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On a conjecture that strengthens the k-factor case of Kundu's k-factor Theorem

James M. Shook[3,4]

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Abstract

In 1974, Kundu showed that for even n if $\pi = (d_1, \ldots, d_n)$ is a non-increasing degree sequence such that $\mathcal{D}_k(\pi) = (d_1 - k, \ldots, d_n - k)$ is graphic, then some realization of π has a k-factor. In 1978, Brualdi and then Busch et al. in 2012, conjectured that not only is there a k-factor, but there is k-factor that can be partitioned into k edge-disjoint 1-factors. Busch et al. showed that if $k \leq 3$, $d_1 \leq \frac{n}{2} + 1$, or $d_n \geq \frac{n}{2} + k - 2$, then the conjecture holds. Later, Seacrest extended this to $k \leq 5$. We explore this conjecture by first developing new tools that generalize edge-exchanges. With these new tools, we can drop the assumption $\mathcal{D}_k(\pi)$ is graphic and show that if $d_{d_1-d_n+k} \geq d_1 - d_n + k - 1$, then π has a realization with k edge-disjoint 1-factors. From this we show that if $d_n \geq \frac{d_1+k-1}{2}$ or $\mathcal{D}_k(\pi)$ is graphic and $d_1 \leq \max\{n/2 + d_n - k, (n + d_n)/2\}$, then the conjecture holds. With a different approach we show the conjecture holds when $\mathcal{D}_k(\pi)$ is graphic and $d_{\min\{\frac{n}{2}, m(\pi)-1\}} > \lceil \frac{n+3k-8}{2} \rceil$ where $m(\pi) = \max\{i : d_i \geq i-1\}$. For $r \leq 2$, Busch et al. and later Seacrest for $r \leq 4$ showed that if $\mathcal{D}_k(\pi)$ is graphic, then there is a realization with a k-factor whose edges can be partitioned into a $(k - r)$-factor and r edge-disjoint 1-factors. We improve this for any $r \leq \max\left\{\min\{k, 4\}, \frac{k+3}{3}\right\}$. As a result, we can show that if $\mathcal{D}_k(\pi)$ is graphic, then there is a realization with at least $2\lfloor \frac{k}{3} \rfloor$ edge-disjoint 1-factors.

1 Introduction

For an undirected graph $G = (V, E)$ with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set E, we let $(deg_G(v_1), \ldots, deg_G(v_n))$ denote a degree sequence of G. We say a sequence $\pi = (d_1, \ldots, d_n)$ is graphic if it is the degree sequence of some graph, and call that graph a realization of π. We let $\mathcal{R}(\pi)$ be the set of realizations of π, and we let $\pi(G)$ be a degree sequence of a graph G and shorten $\mathcal{R}(\pi(G))$ to $\mathcal{R}(G)$. We say a degree sequence $(d_1, \ldots, d_n)$ is non-increasing

[3]National Institute of Standards and Technology, Computer Security Division, Gaithersburg, MD; james.shook@nist.gov.
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if $d_1 \geq \ldots \geq d_n$ and positive if $d_i \geq 1$ for all i. In this paper we will assume all degree sequences are non-increasing and only consider graphs and realizations that have no loops or multi-edges.

In 1974, Kundu [15], followed by Chen [8] in 1988 with a short proof, gave necessary and sufficient conditions for a degree sequence to have a realization with a spanning near regular subgraph. We call a spanning k-regular subgraph a k-factor. Since this paper is only concerned with k-factors we present the regular case in Theorem 1.

Theorem 1 (Regular case of Kundu's k-factor Theorem [15])**.** *Some realization of a degree sequence* $(d_1, \ldots, d_n)$ *has a k-factor if and only if* $(d_1 - k, \ldots, d_n - k)$ *is graphic.*

For a sequence $\pi = (d_1, \ldots, d_n)$, we let $\mathcal{D}_k(\pi)$ denote the sequence $(d_1 - k, \ldots, d_n - k)$ and $\overline{\pi} = (n - 1 - d_n, \ldots, n - 1 - d_1)$. Busch, Ferrara, Hartke, Jacobson, Kaul, and West [7] showed that if both π and $\mathcal{D}_k(\pi)$ are graphic, then for $r \leq \min\{3, k\}$, there is a realization of π with a k-factor that has r edge-disjoint 1-factors. Later, Seacrest [24] improved this to $r \leq \min\{4, k\}$. This naturally leads one to wonder how large can r be? Brualdi [6] and Busch et al. [7], independently, conjectured that $r = k$.

Conjecture 1 ([6] and later in [7])**.** *Some realization of a degree sequence* $(d_1, \ldots, d_n)$ *with even n has k edge-disjoint 1-factors if and only if* $(d_1 - k, \ldots, d_n - k)$ *is graphic.*

Conjecture 1 does not hold for every even order k-regular graph since for some natural number t, the $2t$-regular graph that is the disjoint union of two complete graphs each with $2t + 1$ vertices does not have a 1-factor. However, finding 1-factors in k-regular graphs is well studied [21, 22, 23], and we make use of some of those results here.

For $k \geq 2\lceil \frac{n}{4} \rceil - 1$, the well known 1-factorization conjecture (See [9] by Chetwynd and Hilton) implies that every k-regular graph can be partitioned into k edge-disjoint 1-factors. The same $2t$-regular graph we mentioned before shows the lower bound on k is best possible. In a fantastic paper Csaba et al. proved the 1-factorization conjecture for n sufficiently large.

Theorem 2 ([10])**.** *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Let $n, k \in \mathbb{N}$ be such that $n \geq n_0$ is even and $k \geq 2\lceil \frac{n}{4} \rceil - 1$. Then every k-regular graph G on n vertices can be decomposed into k edge-disjoint 1-factors.*

A positive resolution of the the 1-factorization conjecture would prove Conjecture 1 for large k, and thanks to Csaba et al. we know Conjecture 1 is true for large k and n sufficiently large.

One may think increasing the edge-connectivity of k-regular graphs would produce many edge-disjoint 1-factors. The classic example of this idea is by Berge [4] and expanded on in [5, 14, 20, 25].

Theorem 3 ([4])**.** *All even ordered $(k-1)$-edge-connected k-regular graphs have a 1-factor.*

For large k, we made use of Theorem 3 in [26] to find a realization with a k-factor that has many edge-disjoint 1-factors. However, this approach maybe limited since Mattiolo [18]

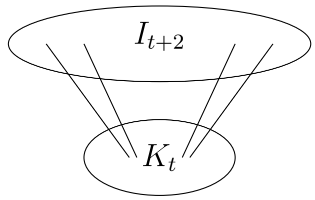

Figure 1: $K_{t+1} * I_{t+2}$

presented k-regular k-edge-connected graphs that cannot be partitioned into a 2-factor and $k-2$ 1-factors.

Along with requiring the connectivity of a graph G and it's complement $\overline{G}$, Ando et al. [1] showed that bounding the difference of the maximum degree and minimum degree of G would yield a 1-factor in either G or $\overline{G}$. Ignoring the connectivity requirement we are able to show that bounding the difference $d_1 - d_n$ for a non-increasing degree sequence $(d_1 \dots, d_n)$ can tell us if there is a realization with many edge-disjoint 1-factors.

Theorem 4. *Let $\pi = (d_1, \dots, d_n)$ be a non-increasing positive degree sequence with even n. For a positive integer $k \leq d_n$, if*

$$d_{d_1-d_n+k} \geq d_1 - d_n + k - 1, \tag{1}$$

then there is some $G \in \mathcal{R}(\pi)$ that has k edge-disjoint 1-factors.

Note that, unlike Conjecture 1, in Theorem 4 we did not require $\mathcal{D}_k(\pi)$ to be graphic. For $k = 1$ and $t \geq 1$, Theorem 4 is best possible since the split graph (See Figure 1) joining every vertex of a complete graph K_{t+1} with every vertex of an independent set I_{t+2} has a non-increasing degree sequence $(d_1, \dots, d_n)$ such that $d_1 = 2t+1$, $d_n = t$, and $d_1 - d_n = t + 1 > d_{d_1-d_n+1} = t$ yet does not have a 1-factor. However, for $k > 1$ we think we can do better. Our motivation for this comes from Corollary 5.

Corollary 5. *Let $\pi = (d_1, \dots, d_n)$ be a non-increasing positive degree sequence with even n. For $k \leq d_n$, if*

$$d_{d_1-d_n+1} \geq d_1 - d_n + k - 1, \tag{2}$$

then there is some realization of π that has a k-factor.

Proof. Assume (2) is true. The Corollary follows directly from Theorem 4 when $k = 1$. Let t be the largest integer such that $\mathcal{D}_t(\pi) = (q_1, \dots, q_n)$, where $q_i = d_i - t$, is graphic. Kundu's k-factor theorem implies that $\mathcal{D}_{t+1}(\pi)$ is not graphic, and therefore, no realization of $\mathcal{D}_t(\pi)$ has a 1-factor. This implies $q_{q_1-q_n+1} < q_1 - q_n$. Since $q_1 - q_n = d_1 - d_n$, we have along with (2) that

$$d_{d_1-d_n+1} - t = q_{q_1-q_n+1} < q_1 - q_n = d_1 - d_n \leq d_{d_1-d_n+1} - (k-1).$$

Which can only be true if $t \geq k$. □

Observe that if Conjecture 1 is true, then Corollary 5 implies there is a realization with k edge-disjoint 1-factors. This naturally motivates Conjecture 2 as a interesting step towards answering Conjecture 1.

Conjecture 2. *Let $\pi = (d_1, \ldots, d_n)$ be a non-increasing positive degree sequence with even n. For a positive integer $k \leq d_n$, if*

$$d_{d_1 - d_n + 1} \geq d_1 - d_n + k - 1, \tag{3}$$

then there is some $G \in \mathcal{R}(\pi)$ that has k edge-disjoint 1-factors.

If we first insist $\mathcal{D}_k(\pi)$ is graphic, then we can use Theorem 4 to prove Theorem 6.

Theorem 6. *Let $\pi = (d_1, \ldots, d_n)$ be a non-increasing positive degree sequence with even n such that $\mathcal{D}_k(\pi)$ is graphic. If*

$$d_{n+1-(d_1-d_n+k)} \leq n - (d_1 - d_n), \tag{4}$$

then π has a realization with k edge-disjoint 1-factors. ‘

Proof. Let $q_i = d_i - k$. We focus on $\mathcal{D}_k(\pi)$ and consider its complement $\overline{\mathcal{D}_k(\pi)} = (\overline{q}_1, \ldots, \overline{q}_n)$ where $\overline{q}_i = n - 1 - q_{n+1-i}$. We have by (4) that

$$q_{n+1-(q_1-q_n+k)} = d_{n+1-(d_1-d_n+k)} - k \leq n - (d_1 - d_n) - k = n - 1 - (q_1 - q_n + k - 1).$$

From this we can show

$$\overline{q}_1 - \overline{q}_n + k - 1 = q_1 - q_n + k - 1 \leq n - 1 - q_{n+1-(q_1-q_n+k)} = \overline{q}_{q_1-q_n+k} = \overline{q}_{\overline{q}_1-\overline{q}_n+k}.$$

Therefore, by Theorem 4, $\overline{\mathcal{D}_k(\pi)}$ has a realization with k edge-disjoint 1-factors. Thus, those k edge-disjoint 1-factors can be added to a realization of $\mathcal{D}_k(\pi)$ to create a realization of π with k edge-disjoint 1-factors. ∎

Note that if Conjecture 2 holds, then (4) can be improved to $d_{n-(d_1-d_n)} \leq n - (d_1 - d_n)$.

