m \mu \mathbb{I}\right) = 0, \tag{2.23}$$

associated with (2.18). Equation (2.23) determines the spectral curve

$$-\frac{1}{2}P_{n+1}\left(P_{n+1}\right)_{xx} + \frac{1}{4}\left(P_{n+1}\right)_x^2 - (u-\lambda)P_{n+1}^2 = \lambda^{2m}\mu^2, \tag{2.24}$$

which takes the more explicit form

$$\lambda^{2n+1} + \sum_{k=0}^{n} h_k \lambda^{n-k} = \lambda^{2m}\mu^2, \tag{2.25}$$

where

$$h_k = -\frac{1}{16}\sum_{i=0}^{n-k}\left[2\gamma_{n-i}(\gamma_{i+k})_{xx} - (\gamma_{n-i})_x(\gamma_{i+k})_x + 4u\gamma_{n-i}\gamma_{i+k}\right] + \frac{1}{4}\sum_{i=1}^{n-k}\gamma_{n-i+1}\gamma_{i+k}.$$

In particular, by (2.11)

$$h_0 = -\frac{1}{16}\sum_{i=0}^{n}\left[2\gamma_{n-i}(\gamma_i)_{xx} - (\gamma_{n-i})_x(\gamma_i)_x + 4u\gamma_{n-i}\gamma_i\right] + \frac{1}{4}\sum_{i=1}^{n}\gamma_{n-i+1}\gamma_i \equiv -\gamma_{n+1} \tag{2.26a}$$

and

$$h_n = -\frac{1}{8}\gamma_n(\gamma_n)_{xx} + \frac{1}{16}(\gamma_n)_x^2 - \frac{1}{4}u\gamma_n^2. \tag{2.26b}$$

In fact, the coefficients $h_0, \ldots, h_n$ of (2.25) are constants of motion of the respective stationary system (2.21), and thus the spectral curve (2.25) describes a common level of them.

Remark 2 *Observe that the l.h.s. of the spectral curve* (2.24) *could be obtained alternatively as follows. By* (2.8) *the* $(n+1)$*-st stationary flow* (2.18) *is given by the condition*

$$2\left(P_{n+1}\right)_x(u-\lambda)+u_xP_{n+1}+\frac{1}{2}\left(P_{n+1}\right)_{3x}=0, \tag{2.27}$$

which can be directly integrated to the form:

$$-\frac{1}{2}P_{n+1}\left(P_{n+1}\right)_{xx}+\frac{1}{4}\left(P_{n+1}\right)_x^2-(u-\lambda)P_{n+1}^2=C(\lambda). \tag{2.28}$$

Here $C(\lambda)=\lambda^{2n+1}+\sum_{k=0}^{n}\varepsilon_k\lambda^{n-k}$ *is an 'integral' sum in* λ *with constant coefficients* $\varepsilon_i=h_i$*. Thus, differentiating the spectral curve* (2.28) *or* (2.24) *one reconstructs the stationary condition* (2.27)*. The equation* (2.28) *plays a crucial role in the construction presented in the article [2].*

2.5 Two integrated representations of stationary KdV systems

Since the KdV hierarchy is bi-Hamiltonian, the $(n+1)$-st stationary flow (2.18) can be written in two ways:

$$\mathcal{K}_{n+1}=\pi_0\gamma_{n+1}=\pi_1\gamma_n=0,$$

which leads to two different integrated representations of the n-th stationary KdV system (2.21). Indeed, integrating the first Hamiltonian structure, $\pi_0\gamma_{n+1}=0$ we find that

$$\gamma_{n+1}+c=0, \tag{2.29}$$

where c is an integration constant. This means that the constraint (2.29) defines a (Hamiltonian) foliation of the stationary manifold $\mathcal{M}_n$, of codimension 1, parameterized by the constant c:

$$\mathcal{M}_n=\bigcup_{c\in\mathbb{R}}\mathcal{M}_{n,c}.$$

In result, the first integrated representation of the n-th stationary KdV system is given by

$$u_{t_1}=\mathcal{K}_1,\qquad u_{t_2}=\mathcal{K}_2,\qquad \ldots,\qquad u_{t_n}=\mathcal{K}_n,\qquad \gamma_{n+1}+c=0, \tag{2.30}$$

which constitutes a system of n ODE's on the $2n$-dimensional leaf $\mathcal{M}_{n,c}$ endowed with the jet coordinates $u,u_x,\ldots,u_{(2n-1)x}$ (the higher derivatives of u with respect to x are not needed as they can all be eliminated by the differential constraint (2.29)). The associated Lax equations are then given by (2.22) with the imposed constraint (2.29). In this case the spectral curve (2.25), choosing $m=0$ and taking into account (2.26a), takes the form

$$\lambda^{2n+1}+c\lambda^n+\sum_{k=1}^{n}H_k\lambda^{n-k}=\mu^2, \tag{2.31}$$

where $H_k:=h_k$.

Integrating the second Hamiltonian structure, $\pi_1\gamma_n=0$, we find another constraint

$$\frac{1}{2}\gamma_n(\gamma_n)_{xx}-\frac{1}{4}(\gamma_n)_x^2+u\gamma_n^2+4\bar{c}=0 \tag{2.32}$$

which defines an alternative foliation of the stationary manifold $\mathcal{M}_n$ parameterized by the constant $\bar{c}$. Thus, (2.32) provides the second representation of the n-th stationary KdV system (2.21):

$$u_{t_1}=\mathcal{K}_1,\qquad u_{t_2}=\mathcal{K}_2,\qquad \ldots,\qquad u_{t_n}=\mathcal{K}_n,\qquad \frac{1}{2}\gamma_n(\gamma_n)_{xx}-\frac{1}{4}(\gamma_n)_x^2+u\gamma_n^2+4\bar{c}=0 \tag{2.33}$$

on the $2n$-dimensional leaf $\bar{\mathcal{M}}_{n,\bar{c}}$ endowed with the same set of jet coordinates $u,u_x,\ldots,u_{(2n-1)x}$ (again, higher order derivatives can be eliminated by (2.32)). The Lax representation of the system (2.33) is again given by (2.22) but now with the imposed condition (2.32). The spectral curve (2.25), taking into account (2.26b) and choosing $m=1$, attains now the form

$$\lambda^{2n}+\bar{c}\lambda^{-1}+\sum_{k=1}^{n}\bar{H}_k\lambda^{n-k}=\lambda\mu^2, \tag{2.34}$$

where $\bar{H}_k:=h_{k-1}$.

