TOWARDS THE GENERALIZED RIEMANN HYPOTHESIS USING ONLY ZEROS OF THE RIEMANN ZETA FUNCTION

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ABSTRACT. For any real $\beta_0 \in [\frac{1}{2}, 1)$, let $\text{GRH}[\beta_0]$ be the assertion that for every Dirichlet character χ and all zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$, one has $\beta \leq \beta_0$ (in particular, $GRH[\frac{1}{2}]$ is the Generalized Riemann Hypothesis). In this paper, we show that the validity of $\text{GRH}[\frac{9}{10}]$ depends only on certain distributional properties of the zeros of the Riemann zeta function $\zeta(s)$. No conditions are imposed on the zeros of nonprincipal Dirichlet L-functions.

CONTENTS

1. Introduction and statement of results

The *Riemann zeta function* is a central object of study in analytic number theory. In terms of the complex parameter $s = \sigma + it$, the zeta function is defined in the half-plane $\sigma > 1$ by two equivalent expressions:

$$
\zeta(s) := \sum_{n \geq 1} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}.
$$

Riemann $\left[6\right]$ $\left[6\right]$ $\left[6\right]$ showed that $\zeta(s)$ extends analytically to a meromorphic function in the whole complex plane, its only singularity being a simple pole at $s = 1$. Moreover, the zeta function satisfies a functional equation relating its values at s and $1 - s$.¹ The *Riemann Hypothesis* (RH) asserts that if $\rho = \beta + i\gamma$ is a zero of $\zeta(s)$ with real part $\beta > 0$, then $\beta = \frac{1}{2}$ $\frac{1}{2}$.

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¹There are many excellent accounts of the theory of the Riemann zeta function; we refer the reader to Titchmarsh [[7](#page-12-1)] and to Borwein *et al* [[1](#page-11-2)] for essential background.

More generally, for a Dirichlet character $\chi \mod q$, the Dirichlet L-function $L(s, \chi)$ is defined for $\sigma > 1$ by:

$$
L(s,\chi) := \sum_{n\geq 1} \chi(n) n^{-s} = \prod_{p \text{ prime}} (1 - \chi(p) p^{-s})^{-1}.
$$

The function $L(s, \chi)$ extends to a meromorphic function (which is entire if χ is nonprincipal), and when χ is primitive it satisfies a simple functional equation relating its values at ^s and 1−s; see, e.g., Bump [[2](#page-11-3), Chapter 1]. The *Generalized Riemann Hypothesis* (GRH), which was perhaps first formulated by Piltz in 1884 (see Davenport [[3](#page-11-4)]), asserts that if $\rho = \beta + i\gamma$ is a zero of $L(s, \chi)$ with $\beta > 0$, then $\beta = \frac{1}{2}$ $\frac{1}{2}$.

For any *principal* character χ_0 mod q one has

$$
L(s, \chi_0) = \zeta(s) \prod_{p \mid q} (1 - p^{-s}),
$$

hence RH is equivalent to GRH for $L(s, \chi_0)$. On the other hand, for *nonprincipal* characters χ , no direct relationship between RH and GRH for $L(s, \chi)$ has been previously established. To establish such a connection, we study the following weak form of the GRH for Dirichlet L-functions.

HYPOTHESIS GRH[β_0]: *Given* $\beta_0 \in \left[\frac{1}{2}\right]$ $(\frac{1}{2}, 1)$ *, the inequality* $\beta \leq \beta_0$ *holds for all zeros* $\rho = \beta + i\gamma$ *of an arbitrary Dirichlet L-function L(s, x).*

Note that $\text{GRH}[\frac{1}{2}]$ is equivalent to the assertion that GRH holds for all Dirichlet L-functions. In the present paper, we show that hypothesis $\text{GRH}[\frac{9}{10}]$ can be reformulated entirely in terms of certain distributional properties of the zeros of the Riemann zeta function.