For a non-increasing degree sequence π, the modified Durfee number is defined as $m(\pi) = \max\{i : d_i \geq i - 1\}$. The modified Durfee number has appeared in the literature many times before and we will make use of it here.

Theorem 7. *Let $\pi = (d_1, \ldots, d_n)$ be a non-increasing positive degree sequence with even n such that $\mathcal{D}_k(\pi)$ is graphic. If $d_{\min\{\frac{n}{2}, m(\pi)-1\}} > \left\lceil \frac{n+3k-8}{2} \right\rceil$ or $\left\lceil \frac{n+5-k}{2} \right\rceil > d_{\max\{\frac{n}{2}+1, n+2-m(\overline{D_k(\pi)})\}}$, then π has a realization with k edge-disjoint 1-factors.*

We used a different strategy in the proof of Theorem 7 than we used in the proof of Theorem 4 and we suspect that the argument can bear more fruit. Currently, if Theorem 4 does not hold, then $m(\pi) \leq \left\lceil \frac{n+3k-8}{2} \right\rceil + 1$ or $\frac{n}{2} + 1 \leq m(\pi) \leq \frac{n}{2} + 6$ when $k = 6$. We would not be surprised if the proof could be modified to improve the bounds on $m(\pi)$ or answer the conjecture for $k = 6$.

In [7] the authors showed that if $\mathcal{D}_k(\pi)$ is graphic and $d_n \geq \frac{n}{2}+k-2$ or $d_1 \leq \frac{n+2}{2}$, then some realization of π has k edge-disjoint 1-factors. We have improved these bounds considerably. Theorem 4 shows that if $d_n \geq \frac{d_1+k-1}{2}$, then $\mathcal{D}_k(\pi)$ is graphic and some realization of π has k edge-disjoint 1-factors. If we first assume $\mathcal{D}_k(\pi)$ is graphic, then we can show

$$d_1 \leq \max\left\{\frac{n}{2}+d_n-k, \frac{n+d_n}{2}\right\}$$

is sufficient. Theorem 6 shows that $d_1 \leq \frac{n+d_n}{2}$ is enough, and the other inequality follows from Theorem 4 after an application of Lemma 1.

Lemma 1 (Proved by Li in [16], but we use the form given by Barrus in [2])**.**

$$m(\pi)+m(\overline{\pi}) = \begin{cases} n+1, & \text{if } d_{m(\pi)} = m(\pi)-1 \\ n, & \text{otherwise.} \end{cases}$$

To see this, we first observe that Theorem 4 says that if $m(\pi) \geq d_1 - d_n + k$, then Conjecture 1 holds. Let $\overline{\mathcal{D}_k(\pi)} = (\overline{q}_1, \ldots, \overline{q}_n)$ where $\overline{q}_i = n-1-d_{n+1-i}+k$ and assume by contradiction that $m(\pi) \leq d_1 - d_n + k - 1$ and $m(\overline{D_k(\pi)}) \leq \overline{q}_1 - \overline{q}_n + k - 1 = d_1 - d_n + k - 1$. Since $m(\overline{\pi}) \leq m(\overline{D_k(\pi)}) \leq m(\overline{\pi}) + k$ we have

$$n \leq m(\pi)+m(\overline{\pi}) \leq m(\pi)+m(\overline{D_k(\pi)}) \leq 2(d_1-d_n+k-1).$$

This implies the contradiction

$$d_1 \geq \frac{m(\pi)+m(\overline{D_k(\pi)})}{2} + d_n - k + 1 \geq \frac{n}{2} + d_n - k + 1.$$

Thus, either $m(\pi) \geq d_1 - d_n + k - 1$ or $m(\overline{D_k(\pi)}) \geq \overline{q}_1 - \overline{q}_n + k - 1$. Applying Theorem 4 to either π or $\overline{D_k(\pi)}$ we can find a realization of π with k edge-disjoint 1-factors. Note that we actually proved the stronger bound

$$d_1 \leq \frac{m(\pi)+m(\overline{D_k(\pi)})}{2} + d_n - k + 1,$$

and if Conjecture 2 is true then we can modify this argument to show Conjecture 1 holds for $d_1 - d_n \leq \frac{m(\pi)+m(\overline{D_k(\pi)})}{2} - 1$.

When $d_n \geq \frac{n}{2}+2$, Hartke and Seacrest [13] showed there is a realization with $f(d_n, n)$ edge-disjoint 1-factors where

$$f(d_n, n) = \left\lfloor \frac{d_n - 2 + \sqrt{n(2d_n - n - 4)}}{4} \right\rfloor.$$

Our lower bound on d_n improves their bound when $d_n - f(d_n, n) \geq d_1 - d_n$. Which is true for the vast majority of possible d_1 for any given n and d_n. To see this, we can use the rough lower bound $d_n > 2f(d_n, n)$ and $d_n \geq n/2+2$ to show any $d_1 \leq 3n/4+3 \leq 3d_n/2 < 2d_n - f(d_n, n)$ will do.

We now turn to the weaker question of how many edge-disjoint 1-factors a k-factor in some realization can have.

In [26] we expanded on a result of Edmonds [12] that gave necessary and sufficient conditions for when a degree sequence has a maximally edge-connected realization. One of the things we showed is that if G is a simple graph with minimum degree two and has a 1-factor F, then there is a realization of $\mathcal{R}(G)$ that is maximally edge-connected with the subgraph $G - E(F)$. This result can be used to require the realization given by Conjecture 1 to be maximally edge-connected when $d_n \geq 2$. In the same paper we proved a more general result that along with Theorem 3 we used to prove a partial result of Conjecture 1.

Theorem 8 ([26])**.** *Let $\pi = (d_1, \ldots, d_n)$ be a non-increasing degree sequence with even n such that $\mathcal{D}_k(\pi)$ is graphic. If $k \geq \frac{d_1}{2} + r - 1$ or $k \geq n - 1 - d_n + 2(r-1)$, then π has a realization with a k-factor that has r edge-disjoint 1-factors.*

If $k \geq \frac{d_1}{2} + r - 1$, then since $d_n \geq k \geq \frac{d_1+2r-2}{2}$, Theorem 4 is stronger than Theorem 8 when $k \leq 2r-1$ and Theorem 2 would be stronger for large k and n. Interestingly, Theorem 3 is traditionally proved with a structure we rely on in this paper. So it maybe possible to directly use the techniques in this paper to prove or even improve Theorem 8.

Seacrest [24] showed that for $r \leq \min\{4, k\}$ there is a k-factor with r edge-disjoint 1-factors. We are able to show there is a k-factor with at least $\left\lfloor \frac{k+3}{3} \right\rfloor$ edge-disjoint 1-factors.

Theorem 9. *Let $\pi = (d_1, \ldots, d_n)$ be a non-increasing positive degree sequence with even n. For a positive integers $k \leq d_n$, if $\mathcal{D}_k(\pi)$ is graphic and*

$$r \leq \max\left\{\min\{k, 4\}, \frac{k+3}{3}\right\},$$

then there is some $G \in \mathcal{R}(\pi)$ that has a k-factor with r edge-disjoint 1-factors.

If some realization of $(n-1-d_1, \ldots, n-1-d_n)$ has a k'-factor then we can make use of Petersen's 2-factor theorem [19] to improve Theorem 9. Recall that Petersen showed that any $2r$-regular graph can be partitioned into r 2-factors.

Theorem 10. *Let $\pi = (d_1, \ldots, d_n)$ be a non-increasing positive degree sequence with even n. For non-negative integers $r \leq k \leq d_n$ and $k' \leq n - 1 - d_1$ such that k' is even and $r \equiv k \mod 2$, if $\mathcal{D}_k(\pi)$ and $\pi' = (d_1 + k', \ldots, d_n + k')$ are graphic and*

$$r \leq \max\left\{\min\{k, 4\}, \frac{k+k'+3}{3}\right\},$$

then there is some $G \in \mathcal{R}(\pi)$ that has a k-factor with r edge-disjoint 1-factors.

Proof. Note that $\mathcal{D}_{k+k'}(\pi') = \mathcal{D}_k(\pi)$. Since π' and $\mathcal{D}_{k+k'}(\pi')$ are graphic and non-increasing we have by Theorem 9 that π' has a realization G with a $(k+k')$-factor F with an r-factor

F_0 made up of r edge-disjoint 1-factors. Since $k+k'-r$ is even we can use Petersen's 2-factor theorem to split $F - E(F_0)$ in to $\frac{k+k'-r}{2}$ 2-factors. We may then select $\frac{k-r}{2}$ of those 2-factors and add them to $G - E(F) + E(F_0)$ to construct a realization of π with a k-factor that has r edge-disjoint 1-factors. □

We can weaken our requirements even more by asking how many edge-disjoint 1-factors can a realization of a graphic sequence π have if $\mathcal{D}_k(\pi)$ is graphic?

Seacrest [24] showed that if $\mathcal{D}_k(\pi)$ is graphic, then there is a realization of π with $\lfloor \frac{k}{2} \rfloor + 2$ edge-disjoint 1-factors. Seacrest did this by first finding a realization with a k-factor that has r edge-disjoint 1-factors for some $r \equiv k \mod 2$. In particular, Seacrest used $r = 3$ when k is odd and $r = 4$ when k is even. Seacrest then took the remaining part of the k-factor and applied Petersen's 2-factor theorem to split it into edge-disjoint 2-factors. Seacrest then visited each 2-factor and performed multi-switches, which are defined similarly to the edge-exchanges given in Section 3 of this paper, to construct at least one additional 1-factor while leaving existing 1-factors and 2-factors intact. This process results in a realization with $\frac{k+r}{2}$ edge-disjoint 1-factors and possibly no k-factor. By Theorem 9 there is an $r \geq \lfloor \frac{k}{3} \rfloor$ such that $k - r$ is even. This implies that if $\mathcal{D}_k(\pi)$ is graphic, then there is a realization of π with at least

$$\frac{k+r}{2} \geq \frac{k + \lfloor \frac{k}{3} \rfloor}{2} \geq 2\left\lfloor \frac{k}{3} \right\rfloor$$

edge-disjoint 1-factors.

We have shown that $\mathcal{D}_k(\pi)$ being graphic results in a realization of π with many edge-disjoint 1-factors. Thus, it was reasonable to pose Conjecture 1. We suspect answering the conjecture for the $k = 6$ case will either lead to a counter example or yield some new tools for larger k. As it is, the conjecture has allowed us to study a generalization of the classic edge-exchange, defined in Section 3, that we used heavily in our work. The idea has been explored before by Seacrest in [24]. However, our presentation is different and our results seem to be new and maybe of independent interest outside of this conjecture.

In Section 2 we present terminology and the Gallai-Edmonds Structure Theorem. In Section 3 we present a generalization of the classic edge-exchange and prove lemmas that we use extensively in our proofs. The proofs of our main results can be found in the rest of the sections.

2 Terminology and Definitions

For notation and definitions not defined here in this paper we refer the reader to [11]. We let K_n denote the complete graph on n vertices. We will denote $\overline{G}$ as the complement of a graph G. We say a graph is trivial if it has a single vertex. For a graph $G = (V, E)$ and disjoint subsets X and Y of V we let $e_G(X, Y)$ be the number of edges with one end in X and the other in Y. For a matching M, if $u \in V(M)$, then we let $\overline{u}_M$ denote that unique neighbor of u in M. Moreover if $U \subseteq V(M)$ we let $\overline{U}_M = \{\overline{u}_M \| \forall u \in U\}$.