Note that the two foliations of the stationary manifold $\mathcal{M}_n$, defined by the constraints (2.29) and (2.32), are not equivalent, and one can show that they are mutually transversal.

Remark 3 *The two integrated representations of (the same) stationary KdV system, as given by* (2.30) *and by* (2.33)*, are not equivalent because the complete form of the evolution equations is only given in jet coordinates on the leaves* $\mathcal{M}_{n,c}$ *and* $\bar{\mathcal{M}}_{n,\bar{c}}$ *by taking into account the respective constraints* (2.29) *and* (2.32) *and their differential consequences. However, both representations become equivalent when considered on the extended* $(2n+1)$*-dimensional phase space, the stationary manifold* $\mathcal{M}_n$*. The equivalence is given by the appropriate Miura map, see Lemma 8.*

Example 4 *The first representation* (2.30) *of the stationary KdV system, given for* $n = 2$*, is constituted by the first two flows from the KdV hierarchy:*

$$u_{t_1} = u_x \equiv K_1, \qquad u_{t_2} = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x \equiv \mathcal{K}_2, \tag{2.35a}$$

and the constraint

$$\frac{1}{16}u_{4x} + \frac{5}{8}uu_{xx} + \frac{5}{16}u_x^2 + \frac{5}{8}u^3 + c = \gamma_3 + c = 0. \tag{2.35b}$$

The Lax representation of the stationary system (2.35) *is given by the Lax equations*

$$\frac{d}{dt_1}\mathbb{V}_3 = [\mathbb{V}_1, \mathbb{V}_3], \qquad \frac{d}{dt_2}\mathbb{V}_3 = [\mathbb{V}_2, \mathbb{V}_3],$$

where the matrices $\mathbb{V}_1$ *and* $\mathbb{V}_2$ *are given by* (2.14a) *and the matrix* $\mathbb{V}_3$ (2.14b) *under the constraint* (2.35b) *takes the form*

$$V_3 = \begin{pmatrix} -\frac{1}{4}u_x\lambda - \frac{1}{16}(u_{3x} + 6uu_x) & \lambda^2 + \frac{1}{2}u\lambda + \frac{1}{8}(u_{xx} + 3u^2) \\ \lambda^3 - \frac{1}{2}u\lambda^2 - \frac{1}{8}(u_{xx} + u^2)\lambda + \frac{1}{8}uu_{xx} - \frac{1}{16}u_x^2 + \frac{1}{4}u^3 + c & \frac{1}{4}u_x\lambda + \frac{1}{16}(u_{3x} + 6uu_x) \end{pmatrix}. \tag{2.36}$$

The associated spectral curve (2.31) *for* $n = 2$ *is given by*

$$\lambda^5 + c\lambda^2 + H_1\lambda + H_2 = \mu^2,$$

where one finds the following (nontrivial) integrals of motion:

$$\begin{aligned} H_1 &= \frac{1}{32}u_xu_{3x} - \frac{1}{64}u_{xx}^2 + \frac{5}{32}uu_x^2 + \frac{5}{64}u^4 + \frac{1}{2}cu, \\ H_2 &= \frac{3}{64}uu_xu_{3x} + \frac{1}{256}u_{3x}^2 - \frac{1}{128}u_x^2u_{xx} + \frac{5}{64}u^3u_{xx} + \frac{15}{128}u^2u_x^2 + \frac{1}{64}uu_{xx}^2 + \frac{3}{32}u^5 + \frac{1}{8}cu_{xx} + \frac{3}{8}cu^2. \end{aligned} \tag{2.37}$$

Example 5 *The second representation* (2.33) *of the stationary KdV system, for* $n = 2$*, is given by the flows:*

$$u_{t_1} = u_x \equiv K_1, \qquad u_{t_2} = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x \equiv \mathcal{K}_2, \tag{2.38a}$$

and the constraint

$$\frac{1}{32}u_{xx}u_{4x} + \frac{3}{32}u^2u_{4x} - \frac{1}{64}u_{3x}^2 - \frac{3}{16}uu_xu_{3x} + \frac{3}{16}u_x^2u_{xx} + \frac{1}{4}uu_{xx}^2 + \frac{15}{16}u^3u_{xx} + \frac{9}{16}u^5 + 4\bar{c} = 0. \tag{2.38b}$$

The Lax representation of the stationary system (2.38) *is given by*

$$\frac{d}{dt_1}\mathbb{V}_3 = [\mathbb{V}_1, \mathbb{V}_3], \qquad \frac{d}{dt_2}\mathbb{V}_3 = [\mathbb{V}_2, \mathbb{V}_3],$$

where the matrices $\mathbb{V}_1$ *and* $\mathbb{V}_2$ *are given by* (2.14a) *and the matrix* $\mathbb{V}_3$ (2.14b) *under the constraint* (2.38b) *takes the form*

$$\mathbb{V}_3 = \begin{pmatrix} -\frac{1}{4}u_x\lambda - \frac{1}{16}(u_{3x} + 6uu_x) & \lambda^2 + \frac{1}{2}u\lambda + \frac{1}{8}(u_{xx} + 3u^2) \\ \lambda^3 - \frac{1}{2}u\lambda^2 - \frac{1}{8}(u_{xx} + u^2)\lambda + * & \frac{1}{4}u_x\lambda + \frac{1}{16}(u_{3x} + 6uu_x) \end{pmatrix}, \tag{2.39}$$

where

$$* = -\frac{u_{3x}^2 + 12uu_xu_{3x} + 36u^2u_x^2 - 256\bar{c}}{32u_{2x} + 96u^2}. \tag{2.40}$$

The associated spectral curve (2.34) *for* $n = 2$ *is given by*

$$\lambda^4 + \bar{c}\lambda^{-1} + \bar{H}_1\lambda + \bar{H}_2 = \lambda\mu^2$$

with the integrals of motion:

$$\begin{aligned} \bar{H}_1 &= -\frac{u_{3x}^2 - 2u_x^2u_{2x} + 12uu_xu_{3x} + 30u^2u_x^2}{32(u_{2x} - 3u^2)} - \frac{1}{8}uu_{2x} - \frac{1}{4}u^3 + \frac{8\bar{c}}{u_{2x} + 3u^2}, \\ \bar{H}_2 &= -\frac{uu_{3x}^2 - 2u_xu_{2x}u_{3x} + 6u^2u_xu_{3x} - 12uu_x^2u_{2x}}{64(u_{2x} + 3u^2)} - \frac{1}{64}u_{2x}^2 - \frac{1}{16}u^2u_{2x} - \frac{3}{64}u^4 + \frac{4\bar{c}u}{u_{2x} + 3u^2}. \end{aligned} \tag{2.41}$$

3 Stäckel systems

In this chapter we gather the necessary information about Stäckel systems.