To state our results, we introduce some notation. If $T > 0$ is not the ordinate of a zero of the zeta function, then $N(T)$ is used to denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ in the rectangle $0 < \beta < 1$, $0 < \gamma < T$, and we define

$$
S(T) := \frac{1}{\pi} \arg \zeta (1/2 + iT).
$$

If $\gamma > 0$ is the ordinate of a zero, then we set

$$
N(\gamma) := \frac{1}{2} \{ N(\gamma^+) + N(\gamma^-) \}, \qquad S(\gamma) := \frac{1}{2} \{ S(\gamma^+) + S(\gamma^-) \}.
$$

Using an explicit form of the well known relation (see, e.g., Montgomery and Vaughan [[5](#page-11-5), Corollary 14.2])

$$
N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(T^{-1}) \qquad (T > 0), \tag{1.1}
$$

one sees that the difference $S(\gamma^+) - S(\gamma^-)$ is an integer for every zero $\rho = \beta + i\gamma$ of $\zeta(s)$ with $\gamma > 0$. Extending the definition of $S(T)$ appropriately, this also holds for complex zeros with $\gamma < 0$. Therefore, writing

$$
\mathbf{e}(u) := \mathrm{e}^{2\pi i u} \qquad (u \in \mathbb{R}),
$$

we can unambiguously define

$$
\mathcal{Z}(\rho) := \lim_{T \to \gamma} \overline{\mathbf{e}(S(T))}
$$
\n(1.2)

(the common value of the left and right limits) for every complex zero ρ of $\zeta(s)$.

Throughout the paper, $C_c^{\infty}(\mathbb{R}^+)$ is the space of smooth functions $f : \mathbb{R}^+ \to \mathbb{C}$ with compact support in \mathbb{R}^+ . Let μ and ϕ denote the Möbius and Euler functions, respectively.

THEOREM 1.1. *Assume RH. Suppose that for every function* $B \in C_c^{\infty}(\mathbb{R}^+)$ *and every rational number* $\xi := m/q$ *with* $0 < m < q$ *and* $(m, q) = 1$ *, the bound*

$$
\sum_{\rho=\frac{1}{2}+i\gamma} \xi^{-i\gamma} \mathcal{Z}(\rho) \mathcal{B}\left(\frac{\gamma}{2\pi X}\right) + \frac{\mu(q)}{\phi(q)} \sum_{n\geqslant 1} \Lambda(n) \mathcal{B}(n/X) \underset{\xi,\mathcal{B},\varepsilon}{\ll} X^{9/10+\varepsilon} \tag{1.3}
$$

holds for any $\varepsilon > 0$, where the sum on the left side runs over all complex zeros $\rho = \frac{1}{2} + i\gamma$ *of* $\zeta(s)$ *, the quantity* $\mathcal{Z}(\rho)$ *is given by* [\(1.2\)](#page-1-0)*, and the implied constant depends only on* ξ , B , and ε . Then, the hypothesis GRH $\left[\frac{9}{10}\right]$ is true.

We reiterate that the hypotheses of Theorem [1.1](#page-2-1) involve only properties of the zeros of the Riemann zeta function. *No conditions are imposed on the zeros of nonprincipal Dirichlet* L*-functions*.

We also have the following converse of Theorem [1.1.](#page-2-1)

THEOREM 1.2. *Assume RH. If* GRH $\left[\frac{9}{10}\right]$ *is true, then for every* $\mathcal{B} \in C_c^{\infty}(\mathbb{R}^+)$ and every rational number $\xi := m/q$ with $0 < m < q$ and $(m, q) = 1$, the bound (1.3) *holds for any* $\varepsilon > 0$.

Throughout the paper, implied constants in the symbols \ll , O , etc., often depend on various parameters (e.g., ξ , \mathcal{B} , ε) as indicated by the notation (e.g., see [\(1.3\)](#page-2-3) above); these constants are independent of all other parameters.

2. APPROACH

Using the explicit formula and assuming RH, we show that a sum of the form

$$
\sum_{n\geqslant 1} \Lambda(n) \mathbf{e}(-n\xi) \mathcal{B}(n/X) \tag{2.1}
$$

with $\xi \in \mathbb{R}^+$ and $\mathcal{B} \in C_c^{\infty}(\mathbb{R}^+)$ is equal to

$$
-\sum_{\rho=\frac{1}{2}+i\gamma} \xi^{-1/2-i\gamma} \mathcal{Z}(\rho) \mathcal{B}\left(\frac{\gamma}{2\pi\xi X}\right) \tag{2.2}
$$

up to an error of at most $O_{\xi,\mathcal{B}}(X^{9/10})$; see Theorem [4.1.](#page-8-1) To handle the integrals that arise in our use of the explicit formula, we apply the method of the stationary phase (however, some care is needed to obtain adequate and explicit estimates for the error terms); see Lemma [3.4,](#page-5-0) which is the main workhorse for the proof of Theorem [4.1.](#page-8-1)