Let G be a graph. We let $o(G)$ be the number of odd components in G. For $S \subseteq V(G)$ we let $\mathrm{def}_G(S) = o(G - S) - |S|$, and let $\mathrm{def}(G) = \max_{S \subseteq V(G)} \mathrm{def}_G(S)$. The Berge-Tutte

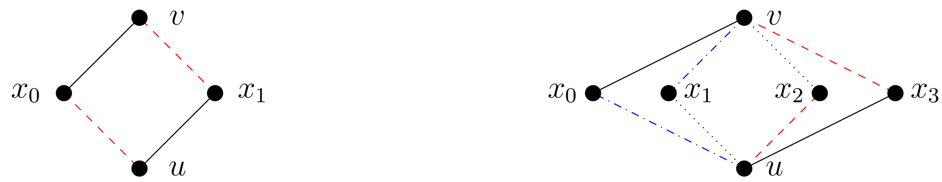

Figure 2: Edge-exchanges with length 2 and length 4, respectively.

Formula [3] says that if G has n vertices, then the maximum size of a matching in G is $\frac{1}{2}(n - \mathrm{def}(G))$. If every subgraph obtained by deleting one vertex from G has a 1-factor, then we say G is factor-critical. If a matching in G covers all but one vertex, then we say the matching is near-perfect.

In a graph G, the Gallai-Edmonds Decomposition of G is a partition of $V(G)$ into three sets A, C, and D such that $D = V(G) - B$ where B is the set of vertices that are in every maximum matching of G and $B = A \cup C$ where A is the set of vertices of B with at least one neighbor in D. The Gallai-Edmonds Structure Theorem (See [27] for a short proof and history.) is an important tool for our work and we present it below before beginning our proofs.

Theorem 11 (Gallai-Edmonds Structure Theorem). *Let A,C,D be the sets in the Gallai-Edmonds Decomposition of a graph G. Let $G_1, \ldots, G_k$ be the components of $G[D]$. If M is a maximum matching in G, then the following properties hold.*

(I) M covers C and matches A into distinct components of $G[D]$.

(II) Each G_i is factor-critical, and M restricts to a near-perfect matching on G_i.

(III) If $\emptyset \neq S \subseteq A$, then $N_G(S)$ has a vertex in at least $|S| + 1$ of $G_1, \ldots, G_k$.

(IV) def$(A) =$ def$(G) = k - |A|$.

3 Edge-Exchanges

The standard operation for passing from one realization G of a degree sequence π to another realization consists of exchanging two edges vx_0 and x_1u from G with two edges x_0u and vx_1 from $\overline{G}$. This operation, see the left side of Figure 2, is commonly called an edge-exchange or a 2-switch. However, we will need a more general form of edge-exchanges and we present it from the perspective of an edge coloring of K_n.

Consider an edge coloring of K_n with natural numbers $\{1, \ldots, t\}$. We let $H_1, \ldots, H_t$ denote the subgraphs of K_n where H_j is formed by all edges colored j. We say the colors of edges vx_0 an x_0u can be exchanged if there exists a natural number l and a list of $2l$ distinct edges

$$(vx_0, x_0u, vx_1, x_1u, \ldots, vx_{l-1}, x_{l-1}u)$$

such that $x_i u$ and vx_{i+1} have the same color for all i modulo l. Indeed if we exchange the colors of vx_i and $x_i u$ for all i modulo l, then we would create another edge coloring of K_n with color classes $H'_1, \ldots, H'_t$ such that $H'_j \in \mathcal{R}(H_j)$. The right hand side of Figure 2 shows, as an example, the exchange $(vx_0, x_0u, vx_1, x_1u, vx_2, x_2u, vx_3, x_3u)$. We often just say two edges can be exchanged when it is clear we mean exchanging their colors. If $x_j u$ and vx_0 are not the same color for $j \leq l-1$, then we call the list a near exchange with length l. For a near exchange or exchange $L = (vx_0, x_0u, vx_1, x_1u, \ldots, vx_{l-1}, x_{l-1}u)$ we let $\mathcal{X}(L) = \{x_0, \ldots, x_{l-1}\}$.

The rest of this section focuses on exchanging edges where the first edge is in H_1. However, in later sections we will want to consider exchanges that start with an edge of H_2 so it is important to point out that all the results in this section still hold when every occurrence of H_1 or H_2 are swapped with each other.

Let L be a vx_0 and x_0u exchange. If for any H_i there is at most one $x_j \in \mathcal{X}(L)$ such that $vx_j \in E(H_i)$, then we call L simplified.

Lemma 2. *Let $H_1, \ldots, H_t$ be the subgraphs formed by coloring every edge of K_n with some integer in $\{1, \ldots, t\}$ such that H_j is a spanning regular graph for $j \geq 3$. For edges $vx_0 \in E(H_1)$ and $x_0u \notin E(H_1)$, if L is an exchange for vx_0 and x_0u, then there exists a simplified vx_0 and x_0u exchange L' with $\mathcal{X}(L') \subseteq \mathcal{X}(L)$.*

Proof. Let $L = (vx_0, x_0u, vx_1, x_1u, \ldots, vx_{l-1}, x_{l-1}u)$ be the shortest counter example. Thus, there is a x_j and a x_t in $\mathcal{X}(L)$ with $j < t$ such that both vx_j and vx_t are in $E(H_i)$ for some i. We have a contradiction since we can create the shorter exchange

$$L' = (vx_0, x_0u, \ldots, vx_{j-1}, x_{j-1}u, vx_t, x_tu, \ldots, vx_{l-1}, x_{l-1}u)$$

with $\mathcal{X}(L') \subseteq \mathcal{X}(L)$. □

Lemma 3. *Let $H_1, \ldots, H_t$ be the subgraphs formed by coloring every edge of K_n with some integer in $\{1, \ldots, t\}$ such that H_j is a spanning regular graph for $j \geq 3$. For edges $vx_0 \in E(H_1)$ and $x_0u \notin E(H_1)$, if vx_0 and x_0u cannot be exchanged, then a longest near edge-exchange using vx_0 and x_0u ends with an edge of H_2.*

Proof. Let $L = (vx_0, x_0u, vx_1, x_1u, \ldots, vx_{l-1}, x_{l-1}u)$ be a longest near edge-exchange. If there is some $x_ju \in E(H_1)$, then $(vx_0, x_0u, \ldots, vx_j, x_ju)$ would be an edge-exchange since $vx_0 \in E(H_1)$. Suppose $x_{l-1}u \in E(H_j)$ for some $j \geq 3$. Since H_j is regular and u and v are incident to the same number of edges of H_j in $\{vx_0, x_0u, \ldots, vx_{l-1}, x_{l-1}u\}$ there must be an $x_l \in N_{H_j}(v) - \mathcal{X}(L)$. However, $(vx_0, x_0u, \ldots, vx_{l-1}, x_{l-1}u, vx_l, x_lu)$ would be a longer near edge-exchange contradicting our choice of l. □

Lemma 4. *Let $H_1, \ldots, H_t$ be the subgraphs formed by coloring every edge of K_n with some integer in $\{1, \ldots, t\}$ such that H_j is a spanning regular graph for $j \geq 3$. For edges $vx_0 \in E(H_1)$ and $x_0u \notin E(H_1)$, if there is a $y \in N_{H_2}(v) \cap N_{H_1}(u)$ or a $y \in N_{H_2}(v) - N_{H_2}(u)$ and a $y' \in N_{H_1}(u) - N_{H_1}(v)$ such that yu and vy' have the same color, then vx_0 and x_0u can be exchanged.*

Proof. Suppose vx_0 and x_0u cannot be exchanged. Let $(vx_0, x_0u, \ldots, vx_{l-1}, x_{l-1}u)$ be a longest near exchange. By Lemma 3 $x_{l-1}u \in E(H_2)$, and thus, there is a smallest j such that $x_ju \in E(H_2)$. However, we have a contradiction since

$$(vx_0, x_0u, \ldots, vx_j, x_ju, vy, yu)$$

is an exchange when $y \in N_{H_2}(v) \cap N_{H_1}(u)$ and otherwise,

$$(vx_0, x_0u, \ldots, vx_j, x_ju, vy, yu, vy', y'u)$$

is an exchange. □

Lemma 5. *Let $H_1, \ldots, H_t$ be the subgraphs formed by coloring every edge of K_n with some integer in $\{1, \ldots, t\}$ such that H_j is a spanning regular graph for $j \geq 3$. For vertices u and v, let $X = \{x_0^{(1)}, \ldots, x_0^{(|X|)}\}$ where $X \subseteq N_{H_1}(v) - N_{H_1}(u)$. If*

$$deg_{H_1}(u) \geq deg_{H_1}(v) - |N_{H_2}(u) \cap N_{H_1}(v)| + |X \cap N_{H_2}(u)|, \tag{5}$$

then there exists a set $L = \{L^{(1)}, \ldots, L^{(|X|)}\}$ such that $L^{(j)} \in L$ is a $vx_0^{(j)}$ and $x_0^{(j)}u$ exchange and

$$\mathcal{X}(L^{(j)}) \cap \mathcal{X}(L^{(i)}) = \emptyset$$

for $j \neq i$.

Proof. Trivially, $\mathcal{X}((vx_0^{(j)}, x_0^{(j)}u)) = \{x_0^{(j)}\}$. Thus, there exists a set $\{L^{(1)}, \ldots, L^{(|X|)}\}$ and an $1 \leq f \leq |X|$, where

$$L^{(j)} = (vx_0^{(j)}, x_0^{(j)}u, vx_1^{(j)}, x_1^{(j)}u, \ldots, vx_{l^{(j)}-1}^{(j)}, x_{l^{(j)}-1}^{(j)}u)$$

is an exchange for $j < f$ and a near exchange for $j \geq f$ with

$$\mathcal{X}(L^{(j)}) \cap \mathcal{X}(L^{(i)}) = \emptyset$$

for all $i \neq j$, such that

$$\sum_{i=1}^{|X|} |\mathcal{X}(L^{(i)})| \tag{6}$$

is maximized.

Let $Y = \bigcup_{i=1}^{|X|} \mathcal{X}(L^{(i)})$. Suppose there exists an $s \geq f$ such that $x_{l^{(s)}}^{(s)}u \in E(H_i)$ for some $i \geq 3$. Since $x_t^{(j)}u$ and $vx_{t+1}^{(j)}$ are the same color for all j and t we know that for each $L^{(j)}$ u is incident with at least as many edges of H_i than v. Since H_i is regular and u is incident with one more edge of H_i than v in $L^{(s)}$ there must be an $x_l \in N_{H_i}(v) - Y$. However, $(vx_0^{(s)}, x_0^{(s)}u, \ldots, vx_{l-1}^{(s)}, x_{l-1}^{(s)}u, vx_l^{(s)}, x_l^{(s)}u)$ would be a longer near edge-exchange contradicting the maximality of (6). Thus, every $x_{l^{(j)}-1}^{j}u \in E(H_2)$.