3.1 Stäckel systems in separation coordinates

Let us consider the spectral curve [27] in the form

$$\sigma(\lambda) + \sum_{k=1}^{n} H_k \lambda^{n-k} = \lambda^m \mu^2, \qquad m \in \mathbb{Z}, \tag{3.1}$$

where $\sigma(\lambda)$ is a (Laurent) polynomial in the variables λ and λ^{-1}. The associated separable systems belong to the so-called Benenti subclass of Stäckel systems [5, 6, 10, 12]. The separation relations are reconstructed by taking n copies of (3.1) with respect to the coordinates $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ on a phase space $M = T^*Q$, where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)^T$ are local coordinates on the configuration space Q and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^T$ are the (fibre) momentum coordinates. Thus, solving the linear system

$$\sigma(\lambda_i) + \sum_{k=1}^{n} H_k \lambda_i^{n-k} = \lambda_i^m \mu_i^2, \qquad i = 1, \ldots, n,$$

with respect to functions $H_k = H_k(\boldsymbol{\lambda}, \boldsymbol{\mu})$ we obtain n quadratic in momenta Hamiltonians on M

$$H_k = \frac{1}{2} \boldsymbol{\mu}^T K_k G_m \boldsymbol{\mu} + V_k, \qquad k = 1, \ldots, n, \tag{3.2}$$

where G_m represents the contravariant metric, defined by the first Hamiltonian H_1, on the configuration space Q. In fact

$$G_m = L^m G_0, \qquad G_0 = 2\,\mathrm{diag}\left(\frac{1}{\Delta_1}, \ldots, \frac{1}{\Delta_n}\right), \qquad \Delta_i = \prod_{j \neq i} (\lambda_i - \lambda_j).$$

Here, K_k are respective Killing tensors and L is a special conformal Killing tensor [15], given by:

$$K_k = (-1)^{k+1} \mathrm{diag}\left(\frac{\partial s_k}{\partial \lambda_1}, \ldots, \frac{\partial s_k}{\partial \lambda_n}\right), \qquad L = \mathrm{diag}(\lambda_1, \ldots, \lambda_n),$$

where s_k are the elementary symmetric polynomials in λ_i. The potential functions V_k are given by

$$V_k = (-1)^{k+1} \sum_{i=1}^{n} \frac{\partial s_k}{\partial \lambda_i} \frac{\sigma(\lambda_i)}{\Delta_i}. \tag{3.3}$$

The Hamiltonians (3.2) are in involution with respect to the Poisson bracket defined by

$$\{\cdot, \cdot\} = \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i} \wedge \frac{\partial}{\partial \mu_i}$$

and moreover, by the very construction, they are all separable in the variables $(\boldsymbol{\lambda}, \boldsymbol{\mu})$. The evolution of any observable ξ with respect to the Hamiltonian H_k has the form $\xi_{t_k} = \{\xi, H_k\}$ and the Hamiltonian evolution equations are

$$\boldsymbol{\lambda}_{t_k} = \{\boldsymbol{\lambda}, H_k\}, \qquad \boldsymbol{\mu}_{t_k} = \{\boldsymbol{\mu}, H_k\}, \qquad k = 1, \ldots, n. \tag{3.4}$$

3.2 Lax representation

As it shown in [9], the Hamiltonian evolution equations (3.4) associated with the spectral curves (3.1) can be represented by the (isospectral) Lax equations

$$\frac{d}{dt_k} \mathbb{L} = [\mathbb{U}_k, \mathbb{L}], \qquad k = 1, \ldots, n, \tag{3.5}$$

with $\mathbb{L}$ and $\mathbb{U}_k$ being 2×2 traceless matrices depending rationally on the spectral parameter λ. The Lax matrix $\mathbb{L}$ has the form

$$\mathbb{L} = \begin{pmatrix} \boldsymbol{v} & \boldsymbol{u} \\ \boldsymbol{w} & -\boldsymbol{v} \end{pmatrix}, \tag{3.6}$$

where in the separation coordinates $(\boldsymbol{\lambda},\boldsymbol{\mu})$ the entries are[1]

$$\boldsymbol{u} = \prod_{k=1}^{n}(\lambda-\lambda_k) \equiv \lambda^n + \sum_{k=1}^{n}(-1)^k s_k\lambda^{n-k},$$

$$\boldsymbol{v} = \sum_{k=1}^{n}(-1)^{k+1}\left[\sum_{i=1}^{n}\frac{\partial s_k}{\partial\lambda_i}\frac{\lambda_i^m\mu_i}{\Delta_i}\right]\lambda^{n-k}$$

and

$$\boldsymbol{w} = \frac{1}{\boldsymbol{u}}\left[\lambda^m\Big(\sigma(\lambda)+\sum_{k=1}^{n}H_k\lambda^{n-k}\Big)-\boldsymbol{v}^2\right]. \tag{3.7}$$

In fact, $\boldsymbol{w}$ is defined so that the spectral curve (3.1) can be reconstructed from the characteristic equation for $\mathbb{L}$, since

$$0 = \det\big[\mathbb{L}-\lambda^m\mu\mathbb{I}\big] = -\lambda^m\Big(\sigma(\lambda)+\sum_{k=1}^{n}H_k\lambda^{n-k}-\lambda^m\mu^2\Big).$$