Given the close relationship (under RH) between the sums (2.1) and (2.2) , the estimate [\(1.3\)](#page-2-3) for rational numbers $\xi := m/q$ allows us to deduce an equally strong bound

$$
\sum_{n\geqslant 1} \Lambda(n)\chi(n)\mathcal{B}(n/X) \underset{q,\mathcal{B},\varepsilon}{\ll} X^{9/10+\varepsilon}.\tag{2.3}
$$

for any primitive Dirichlet character χ modulo $q > 1$; see the proof of Theo-rem [1.1](#page-2-1) in §[5.](#page-9-0) Since $\mathcal{B} \in C_c^{\infty}(\mathbb{R}^+)$ is arbitrary, $\overline{GRH[\frac{9}{10}]}$ holds for the function $L(s, \chi)$. Conversely, under GRH $\left[\frac{9}{10}\right]$, it follows that (2.3) holds for every primitive character χ mod q. This leads to similar estimates for the sums [\(2.1\)](#page-2-4) and (2.2) ; see the proof of Theorem [1.2](#page-2-2) in §[5.](#page-9-0)

3. Integral bounds

Our aim in this section is to establish certain integral bounds that are needed in the proof of Theorem [4.1](#page-8-1) (see $\S4$).

For each $\mathcal{B} \in C_c^{\infty}(\mathbb{R}^+)$, let $X_{\mathcal{B}} \geq 10$ be a number large enough so that

$$
1 \notin X \operatorname{supp}(\mathcal{B}) := \{ Xu : u \in \operatorname{supp}(\mathcal{B}) \} \qquad (X > X_{\mathcal{B}}).
$$

The value of $X_{\mathcal{B}}$ is held fixed throughout the paper.

LEMMA 3.1. *Let* $\xi \in \mathbb{R}^+$, $\mathcal{B} \in C_c^{\infty}(\mathbb{R}^+)$, and $X > X_{\mathcal{B}}$. Then \int \mathbb{R}^+ $\Big(1 -$ 1 u^3-u $\Big)$ e(-u ξ) $\mathcal{B}(u/X)$ du $\underset{\xi,\mathcal{B}}{\ll} X^{-1}$.

Proof. Put

$$
g(u) := \left(1 - \frac{1}{u^3 - u}\right) \mathcal{B}(u/X) \qquad (u \in \mathbb{R}^+).
$$

Then g and all of its derivatives are smooth and supported on the set $X \text{supp}(\mathcal{B})$ (note that $\mathcal{B}(u/X) = 0$ for $u \in \{0, \pm 1\}$ since $\mathcal{B} \in C_c^{\infty}(\mathbb{R}^+)$ and $X > X_{\mathcal{B}}$). In particular, $g''(u) = 0$ unless $u \approx_{\mathcal{B}} X$, in which case one has $g''(u) \ll_{\mathcal{B}} X^{-2}$. Integrating by parts twice, we have (since g and g' vanish at 0 and ∞)

$$
\int_{\mathbb{R}^+} \mathbf{e}(-u\xi)g(u) du = \int_{\mathbb{R}^+} \frac{\mathbf{e}(-u\xi)}{(-2\pi i\xi)^2} g''(u) du \ll \xi^{-2} X^{-2} \operatorname{meas}(X \operatorname{supp}(\mathcal{B})),
$$

and the lemma follows. \square

The next two technical lemmas are needed in the proof of Lemma [3.4](#page-5-0) below.