We can rewrite (5) so that

$$deg_{H_2}(v) \geq deg_{H_2}(u) - |N_{H_2}(u) \cap N_{H_1}(v)| + |X \cap N_{H_2}(u)| = |N_{H_2}(u) - N_{H_1}(v)| + |X \cap N_{H_2}(u)|.$$

Since $Y \cap N_{H_2}(u) \cap N_{H_1}(v) = X \cap N_{H_2}(u)$ we have that

$$Y \cap N_{H_2}(u) \subseteq N_{H_2}(u) - (N_{H_1}(v) - X).$$

Furthermore, we have

$$|\mathcal{X}(L^{(j)}) \cap N_{H_2}(u)| = |\mathcal{X}(L^{(j)}) \cap N_{H_2}(v)|$$

for $j < f$ and

$$|\mathcal{X}(L^{(j)}) \cap N_{H_2}(u)| = |\mathcal{X}(L^{(j)}) \cap N_{H_2}(v)| + 1$$

for $j \geq f$. Therefore, there is an $x_l \in N_{H_2}(v) - Y$ since

$$deg_{H_2}(v) \geq |N_{H_2}(u) - N_{H_1}(v)| + |X \cap N_{H_2}(u)| \geq |Y \cap N_{H_2}(u)| > |Y \cap N_{H_2}(v)|.$$

However, we have a contradiction to (6) since

$$(vx_0, x_0u, \ldots, vx_{l-1}, x_{l-1}u, vx_l, x_lu)$$

would be a longer near exchange than $L^{(f)}$. □

Lemma 6. *Let $H_1, \ldots, H_t$ be the subgraphs formed by coloring every edge of K_n with some integer in $\{1, \ldots, t\}$ such that H_j is a spanning regular graph for $j \geq 3$. For vertices u and v, let $X = \{x_0^{(1)}, \ldots, x_0^{(|X|)}\}$ where $X \subseteq N_{H_1}(v) - N_{H_1}(u)$ such that $|X \cap N_{H_2}(u)| \leq |N_{H_2}(v) - N_{H_2}(u)|$. If*

$$|X - N_{H_2}(u)| + |N_{H_2}(v) - N_{H_2}(u)| > |N_{H_2}(u) - N_{H_1}(v) - N_{H_2}(v)|,$$

then there exists a $x_0^{(j)} \in X$ such that $vx_0^{(j)}$ and $x_0^{(j)}u$ can be colored exchange.

Proof. Trivially, $\mathcal{X}((vx_0^{(j)}, x_0^{(j)}u)) = \{x_0^{(j)}\}$. Thus, there exists a set $\{L^{(1)}, \ldots, L^{(|X|)}\}$, where

$$L^{(j)} = (vx_0^{(j)}, x_0^{(j)}u, vx_1^{(j)}, x_1^{(j)}u, \ldots, vx_{l^{(j)}-1}^{(j)}, x_{l^{(j)}-1}^{(j)}u)$$

is a near exchange for $1 \leq j \leq |X|$ with

$$\mathcal{X}(L^{(j)}) \cap \mathcal{X}(L^{(i)}) = \emptyset$$

for all $i \neq j$, such that

$$\sum_{i=1}^{|X|} |\mathcal{X}(L^{(i)})| \tag{7}$$

is maximized.

Let $Y = \bigcup_{i=1}^{|X|} \mathcal{X}(L^{(i)})$. Suppose there exists an s such that $x_{l^{(s)}}^{(s)}u \in E(H_i)$ for some $i \geq 3$. Since $x_t^{(j)}u$ and $vx_{t+1}^{(j)}$ are the same color for all j and t we know that for each $L^{(j)}$ u is incident with at least as many edges of H_i than v. Since H_i is regular and u is incident with one more edge of H_i than v in $L^{(s)}$ there must be an $x_l \in N_{H_i}(v) - Y$. However,

$(vx_0^{(s)}, x_0^{(s)}u, \ldots, vx_{l-1}^{(s)}, x_{l-1}^{(s)}u, vx_l^{(s)}, x_l^{(s)}u)$ would be a longer near edge-exchange contradicting the maximality of (6). Thus, every $x_{l(j)-1}^{j}u \in E(H_2)$.

If there is an $x_l \in N_{H_2}(v) - N_{H_2}(u)$ not in Y, then

$$(vx_0^{(1)}, x_0^{(1)}u, vx_1^{(1)}, x_1^{(1)}u, \ldots, vx_{l^{(1)}-1}^{(1)}, x_{l^{(1)}-1}^{(1)}u, vx_l, x_lv)$$

is a longer near exchange that contradicts (7). Thus, $N_{H_2}(v) - N_{H_2}(u) \subseteq Y$.

Given a $z \in (X - N_{H_2}(u)) \cup (N_{H_2}(v) - N_{H_2}(u))$ there is an $L^{(s)}$ and a j such that $x_j^{(s)} = z$. Thus, there is a smallest $f \geq 1$ such that $x_{j+f}^{(s)} \in N_{H_2}(u)$. By the minimality of f we know that $x_{j+f}^{(s)} \notin N_{H_2}(v)$. If $x_{j+f}^{(s)} \in N_{H_1}(v)$, then

$$(vx_0^{(s)}, x_0^{(s)}u, vx_1^{(s)}, x_1^{(s)}u, \ldots, vx_{j+f-1}^{(s)})$$

would be an exchange. This implies that $x_{j+f-1}^{(s)} \in N_{H_2}(u) - N_{H_1}(v) - N_{H_2}(v)$. We may further note that $x_{j+i}^{(s)} \notin (X - N_{H_2}(u)) \cup (N_{H_2}(v) - N_{H_2}(u))$ for any $1 \leq i \leq f$ since $vx_{j+i}^{(s)} \in N_{H_1}(v)$ or $x_{j+i}^{(s)} \in N_{H_2}(v)$ which implies $x_{j+i-1}^{(s)} \in N_{H_2}(u)$. Thus, we have the contradiction

$$|(X - N_{H_2}(u)) \cup (N_{H_2}(v) - N_{H_2}(u))| \leq |N_{H_2}(u) - N_{H_1}(v) - N_{H_2}(v)|. \qquad \square$$

4 Proof of Theorem 4

Proof. We will assume every $G \in \mathcal{R}(\pi)$ has vertex set $V = \{v_1, \ldots, v_n\}$ such that $deg_G(v_i) = d_i$. Let $r \leq k$ be the largest integer such that there is a realization of π with r edge-disjoint 1-factors. By contradiction we assume $r \leq k-1$. Let $\mathcal{G}$ be the set of tuples of the form (G, F, t) where $G \in \mathcal{R}(\pi)$, F is an r-factor of G whose edges can be partitioned into r 1-factors, and $v_t \notin V(M)$ for some maximum matching M of $G - E(F)$.

(C1) We choose a $(G, F, t) \in \mathcal{G}$ such that $\text{def}(G - E(F))$ is minimized, and

(C2) subject to $(C1)$, we minimize t.

Let $M_1, \ldots, M_r$ be a partition of $E(F)$ into r edge-disjoint 1-factors. Furthermore, we let M be a maximum matching of $G - E(F)$ that misses v_t. Let $Q = \{v_1, \ldots, v_{t-1}\}$.

Let $H_1 = G - E(F)$ and $H_2 = \overline{G}$, and note that $H_1, H_2, H_3, \ldots, H_{r+2}$ where $H_i = M_{i-2}$ for $i \geq 3$ represent a coloring of the edges of K_n. Thus, any edge-exchange involving any H_i corresponds to a $(G', F', t') \in \mathcal{G}$.

Let A,C,D be a Gallai-Edmonds Decomposition of H_1. We know that D is not empty since our assumption is that H_1 does not have a matching. We let $D' \subseteq D$ be the largest set such that for every $u \in D'$ there is a matching M_u in H_1 that misses both u and v_t such that $E(M_u - V(D')) = E(M - V(D'))$. We know D' is not empty since M misses v_t and some other vertex. Let $A' \subseteq A$ be all vertices in A adjacent in H_1 to a vertex in D'.

Claim 4.1. $N_{H_1}(u) \subseteq A' \subseteq Q$ for all $u \in D'$.

Proof. Suppose there is a $v \in N_{H_1}(u)$ not in Q. Consider a maximum matching M_u of H_1 that misses both u and v_t. By Lemma 4 we may exchange $v_t u$ and uv to find a $(G', F', t') \in \mathcal{G}$. However, we have a contradiction since $M_u + \{v_t u\}$ is a maximum matching in $G' - F'$ that violates $(C1)$. □

Claim 4.2. $\overline{w}_M \in D'$ for every $w \in A'$

Proof. Suppose there is a $w \in A'$ such that $\overline{w}_M \in D - D'$. By definition of D', for any $u \in N_{H_1}(w) \cap D'$ there is a M_u that misses u and v_t such that $E(M_u - V(D')) = E(M - V(D'))$. However, since $M' = M_u - \{w\overline{w}_M\} + \{wu\}$ is a maximum matching that misses v_t with $E(M' - V(D' \cup \{w_M\})) = E(M - V(D' \cup \{w_M\}))$ we have that $D' \cup \{w_M\}$ is a larger set than D'. □

Claim 4.1 implies every component of $H_1[D']$ is a single vertex. By Claim 4.2 and (III) we have that $|D'| > |A'|$. Therefore,

$$e_{H_1}(D', A') \geq |D'|(d_n - r) > |A'|(d_n - r)$$

and by the pigeon hole principle there is some vertex $s \in A'$ adjacent in H_1 to at least $d_n - r + 1$ vertices in D'.

Claim 4.3. Q is complete in G to $A' \cup N_{H_1}(v_t)$.

Proof. Suppose there is a $w \in Q$ and $v \in A' \cup N_{H_1}(v_t)$ that are not adjacent in G, and let $u \in N_{H_1}(v) \cap D'$. By definition of D there is a maximum matching M_u of H_1 that misses u. By (II) u and $\overline{w}_{M_u}$ are in separate components of $H_1[D]$, and therefore, $u\overline{w}_{M_u} \notin E(H_1)$. By Lemma 4 and Lemma 2 there exists a simplified $\overline{w}_{M_u}w$ and $u\overline{w}_{M_u}$ exchange that when exchanged creates another $(G', F', t') \in \mathcal{G}$ with the matching $M_u - \{\overline{w}_{M_u}w\} + \{\overline{w}_{M_u}u\}$ of $G' - E(F')$ that violates (C2). □

Since $s \in A' \subseteq Q$ we have by Claim 4.3 that s is adjacent in G to every vertex in $Q - \{s\} \cup N_{H_1}(v_t)$. Thus,

$$\begin{aligned}|Q - \{s\}| &\leq deg_G(s) - |N_{H_1}(s) - Q| - |N_F(s) - Q| \\ &\leq d_1 - |N_{H_1}(s) \cap (D' - \{v_t\})| - |N_{H_1}(s) \cap \{v_t\}| - |N_F(s) - Q| \\ &\leq d_1 - (d_n - r + 1) - |N_{H_1}(s) \cap \{v_t\}| - |N_F(s) - Q| \\ &= d_1 - d_n + r - 1 - |N_{H_1}(s) \cap \{v_t\}| - |N_F(s) - Q|.\end{aligned}$$

Since $s \in Q$ we have that $|Q| \leq d_1 - d_n + r - |N_{H_1}(s) \cap \{v_t\}| - |N_F(s) - Q|$. However, since $|Q| = t - 1$ we have by (1) that

$$d_t = d_{|Q|+1} \geq d_{d_1 - d_n + r + 1} \geq d_{d_1 - d_n + k} \geq d_1 - d_n + k - 1 \geq d_1 - d_n + r.$$

Suppose $N_{H_1}(s) \cap D' \subseteq N_{H_2}(v_t)$. Let $X = \{x\}$ for some $x \in N_{H_1}(s) \cap D'$. We let M_x be a maximum matching in H_1 that misses both x and v_t. Since $deg_{H_1}(v_t) \geq d_1 - d_n$ and $|N_{H_2}(v_t) \cap N_{H_1}(s)| \geq d_n - r + 1$, and $|X \cap N_{H_2}(v_t)| = 1$ we have that

$$deg_{H_1}(s) - |N_{H_2}(v_t) \cap N_{H_1}(s)| + |X \cap N_{H_2}(v_t)| \leq d_1 - r - (d_n - r + 1) + 1 = d_1 - d_n \leq deg_{H_1}(v_t).$$

Therefore, by Lemma 5 and Lemma 2 there exist a simplified $v_t x$ and xs exchange that when exchanged creates another $(G', F', t') \in \mathcal{G}$ such that M_x is a matching of $G' - E(F')$. However, this violates (C2) since $M_x + \{v_t x\}$ is a larger matching of $G' - E(F')$. Thus, $N_{H_1}(s) \cap D' \cap N_F(v_t) \neq \emptyset$.