One can show that the expression in the quadratic bracket in (3.7) factorizes so that $\boldsymbol{w}$ takes the form of a Laurent polynomial in λ:

$$\boldsymbol{w} = \lambda^m\left[\frac{\sigma(\lambda)-\lambda^{-m}\boldsymbol{v}^2}{\boldsymbol{u}}\right]_+. \tag{3.8}$$

Here, the operation $[\cdot]_+$ means the projection on the uniquely defined quotient of the division of an analytic function A over a (pure) polynomial $\boldsymbol{u}$ such that the following decomposition holds:

$$A = \left[\frac{A}{\boldsymbol{u}}\right]_+\boldsymbol{u}+r,$$

where the (unique) remainder r is a lower degree polynomial than the polynomial $\boldsymbol{u}$, see for details [9]. In particular when A is a Laurent polynomial we have

$$\left[\frac{A}{\boldsymbol{u}}\right]_+ \equiv \left[\frac{[A]_{\geqslant 0}}{\boldsymbol{u}}\right]_{\geqslant 0}+\left[\frac{[A]_{<0}}{\boldsymbol{u}}\right]_{<0},$$

where $[\cdot]_{\geqslant 0}$ is the projection on the part consisting of non-negative degree terms in the expansion into Laurent series at ∞ and $[\cdot]_{<0}$ is the projection on the part consisting of negative degree terms in the expansion into Laurent series at 0.

Further, the generating matrices $\mathbb{U}_k$ are defined by

$$\mathbb{U}_k := \left[\frac{\boldsymbol{u}_k\mathbb{L}}{\boldsymbol{u}}\right]_+ \equiv \begin{pmatrix} \left[\frac{\boldsymbol{u}_k\boldsymbol{v}}{\boldsymbol{u}}\right]_+ & \boldsymbol{u}_k \\ \left[\frac{\boldsymbol{u}_k\boldsymbol{w}}{\boldsymbol{u}}\right]_+ & -\left[\frac{\boldsymbol{u}_k\boldsymbol{v}}{\boldsymbol{u}}\right]_+ \end{pmatrix}, \qquad k=1,\ldots,n, \tag{3.9}$$

where

$$\boldsymbol{u}_k := \left[\frac{\boldsymbol{u}}{\lambda^{n-k+1}}\right]_+ \equiv \lambda^{k-1}+\sum_{i=1}^{k-1}(-1)^k s_k\lambda^{k-i-1}.$$

The evolution of the Lax matrix (3.6) with respect to Hamiltonian equations (3.4), and consequently the Lax equations (3.5), can be directly derived from the following useful relations:

$$\{\boldsymbol{u},H_k\} = -2\boldsymbol{u}_k\boldsymbol{v}+2\boldsymbol{u}\left[\frac{\boldsymbol{u}_k\boldsymbol{v}}{\boldsymbol{u}}\right]_+,$$

$$\{\boldsymbol{v},H_k\} = \boldsymbol{u}_k\boldsymbol{w}-\boldsymbol{u}\left[\frac{\boldsymbol{u}_k\boldsymbol{w}}{\boldsymbol{u}}\right]_+,$$

$$\{\boldsymbol{w},H_k\} = -2\boldsymbol{w}\left[\frac{\boldsymbol{u}_k\boldsymbol{v}}{\boldsymbol{u}}\right]_+ +2\boldsymbol{v}\left[\frac{\boldsymbol{u}_k\boldsymbol{w}}{\boldsymbol{u}}\right]_+,$$

[1] In the construction of the Lax equations in [9] there is some freedom. In the present article we choose, using for a moment the notation from [9], $g(\lambda)=\frac{1}{2}f(\lambda)=\lambda^m$.

which were obtained in [9]. Moreover, considering (3.10) for $k=1$ and observing that $\boldsymbol{u}_1=1$ and $\left[\frac{\boldsymbol{v}}{\boldsymbol{u}}\right]_+=0$ we can rewrite the matrices (3.6) and (3.9) in the form:

$$\mathbb{L}=\begin{pmatrix} -\frac{1}{2}\dot{\boldsymbol{u}} & \boldsymbol{u} \\ -\frac{1}{2}\ddot{\boldsymbol{u}}+\boldsymbol{u}Q & \frac{1}{2}\dot{\boldsymbol{u}} \end{pmatrix} \tag{3.11a}$$

and

$$\mathbb{U}_k=\begin{pmatrix} -\frac{1}{2}\dot{\boldsymbol{u}}_k & \boldsymbol{u}_k \\ -\frac{1}{2}\ddot{\boldsymbol{u}}_k+\boldsymbol{u}_kQ & \frac{1}{2}\dot{\boldsymbol{u}}_k \end{pmatrix}, \tag{3.11b}$$

where $Q\equiv\left[\frac{\boldsymbol{w}}{\boldsymbol{u}}\right]_+$. Here, the dot means the derivative with respect to the first Hamiltonian flow, i.e. $\dot{\xi}\equiv\xi_{t_1}$, and one can see that $\dot{\boldsymbol{u}}_k=\left[\frac{\boldsymbol{u}_k\dot{\boldsymbol{u}}}{\boldsymbol{u}}\right]_+$. Now, the connection between (3.11) and (2.13) is apparent.

3.3 Stäckel systems in Viète's coordinates

From the point of view of expressing the Hamiltonian systems (3.4) in the Lax form the most practical are the so-called Viète's coordinates, defined as

$$q_i=(-1)^i s_i,\qquad p_i=-\sum_{k=1}^{n}\frac{\lambda_k^{n-i}\mu_k}{\Delta_k},\qquad i=1,\ldots,n. \tag{3.12}$$

Since the above transformation is a point transformation on M, these coordinates are also canonical with respect to the same Poisson bracket $\{\cdot,\cdot\}$ which now reads $\{\cdot,\cdot\}=\sum_i\frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial p_i}$. Let $\boldsymbol{p}=(p_1,\ldots,p_n)^T$ and $\boldsymbol{q}=(q_1,\ldots,q_n)^T$. In this coordinates the geodesic part of the Hamiltonians H_k is always polynomial function of their arguments and the potentials V_k are either polynomials or rational functions. In Viète's coordinates the Hamiltonians (3.2) take the form

$$H_k=\frac{1}{2}\boldsymbol{p}^TK_kG_m\boldsymbol{p}+V_k,\qquad k=1,\ldots,n,$$

and the respective Hamiltonian evolution equations are

$$\boldsymbol{q}_{t_k}=\{\boldsymbol{q},H_k\},\qquad \boldsymbol{p}_{t_k}=\{\boldsymbol{p},H_k\},. \tag{3.13}$$