LEMMA 3.2. *Fix* $u_0 \in \mathbb{R}^+$, and let $\mathcal{L}, \mathcal{D} : C_c^{\infty}(\mathbb{R}^+) \to C_c^{\infty}(\mathbb{R}^+)$ be the linear *operators defined by*

$$
\mathcal{L}F(u) := \frac{uF(u)}{(u_0 - u)} \qquad and \qquad \mathcal{D}F(u) := F'(u) \qquad (u \in \mathbb{R}^+).
$$

For any integer $k \geqslant 0$, let $(D\mathcal{L})^k$ and $\mathcal{L}(D\mathcal{L})^k$ be the linear operators given by

$$
[\mathcal{DL}]^k := \underbrace{\mathcal{D} \circ \mathcal{L} \circ \cdots \circ \mathcal{D} \circ \mathcal{L}}_{k \text{ copies each of } \mathcal{D} \text{ and } \mathcal{L}} \qquad \text{and} \qquad \mathcal{L}[\mathcal{DL}]^k := \mathcal{L} \circ [\mathcal{DL}]^k.
$$

Then for every $F \in C_c^{\infty}(\mathbb{R}^+)$ *and every integer* $k \geq 0$ *we have*

$$
[\mathcal{DL}]^{k}F(u) \underset{F,k}{\ll} \max\left\{|u-u_{0}|^{-k}, |u-u_{0}|^{-2k}\right\},\
$$

$$
\mathcal{L}[\mathcal{DL}]^{k}F(u) \underset{F,k}{\ll} \max\left\{|u-u_{0}|^{-k}, |u-u_{0}|^{-2k-1}\right\},\
$$

for all $u \in \mathbb{R}^+$ *,* $u \neq u_0$ *.*

Proof. Let $F \in C_c^{\infty}(\mathbb{R}^+)$ be fixed in what follows. For any integers $A, B \geq 0$ and a real number $C > 0$, let $\mathcal{V}(A, B, C)$ denote the set of functions $G \in C_c^{\infty}(\mathbb{R}^+)$ of the form $\sqrt{ }$

$$
G(u) = \sum_{\substack{h,i,j \geqslant 0 \\ h \leqslant A \leqslant j \\ i+j=h+B}} c_{h,i,j} \frac{u^h F^{(i)}(u)}{(u_0 - u)^j},
$$

where the sum runs over nonnegative integers h, i, j , and the coefficients $c_{h,i,j}$ are complex numbers satisfying $|c_{h,i,j}| \leq C$. If F is supported on the interval $[a, b]$, where $0 < a < b < \infty$, then the trivial bound

$$
|G(u)| \leq C'_{A,B} C \max\{b^A, 1\} \max_{0 \leq i \leq A+B} |F^{(i)}(u)| \max_{A \leq j \leq A+B} |u - u_0|^{-j} \tag{3.1}
$$

holds for all $G \in \mathcal{V}(A, B, C)$, where

$$
C'_{A,B} := \left| \{ (h, i, j) : h, i, j \ge 0, \ h \le A \le j, \ i+j=h+B \} \right| \le \sum_{h=0}^{A} \sum_{j=A}^{A+B} 1 \underset{A,B}{\ll} 1.
$$

Next, noting that

$$
\mathcal{L}G(u) = \sum_{\substack{h,i,j \geq 0 \\ h \leq A \leq j \\ i+j=h+B}} c_{h,i,j} \frac{u^{h+1} F^{(i)}(u)}{(u_0 - u)^{j+1}} = \sum_{\substack{h \geq 1, i,j \geq 0 \\ h \leq A+1 \leq j \\ i+j=h+B}} c_{h-1,i,j-1} \frac{u^h F^{(i)}(u)}{(u_0 - u)^j},
$$

it follows that

$$
\mathcal{L}: \mathcal{V}(A, B, C) \to \mathcal{V}(A+1, B, C). \tag{3.2}
$$

Similarly, we write $\mathcal{D}G(u) = G_1(u) + G_2(u) + G_3(u)$, where the functions G_j are defined as follows. First, we have

$$
G_1(u) := \sum_{\substack{h \ge 1, i, j \ge 0 \\ k \le A \le j \\ i+j=h+B}} c_{h,i,j} \frac{hu^{h-1} F^{(i)}(u)}{(u_0 - u)^j} = \sum_{\substack{h,i,j \ge 0 \\ h+1 \le A \le j \\ i+j=h+B+1}} (h+1) c_{h+1,i,j} \frac{u^h F^{(i)}(u)}{(u_0 - u)^j}.
$$