Let $X = N_{H_1}(s) \cap D' \cap N_F(v_t)$, and observe $X = X - N_{H_2}(v_t)$ and $N_F(s) \cap N_{H_2}(v_t) = N_{H_2}(v_t) - N_{H_1}(s) - N_{H_2}(s)$. If $|X| + |N_{H_2}(s) - N_{H_2}(v_t)| > |N_F(s) \cap N_{H_2}(v_t)|$, then by Lemma 6 and Lemma 2 for some $x \in X$ there exists a simplified $v_t x$ and xs exchange that when exchanged creates another $(G', F', t') \in \mathcal{G}$. Note that since the exchange was simplified $M - \{sx\}$ is a matching of $G' - E(F')$. If $x \notin V(M)$, then $M + \{v_t x\}$ is a matching of $G' - E(F')$ that contradicts (C1). If $x \in V(M)$, then since $\overline{x}_M \in Q$ we have that $M - \{x\overline{x}_M\} + \{v_t x\}$ is a matching of $G' - E(F')$ that violates (C2). Thus, we are left with the case

$$|N_F(s) \cap N_{H_2}(v_t)| \geq |X| + |N_{H_2}(s) - N_{H_2}(v_t)| \geq |X| + |N_{H_2}(s) \cap N_F(v_t)|. \tag{8}$$

From (8) we have

$$\begin{aligned} |N_F(s) - N_{H_1}(v_t)| &= |N_F(s) \cap N_F(v_t)| + |N_F(s) \cap N_{H_2}(v_t)| \\ &\geq |N_F(s) \cap N_F(v_t)| + |X| + |N_{H_2}(s) \cap N_F(v_t)| \\ &\geq |N_F(s)| - |N_{H_1}(s) \cap N_F(v_t)| + |X| \\ &= r - |N_{H_1}(s) \cap N_F(v_t)| + |X| \end{aligned} \tag{9}$$

Observe

$$|N_{H_1}(s) - N_{H_1}(v_t)| = |N_{H_1}(s) \cap N_F(v_t)| + |N_{H_1}(s) \cap N_{H_2}(v_t)|, \tag{10}$$

and

$$|N_{H_1}(s) \cap N_{H_2}(v_t)| + |X| \geq |(N_{H_1}(s) \cap D') - N_{H_1}(v_t)| \geq d_n - r + 1. \tag{11}$$

From (10) and (11) we have

$$|N_{H_1}(s) - N_{H_1}(v_t)| \geq |N_{H_1}(s) \cap N_F(v_t)| + d_n - r + 1 - |X|. \tag{12}$$

Thus, combining (9) and (12) we have

$$|N_{H_1}(s) - N_{H_1}(v_t)| + |N_F(s) - N_{H_1}(v_t)| \geq d_n + 1. \tag{13}$$

Since $deg_{H_1}(v_t) \geq d_1 - d_n$ and Claim 4.1 says that $N_{H_1}(v_t) \subseteq N_G(s)$ we can use (13) to show our final contradiction

$$\begin{aligned} d_1 \geq deg_G(s) &\geq deg_{H_1}(v_t) + |N_{H_1}(s) - N_{H_1}(v_t)| + |N_F(s) - N_{H_1}(v_t)| \\ &\geq d_1 - d_n + d_n + 1 = d_1 + 1. \end{aligned}$$

□

5 Proof of Theorem 7

Proof. We assume every $G \in \mathcal{R}(\pi)$ has vertex set $V = \{v_1, \ldots, v_n\}$ such that $deg_G(v_i) = d_i$. Suppose $\lceil \frac{n+5-k}{2} \rceil > d_{\frac{n}{2}+1}$. Since $\mathcal{D}_k(\pi)$ is graphic Kundu's k-factor Theorem says π has a realization with a k-factor. Moreover, by Theorem 9 and Petersen's 2-factor theorem π has a realization with a k-factor whose edges can be partitioned into 1-factors and 2-factors. We let r be the largest natural number such that $k + r$ is even and there is a $G \in \mathcal{R}(\pi)$ with a k-factor F whose edges can be partitioned into graphs $F_1, \ldots, F_{\frac{k+r}{2}}$ such that F_i is a 1-factor when $i \leq r$ or a 2-factor when $i > r$. By contradiction we assume $r \leq k - 2$. Note that for $i > r$ every F_i must have at least two odd cycles as components. Otherwise, we could split F_i into two 1-factors contradicting our choice of r.

We let $(H, H_1, H_2, H_3, \ldots, H_q)$ correspond to an edge coloring of K_n with the natural numbers $\{1, \ldots, q\}$ such that $H = K_n - E(H_2)$ and H_i is the subgraph induced by the edges colored i. We let $\mathcal{G}$ be the set of such tuples where $H_1 \in \mathcal{R}(G - E(F))$, $H_2 \in \mathcal{R}(\overline{G})$, and $H_i \in \mathcal{R}(F_{i-2})$ for all $3 \leq i \leq q$. We further consider the subset $\mathcal{G}' \subseteq \mathcal{G}$ to be all tuples such that the number of cycles in H_q is minimized.

For the rest of this proof we will assume all color exchanges are simplified.

Claim 5.1. For $(H, H_1, \ldots, H_q) \in \mathcal{G}'$, let $A = a_0 \ldots a_{|A|-1} a_0$ and $B = b_0 \ldots b_{|B|-1} b_0$ be distinct cycles of H_q. For some a_i and b_j if $L = \{a_i x_0, x_0 b_j, \ldots, a_i x_{l-1}, x_{l-1} b_j\}$ is a color exchange, then $\mathcal{X}(L) \cap (V(A) \cup V(B)) = \emptyset$.

Proof. Suppose the claim is false and there is an edge $a_i x_s$ of A and an edge $x_{s+1} b_j$ of B. After, performing the exchange we denote the resulting tuple as $(H', H'_1, \ldots, H'_q)$. Since L is simplified $a_i x_s$ and $x_{s+1} b_j$ are the only edges of H_q exchanged. Without loss of generality we assume $x_s = a_{i'}$ and $x_{s+1} = b_{j'}$ with $i' = i + 1 \mod |A|$ and $j' = j + 1 \mod |B|$. With this we can see $a_{i'} \ldots a_{|A|-1} a_0 \ldots a_i b_{j'} \ldots b_{|B|-1} b_0 \ldots b_j a_{i'}$ is a cycle in H'_q that combines the vertices of A and B and leaves all other cycles alone. However, this implies the contradiction that H'_q has fewer cycles than H_q. □

We choose an arbitrary $(H, H_1, \ldots, H_q) \in \mathcal{G}'$. We let f be the largest index such that $v_f v_f^-$ is an edge of H_q with $deg_H(v_f) \geq deg_H(v_f^-)$ and denote the cycle containing it by A. Since H_q does not have a 1-factor there is an odd cycle C that is distinct from A. Furthermore, we know there is $v_t \in V(C)$ with neighbors v_t^- and v_t^+ along C in H_q such that $deg_G(v_t) \leq deg_G(v_t^+)$. We choose a v_t such that t is minimized, and by our choice of v_f we know that $t < f$. There is an odd cycle D in H_q that is not C. Like v_t in C, D has vertices v_s, v_s^+, and v_s^- such that $deg_{H_q}(v_s^-) \leq deg_{H_q}(v_s) \leq deg_{H_q}(v_s^+)$. By our choice of v_f we know that either $f = s$ or $f < s$.

By the minimally of our choice of f we know that $\{v_{f+1}, \ldots, v_n\}$ is an independent set in H_q. Thus, $e_{H_q}(\{v_1, \ldots, v_f\}, \{v_{f+1}, \ldots, v_n\}) \geq 2(n-f)$. Note that v_s, $v^+(s)$, v_t, and $v^+(t)$ are adjacent in H_q to vertices in $\{v_1, \ldots, v_f\}$. We therefore, have $e_{H_q}(\{v_1, \ldots, v_f\}, \{v_{f+1}, \ldots, v_n\}) \leq 2f - |\{v_s, v^+(s), v_t, v^+(t)\}| = 2(f-2)$. Combining these two bounds and solving for f we have that $f \geq \frac{n}{2} + 1$.

Claim 5.2. $\{v_f v_t^+, v_f v_t, v_f^- v_t^+, v_f^- v_t\} \subseteq \bigcup_{3 \leq i < q} E(H_i)$.

Proof. Note that $deg_H(v_f^-) \leq deg_H(v_f) \leq deg_H(v_t) \leq deg_H(v_t^+)$. If $v_f v_t$ is an edge of H_1, then we have a contradiction to Claim 5.1 since $v_t^+ v_t$ and $v_t v_f$ can be exchanged. If $v_f v_t$ is an edge of H_2, then we have a contradiction to Claim 5.1 since $v_f^- v_f$ and $v_f v_t$ can be exchanged. Similar arguments prove the claim for $v_f v_t^+$, $v_f^- v_t^+$, and $v_f^- v_t$. □

Claim 5.3. $N_{H_1}(v_t) \subseteq N_H(v_f) \cap \{v_1, \ldots, v_f\}$.

Proof. Let $v_i \in N_{H_1}(v_t)$. Suppose $v_i \in V(A)$. If $i > t$, then by Lemma 5 we have a contradiction to Claim 5.1 since $v_t^+ v_t$ and $v_t v_i$ can be exchanged. Thus, $f > t > i$. If $v_i \in N_{H_2}(v_f)$, then we can exchange the edges $v_t v_f$ and $v_f v_i$ to create a tuple $(H_1', \ldots, H_q') \in \mathcal{G}'$ such that $H_q = H_q'$. However, we have a contradiction to Claim 5.1 since $v_f^- v_f$ and $v_f v_t$ can be exchanged with respect to this new tuple. Thus, $v_i \subseteq N_H(v_f) \cap \{v_1, \ldots, v_f\}$ when $v_i \in V(A)$.

Suppose $v_i \notin V(A)$. If $i > f$, then we can exchange the edges $v_f v_t$ and $v_t v_i$ to create a tuple $(H_1', \ldots, H_q') \in \mathcal{G}'$ such that $H_q = H_q'$. However, we have a contradiction to Claim 5.1 since $v_t^+ v_t$ and $v_t v_f$ can be exchanged with respect to this new tuple. Thus, we are left with the case $i < f$ and $v_i \in N_{H_2}(v_f)$. Here we have a contradiction to Claim 5.1 since $v_f^- v_f$ and $v_f v_i$ can be exchanged. Thus, $v_i \subseteq N_H(v_f) \cap \{v_1, \ldots, v_f\}$ when $v_i \notin V(A)$. □

Since both v_t and v_f are not in $N_{H_1}(v_t)$ we have by Claim 5.3 that $|N_{H_1}(v_t)| \leq f - 2$. For $v_i \in \{v_{f-1}, \ldots, f_n\}$, this implies

$$\overline{q}_{n+1-i} = n - 1 - d_i + k \geq n - 1 - |N_{H_1}(v_t)| \geq n + 1 - f.$$

Thus, $m(\overline{D_k(\pi)}) \geq n + 2 - f$ and therefore, $f \geq \max\{\frac{n}{2} + 1, n + 2 - m(\overline{D_k(\pi)})\}$.