For $\sigma(\lambda)=\sum_i\alpha_i\lambda^i$ the potential functions (3.3) are given by $V_k=\sum_i\alpha_i\mathcal{V}_k^{(i)}$, where the so-called elementary separable potentials $\mathcal{V}_k^{(i)}$ can be explicitly constructed by the recursion formula [13]

$$\mathcal{V}^{(i)}=R^i\mathcal{V}^{(0)},\qquad \mathcal{V}^{(i)}=\left(\mathcal{V}_1^{(i)},\ldots,\mathcal{V}_n^{(i)}\right)^T,\qquad \mathcal{V}^{(0)}=(0,\ldots,0,-1)^T,$$

where

$$R=\begin{pmatrix} -q_1 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \vdots & 0 & 0 & 1 \\ -q_n & 0 & 0 & 0 \end{pmatrix},\qquad R^{-1}=\begin{pmatrix} 0 & 0 & 0 & -\frac{1}{q_n} \\ 1 & 0 & 0 & \vdots \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & -\frac{q_{n-1}}{q_n} \end{pmatrix}. \tag{3.14}$$

In Viète's coordinates the metric G_0 for $m=0$ has the form

$$(G_0)^{ij}=\begin{cases} 2q_{i+j-n-1} & \text{if}\quad i+j\geq n+1, \\ 0 & \text{otherwise}, \end{cases}$$

and the metrics for arbitrary m are given by $G_m=L^mG_0$, where in Viète's coordinates the special conformal Killing tensor L has the matrix representation identical to R (3.14) [12]. Moreover, the Killing tensors K_r, for $r=1,\ldots,n$, are given by

$$(K_r)^i_j=\begin{cases} q_{i-j+r-1} & \text{if}\quad i\leqslant j\quad\text{and}\quad r\leqslant j, \\ -q_{i-j+r-1} & \text{if}\quad i>j\quad\text{and}\quad r>j, \\ 0 & \text{otherwise}. \end{cases}$$

Notice that $(K_1)^i_j = \delta^i_j$. For convenience, in the above definitions we set $q_0 = 1$ and $q_l = 0$ for $l < 0$ or $l > n$.

In Viète's coordinates $\boldsymbol{u}$ in the Lax matrix (3.6) is simply given by

$$\boldsymbol{u} = \lambda^n + \sum_{k=1}^{n} q_k \lambda^{n-k}, \tag{3.15}$$

and by simple calculation, involving the change of coordinates for the metric G_m, and observation that $\mu_i = \sum_{k=1}^{n} (-1)^k \frac{\partial s_k}{\partial \lambda_i} p_k$, one finds (again) that $\boldsymbol{v}$ has the form

$$\boldsymbol{v} = -\frac{1}{2} \sum_{k=1}^{n} \Big[\sum_{l=1}^{n} (G_m)^{kl} p_l \Big] \lambda^{n-k} \equiv -\frac{1}{2} \dot{\boldsymbol{u}}.$$

Finally $\boldsymbol{w}$ can be obtained from the formula (3.7) or (3.8).

4 Stäckel representations of KdV stationary systems

The first Stäckel representation of the n-th KdV stationary system is given by the spectral curve (2.31),

$$\lambda^{2n+1} + c\lambda^n + \sum_{k=1}^{n} H_k \lambda^{n-k} = \mu^2, \tag{4.1}$$

which is a special case of the general case (3.1) with $m = 0$ and $\sigma(\lambda) = \lambda^{2n+1} + c\lambda^n$. In this case, the Hamiltonians H_k in the Viète's coordinates $(\boldsymbol{q}, \boldsymbol{p})$ are given by

$$H_k = \frac{1}{2} \boldsymbol{p}^T K_k G_0 \boldsymbol{p} + \mathcal{V}_k^{(2n+1)} + c\mathcal{V}_k^{(n)}, \qquad k = 1, \ldots, n. \tag{4.2}$$

The second representation is associated with the spectral curve (2.34)

$$\lambda^{2n} + \bar{c}\lambda^{-1} + \sum_{k=1}^{n} \bar{H}_k \lambda^{n-k} = \lambda\mu^2, \tag{4.3}$$

which is a special case of (3.1) with $m = 1$ and $\sigma(\lambda) = \lambda^{2n} + \bar{c}\lambda^{-1}$. In this case the Hamiltonians $\bar{H}_k$ are

$$\bar{H}_k = \frac{1}{2} \boldsymbol{p}^T K_k G_1 \boldsymbol{p} + \mathcal{V}_k^{(2n)} + \bar{c}\mathcal{V}_k^{(-1)}, \qquad k = 1, \ldots, n. \tag{4.4}$$

The Lax representation (3.5) of respective Hamiltonian flows (3.13) are generated by the Lax operator (3.6) or equivalently (3.11a) with the same, in both representations, Q term:

$$Q \equiv \left[\frac{\boldsymbol{w}}{\boldsymbol{u}} \right]_+ = \left[\frac{\lambda^{2n+1}}{\boldsymbol{u}^2} \right]_{\geqslant 0} = \lambda - 2q_1. \tag{4.5}$$

One obtains (4.5) using the formula (3.8) for $\boldsymbol{w}$ and observing that, in this particular cases, only the term with the highest degree in $\sigma(\lambda)$ contributes to the form of Q.

