GENERALIZATIONS OF GRADED PRIME IDEALS OVER GRADED NEAR RINGS

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Abstract

This paper considers graded near-rings over a monoid G as a generalizations of the graded rings over groups, introduce certain innovative graded weakly prime ideals and graded almost prime ideals as a generalizations of graded prime ideals over graded near-rings, and explore their various properties and their generalizations in graded near-rings.

1 INTRODUCTION

Throughout this article, G will be an abelian group with identity e and R be a commutative ring with nonzero unity 1 element. R is called a G-graded ring if $R = \bigoplus_{g \in G} R_g$ with the property $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are called homogeneous of degree g. If $x \in R$, then x can be written uniquely as $\sum_{g \in G} x_g$, where x_g is the component of x in R_g . The set of all homogeneous elements of R is $h(R) = \bigcup_{g \in G} R_g$. Let P be an ideal of a G-graded ring R. Then P is called a graded ideal if $P = \bigoplus_{g \in G} P_g$, i.e., for $x \in P$ and $x = \sum_{g \in G} x_g$ where $x_g \in P_g$ for all $g \in G$. An ideal of a G-graded ring is not necessary graded ideal (see [1]). The concept of graded prime ideals and its generalizations have an indispensable role in commutative G-graded rings. Near-rings are generalizations of rings in which addition is not necessarily abelian and and a graded rings.

only one distributive law holds. They arise in a natural way in the study of mappings on groups: the set M(G) of all maps of a group (G; +) into itself endowed with point-wise addition and composition of functions is a near-ring. For general background on the theory of near-rings, the monographs written by Pilz [7] and Meldrum [6] should be referred. The definition of a near-ring $(N, +, \times)$ is a set N with two binary + and × that satisfy the following axioms:

(1) (N, +) is a group.

(2) (N, \times) is semi-group. (semi-group: a set together with an associative binary operation).

(3) × is right distributive over + (i.e. $(a + b) \times y = ay + by$).

The graded rings were introduced by Yoshida in [8]. Also, graded near-rings were introduced and studied by Dumitru, Nastasescu and Toader in [4]. Let G be a multiplicatively monoid (an algebraic structure with a single associative binary operation) with identity. A near-ring N is called a G-graded near-ring if there exists a family of additive normal subgroups $\{N_{\sigma}\}$ of N satisfying that:

(1)
$$N = \bigoplus_{\sigma \in G} N_{\sigma}.$$

(2) $N_{\sigma}N_{\tau} \subseteq N_{\sigma\tau}$ for all $\sigma, \tau \in G$.

A graded ideal P of a G-graded ring R is said to be graded prime ideal of R if $ab \in P$,

where $a, b \in h(R)$, then $a \in P$ or $b \in P$. Graded prime ideals have been generalized to graded weakly prime ideals and graded almost prime ideals. In [2], a graded ideal P of R is said to be graded weakly prime ideal of R if $0 \neq ab \in P$, where $a, b \in h(R)$, then $a \in P$ or $b \in P$. We say that a graded ideal P of R is a graded almost prime ideal of R if $ab \in P - [P^2 \cap R]$, where $a, b \in h(R)$, then $a \in P$ or $b \in P$ (see [5]).

Bataineh, Al-Shorman and Al-Kilany in [3], defined the concept of graded prime ideals over graded near-rings. A graded ideal P of a graded near-ring N is said to be a graded prime ideal of N if whenever $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$, for any graded ideals I and J in N. In Section Tow, we introduced the concept of graded weakly prime ideals in graded near-rings. We say that P is a graded weakly prime ideal of N if whenever $\{0\} \neq IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$, for any graded ideals I and J in N. In Section Three, we introduce the concept of graded almost prime ideals in graded near-ring. We say that P is a graded almost prime ideal of N if whenever $IJ \subseteq P$ and $IJ \not\subseteq (P^2 \cap N)$, then either $I \subseteq P$ or $J \subseteq P$, for any graded ideals I and J in N.

2 GRADED WEAKLY PRIME IDEALS OVER GRADED NEAR RINGS

In this section, we introduce graded weakly prime ideals graded over near-rings concept and study their basic properties.

Definition 2.1. Let G be a multiplucative monoid group with identity element and N be a G-graded near-ring. A graded ideal P of N is called graded weakly prime ideal of N if whenever $\{0\} \neq IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$, for any graded ideals I and J in N.

Example 2.2. Consider the ring $(\mathbf{Z}_{12}, +, \times)$ is a near-ring with $G = \{0, 1\}$ is a group under (+), where (+) defined as 0+0=0, 0+1=1, 1+0=1 and 1+1=1.

Let N be a G-graded near-ring defined by $N_0 = \mathbb{Z}_{12}$ and $N_1 = \{0\}$. Note that the graded ideals $P_1 = \{0\}, P_2 = \{0, 2, 4, 6, 8, 10\}$ and $P_3 = \{0, 3, 6, 9\}$ are graded weakly prime ideals of N.

Remark 2.3. Every graded prime ideals over graded near-rings is a graded weakly prime ideals over graded near-rings. However, the converse is not true. For Example 2.2, P_1 is a graded weakly prime ideal of N but not graded prime ideal of N.

The following theorem and corollary state that graded weakly prime ideals of N are graded prime ideals of N when certain conditions are met.

Theorem 2.4. Let N be a G-graded near-ring and P be a graded weakly prime ideal of N. If P is not graded prime ideal of N, then $P^2 \cap N = \{0\}$.

Proof. Suppose that $P^2 \cap N \neq \{0\}$. It is observed that P is graded prime ideal of N. Let I and J be a graded ideals of N such that $IJ \subseteq P$. If $IJ \neq \{0\}$, then $I \subseteq P$ or $J \subseteq P$ since P is a graded weakly prime ideal of N. So, it could be assumed that $IJ = \{0\}$. Since $P^2 \cap N \neq \{0\}$, so there exists $p, q \in P$ such that $< q > \neq 0$ and so $(I +)(J + < q >) \neq \{0\}$. Suppose that $(I +)(J + < q >) \not\subseteq P$, then there exists $i \in I$, $j \in J$, $p_0 \in$ and $q_0 \in < q >$ such that $(i + p_0)(j + q_0) \notin P$ which implies that $i(j + q_0) \notin P$, but $i(j + q_0) = i(j + q_0) - ij \in P$ since $IJ = \{0\}$. This is a

contradiction. Thus, $\{0\} \neq (I + \langle p \rangle) \subseteq P$ which implies that $I \subseteq P$ or $J \subseteq P$.