We now turn to finding a lower bound for d_f.

Claim 5.4. If $v_i v_j \in E(H_q)$ such that $v_i \in N_{H_2}(v_t)$, then $v_j \in N_H(v_f) \cap \{v_1, \ldots, v_f\}$.

Proof. Suppose $v_j \notin N_H(v_f) \cap \{v_1, \ldots, v_f\}$. We first assume $v_i \notin V(C)$. By Claim 5.1 $v_j v_i$ and $v_i v_t$ cannot be exchanged. Therefore, by Lemma 5 $j < t$. If $v_j \in N_{H_2}(v_f)$, then we can exchange the edges $v_t v_f$ and $v_f v_j$ to create a tuple $(H_1', \ldots, H_q') \in \mathcal{G}'$ such that $H_q = H_q'$. However, we have a contradiction to Claim 5.1 since $v_f^- v_f$ and $v_f v_t$ can be exchanged with respect to this new tuple. Thus, $v_j \in N_H(v_f) \cap \{v_1, \ldots, v_f\}$.

We are left with the case $v_j \in V(C)$. If $i < f$, then we can exchange the edges $v_f v_t$ and $v_t v_i$ to create a tuple $(H_1', \ldots, H_q') \in \mathcal{G}'$ such that $H_q = H_q'$. However, we have a contradiction to Claim 5.1 since $v_f^- v_f$ and $v_f v_t$ can be exchanged with respect to this new tuple. Thus, $i > f$ and therefore, $j < f$ by the minimally of f. By Claim 5.1 we know that $v_i v_j$ and $v_j v_f$ cannot be exchanged. Thus, by Lemma 5 $v_j \in N_H(v_f)$. Thus, $v_j \in N_H(v_f) \cap \{v_1, \ldots, v_f\}$. □

We let $W = N_H(v_f) \cap \{v_1, \ldots, v_f\}$. By Claim 5.3 we know that $N_{H_1}(v_t) \subseteq W$, and we know that v_t is adjacent in $H - E(H_1)$ to v_f, v_f^-, and v_t^+. Thus,

$$d_f = |N_H(v_f)| \geq |N_{H_1}(v_t) \cup \{v_f^-, v_t^+, v_t\}| \geq d_t - k + 3.$$

From Claim 5.4 we know that if $v_i \in N_{H_2}(v_t)$, then $N_{H_q}(v_i) \subseteq W$. Thus,

$$e_{H_q}(N_{H_2}(v_t), W) \geq 2|N_{H_2}(v_t)| = 2(n-1-d_t).$$

On the other hand, v_t^+ and v_s^+ are in W and each of them are adjacent in H_q to at least one vertex not in $N_{H_2}(v_t)$, and v_t is adjacent in H_q to two vertices not in $N_{H_2}(v_t)$. Thus,

$$e_{H_q}(N_{H_2}(v_t), W) \leq 2|W| - 4.$$

Combining we have

$$2|W| - 4 \geq e_{H_q}(N_{H_2}(v_t), W) \geq 2(n-1-d_t).$$

Solving for $|W|$ we have

$$|W| \geq n + 1 - d_t.$$

Since $v_f^- \notin W$ we have that $|W| \leq d_f - 1$. Therefore,

$$\max\{n + 2 - d_t, d_t - k + 3\} \leq d_f.$$

Letting $n + 2 - d_t = d_t - k + 3$ we have that $\max\{n + 2 - d_t, d_t - k + 3\}$ is minimized when $2d_t - (n-1) = k$. Thus,

$$\left\lceil \frac{n+5-k}{2} \right\rceil \leq d_f.$$

Using the lower bound on f we have the contradiction

$$\left\lceil \frac{n+5-k}{2} \right\rceil \leq d_f \leq d_{\max\{\frac{n}{2}+1, n+2-m(\overline{D_k(\pi)})\}}.$$

We let $l = \max\{\frac{n}{2} + 1, n + 2 - m(\pi)\}$ and suppose

$$\left\lceil \frac{n+5-k}{2} \right\rceil \leq \overline{q}_l.$$

We have that $\overline{q}_l = n - 1 - d_{n+1-l} + k \geq \left\lceil \frac{n+5-k}{2} \right\rceil$. Solving for d_{n+1-1} and realizing $n+1-l = \min\{\frac{n}{2}, m(\pi) - 1\}$ we have the contradiction

$$d_{\min\{\frac{n}{2}, m(\pi)-1\}} \leq n + k - 1 - \left\lceil \frac{n+5-k}{2} \right\rceil = \left\lceil \frac{n+3k-8}{2} \right\rceil.$$

Thus,

$$\overline{q}_l < \left\lceil \frac{n+5-k}{2} \right\rceil$$

and by the first part of this theorem some realization of $\overline{D_k(\pi)}$ has k-edge-disjoint 1-factors. Thus, some realization of π has k edge-disjoint 1-factors. □

6 Proof of theorem 9

Proof. Let $\mathcal{G}$ be the set of tuples of the form (G, F, r, F_0) where $G \in \mathcal{R}(\pi)$, F is a k-factor of G, F_0 is a spanning $(k-r)$-factor of F, and $E(F - E(F_0))$ can be partitioned into r edge-disjoint 1-factors.

(C1) We choose a $(G, F, r, F_0) \in \mathcal{G}$ such that r is maximized, and

(C2) subject to $(C1)$, we minimize $\text{def}(F_0)$.

Let $k' = k - r$, and by contradiction we assume $r \leq \left\lceil \frac{k'}{2} \right\rceil$. Let $M_1, \ldots, M_r$ be a partition of $E(F - E(F_0))$ into r edge-disjoint 1-factors.

Let $H_1 = G - E(F)$, $H_2 = \overline{G}$, $H_3 = F_0$, and note that $H_1, H_2, H_3, \ldots, H_{r+2}$ where $H_i = M_{i-3}$ for $i \geq 4$ represent a coloring of the edges of K_n. Thus, any exchange involving any H_i in K_n corresponds to some $(G', F', r', F_0') \in \mathcal{G}$.

Let A,C,D be a Gallai-Edmonds Decomposition of F_0. Since F_0 does not have a 1-factor we know that D is not empty. Let $\mathcal{D} = \{D_1, \ldots, D_{|\mathcal{D}|}\}$ be the components of $F_0[D]$.

In [17] Lovász showed every non-trivial factor critical graph has an odd cycle. Since (II) says every non-trivial component in $\mathcal{D}$ is factor critical we know each one also has an odd cycle. We let $L_i \subseteq V(D_i)$ be the largest such set where for every $u \in L_i$ there is an odd cycle in D_i with distinct edges uu^+ and uu^- such that $deg_G(u^+) \geq deg_G(u) \geq deg_G(u^-)$. We let l_i be the largest number of edge disjoint odd cycles in D_i, and note that $|L_i| \geq l_i$.

Claim 6.1. Suppose $u \in V(D_j)$ and $x \in V(D_i)$ such that there exists a maximum matching M of F_0 that misses both u and x. If $uy \in E(H_f)$ for some $y \in N_{D_i}(x)$, then $xv \notin E(H_f)$ for all $v \in N_{D_j}(u)$.

Proof. If there does exists such a $v \in N_{D_j}(u)$, then we may exchange the edges uv and xy with the edges xv and uy of H_f to find a $(G', F', r, F_0') \in \mathcal{G}$ such that $M + \{ux\}$ is a larger matching in F_0' that contradicts (C2). ☐

For $u \in V(D_i)$ we let $W_i^+[u]$ be all $v \in N_{F_0}[u] \cap V(D_i)$ such that $deg_G(u) \leq deg_G(v)$. Similarly, we let $W_i^-[u]$ be all $v \in N_{F_0}[u] \cap V(D_i)$ such that $deg_G(v) \leq deg_G(u)$.

Claim 6.2. Suppose $v_2 \in V(D_j)$ and $u_1 \in V(D_i)$ such that there exists a maximum matching M of F_0 that misses both v_2 and u_1. If $deg_G(v_1) \geq deg_G(v_2) \geq deg_G(u_1) \geq deg_G(u_2)$ for $v_1 \in W_j^+[v_2]$ and $u_2 \in W_i^-[u_1]$, then none of $\{u_1v_1, u_1v_2, u_2v_1, u_2v_2\}$ are edges of H_1 or H_2.

Proof. By contradiction suppose there is a $v_s \in \{v_1, v_2\}$ and a $u_t \in \{u_1, u_2\}$ such that $v_su_t \in E(H_1) \cup E(H_2)$. Since D_1 and D_i are factor critical and M restricts to a near perfect matching on D_i and D_j we can find a maximum matching M' of F_0 that misses both v_s and u_t. If $v_su_t \in E(H_1)$, then since $deg_G(v_1) \geq deg_G(v_2) \geq deg_G(u_1) \geq deg_G(u_2)$ and by Lemma 5 and Lemma 2 there exists a simplified v_1v_2 and v_su_t edge-exchange that when exchanged creates another $(G', F', r, F_0') \in \mathcal{G}$ with a larger matching $M' + \{v_su_t\}$ that contradicts (C2). If $v_su_t \in E(H_2)$, then since $deg_G(v_1) \geq deg_G(v_2) \geq deg_G(u_1) \geq deg_G(u_2)$ and by Lemma 5 and Lemma 2 there exists a simplified u_1u_2 and v_su_t edge-exchange that when exchanged creates another $(G', F', r, F_0') \in \mathcal{G}$ with a larger matching $M' + \{v_s u_t\}$ that contradicts (C2). $\square$

Claim 6.3. Suppose $u \in L_i$ and $v \in L_j$ such that there exists a maximum matching M of F_0 that misses both u and v. If $deg_G(u) \geq deg_G(v)$, then every vertex in $W_j^-[v]$ is adjacent in $F - E(F_0)$ to every vertex in $W_1^+[u]$.

Proof. The claim follows from Claim 6.2 since $deg_G(u^+) \geq deg_G(u) \geq deg_G(v) \geq deg_G(v^-)$ for every $u^+ \in W_1^+[u]$ and $v^- \in W_j^-[v]$. $\square$

Claim 6.4. Suppose $u \in L_i$ and $v \in L_j$ such that there exists a maximum matching M of F_0 that misses both u and v. If $deg_G(u) \geq deg_G(v)$, then for any $u^+ \in W_i^+[u]$ and $v^- \in W_j^-[v]$ with $\{u, v\} \neq \{v^-, u^+\}$ the edges uv^- and vu^+ are in distinct H_s for $s \geq 3$.