The equivalence between the appropriate Stäckel representations and the representations (2.30) and (2.33) of n-th stationary KdV system (2.21) is apparent on the level of Lax equations (2.22) (valid under the respective constraints) and (3.5) if we make the following identifications:

$$\mathbb{L} \equiv \mathbb{V}_{n+1}, \qquad \mathbb{U}_k \equiv \mathbb{V}_k, \qquad k = 1, \ldots, n,$$

and

$$\boldsymbol{u} \equiv P_{n+1}, \qquad \boldsymbol{u}_k \equiv P_k, \qquad k = 1, \ldots, n. \tag{4.6}$$

The transformation between the jet coordinates on the respective leaves: $\mathcal{M}_{n,c}$ for $m = 0$ and $\bar{\mathcal{M}}_{n,\bar{c}}$ for $m = 1$, of the representations (2.30) and (2.33) of n-th stationary KdV system and Viète's coordinates $(\boldsymbol{q}, \boldsymbol{p})$ is given, through the KdV co-symmetries γ_i, as

$$q_i = \frac{1}{2}\gamma_i, \qquad p_i = \frac{1}{2} \sum_{j=1}^{n} (G_m^{-1})_{ij} (\gamma_j)_x, \qquad i = 1, \ldots, n, \qquad (m = 0, 1), \tag{4.7}$$

where we make the identification $t_1 \equiv x$. The transformation (4.7) is a direct consequence of comparison of (3.15) with (2.7) through (4.6) and the first Hamiltonian flow (3.13) on $\boldsymbol{q}$: $\dot{\boldsymbol{q}} = G_m \boldsymbol{p}$. Notice that $q_1 = \frac{1}{2}u$. Obviously, one can obtain the transformation between jet coordinates and separation variables from (4.7) using the change of coordinates (3.12).

Summing up the obtained results, we get the first part of Theorem 1, the remaining part is included in Lemma 8.

Example 6 *The Stäckel representation of the first KdV stationary system on $\mathcal{M}_{2,c}$ (for $n = 2$) is generated by the spectral curve*

$$\lambda^5 + c\lambda^2 + H_1\lambda + H_2 = \mu^2.$$

The Hamiltonians (4.4) *in Viète's coordinates* $(\boldsymbol{q}, \boldsymbol{p})$ *are*

$$\begin{aligned} H_1 &= \frac{1}{2}\boldsymbol{p}^T G_0 \boldsymbol{p} + \mathcal{V}_1^{(5)} + c\mathcal{V}_1^{(2)} = 2p_1p_2 + q_1p_2^2 - q_1^4 - q_2^2 + 3q_1^2q_2 + cq_1, \\ H_2 &= \frac{1}{2}\boldsymbol{p}^T K_2 G_0 \boldsymbol{p} + \mathcal{V}_1^{(5)} + c\mathcal{V}_1^{(2)} = p_1^2 + (q_1^2 - q_2)p_2^2 + 2q_1p_1p_2 - q_1^3q_2 + 2q_1q_2^2 + cq_2, \end{aligned} \tag{4.8}$$

where

$$G_0 = 2\begin{pmatrix} 0 & 1 \\ 1 & q_1 \end{pmatrix}, \qquad K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & 1 \\ -q_2 & q_1 \end{pmatrix}.$$

The related Lax operator (3.6) *has the form*

$$\mathbb{L} = \begin{pmatrix} -p_2\lambda - p_1 - q_1p_2 & \lambda^2 + q_1\lambda + q_2 \\ \lambda^3 - q_1\lambda^2 + (q_1^2 - q_2)\lambda - p_2^2 - q_1^3 + 2q_1q_2 + c & p_2\lambda + p_1 + q_1p_2 \end{pmatrix}, \tag{4.9}$$

and the generating matrices (3.9) *are*

$$\mathbb{U}_1 = \begin{pmatrix} 0 & 1 \\ \lambda - 2q_1 & 0 \end{pmatrix}, \qquad \mathbb{U}_2 = \begin{pmatrix} -p_2 & \lambda + q_1 \\ \lambda^2 - q_1\lambda + q_1^2 - 2q_2 & p_2 \end{pmatrix}. \tag{4.10}$$

The respective Hamiltonian flows can be now obtained directly from the Hamiltonian equations (3.13) *or equivalently Lax equations* (3.5)*. Thus, the first Hamiltonian flow has the form*

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}_{t_1} = \begin{pmatrix} 2p_2 \\ 2p_1 + 2q_1p_2 \\ -p_2^2 + 4q_1^3 - 6q_1q_2 - c \\ 2q_2 - 3q_1^2 \end{pmatrix} \tag{4.11}$$

and the second one is given by

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}_{t_2} = \begin{pmatrix} 2p_1 + 2q_1p_2 \\ 2(q_1^2 - q_2)p_2 + 2q_1p_1 \\ -2q_1p_2^2 - 2p_1p_2 + 3q_1^2q_2 - 2q_2^2 \\ p_2^2 + q_1^3 - 4q_1q_2 - c \end{pmatrix}. \tag{4.12}$$

The transformation to the jet coordinates is given by (4.7)*, thus*

$$\begin{aligned} q_1 &= \frac{1}{2}\gamma_1 \equiv \frac{1}{2}u, \qquad q_2 = \frac{1}{2}\gamma_2 \equiv \frac{1}{8}u_{xx} + \frac{3}{8}u^2, \\ p_1 &= \frac{1}{4}(\gamma_2)_x - \frac{1}{4}u(\gamma_1)_x \equiv \frac{1}{16}u_{3x} + \frac{1}{4}uu_x, \qquad p_2 = \frac{1}{4}(\gamma_1)_x \equiv \frac{1}{4}u_x. \end{aligned} \tag{4.13}$$

Substituting (4.13) *to the first* (4.11) *and the second flow* (4.12) *we obtain the equalities*

$$u_{t_1} = (\gamma_1)_x, \qquad u_{t_2} = (\gamma_2)_x, \qquad \gamma_3 + c = 0,$$

which constitute the first representation of the 2*-th stationary KdV system* (2.35)*. Substituting* (4.13) *to* (4.9) *and* (4.10) *one reconstructs the respective Lax matrices* (2.14a) *and* (2.36)*. Substituting* (4.13) *to* (4.8) *one reconstructs the integrals of motion* (2.37)*.*