Corollary 2.5. Let N be a G-graded near-ring and let P be a graded ideal of N such that $P^2 \cap N \neq \{0\}$. Then P is graded prime ideal of N if and only if P is graded weakly prime ideal of N.

Proof. Let P be a graded ideal of N such that $P^2 \cap N \neq \{0\}$. By Theorem 2.4, if P is a graded weakly prime ideal of N, than P is a graded prime ideal of N. Also, by Remark 2.3, if P is a graded prime ideal of N, then P is a graded weakly prime ideal of N.

Remark 2.6. It is not necessary that P is graded weakly prime ideal of N such that $P^2 \cap N = \{0\}$. Let N be a graded near-ring which is defined in Example 2.2 and let $P = \{0, 6\}$, Note that $P^2 \cap N = \{0\}$, but P is not graded weakly prime ideal of N.

The next proposition gives an interesting case where graded weakly prime ideals lead to graded prime ideals in a graded near-ring.

Proposition 2.7. Let N be a G-graded near-ring and P be a graded ideal of N. If P is a graded weakly prime ideal of N and $(\{0\} : P) \subseteq P$, then P is a grade prime ideal of N.

Proof. Suppose that P is not a graded prime ideal of N, then there exists $I \not\subseteq P$ and $J \not\subseteq P$ satisfying that $IJ \subseteq P$, where I and J are two graded ideals of N. If $IJ \neq \{0\}$, then it is completed. So, it is assume that $IJ = \{0\}$. Note that $IJ \subseteq P$ since if an element belongs to IP, then it belongs to both N and P. Consider $I(J+P) \subseteq P$ if $I(J+P) \neq \{0\}$, then either $I \subseteq P$ or $J \subseteq P$, this is a contradiction. Otherwise, $I(J+P) = \{0\}$, then $IP = \{0\}$ implies $I \subseteq (\{0\} : P) \subseteq P$.

Theorem 2.8. Let N be a G-graded near-ring and P be a graded weakly prime ideal of N. If $IJ = \{0\}$ with $I \not\subseteq P$ and $J \not\subseteq P$ where I and J are two graded ideals of N, then IP = PJ.

Proof. Suppose that there exists $p \in P$ and $i \in I$ such that ip = 0. Then $\{0\} \neq I(J+) \subseteq P$. But $I \not\subseteq P$ and $J+ \not\subseteq P$, which contradicts that P being graded weakly prime ideal of N.

Lemma 2.9. Let N be a G-graded near-ring. If P, I and J are graded ideals of N such that $P = I \cup J$, then P equals I or J.

Proof. Suppose P does not equal I nor J. Let $x \in P$ such that $x \in I$ but $x \notin J$ and $y \in P$ such that $y \in J$ but $y \notin I$. Since P is a graded ideal of N, $x - y \in P$. This implies $x - y \in I$ or $x - y \in J$. If $x - y \in I$, then $y \in I$ since I is a graded ideal of N, this is a contradiction. If $x - y \in J$, then $x \in J$ since J is a graded ideal of N, this is a contradiction. Therefore, P equals either I or J.

Proposition 2.10. Let N be a G-graded near-ring and P be a graded ideal of N. Then the following are equivalent:

(1) For x, y and $z \in N$ with $0 \neq x (\langle y \rangle + \langle z \rangle) \subseteq P$, $x \in P$ or y and $z \in P$.

(2) For $x \in N$ but $x \notin P$ we have $(P : \langle x \rangle + \langle y \rangle) = P \cup (0 : \langle x \rangle + \langle y \rangle)$ for any $y \in N$.

(3) For $x \in N$ but $x \notin P$ we have $(P : \langle x \rangle + \langle y \rangle) = P$ or $(P : \langle x \rangle + \langle y \rangle) = (0 : \langle x \rangle + \langle y \rangle)$ for any $y \in N$. (4) P is a graded weakly prime ideal of N.

Proof. (1) \Rightarrow (2): Let $t \in N$ and $t \in (P : \langle x \rangle + \langle y \rangle)$ for any y and x belongs to N but $x \notin P$. Then $t(\langle x \rangle + \langle y \rangle) \subseteq P$. If $t(\langle x \rangle + \langle y \rangle) = 0$. Then $t \in (0 : \langle x \rangle + \langle y \rangle)$. Otherwise $0 \neq t(\langle x \rangle + \langle y \rangle) \subseteq P$. Thus, $t \in P$ by hypothesis.

(2) \Rightarrow (3): It is following directly from Lemma 2.9.

 $(3) \Rightarrow (4)$: Let I and J be a graded ideal of N such that $IJ \subseteq P$. Suppose that $I \not\subseteq P$ and $J \not\subseteq P$. Then there exist $j \in J$ with $j \notin P$. Now, it is claimed that $IJ = \{0\}$. Let $j_1 \in J$, then $I(\langle j \rangle + \langle j_1 \rangle \subseteq P)$, which implies $I \subseteq (P : \langle j \rangle + \langle j_1 \rangle)$. Then by assumption, $I(\langle j \rangle + \langle j_1 \rangle) = 0$ which gives $Ij_1 = \{0\}$. Thus $IJ = \{0\}$ and hence P is a graded weakly prime ideal of N.

(4) \Rightarrow (1): If $0 \neq x (\langle y \rangle + \langle z \rangle) \subseteq P$, then $\{0\} \neq \langle x \rangle (\langle y \rangle + \langle z \rangle) \subseteq P$. Since P is a graded weakly prime ideal of N, there is $\langle x \rangle \subseteq P$ or $\langle y \rangle + \langle z \rangle \subseteq P$. By assumption, x, y and $z \in N$. Hence $x \in P$ or y and $z \in P$.

Theorem 2.11. Let N be a G-graded near-ring and P be a graded ideal of N. Then the following are equivalent:

(1) P is a graded weakly prime ideal of N.