Proof. This follows from Claim 6.1 and Claim 6.2. $\square$

If there exists a $z_1 \in \bigcup_{D_i \in \mathcal{D}} L_i$ such that $e_{F_0}(z_1, A) \leq \lfloor \frac{k'}{2} \rfloor$, then we choose such a z_1 so that $deg_G(z_1)$ is maximized. Otherwise, we choose a $z_1 \in D$ such that $e_{F_0}(z_1, A)$ is minimized. Without loss of generality we may assume $z_1 \in V(D_1)$. We may also assume $D_2 \in \mathcal{D}$ is a component with $e_{F_0}(D_2, A)$ minimized such that there is a maximum matching M_v of F_0 that misses both z_1 and some vertex in $v \in V(D_2)$. If D_2 is not trivial, then we choose a $z_2 \in L_2$. Otherwise, we let z_2 be the only vertex in D_2. Since M_v restricts to a near-perfect matching on D_2 we can use the fact that D_2 is factor critical to find a maximum matching M of F_0 that misses z_1 and z_2.

We let $D' \subseteq D$ be the largest set such that for every $u \in D'$ there is a maximum matching M_u of F_0 such that $E(M_u) - E(D') = E(M) - E(D')$. Note that $z_2 \in D'$, and since every component in $\mathcal{D}$ is factor critical and M restricts to a near matching on each of them we may conclude that for $D_j \in \mathcal{D}$, if $V(D_j) \cap D' \neq \emptyset$, then $V(D_j) \subseteq D'$. We may assume without loss of generality that $\mathcal{D}' = \{D_2, \ldots, D_{|\mathcal{D}'|}\}$ are the set of components of $F_0[D']$ and $e_{F_0}(D_2, A) \leq \ldots \leq e_{F_0}(D_{|\mathcal{D}'|+1}, A)$. Let $S = N_{F_0}(D') \cap A$. Suppose $\overline{w}_M \notin D'$ for some $w \in S$. Let $v \in N_{F_0}(w) \cap D'$. By the definition of D' there is a maximum matching M_v of F_0 that misses both z_1 and v such that $w\overline{w}_M \in E(M_v)$. This is a contradiction since $E(M_v - \{w\overline{w}_M\} + \{wv\}) - E(D' \cup \{\overline{w}_M\}) = E(M) - E(D' \cup \{w_M\})$.

Let t be the largest integer such that $e_{F_0}(D_{t+1}, S) < k'$. Since $|\mathcal{D}'| \geq |S| + 1$ by (III) we may conclude that $e_{F_0}(D_2, S) < k'$, and therefore, $t \geq 1$. Otherwise, we have the contradiction

$$k'|S| \geq \sum_{i=2}^{|\mathcal{D}'|+1} e_{F_0}(D_i, S) \geq k'|\mathcal{D}'| \geq k'(|S| + 1).$$

Furthermore, since F_0 is k'-regular every such component in $\mathcal{D}'$ must be non-trivial. Thus, both D_1 and D_2 have odd cycles, and therefore, by our choice of z_1 and z_2 we know that $z_1 \in L_1$ and $z_2 \in L_2$.

We pause here to recognize that applying Claim 6.2 and Claim 6.4 with z_1 and z_2 we may conclude that $r \geq 4$. This proves this theorem and Conjecture 1 for $k \leq 5$.

For the rest of the proof we will need a $z_1^- \in W_1^-[z_1]$ such that $deg_G(z_1^-) \leq deg_G(z')$ for every $z' \in W_1^-[z_1]$.

Claim 6.5. If $e_{F_0}(z_1, A) \leq \lfloor \frac{k'}{2} \rfloor$ and there exists a $D_j \in \mathcal{D}'$ with a $v \in L_j$ such that $deg_G(v) \leq deg_G(z_1)$, then for every $v^+ \in W_j^+[v]$ at least one of v^+z_1 or $v^+z_1^-$ is an edge of $F - E(F_0)$.

Proof. Suppose there is a $v^+ \in W_j^+[v]$ not adjacent in $F - E(F_0)$ to z_1^- nor z_1. If $deg_G(z_1^-) \geq deg_G(v)$, then $deg_G(v^-) \leq deg_G(v) \leq deg_G(z_1^-) \leq deg_G(z') \leq deg_G(z_1)$ for every $z' \in W_1^-[z_1]$ and $v^- \in W_j^-[v]$. Thus, by Claim 6.2 and Claim 6.3 v must be adjacent in $F - E(F_0)$ to every vertex in $W_1^-[z_1]$ and $W_1^+[z_1]$. However, this implies $r \geq \lceil \frac{k'}{2} \rceil + 1$. If $deg_G(v^+) < deg_G(z_1)$, then $deg_G(z_1^+) \geq deg_G(z_1) \geq deg_G(v^+) \geq deg_G(v)$ for every $z_1^+ \in W_1^+[z_1]$. Thus, by Claim 6.2 z_1 is adjacent in $F - E(F_0)$ to v^+. We are left with the case $deg_G(v^+) \geq deg_G(z_1) \geq deg_G(v) \geq deg_G(z_1^-)$. Let M' be a matching of F_0 that misses both z_1^- and v^+. If $z_1^-v^+ \in E(H_1)$, then by Lemma 4 and Lemma 2 there exists a simplified vv^+ and $v^+z_1^-$ edge-exchange that when exchanged creates another $(G', F', r, F_0') \in \mathcal{G}$ with a larger matching $M' + \{v^+z_1^-\}$ of F_0' that contradicts (C2). If $z_1^-v^+ \in E(H_2)$, then by Lemma 4 and Lemma 2 there exists a simplified $z_1z_1^-$ and $v^+z_1^-$ edge-exchange that when exchanged creates another $(G', F', r, F_0') \in \mathcal{G}$ with a larger matching $M' + \{v^+z_1^-\}$ of F_0' that contradicts (C2). □

We observe that

$$k'|S| \geq \sum_{i=2}^{|S|+2} e_{F_0}(D_i, S) \geq k'(|S| + 1 - t) + \sum_{i=2}^{t+1} e_{F_0}(D_i, S).$$

Thus,

$$\sum_{i=2}^{t+1} e_{F_0}(D_i, S) \leq k'(t - 1). \tag{14}$$

In the case $t = 1$ we have that $e_{F_0}(D_2, S) = 0$. Thus, $e_{F_0}(z_1, A) \leq \lfloor \frac{k'}{2} \rfloor$ and $deg_G(z_2) \leq deg_G(z_1)$ by our choice of z_1. By Claim 6.3 z_1 is adjacent in $F - E(F_0)$ to every vertex in $W_2^-[z_2]$ and z_2 is adjacent in $F - E(F_0)$ to every vertex in $W_1^+[z_1]$. By Claim 6.4 all those edges are in distinct 1-factors. Thus,

$$|W_1^+(z_1)| + |W_2^-[z_2]| \leq |N_{F-E(F_0)}(z_1) \cap V(D_2)| + |W_1^+(z_1)| \leq r \leq \left\lceil \frac{k'}{2} \right\rceil.$$

Since $|W_1^+(z_1)| \geq 1$ we have that

$$|N_{F-E(F_0)}[z_1] \cap V(D_2)| \leq \left\lceil \frac{k'}{2} \right\rceil - 1. \tag{15}$$

Since $e_{F_0}(z_2, S) = 0$ we have by Claim 6.4 and Claim 6.5 that $N_{F_0}[z_2] \subseteq N_{F-E(F_0)}(z_1) \cup N_{F-E(F_0)}(z_1^-)$. By Claim 6.5 and (15) we have the contradiction

$$\begin{aligned} r &\geq |N_{F-E(F_0)}(z_1^-) \cap V(D_2)| \\ &\geq |N_{F_0}[z_2]| - |N_{F-E(F_0)}(z_1) \cap N_{F_0}[z_2]| \\ &\geq k' + 1 - \left(\left\lceil \frac{k'}{2} \right\rceil - 1 \right) \\ &= \left\lfloor \frac{k'}{2} \right\rfloor + 2 = \left\lceil \frac{k'}{2} \right\rceil + 1. \end{aligned}$$

Thus, we may assume $t \geq 2$.

For any $D_i \in \mathcal{D}'$ we let J_i be a set of l_i edges that can be removed from D_i such that the resulting graph is bipartite. Let X_i and Y_i be an independent vertex sets that partition $D_i - E(J_i)$. We may assume $|X_i| \geq |Y_i| + 1$ since D_i has an odd number of vertices. Since each edge of J_i is an odd cycle we know that no edge of J_i is incident with both X_i and Y_i. Let $w_i = |V(J_i) \cap X_i|$ and $w_i' = |V(J_i) \cap Y_i|$. Thus,

$$k'|Y_i| - 2w_i' - e_{F_0}(Y_i, S) \geq e_{F_0}(X_i, Y_i) \geq k'|X_i| - 2w_i - e_{F_0}(X_i, S) \geq k'(|Y_i| + 1) - 2w_i - e_{F_0}(X_i, S).$$

Since $l_i = w_i + w_i'$ we have after some rearranging and reduction that

$$0 \geq k' + 2(2w_i' - l_i) + e_{F_0}(Y_i, S) - e_{F_0}(X_i, S) \geq k' - 2l_i - e_{F_0}(X_i, S).$$

Thus, we have

$$0 \geq \sum_{i=2}^{t+1} (k' - 2l_i - e_{F_0}(X_i, S)) = k't - 2\sum_{i=2}^{t+1} l_i - \sum_{i=2}^{t+1} e_{F_0}(X_i, S). \tag{16}$$

Letting $l \leq \min\{l_2, \ldots, l_{t+1}\}$ we can further bound and rearrange (16) so that

$$\sum_{i=2}^{t+1} e_{F_0}(X_i, S) \geq tk' - 2tl. \tag{17}$$

We may now use (17) to bound (14) from below.

$$k'(t-1) \geq \sum_{i=2}^{t+1} e_{F_0}(D_i, S) \geq \sum_{i=2}^{t+1} e_{F_0}(X_i, S) \geq k't - 2tl.$$

Therefore, $l \geq \left\lceil \frac{k'}{2t} \right\rceil$.

We let $\mathcal{D}'' \subseteq \{D_2, \ldots, D_{t+1}\}$ be the largest set such for any $D_i \in \mathcal{D}''$ no vertex in $V(D_i)$ is incident with l edge-disjoint odd cycles. This implies $|L_i| \geq 2$ for every such component. Furthermore, since $e_{F_0}(D_i, S) \leq k' - 1$ for every $D_i \in \mathcal{D}''$ there must be a vertex in L_i that is adjacent in F_0 to at most $\frac{k'-1}{2}$ vertices in S and therefore, at least $k' - \left\lfloor \frac{k'-1}{2} \right\rfloor = \left\lceil \frac{k'+1}{2} \right\rceil$ vertices in D_i. We let $t' = |\mathcal{D}''|$. Suppose $t' = 0$. By the definition of L_i for every $D_i \in \{D_2, \ldots, D_{t+1}\}$ any $v_i \in L_i$ has $|W_i^-[v_i]| \geq l+1$. Therefore, by Claim 6.4 z_1 is adjacent in $F - F_0$ to $l+1$ vertices in every component in $\{D_2, \ldots, D_{t+1}\}$. However, this implies the contradiction

$$r \geq t(l+1) \geq t\left(\left\lceil \frac{k'}{2t} \right\rceil + 1\right) \geq \left\lceil \frac{k'}{2} \right\rceil + t.$$

Therefore, $t' \geq 1$, and by our choice of z_1 we have $e_{F_0}(z_1, A) \leq \lfloor \frac{k'}{2} \rfloor$. Therefore, for every $D_i \in \mathcal{D}''$ we may identify a $v_i \in L_i$ that is adjacent in F_0 to at least $\lceil \frac{k'+1}{2} \rceil$ vertices in D_i. By Claim 6.4 and Claim 6.5 $N_{F_0}[v_i] \subseteq N_{F-E(F_0)}(z_1) \cup N_{F-E(F_0)}(z_1^-)$ for every v_i. Thus,

$$e_{F-E(F_0)}(z_1, N_{F_0}[v_i]) + e_{F-E(F_0)}(z_1, N_{F_0}[v_i]) \geq |N_{F_0}[v_i]| \geq \left\lceil \frac{k'+1}{2} \right\rceil + 1.$$