Example 7 *The Stäckel representation of the second KdV stationary system on* $\bar{\mathcal{M}}_{2,\bar{c}}$ *(for* $n=2$*) is generated by the spectral curve*

$$\lambda^4+\bar{c}\lambda^{-1}+\bar{H}_1\lambda+\bar{H}_2=\lambda\mu^2.$$

Then, in Viète's coordinates $(\boldsymbol{q},\boldsymbol{p})$ *we find the following Hamiltonians*

$$\begin{aligned}\bar{H}_1&=\frac{1}{2}\boldsymbol{p}^TG_1\boldsymbol{p}+\mathcal{V}_1^{(4)}+\bar{c}\mathcal{V}_1^{(-1)}=p_1^2-q_2p_2^2+q_1^3-2q_1q_2+\frac{\bar{c}}{q_2},\\ \bar{H}_2&=\frac{1}{2}\boldsymbol{p}^TK_2G_1\boldsymbol{p}+\mathcal{V}_2^{(4)}+\bar{c}\mathcal{V}_2^{(-1)}=-2q_2p_1p_2-q_1q_2p_2^2+q_1^2q_2-q_2^2+\frac{\bar{c}q_1}{q_2},\end{aligned}\tag{4.14}$$

where

$$G_1=2\begin{pmatrix}1&0\\0&-q_2\end{pmatrix}$$

and the Killing tensors are the same as in the previous example.

The related Lax operator and the generating matrices are given by

$$\mathbb{L}=\begin{pmatrix}-p_1\lambda+q_2p_2&\lambda^2+q_1\lambda+q_2\\ \lambda^3-q_1\lambda^2+(q_1^2-q_2)\lambda+(\bar{c}-q_2^2p_2^2)q_2^{-1}&p_1\lambda-q_2p_2\end{pmatrix}\tag{4.15}$$

and

$$\mathbb{U}_1=\begin{pmatrix}0&1\\ \lambda-2q_1&0\end{pmatrix},\qquad \mathbb{U}_2=\begin{pmatrix}-p_1&\lambda+q_1\\ \lambda^2-q_1\lambda+q_1^2-2q_2&p_1\end{pmatrix}.\tag{4.16}$$

The respective Hamiltonian flows are

$$\begin{pmatrix}q_1\\q_2\\p_1\\p_2\end{pmatrix}_{t_1}=\begin{pmatrix}2p_1\\-2q_2p_2\\-3q_1^2+2q_2\\p_2^2+2q_1+\bar{c}q_2^{-2}\end{pmatrix}\tag{4.17}$$

and

$$\begin{pmatrix}q_1\\q_2\\p_1\\p_2\end{pmatrix}_{t_2}=\begin{pmatrix}-2q_2p_2\\-2q_1q_2p_2-2q_2p_1\\q_2p_2^2-2q_1q_2-\bar{c}q_2^{-1}\\2p_1p_2+q_1p_2^2-q_1^2+2q_2+\bar{c}q_1q_2^{-2}\end{pmatrix}.\tag{4.18}$$

The transformation to the jet coordinates (4.7) *takes the form*

$$\begin{aligned}q_1&=\frac{1}{2}\gamma_1\equiv\frac{1}{2}u, && q_2=\frac{1}{2}\gamma_2\equiv\frac{1}{8}u_{xx}+\frac{3}{8}u^2,\\ p_1&=\frac{1}{4}(\gamma_1)_x\equiv\frac{1}{4}u_x, && p_2=-\frac{1}{2}\frac{(\gamma_2)_x}{\gamma_2}\equiv-\frac{1}{2}\frac{u_{3x}+6uu_x}{u_{xx}+3u^2}.\end{aligned}\tag{4.19}$$

Substituting (4.19) *to the first* (4.17) *and the second flow* (4.18) *we obtain the equalities*

$$u_{t_1}=(\gamma_1)_x,\qquad u_{t_2}=(\gamma_2)_x,\qquad \frac{1}{2}\gamma_2(\gamma_2)_{xx}-\frac{1}{4}(\gamma_2)_x^2+u\gamma_2^2+4\bar{c}=0,$$

which constitute the second representation of the 2*-th stationary KdV system* (2.38)*. Substituting* (4.19) *to* (4.15), (4.16) *and* (4.14) *one reconstructs the respective Lax matrices* (2.14a), (2.39) *and the integrals of motion* (2.41).

5 Miura map and bi-Hamiltonian representations of stationary systems

We know that the Stäckel systems associated with the curves (4.1) and (4.3) are two different representations of the same n-th stationary KdV system on leaves of two mutually transversal Hamiltonian foliations of the stationary manifold $\mathcal{M}_n$. These representations are non-equivalent unless we extend the considered phase space of each of these systems to the $(2n+1)$-dimensional phase space $\mathcal{M}_n \cong M \oplus \mathbb{R}$, where M now plays the role of the respective leaf for both foliations. Consider thus the curve (4.1)

$$\lambda^{2n+1} + \sum_{k=0}^{n} H_k \lambda^{n-k} = \mu^2, \tag{5.1}$$

on the phase space $\mathcal{M}_n$, parameterized by the extended Viète's coordinates $(\boldsymbol{q}, \boldsymbol{p}, c)$. It leads to Hamiltonians (4.2), $H_k = H_k(\boldsymbol{q}, \boldsymbol{p}, c)$, explicitly depending on the additional trivial integral of motion, $H_0 := c$.

Consider also the curve (4.3)

$$\bar{\lambda}^{2n} + \sum_{k=1}^{n+1} \bar{H}_k \bar{\lambda}^{n-k} = \bar{\lambda}\bar{\mu}^2, \tag{5.2}$$

yielding the Hamiltonians (5.2), $\bar{H}_k = \bar{H}_k(\boldsymbol{Q}, \boldsymbol{P}, \bar{c})$, explicitly depending on the additional trivial integral of motion, $\bar{H}_{n+1} := \bar{c}$. These Hamiltonians are defined on the same extended phase space $\mathcal{M}_n$, parameterized this time by the extended Viète's coordinates $(\boldsymbol{Q}, \boldsymbol{P}, \bar{c})$. Having established this notation, we can now formulate the following lemma.