(2) For any ideals I and J in N with $P \subset I$ and $P \subset J$, then there is either $IJ = \{0\}$ or $IJ \not\subseteq P$.

(3) For any ideals I and J in N with $I \not\subseteq P$ and $J \not\subseteq P$, then there is either $IJ = \{0\}$ or $IJ \not\subseteq P$.

Proof. (1) \Rightarrow (2): Let I an J be two graded ideals of N with $P \subset I$, $P \subset J$ and $IJ \neq \{0\}$. Take $i \in I$ and $j \in J$ with $i \notin P$ and $j \notin P$, which implies that $\{0\} \neq \langle i \rangle \langle j \rangle \not\subseteq P$ and hence $\{0\} \neq IJ \not\subseteq P$.

 $\begin{array}{ll} (2) \Rightarrow (3): \ Let \ I \ and \ J \ be \ a \ graded \ ideals \ of \ N \ with \ I \ \not\subseteq \ P \ and \ J \ \not\subseteq \ P. \ Then \ there \\ exists \ i_1 \in \ I \ and \ j_1 \in \ J \ such \ that \ i_1 \not\in \ P \ and \ j_1 \not\in \ P. \ Suppose \ that \ < i > < j > \neq \{0\} \\ for \ some \ i \in \ I \ and \ j \in \ J. \ Then \ (P+ < i > + < i_1 >)(P+ < j > + < j_1 >) \neq \{0\} \\ and \ P \subset (P+ < i > + < i_1 >) \ and \ P \subset (P+ < j > + < j_1 >). \ By \ hypothesis, \\ (P+ < i > + < i_1 >)(P+ < j > + < j_1 >). \ By \ hypothesis, \\ (P+ < i > + < i_1 >)(P+ < j > + < j_1 >) \not\subseteq \ P. \ So, \ < i > (P+ < j > + < j_1 >)+ < \\ i_1 > (P+ < j > + < j_1 >) \not\subseteq \ P. \ Hence \ there \ exists \ i' \in < i >, \ i_1' \in < i_1 >, \ j', \ j'' \in < j >, \\ j'_1, \ j''_1 \in < j_1 > \ and \ p_1, \ p_2 \in \ P \ such \ that \ i'(p_1 + j' + j'_1) + i'_1(p_2 + j'' + j''_1) + i'_1(j'' + j''_1) \notin \ P. \ Thus \\ i'(p_1 + j' + j'_1) - i'(j' + j'_1) + i'(j' + j'_1) + i'_1(p_2 + j'' + j''_1) - i'_1(j'' + j''_1) \notin \ P. \ Which \ implies \\ neither \ i'(j' + j'_1) \ nor \ i'_1(j'' + j''_1) \ belongs \ to \ P. \ Therefore, \ IJ \not\subseteq P. \end{array}$

 $(3) \Rightarrow (1)$: Follows directly from the definition of graded weakly prime ideals of N.

Proposition 2.12. Let N be a G-graded near-ring, A be a totally ordered set and $(P_a)_{a \in A}$ be a family of graded weakly prime ideals of N with $P_a \subseteq P_b$ for any $a, b \in A$ with $a \leq b$. Then $P = \bigcap_{a \in A} P_a$ is a graded weakly prime ideal of N.

Proof. Let I and J be two graded ideals of N with $\{0\} \neq IJ \subseteq P$, which implies for all $a \in A$ there is $IJ \subseteq P_a$. If there exists $a \in A$ such that $I \not\subseteq P_a$, then $J \subseteq P_a$. Hence for all $a \leq b$ there is $J \subseteq P_b$. If there exists c < a such that $j \not\subseteq P_c$, then $I \subseteq P_c$ and then $I \subseteq P_a$, this is a contradiction. Hence for any $a \in A$, there is $J \subseteq P_a$. Therefore, $J \subseteq P$.

Proposition 2.13. Let N be a G-graded near-ring and P be an intersection of some graded weakly prime ideals of N. Then for any graded ideal I of N satisfying that $\{0\} \neq I^2 \subseteq P$ there is $I \subseteq P$.

Proof. Let P_a be a set of graded weakly prime ideals of N, P be the intersection of P_a and I be a graded ideal of N such that $\{0\} \neq I^2 \subseteq P$. Then I^2 is subset of each P_a since P_a is graded weakly prime ideal of N there is $I \subseteq P_a$. Therefore, $I \subseteq P$.

Next Example and Theorem 2.16, shows that the pre-image of a surjective homomorphism map of graded weakly prime ideal of N is not necessary to be graded weakly prime ideal of N, while the image of a surjective homomorphism map of graded weakly prime ideal of N which contains the kernal is graded weakly prime ideal of N.

Example 2.14. Let G be the multiplicatively monoid which defined in Example 2.2 and $N = \mathbb{Z}_8$ and $M = \mathbb{Z}_4$ be two G-graded near-rings where $N_0 = \mathbb{Z}_8$, $N_1 = \{0\}$, $M_0 = \mathbb{Z}_4$ and $M_1 = \{0\}$. Consider $\phi : N \to M$ where $\phi(x) = X$ is surjective homomorphism map. However, $\{0\}$ is a graded weakly prime ideal in M although $\phi^{-1}(\{0\}) = \{0, 4\}$ is not graded weakly prime ideal of N.

Lemma 2.15. Let N and M be two G-graded near-rings and ϕ be a surjective homomorphism from N into M. For any two graded ideals I and J of N if $IJ \neq \{0\}$, then $\phi^{-1}(I)\phi^{-1}(J) \neq \{0\}$.

Proof. Let I and J be two graded ideals of N such that $IJ \neq \{0\}$. Suppose that $\phi^{-1}(I)\phi^{-1}(J) = \{0\}$, then $\phi^{-1}(I)\phi^{-1}(J) = \phi^{-1}(IJ) = \{0\}$. Therefore, $\phi(\{0\}) = IJ$ which contradict with the fact that the image of zero is zero for any homomorphism map. Hence $\phi^{-1}(I)\phi^{-1}(J) \neq \{0\}$.