We now have

$$e_{F-E(F_0)}(z_1, D') \geq \sum_{j=2}^{t} e_{F-E(F_0)}(z_1, V(D_j) \geq (l+1)(t-t') + \sum_{D_i \in \mathcal{D}''} e_{F-E(F_0)}(z_1, N_{F_0}[v_i])$$

and

$$e_{F-E(F_0)}(z_1^-, D') \geq \sum_{j=2}^{t} e_{F-E(F_0)}(z_1^-, V(D_j)) \geq l(t'-t'') + \sum_{D_i \in \mathcal{D}''} \left(\left\lceil \frac{k'+1}{2} \right\rceil + 1 - e_{F-E(F_0)}(z_1, N_{F_0}[v_i]) \right).$$

Combining the two equations we have

$$\begin{aligned} 2r &\geq e_{F-E(F_0)}(z_1, D') + e_{F-E(F_0)}(z_1^-, D') \\ &\geq 2(t-t')l + (t-t') + t'\left(\left\lceil \frac{k'+1}{2} \right\rceil + 1\right) \\ &\geq 2(t-t')\left\lceil \frac{k'}{2t} \right\rceil + t - t' + t'\left(\left\lceil \frac{k'+1}{2} \right\rceil + 1\right). \end{aligned} \tag{18}$$

If $t' \geq 2$, then (18) implies the contradiction $r \geq \lceil \frac{k'+1}{2} \rceil + 1$. If $t' = 1$, then since $t \geq 2$ we have that $2(t-t') \geq t$, and therefore, $2(t-t')\lceil \frac{k'}{2t'} \rceil \geq t\lceil \frac{k'}{2t} \rceil \geq \lceil \frac{k'}{2} \rceil$. Using this last inequality in (18) we have our final contradiction

$$2r \geq 2\left(\left\lceil \frac{k'}{2} \right\rceil + 1\right).$$

□

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A Excluding Edges

For a graph F, with vertex set V, we let $\mathcal{R}(\pi, F) \subseteq \mathcal{R}(\pi)$ be the set of all realizations whose set of edges include $E(F)$. We have $\mathcal{R}(\pi) = \mathcal{R}(\pi, \emptyset)$, and we write $\mathcal{R}(G, F)$ for $\mathcal{R}(\pi(G), F)$.

With a similar proof technique as Theorem 4 we can fix a graph F and ask when there is a $G \in \mathcal{R}(\pi, F)$ such that $G - F$ has a 1-factor.

Theorem 12. *Let $\pi = (d_1, \ldots, d_n)$ be a non-increasing positive degree sequence, and let F be a subgraph of some realization of π. For $r = \Delta(F)$, If*

$$d_{d_1 - d_n + 2r + 1} \geq d_1 - d_n + 2r, \tag{19}$$

then there exist a $G \in \mathcal{R}(\pi, F)$ such that def$(G - E(F)) \leq 1$.

Corollary 13. *Let $\pi = (d_1, \ldots, d_n)$ be a non-increasing positive degree sequence with even n. If*

$$d_{d_1-d_n+2r+1} \geq d_1 - d_n + 2r,$$

then there is some realization of π that has $r+1$ edge-disjoint 1-factors. Moreover, one of those 1-factors can be chosen to be any 1-factor of any realization of π.

Proof. Since

$$d_{d_1-d_n+2i+1} \geq d_{d_1-d_n+2r+1} \geq d_1 - d_n + 2r \geq d_1 - d_n + 2i$$

for every $i \leq r$ the Corollary follows by induction using Theorem 12 and starting with any chosen 1-factor for $i = 0$. □

With essentially the same proof technique as Theorem 6 we have the following.

Corollary 14. *Let $\pi = (d_1, \ldots, d_n)$ be a non-increasing positive degree sequence with even n such that $\mathcal{D}_{r+1}(\pi)$ is graphic. If*

$$d_{n+1-(d_1-d_n+2r+1)} \leq n - (d_1 - d_n + r),$$

then π has a realization that has $r+1$ edge-disjoint 1-factors. Moreover, one of those 1-factors can be chosen to be any 1-factor of any realization of π.

A.1 Proof of Theorem 12

Proof. We first carefully chose a realization of $\mathcal{R}(\pi, F)$.

(C1) We choose a $G \in \mathcal{R}(\pi, F)$ such that $\text{def}(G - E(F))$ is minimized, and

(C2) subject to $(C1)$, we choose a maximum matching M of $G - E(F)$ that maximizes

$$\sum_{x \notin V(M)} deg_G(x).$$

Let $H = G - E(F)$, and by contradiction we assume $\text{def}(H) \geq 2$. From (19) we have $d_n \geq 2r$, and thus, $n \geq d_1+1 \geq d_1-d_n+2r+1$. Let A,C,D be a Gallai-Edmonds Decomposition of H with $H_1, \ldots, H_k$ being the components of $H[D]$. Let $Z = \{z_1, \ldots, z_{\text{def}(H)}\}$ be the vertices in D not in M such that $deg(z_1) \geq \ldots \geq deg(z_{\text{def}(H)})$. By (II) we may assume without loss of generality that $z_i \in V(H_i)$ for $i \leq \text{def}(H)$.

Let $D' \subseteq D$ denote the largest set of vertices such for any $u \in D'$ there exists some maximum matching, denoted by M_u, that avoids both u and z_1. Since M avoids both z_1 and z_2 we have $z_2 \in D'$. Let $u \in V(H_i) \cap D'$ and $v \in V(H_i)$. Since H_i is factor-critical there is a near-perfect matching M' of H_i that avoids v. Since M_u restricts to a near-perfect matching of H_i we can construct the maximum matching $M_u - E(M_u[H_i]) + E(M')$ of H to show that $v \in D'$. Thus, D' is the vertex union of a set S of components of D. We let $A' = N_H(D') \cap A$.

We now show $e_H(H_i, A) \geq d_n - r$ for all $H_i \in S$. We choose an arbitrary $H_i \in S$, and let $T_i \subseteq V(H_i)$ denote the set of vertices adjacent in H to the fewest vertices in A. For $t \geq 1$, if each vertex in T_i is adjacent in H to t vertices in A, then for any $u \in T_i$ we have $|V(H_i)| \geq deg_G(u) - r - t + 1 \geq d_n - r - t + 1$, and therefore,

$$e_H(V(H_i), A) \geq t|V(H_i)| \geq t(d_n - r - t + 1) \geq d_n - r.$$

We now consider the case $e_H(T_i, A) = 0$. If some $u \in V(H_i)$ is not adjacent to some $v \in N_H(z_1)$, then z_1 must be adjacent in F to every vertex in $N_{H_i}(u)$. Otherwise, we can use some $x \in N_{H_i}(u)$ not adjacent to z_1 to exchange the edges ux and z_1v with the non-edges uv and z_1x to create a realization and matching $M_x + \{z_1x\}$ that violates (C1). Let $u \in T_i$. If u is adjacent in F to z_1, then then u can not be adjacent in F to some vertex $v \in N_H(z_1)$ since $|N_H(z_1)| \geq deg(z_1) - r \geq r$. However, this is a contradiction since z_1 would be adjacent in F to at least $r+1$ vertices in $N_{H_i}[u]$. Thus, every vertex in $N_{H_i}(u)$ is adjacent to every vertex in $N_H(z_1)$. If u is also adjacent to every vertex in $N_H(z_1)$, then since $|N_{H_i}[u]| \geq d_n - r + 1 \geq r + 1$ every vertex in $N_H(z_1)$ would be adjacent in H to at least one vertex in $N_{H_i}[u]$. Thus, $e_H(V(H_i), A) \geq |N_H(z_1)| \geq d_n - r$. If u is not adjacent to some $v \in N_H(Z_1)$, then every vertex in $N_{H_i}(u)$ is adjacent to every vertex in $N_H[z_1]$. Since $|N_H[z_1]| \geq d_n - r + 1 \geq r + 1$ every vertex in $N_{H_i}(u)$ is adjacent in H to some vertex in $A \cap N_H(z_1)$. Thus, $e_H(V(H_i), N_H(z_1) \cap A) \geq |N_{H_i}(u)| \geq d_n - r$.

For $u \in D'$, if there is a $w \in N_H(u) \cap A'$ and $\overline{w}_{M_u} \notin D'$, then we have a contradiction since $M_u - \{w\overline{w}_{M_u}\} + \{wu\}$ would be a matching that misses z_1 and $\overline{w}_{M_u}$. Thus, no such w exists and $\overline{A'}_M \subset D'$. Therefore, $|S| \geq |\overline{A'}_M| + |\{H_2\}| > |A'|$ since $H_2 \in S$.

Since

$$e_H(D', A') \geq \sum_{H_i \in S} e_H(H_i, A') \geq |S|(d_n - r) > |A'|(d_n - r)$$

we have by the pigeon hole principle that some vertex $s \in A'$ is adjacent to at least $d_n - r + 1$ vertices in D'.

Suppose $deg(z_1) < d_{d_1 - d_n + 2r + 1}$, and let Q be the set of vertices in G with degree at least $d_{d_1 - d_n + 2r + 1}$. For every vertex in D we know there is a matching that avoids it. Therefore, $D \cap Q = \emptyset$ by (C2). Thus, s is adjacent to at most $d_1 - (d_n - r + 1) = d_1 - d_n + r - 1$ vertices in Q. Since $|Q| \geq d_1 - d_n + 2r + 1$ and with the possibility $s \in Q$ we may conclude that s must not be adjacent to at least

$$|Q - \{s\} - N_G(s)| \geq d_1 - d_n + 2r + 1 - 1 - (d_1 - d_n + r - 1) = r + 1$$

vertices in Q. Let $P = Q - N_G(s)$, and choose some $x \in N_H(s) \cap D'$. By definition of A we have that $\overline{P}_{M_x} \subseteq D \cup C$, and therefore, by the definition of D and C we know that x is not adjacent in H to vertices in $\overline{P}_{M_x}$. Since $|\overline{P}_{M_x}| = |P| \geq d_n - r + 1 \geq r + 1$ there must be a $w \in P$ such that $\overline{w}_{M_x}$ is not adjacent to x. However, we may exchange the edges xs and $w\overline{w}_{M_x}$ for the non-edges sw and $x\overline{w}_{M_x}$ to create a realization and matching $M_x - \{w\overline{w}_{M_x}\} + \{x\overline{w}_{M_x}\}$ that violates (C2). Thus, $deg(z_1) \geq d_{d_1 - d_n + 2r + 1}$.

Since $d_n - r + 1 \geq r + 1$ we have that z_1 is not adjacent in G to some $x \in N_H(s) \cap D'$. If s is not adjacent to some $v \in N_H(z_1)$, then we may exchange the edges xs and z_1v with the non-edges sv and xz_1 to create a realization and matching $M_x + \{xz_1\}$ that violates (C1). Thus, s must be adjacent in G to every vertex in $N_H(z_1)$. However, this implies the contradiction

$$deg_G(s) > d_n - r + deg(z_1) - r \geq d_n - r + d_{d_1 - d_n + 2r+1} - r \geq d_1. \qquad \square$$