Lemma 8 *The Hamiltonian equations* (3.13) *associated with the Stäckel systems defined by the curves* (5.1) *and* (5.2) *are related by the Miura map on* $\mathcal{M}_n$

$$\begin{aligned} q_i &= Q_i, \qquad i = 1, ..., n \\ p_1 &= -\big(Q_1P_1 + Q_2P_2 + \ldots + Q_nP_n\big), \qquad p_i = P_{i-1}, \qquad i = 2, ..., n, \\ c &= \bar{H}_1(\boldsymbol{Q}, \boldsymbol{P}, \bar{c}) \end{aligned} \tag{5.3}$$

and its inverse

$$\begin{aligned} Q_i &= q_i, \qquad i = 1, ..., n \\ P_i &= p_{i+1}, \qquad i = 1, ..., n-1, \qquad P_n = -\frac{1}{q_n}\big(p_1 + q_1p_2 + \ldots + q_{n-1}p_n\big), \\ \bar{c} &= H_n(\boldsymbol{q}, \boldsymbol{p}, c). \end{aligned} \tag{5.4}$$

Proof. It is simple to observe that the curves (5.1) amd (5.2) are related by the relations

$$\lambda = \bar{\lambda}, \qquad \mu = \bar{\lambda}\bar{\mu} \tag{5.5}$$

and the following identification between Hamiltonians:

$$\begin{aligned} c \equiv H_0(\boldsymbol{q}, \boldsymbol{p}, c) &= \bar{H}_1(\boldsymbol{Q}, \boldsymbol{P}, \bar{c}), \\ H_i(\boldsymbol{q}, \boldsymbol{p}, c) &= \bar{H}_{i+1}(\boldsymbol{Q}, \boldsymbol{P}, \bar{c}), \qquad i = 1, \ldots, n-1, \\ H_n(\boldsymbol{q}, \boldsymbol{p}, c) &= \bar{H}_{n+1}(\boldsymbol{Q}, \boldsymbol{P}, \bar{c}) \equiv \bar{c}. \end{aligned} \tag{5.6}$$

Let us notice that the map (5.5) translates immediately to the transformation between separation variables and so the Viète's ones too, where the coordinates on the configuration space are preserved, hence $\boldsymbol{q} = \boldsymbol{Q}$. Now, the connection between coordinates on the extended phase space is consequence of (5.6) and the equivalence $\dot{\boldsymbol{q}} = \dot{\boldsymbol{Q}}$. Thus, from (3.13) for $k = 1$ we have

$$\boldsymbol{p} = G_0^{-1}G_1\boldsymbol{P} \equiv R^T\boldsymbol{P} \qquad \Longleftrightarrow \qquad \boldsymbol{P} = G_1^{-1}G_0\boldsymbol{P} \equiv (R^{-1})^T\boldsymbol{p},$$

where R, given by (3.14), is the matrix representation, valid only in Viète's coordinates, of the special conformal Killing tensor L [12]. Alternatively, the Miura map can be constructed through the respective changes of coordinates from the separation variables to Viète's coordinates (3.12) using the relations (5.5). ■

The Miura map (5.3), or equivalently its inverse (5.4), represents the non-canonical transformation between two sets $(\boldsymbol{q}, \boldsymbol{p}, c)$ and $(\boldsymbol{Q}, \boldsymbol{P}, \bar{c})$ of canonical coordinates on the extended phase space $\mathcal{M}_n$. In consequence both Stäckel representations are bi-Hamiltonian on $\mathcal{M}_n$ [8] and the Miura map transforms the canonical Poisson structure in parametrization $(\boldsymbol{Q}, \boldsymbol{P}, \bar{c})$ of one Stäckel system into non-canonical Poisson structure in parametrization $(\boldsymbol{q}, \boldsymbol{p}, c)$ of the second one.

Example 9 *Consider again the case $N = n = 2$. The Miura map* (5.3) *on the stationary manifold $\mathcal{M}_2$ has the form:*

$$\begin{aligned} q_1 &= Q_1, \qquad q_2 = Q_2, \qquad p_1 = -Q_1P_1 - Q_2P_2, \qquad p_2 = P_1, \\ c &= \bar{H}_1(\boldsymbol{Q}, \boldsymbol{P}, \bar{c}) \equiv P_1^2 - Q_2P_2^2 + Q_1^3 - 2Q_1Q_2 + \frac{\bar{c}}{Q_2}, \end{aligned} \tag{5.7}$$

where $\bar{H}_1$ is given by (4.14). This yields the bi-Hamiltonian representation of Stäckel systems, from Example 6, on the extended phase space parametrized by (q_1, q_2, p_1, p_2, c):

$$\pi_0 dc = 0, \qquad \pi_0 dH_1 = \pi_1 dc, \qquad \pi_0 dH_2 = \pi_1 dH_1, \qquad 0 = \pi_1 dH_2,$$

where π_0 is a canonical Poisson tensor, with matrix representation:

$$\pi_0 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and π_1 is a non-canonical one, generated by the Miura map (5.7), with matrix representation:

$$\pi_1 = \begin{pmatrix} 0 & 0 & -q_1 & 1 & 2p_2 \\ 0 & 0 & -q_2 & 0 & 2p_1 + 2q_1p_2 \\ q_1 & q_2 & 0 & -p_2 & -p_2^2 + 4q_1^3 - 6q_1q_2 - c \\ -1 & 0 & p_2 & 0 & 2q_2 - 3q_1^2 \\ -2p_2 & -2p_1 - 2q_1p_2 & p_2^2 - 4q_1^3 + 6q_1q_2 + c & -2q_2 + 3q_1^2 & 0 \end{pmatrix},$$

where in the last column naturally appear components of the Hamiltonian vector field (4.11).

The bi-Hamiltonian representation of Stäckel systems from Example 7 can be constructed in a similar fashion.

6 Conclusions and further research

In this article we have systematized the already existing knowledge about stationary KdV systems, but also added new facts: we have shown explicit maps between the jet variables of each stationary KdV system and separation coordinates (via Viete's coordinates) of its two Stäckel representations on $2n$-dimensional phase space. These two non-equivalent Stäckel representations of the same stationary KdV system, when considered on the extended $(2n+1)$-dimensional phase space $\mathcal{M}_n$, are connected by a Miura map, which leads to bi-Hamiltonian formulation of the stationary KdV system, known from the literature. Thus, we have proved that different Stäckel representations of stationary KdV systems are related with different foliations of stationary manifolds $\mathcal{M}_n$.

Based on results of this article, we will attempt to develop a similar theory for the coupled (N-field) KdV hierarchy. This will be a topic of an upcoming article.

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