Theorem 2.16. Let N and M be two G-graded near-rings and ϕ be a surjective homomorphism from N into M. Then the image of graded weakly prime ideal of N which contains the kernal of ϕ is a graded weakly prime ideal of M.

Proof. Suppose that $\{0\} \neq IJ \subseteq \phi(P)$ where I and J are graded ideals of M and P is a graded weakly prime ideals of N. By Lemme 2.15, $\phi^{-1}(I)\phi^{-1}(J) \neq \{0\}$. Hence $\{0\} \neq \phi^{-1}(I)\phi^{-1}(J) \subseteq P + Ker(\phi) \subseteq P$. However, $\phi^{-1}(I)\phi^{-1}(J) \subseteq N$ then $\phi^{-1}(I)\phi^{-1}(J) \subseteq P$ since P is a graded weakly prime ideal of N, so $\phi^{-1}(I) \subseteq P \phi^{-1}(J) \subseteq P$. Therefore, $I \subseteq \phi(P)$ or $J \subseteq \phi(P)$. Hence $\phi(P)$ is a graded weakly prime ideal of M.

Next Example and Theorem 2.18, if $I \subseteq P$ and $\pi : N \to \overline{N} := N/I$ is the canonical epimorphism, then $\pi(P)$ is graded weakly prime ideal of \overline{N} if P is graded weakly prime ideal of N while it is not necessary that P is graded weakly prime ideal in N if $\pi(P)$ is graded weakly prime ideal of \overline{N} .

Example 2.17. Let $N = \mathbb{Z}_{18}$ be a G-graded near-ring where $N_0 = \mathbb{Z}_{18}$ and $N_1 = \{0\}$. Consider $\pi : N \to \overline{N} := N/I$, where $\pi(x) = x$ and $I = \{0, 9\}$. It is easily to check hat $\pi(\{0, 9\}) = \overline{0}$ is a graded weakly prime ideal of \overline{N} . However, $I \subseteq \{0, 9\}$ is not graded weakly prime ideal of N.

Theorem 2.18. Let N be a G-graded near-ring and P, I be a graded ideals of N with $I \subseteq P$. Consider $\pi : N \to \overline{N} := N/I$ is the canonical epimorphism. If P is a graded weakly prime ideal of N, then $\pi(P)$ is a graded weakly prime ideal of \overline{N} .

Proof. Let J and K be a graded ideals of N with $\{0\} \neq KJ \subseteq P$, so $\pi(J)$ and $\pi(K)$ are graded ideals of \overline{N} with $\{0\} \neq \pi(J)\pi(K) = \pi(JK) \subseteq \pi(P)$. Since $\{0\} \neq \pi(J)\pi(K)$ then by Lemma 2.15 $\{0\} \neq \pi^{-1}(\pi(J))\pi^{-1}(\pi(K))$ and then $\pi^{-1}(\pi(J))\pi^{-1}(\pi(K)) = JK \subseteq \pi^{-1}(\pi(P)) = P + I = P$. Thus $J \subseteq P$ or $K \subseteq P$. Therefore. $J = \pi^{-1}(\pi(J)) \subseteq P \subseteq \pi^{-1}(\pi(P))$ so $\pi(J) \subseteq \pi(P)$ or $\pi(K) \subseteq \pi(P)$. Thus $\pi(P)$ is a grade weakly prime ideal of \overline{N} .

Note that, from the definition of graded weakly prime ideals, for any graded ideal of N with $I^2 \subseteq P$ where P is a graded weakly prime ideal of N, if $I \not\subseteq P$ then $I^2 = \{0\}$. If there is another special cases that guarantee $I^2 = \{0\}$. The Theorem 2.20 gives one case of them but before state it the following lemma is presented.

Lemma 2.19. Let P be a graded weakly prime ideal of N. If \overline{I} is a graded ideal of N/P with $\overline{I}\overline{J} = \{0\}$ for some non zero graded ideal \overline{J} of N/P. Then there is either $I \subseteq P$ or $PJ = \{0\}$.

Proof. Suppose that $I \not\subseteq P$ and let $p \in P$. Then $(\langle p \rangle + I) \not\subseteq P$ and $(\langle p \rangle + I)J \subseteq P$ which implies $(\langle p \rangle + I)J = \{0\}$ but P is a graded weakly prime ideal of N. Thus $\langle p \rangle J = \{0\}$ and hence $PJ = \{0\}$.

Theorem 2.20. Let N be a G-graded near-ring and P be a graded weakly prime ideal of N with $P^2 = \{0\}$. If I is a graded ideal of N and $I^2 \subseteq P$, then $I^2 = \{0\}$.

Proof. Let P is a graded weakly prime ideal of N and for any $x, y \in I$, there is $\langle x \rangle \langle y \rangle \subseteq I^2 \subseteq P$. Now, the claim is that $\langle x \rangle \langle y \rangle = \{0\}$. Suppose not, then since P is graded weakly prime ideal of N, there is $x \in P$ or $y \in P$. If both $x, y \in P$, then $\langle x \rangle \langle y \rangle \subseteq P^2 = \{0\}$. So, it is assumed that only one of them x or y belongs to P. Take $x \in P$ since $\langle y \rangle \langle y \rangle \subseteq I^2 \subseteq P$ and by Lemma 2.19 we have $\langle x \rangle \langle y \rangle \subseteq P \langle y \rangle = \{0\}$ which implies $I^2 = \{0\}$.

Recall that, if N and M is a G-graded near-rings, then $N \times M$ is a G-graded near-ring.

Theorem 2.21. Let N and M be a G-graded near-rings and P be a graded ideal of N. Then P is a graded weakly prime ideal of N if and only if $P \times M$ is a graded weakly prime ideal of $N \times M$.

Proof. (\Rightarrow) Let P be a graded weakly prime ideal of N and $I \times M$, $J \times M$ be a graded ideal of $N \times M$ such that $\{0\} \neq (I \times M)(J \times M) \subseteq P \times M$. Then $\{0\} \neq (I \times M)(J \times M) = (IJ \times MM) \subseteq P \times M$. So, $\{0\} \neq IJ \subseteq P$ but P is a graded weakly prime ideal of N then $I \subseteq P$ or $J \subseteq P$. Therefore, $I \times M \subseteq P \times M$ or $J \times M \subseteq P \times M$. Thus $P \times M$ is a graded weakly prime ideal of $N \times M$.

(\Leftarrow) Suppose that $P \times M$ is a graded weakly prime ideal of $N \times M$ and Let I, J be a graded ideals of N such that $\{0\} \neq IJ \subseteq P$. Then $\{0\} \neq (I \times M)(J \times M) \subseteq P \times M$. By assumption we have $I \times M \subseteq P \times M$ or $J \times M \subseteq P \times M$. So $I \subseteq P$ or $J \subseteq P$. Thus P is a graded weakly prime ideal of N.

Corollary 2.22. Let N and M be two G-graded near-rings. If every graded ideal of N and M is a product of graded weakly prime ideals, then every graded ideal of $N \times M$ is a product of graded weakly prime ideals.

Proof. Let I be a graded ideal of N and J be a graded ideal of M such that $I = I_1...I_n$ and $J = J_1...J_m$ where I_i and J_j is a graded weakly prime ideal of N and M respectively. If the graded ideal is of the form $I \times M$ then $I \times M = (I_1...I_n) \times M$ can be written as $(I_1 \times M)...(I_n \times M)$ which is by Theorem 2.21 a product of graded weakly prime ideals. Similarly, if the graded ideal is of the form $N \times J$, then it is a product of graded weakly prime ideals. If the graded ideal is of the form $I \times J$ then it can be written as $(I_1...I_n) \times (J_1...J_m) =$ $((I_1...I_2) \times M)(N \times (J_1...J_m)) = (I_1 \times M)...(I_n \times M)(N \times J_1)...(N \times J_m)$ which is a product of graded weakly prime ideals.

Theorem 2.23. Let N and M be two G-graded near-rings. Then a graded ideal P of $N \times M$ is graded weakly prime if and only if it has one of the following two forms: (i) $I \times M$, where I is a graded weakly prime ideal of N. (ii) $N \times J$, where J is a graded prime ideal of M.

Proof. Let P be a graded ideal of $N \times M$. Then P has one of the following three forms (i) $I \times M$ where I is a graded ideal of N or (ii) $N \times J$, where J is proper ideal of M or $I \times J$, where $I \neq N$ and $J \neq M$. If P is of the form $I \times M$ or of the form $N \times J$ then by Theorem 2.21, P is graded weakly prime ideal of $N \times N$ if and only if both I and J are graded weakly prime ideals of N and M respectively. Let $P = I \times J$ be a graded weakly prime ideal of $N \times M$ with $I \neq N$ and $J \neq M$. Suppose $x \in I$. Then $\langle x \rangle \times \{0\} \subseteq P$ This implies that either $\langle x \rangle \times M \subseteq P$ or $\langle N \times \{0\} \subseteq P$. If $\langle x \rangle \times M \subseteq P$, then M = J and if $\langle N \times \{0\} \subseteq P$, then N = I this is a contradiction. Hence $I \times J$ can not be graded weakly prime ideal of $N \times M$ if both I and J are graded ideals.

Theorem 2.24. Let N and M be two G-graded near-rings. Then $P = \{0\} \times \{0\}$ is a graded weakly prime ideal of $N \times M$.

Proof. Let $I = \{0\}$ be a graded ideal of N. Suppose that $x \in I - \{0\}$ then $\langle x \rangle \times \{0\} \subseteq P$ and $\langle x \rangle \times \{0\} \neq \{0\}$. This implies that either $\langle x \rangle \times M \subseteq P$ or $N \times \{0\} \subseteq P$ then N = I this is a contradiction. So $I - \{0\}$ is empty. Similarly if $J = \{0\}$ where J is a graded ideal of M. Therefore, P is a graded weakly prime ideal of $N \times M$.

Proposition 2.25. Let N be a G-graded near-ring and P, I be two graded ideals of N. If P and I are graded weakly prime ideal of N, then $P \cup I$ is a graded weakly prime ideal of N.

Proof. Let J and K be a two graded ideals of N such that $\{0\} \neq JK \subseteq P \cup I$. Then $JK \subseteq P$ or $JK \subseteq I$. Since $JK \neq \{0\}$ and if $JK \subseteq P$, then $J \subseteq P$ or $K \subseteq P$ since P is a graded weakly prime ideal of N. Hence $J \subseteq P \cup I$ or $K \subseteq P \cup I$. If $JK \subseteq I$, then $J \subseteq I$ or $K \subseteq I$ since I is a graded weakly prime ideal of N. thus $J \subseteq P \cup I$ or $K \subseteq P \cup I$. Therefore, $P \cup I$ is a graded weakly prime ideal of N.

3 GRADED ALMOST PRIME IDEALS OVER GRADED NEAR RINGS

In this section, we introduce graded almost prime ideals over near-rings concept and study their basic properties.

Definition 3.1. Let G be a multiplucative monoid group with identity element and N be a G-graded near-ring. A graded ideal P of N is called graded almost prime ideal of N if whenever $IJ \subseteq P$ and $IJ \not\subseteq (P^2 \cap N)$, then either $I \subseteq P$ or $J \subseteq P$, for any graded ideals I and J in N.

Example 3.2. Consider a G-graded near-ring which is defined in Example 2.2. Not that $P_4 = \{0, 4, 8\}$ is a graded almost prime ideal of N but not graded weakly prime ideal of N. However, $P_5 = \{0, 6\}$ is neither graded weakly prime ideal of N nor graded almost prime ideal of N.

In the previous section, it was observed that if P is graded prime ideal then it is graded weakly prime ideal but the converse is not true for example P1 in Example 2.2 is not graded prime ideal of N but it is graded weakly prime ideal of N. Also, by Example 3.2, a graded almost prime ideal of N may not implies a graded weakly prime ideal of N . Now, the question is: Does a graded weakly prime ideal give a graded almost prime ideal? The next theorem answers this question.

Theorem 3.3. Let N be a G-graded near-ring and P be a graded ideal of N. If P is a graded weakly prime ideal of N, then P is a graded almost prime ideal of N.

Proof. Let P be a graded weakly prime ideal of N and I, J be two graded ideals of N such that $IJ \subseteq P$ and $IJ \not\subseteq (P^2 \cap N)$. If P is a graded prime ideal of N then P is a graded almost prime ideal of N. Otherwise, $(P^2 \cap N) = \{0\}$ by Theorem 2.4 $IJ \neq \{0\}$ since $IJ \not\subseteq (P^2 \cap N) = \{0\}$. But P is a graded weakly prime ideal of N. Therefore, either $I \subseteq P$ or $J \subseteq P$ which means that P is a graded almost prime ideal of N.

Proposition 3.4. Let N be a G-graded near-ring and P be a graded prime ideal of N. If P is a graded almost prime ideal of N and $((P^2 \cap N) : P) \subseteq P$, then P is a graded prime ideal of N.

Proof. Suppose that P is not graded prime ideal of N. Then there exist $I \not\subseteq P$ and $J \not\subseteq P$ satisfying that $IJ \subseteq P$ where I and J are two graded ideals of N. If $IJ \not\subseteq (P^2 \cap N)$ we are done. So, it is assumed $IJ \subseteq (P^2 \cap N)$. Consider $I(J+P) \subseteq P$ if $I(J+P) \not\subseteq (P^2 \cap N)$ then there is $I \subseteq P$ or $J \subseteq P$ this is a contradiction. Otherwise, $I(J+P) \subseteq (P^2 \cap N)$ then $IP \subseteq (P^2 \cap N)$ which implies $I \subseteq ((P^2 \cap N) : P) \subseteq P$ which is a contradiction. Thus P is a graded prime ideal of N.

Next, some equivalent conditions are given for a graded ideal to be graded almost prime ideal in the G-graded near-ring.

Theorem 3.5. Let N be a G-graded near-ring and P be a graded ideal of N. Then the following are equivalent: (1) For x, y and $z \in N$ with $x(< y > + < z >) \subseteq P$ and $x(< y > + < z >) \not\subseteq (P^2 \cap N)$ there is $x \in P$ or y and $z \in P$. (2) For $x \in N$ but $x \notin P$, $(P :< x > + < y >) = P \cup ((P^2 \cap N) :< x > + < y >)$ for any $y \in N$. (3) For $x \in N$ but $x \notin P$ we have (P :< x > + < y >) = P or $(P :< x > + < y >) = ((P^2 \cap N) :< x > + < y >) = ((P^2 \cap N) :< x > + < y >)) = ((P^2 \cap N) :< x > + < y >)$ for any $y \in N$. (4) P is a graded almost prime ideal of N. **Proof.** (1) \Rightarrow (2): Let $t \in N$ and $t \in (P : \langle x \rangle + \langle y \rangle)$ for any y and x belongs to N but $x \notin P$. Then $t(\langle x \rangle + \langle y \rangle) \subseteq P$. If $t(\langle x \rangle + \langle y \rangle) \subseteq (P^2 \cap N)$. Then $t \in ((P^2 \cap N) : \langle x \rangle + \langle y \rangle)$. Otherwise, we get $t(\langle x \rangle + \langle y \rangle) \not\subseteq (P^2 \cap N)$. Thus, $t \in P$ by hypothesis.

(2) \Rightarrow (3): It is following directly from Lemma 2.9.

 $(3) \Rightarrow (4)$: Let I and J be a graded ideal of N such that $IJ \subseteq P$ and $IJ \not\subseteq (P^2 \cap N)$. Suppose that $I \not\subseteq P$ and $J \not\subseteq P$. Then there exist $j \in J$ with $j \notin P$. Now, it is claimed that $IJ \subseteq (P^2 \cap N)$. Let $j_1 \in J$, then $I(\langle j \rangle + \langle j_1 \rangle \subseteq P)$, which implies $I \subseteq (P : \langle j \rangle + \langle j_1 \rangle)$. Then by assumption, $I(\langle j \rangle + \langle j_1 \rangle) \subseteq (P^2 \cap N)$ which gives $Ij_1 \subseteq (P^2 \cap N)$. Thus $IJ \subseteq (P^2 \cap N)$ and hence P is a graded almost prime ideal of N.

(4) \Rightarrow (1): If $x(\langle y \rangle + \langle z \rangle) \subseteq P$ and $x(\langle y \rangle + \langle z \rangle) \not\subseteq (P^2 \cap N)$, then {0} $\neq \langle x \rangle (\langle y \rangle + \langle z \rangle) \subseteq P$. Since P is a graded almost prime ideal of N, there is $\langle x \rangle \subseteq P$ or $\langle y \rangle + \langle z \rangle \subseteq P$. By assumption, x, y and $z \in N$. Hence $x \in P$ or y and $z \in P$.

Theorem 3.6. Let N be a G-graded near-ring and P be a graded ideal of N. Then the following are equivalent:

(1) P is a graded almost prime ideal of N.

(2) For any ideals I and J in N with $P \subset I$ and $P \subset J$, then there is either $IJ \subseteq (P^2 \cap N)$ or $IJ \not\subseteq P$.

(3) For any ideals I and J in N with $I \not\subseteq P$ and $J \not\subseteq P$, then there is either $IJ \subseteq (P^2 \cap N)$ or $IJ \not\subseteq P$.

Proof.(1) \Rightarrow (2): Let I an J be two graded ideals of N with $P \subset I$, $P \subset J$ and $IJ \not\subseteq (P^2 \cap N)$. Take $i \in I$ and $j \in J$ with $i \notin P$ and $j \notin P$, which implies that $\langle i \rangle \langle j \rangle \not\subseteq P$ and hence $IJ \not\subseteq (P^2 \cap N)$ and $IJ \not\subseteq P$.

 $\begin{array}{l} (2) \Rightarrow (3): \ Let \ I \ and \ J \ be \ a \ graded \ ideals \ of \ N \ with \ I \not\subseteq P \ and \ J \not\subseteq P. \ Then \ there \ exists \\ i_1 \in I \ and \ j_1 \in J \ such \ that \ i_1 \not\in P \ and \ j_1 \not\in P. \ Suppose \ that < i > < j > (P^2 \cap N) \ for \\ some \ i \in I \ and \ j \in J. \ Then \ (P+ < i > + < i_1 >)(P+ < j > + < j_1 >) \not\subset (P^2 \cap N) \\ and \ P \subset (P+ < i > + < i_1 >), \ P \subset (P+ < j > + < j_1 >). \ By \ hypothesis, \\ (P+ < i > + < i_1 >)(P+ < j > + < j_1 >) \not\subseteq P. \ So, < i > (P+ < j > + < j_1 >)+ < \\ i_1 > (P+ < j > + < j_1 >) \not\subseteq P. \ Hence \ there \ exists \ i' \in < i >, \ i'_1 \in < i_1 >, \ j', \ j'' \in < j >, \\ j'_1, \ j''_1 \in < j_1 > \ and \ p_1, \ p_2 \in P \ such \ that \ i'(p_1 + j' + j'_1) + i'_1(p_2 + j'' + j''_1) \neq P. \ Thus \\ i'(p_1 + j' + j'_1) - i'(j' + j'_1) + i'(j' + j'_1) + i'_1(p_2 + j'' + j''_1) - i'_1(j'' + j''_1) \notin P. \\ But \ i'(p_1 + j' + j'_1) - i'(j' + j'_1) \in P \ and \ i'_1(p_2 + j'' + j''_1) - i'_1(j'' + j''_1) \in P. \ Which \ implies \\ neither \ i'(j' + j'_1) \ nor \ i'_1(j'' + j''_1) \ belongs \ to \ P. \ Therefore, \ IJ \not\subseteq P. \end{array}$

 $(3) \Rightarrow (1)$: Follows directly from the definition of graded almost prime ideals of N.

Proposition 3.7. Let N be a G-graded near-ring, A be a totally ordered set and $(P_a)_{a \in A}$ be a family of graded almost prime ideals of N with $P_a \subseteq P_b$ for any $a, b \in A$ with $a \leq b$. Then $P = \bigcap_{a \in A} P_a$ is a graded almost prime ideal of N.

Proof. Let I and J be two graded ideals of N with $IJ \subseteq P$ but $IJ \not\subseteq (P^2 \cap N)$, which implies for all $a \in A$ there is $IJ \subseteq P_a$. If there exists $a \in A$ such that $I \not\subseteq P_a$, then $J \subseteq P_a$. Hence for all $a \leq b$ there is $J \subseteq P_b$. If there exists c < a such that $j \not\subseteq P_c$, then $I \subseteq P_c$ and then $I \subseteq P_a$, this is a contradiction. Hence for any $a \in A$, there is $J \subseteq P_a$. Therefore, $J \subseteq P$. **Proposition 3.8.** Let N be a G-graded near-ring and P be an intersection of some graded almost prime ideals of N. Then for any graded ideal I of N satisfying that $I^2 \subseteq P$ but $I^2 \not\subseteq (P^2 \cap N)$ we have $I \subseteq P$.

Proof. Let P_a be a set of graded almost prime ideals of N, P be the intersection of P_a and I be a graded ideal of N such that $I^2 \subseteq P$ but $I^2 \not\subseteq (P^2 \cap N)$. Then I^2 is subset of each P_a since P_a is graded almost prime ideal of N there is $I \subseteq P_a$. Therefore, $I \subseteq P$.

Lemma 3.9. Let N and M be two G-graded near-rings and ϕ be a surjective homomorphism from N into M. For any two graded ideals P, I and J of N if $IJ \not\subseteq P$, then $\phi^{-1}(I)\phi^{-1}(J) \not\subseteq \phi^{-1}(P)$.

Proof. If $I \not\subseteq P$, then $\phi^{-1}(I) \not\subseteq \phi^{-1}(P)$ since $\phi(I) \subseteq \phi(P)$, then $I = \phi(\phi^{-1}(I)) \subseteq \phi(\phi^{-1}(P)) \subseteq P$. Hence if $IJ \not\subseteq P$, then $\phi^{-1}(IJ) = \phi^{-1}(I)\phi^{-1}(J) \not\subseteq \phi^{-1}(P)$.

Theorem 3.10. Let N and M be two G-graded near-rings and ϕ be a surjective homomorphism from N into M. Then the image of graded almost prime ideal of N which contains the kernal of ϕ is a graded almost prime ideal of M.

Proof. Suppose that $IJ \subseteq \phi(P)$ and $IJ \not\subseteq ((\phi(P))^2 \cap N)$ where I and J be two graded ideals of N and P is a graded almost prime ideal of N. By Lemma 3.9 $\phi^{-1}(I)\phi^{-1}(J) \not\subseteq$ $(P^2 \cap N)$. Hence $\phi^{-1}(I)\phi^{-1}(J) \subseteq P + Ker(\phi) = P$. Since P is a graded almost prime ideal of N then $\phi^{-1}(I) \subseteq P$ or $\phi^{-1}(J) \subseteq P$. Therefore, $I \subseteq \phi(P)$ or $J \subseteq \phi(P)$. Hence $\phi(P)$ is a graded almost prime ideal of M.

Theorem 3.11. Let N be a G-graded near-ring and P, I be a graded ideals of N with $I \subseteq P$. Consider $\pi : N \to \overline{N} := N/I$ is the canonical epimorphism. If P is a graded almost prime ideal of N, then $\pi(P)$ is a graded almost prime ideal of \overline{N} .

Proof. Let P be a graded almost prime ideal of N and $\pi(J), \pi(K)$ be a graded ideal of \overline{N} with $\pi(J)\pi(K) = \pi(JK) \subseteq \pi(P)$ and $\pi(J)\pi(K) = \pi(JK) \not\subseteq ((\pi(P))^2 \cap N)$. Since $\pi(J)\pi(K) = \pi(JK) \not\subseteq ((\pi(P))^2 \cap N)$ then by Lemma 3.9 $JK \not\subseteq P^2$. Therefore, $\pi^{-1}(\pi(J))\pi^{-1}(\pi(K)) = JK \subseteq \pi^{-1}(\pi(P)) = P + I = P$. Hence $J \subseteq P$ or $K \subseteq P$ and hence $\pi(J) \subseteq of\pi(P)$ or $\pi(K) \subseteq of\pi(P)$. Thus $\pi(P)$ is a graded almost prime ideal of \overline{N} .

Lemma 3.12. Let P be a graded almost prime ideal of N. If \overline{I} is a graded ideal of N/P with $\overline{I}\overline{J} = \{0\}$ for some non zero graded ideal \overline{J} of N/P. Then there is either $I \subseteq P$ or $PJ \subseteq (P^2 \cap N)$.

Proof. Suppose that $I \not\subseteq P$ and let $p \in P$. Then $(\langle p \rangle + I) \not\subseteq P$ and $(\langle p \rangle + I)J \subseteq P$ which implies $(\langle p \rangle + I)J \subseteq (P^2 \cap N)$ but P is a graded almost prime ideal of N. Thus $\langle p \rangle J \subseteq (P^2 \cap N)$ and hence $PJ \subseteq (P^2 \cap N)$.

Theorem 3.13. Let N be a G-graded near-ring and P be a graded almost prime ideal of N. If I is a graded ideal of N and $I^2 \subseteq P$, then $I^2 \subseteq (P^2 \cap N)$.

Proof. Let P is a graded almost prime ideal of N and for any $x, y \in I$ we have $\langle x \rangle \langle y \rangle \subseteq I^2 \subseteq P$. Now, the claim is that $\langle x \rangle \langle y \rangle \subseteq (P^2 \cap N)$. Suppose not, then since P is graded almost prime ideal of N, there is $x \in P$ or $y \in P$. So, it is assumed

that only one of them x or y belongs to P. Take $x \in P$ since $\langle y \rangle \langle y \rangle \subseteq I^2 \subseteq P$ and by Lemma 3.12 we have $\langle x \rangle \langle y \rangle \subseteq P \langle y \rangle \subseteq (P^2 \cap N)$ which implies $I^2 \subseteq (P^2 \cap N)$.

Previous theorem is important in some G-graded near-ring with unique maximal in N (as N is a G-graded near-ring). If this maximal ideal is graded ideal and satisfying that $MM = M^2 \cap N$ like the G-graded near-ring $N = (\mathbf{Z}_{16}, +, \times)$ with G defined as Example 2.2 where $N_0 = \mathbf{Z}_{16}$ and $N_1 = \{0\}$. Note that N as a G-graded ring has unique Maximal ideal $M = \{0, 2, 4, 6, 8, 10, 12, 14\}$ and M satisfies the property $MM = (M^2 \cap N)$. The importance of such G-graded near-rings is explained in the following theorem.

Theorem 3.14. Let N be a G-graded near-ring. If M is the unique maximal ideal of N with $MM = (M^2 \cap N)$, then for any graded ideal P of N with $M^2 \cap N \subseteq P$. We have P is a graded almost prime ideal of N if and only if $(M^2 \cap N) = (P^2 \cap N)$.

Proof. (\Rightarrow) Let P be a graded almost prime ideal of N. Then there is $M^2 \cap N = MM \subseteq (P^2 \cap N)$ by Theorem 3.13. Also, $(P^2 \cap N) = (M^2 \cap N)$ since M is the unique Maximal ideal of N.

 (\Leftarrow) Let $(M^2 \cap N) = (P^2 \cap N)$ we claim that P is a graded almost prime ideal of N. Let I and J be two graded ideals of N with $IJ \subseteq P$. Since M is the unique Maximal ideal of N so $I \subseteq M$ and $J \subseteq M$. Therefore, $IJ \subseteq MM = (M^2 \cap N) = (P^2 \cap N)$. Hence P is a graded almost prime ideal of N.

Theorem 3.15. Let N and M be a G-graded near-rings and P be a graded ideal of N. Then P is a graded almost prime ideal of N if and only if $P \times M$ is a graded almost prime ideal of $N \times M$.

Proof. (\Rightarrow) Let P be a graded almost prime ideal of N and $I \times M$, $J \times M$ be a graded ideal of $N \times M$ such that $(I \times M)(J \times M) \subseteq P \times M$ and $(I \times M)(J \times M) \not\subseteq ((P \times M)^2 \cap (N \times M))$. Then $(I \times M)(J \times M) = (IJ \times MM) \subseteq P \times M$ and $(I \times M)(J \times M) = (IJ \times MM) \not\subseteq ((P^2 \cap N) \times M)$. So, $IJ \subseteq P$ and $IJ \not\subseteq (P^2 \cap N)$ but P is a graded almost prime ideal of N then $I \subseteq P$ or $J \subseteq P$. Therefore, $I \times M \subseteq P \times M$ or $J \times M \subseteq P \times M$. Thus $P \times M$ is a graded almost prime ideal of $N \times M$.

(\Leftarrow) Suppose that $P \times M$ is a graded almost prime ideal of $N \times M$ and Let I, J be a graded ideals of N such that $IJ \subseteq P$ and $IJ \not\subseteq (P^2 \cap N)$. Then $(I \times M)(J \times M) \subseteq P \times M$ and $(I \times M)(J \times M) \not\subseteq ((P \times M)^2 \cap (N \times M))$. By assumption we have $I \times M \subseteq P \times M$ or $J \times M \subseteq P \times M$. So $I \subseteq P$ or $J \subseteq P$. Thus P is a graded almost prime ideal of N.

Corollary 3.16. Let N and M be two G-graded near-rings. If every graded ideal of N and M is a product of graded almost prime ideals, then every graded ideal of $N \times M$ is a product of graded almost prime ideals.

Proof. Let I be a graded ideal of N and J be a graded ideal of M such that $I = I_1...I_n$ and $J = J_1...J_m$ where I_i and J_j is a graded almost prime ideal of N and M respectively. If the graded ideal is of the form $I \times M$ then $I \times M = (I_1...I_n) \times M$ can be written as $(I_1 \times M)...(I_n \times M)$ which is by Theorem 3.15 a product of graded almost prime ideals. Similarly, if the graded ideal is of the form $N \times J$, then it is a product of graded almost prime ideals. If the graded ideal is of the form $I \times J$ then it can be written as $(I_1...I_n) \times (J_1...J_m) =$ $((I_1...I_2) \times M)(N \times (J_1...J_m)) = (I_1 \times M)...(I_n \times M)(N \times J_1)...(N \times J_m)$ which is a product of graded almost prime ideals.

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