Since the coefficients $(h+1)c_{h+1,i,j}$ do not exceed AC in absolute value, it is clear that $G_1 \in \mathcal{V}(A, B+1, AC)$. Next, we have

$$
G_2(u) := \sum_{\substack{h,i,j \ge 0 \\ h \le A \le j \\ i+j=h+B}} c_{h,i,j} \frac{u^h F^{(i+1)}(u)}{(u_0 - u)^j} = \sum_{\substack{h,j \ge 0, i \ge 1 \\ h \le A \le j \\ i+j=h+B+1}} c_{h,i+1,j} \frac{u^h F^{(i)}(u)}{(u_0 - u)^j},
$$

and clearly $G_2 \in \mathcal{V}(A, B + 1, C)$. Finally, we have

$$
G_3(u) := \sum_{\substack{h,i,j \geq 0 \\ h \leq A \leq j \\ i+j=h+B}} c_{h,i,j} \frac{j u^h F^{(i)}(u)}{(u_0 - u)^{j+1}} = \sum_{\substack{h,i \geq 0, j \geq 1 \\ h \leq A \leq j-1 \\ i+j=h+B+1}} (j-1) c_{h,i,j-1} \frac{u^h F^{(i)}(u)}{(u_0 - u)^j},
$$

As the coefficients $(j-1)c_{h,i,j-1}$ do not exceed $(A+B)C$ in absolute value, we have $G_3 \in \mathcal{V}(A, B+1, (A+B)C)$. Consequently, $\mathcal{D}G \in \mathcal{V}(A, B+1, (2A+B+1)C)$, and therefore

$$
\mathcal{D}: \mathcal{V}(A, B, C) \to \mathcal{V}(A, B+1, (2A+B+1)C). \tag{3.3}
$$

Observe that F itself lies in $\mathcal{V}(0, 0, 1)$. Therefore, an inductive argument using (3.2) and (3.3) shows that for every integer $k \geq 0$ we have

$$
[\mathcal{DL}]^k F \in \mathcal{V}(k, k, k! \cdot 3^k)
$$
 and $\mathcal{L}[\mathcal{DL}]^k F \in \mathcal{V}(k+1, k, k! \cdot 3^k).$

Using (3.1) , the result follows.

LEMMA 3.3. For any positive number λ , we have

$$
\int_{\mathbb{R}} e^{-iu^2} du = \frac{e^{-i\lambda^2}}{i\lambda} + i \int_{\mathbb{R}} e^{-iu^2} u^{-2} du \ll \lambda^{-1}.
$$

$$
(|u| > \lambda)
$$

The next result is our primary tool; its proof is based on the well known stationary phase method.

LEMMA 3.4. Let $\xi \in \mathbb{R}^+$, $\mathcal{B} \in C_c^{\infty}(\mathbb{R}^+)$, and $X > X_{\mathcal{B}}$. Suppose \mathcal{B} is supported *on the interval* $[a, b]$ *, where* $0 < a < b < \infty$ *. For any real number* $\gamma \neq 0$ *, consider the integral* $\mathcal{I}(\gamma)$ *defined by*

$$
\mathcal{I}(\gamma) := \int_{\mathbb{R}^+} \mathbf{e}(-u\xi) \mathcal{B}(u/X) u^{-1/2 + i\gamma} du.
$$

Put

$$
a_{\star} := \frac{1 + 2a - \sqrt{1 + 4a}}{2}, \qquad b_{\star} := \frac{1 + 2b + \sqrt{1 + 4b}}{2}, \qquad \gamma_{\star} := \frac{\gamma}{2\pi\xi X}.
$$
 (3.4)

Then, for $\gamma_{\star} \notin [a_{\star}, b_{\star}]$ *the bound*

$$
\mathcal{I}(\gamma) \underset{\xi, \mathcal{B}}{\ll} X^{1/2} \max \left\{ X^{-2} |\gamma|^{-2}, |\gamma|^{-4} \right\}
$$

holds, whereas for $\gamma_{\star} \in [a_{\star}, b_{\star}]$ *we have the estimate*

$$
\mathcal{I}(\gamma) = \xi^{-1/2 - i\gamma} \mathbf{e} \left(\frac{\gamma}{2\pi} \log \frac{\gamma}{2\pi e} + \frac{7}{8} \right) \mathcal{B}(\gamma_\star) + O_{\xi, \mathcal{B}}(X^{-1/10}).
$$

Proof. Making the change of variables $u \mapsto Xu$, we have

$$
\mathcal{I}(\gamma) = X^{1/2 + i\gamma} \int_{\mathbb{R}^+} e^{iX f(u)} g(u) \, du = X^{1/2 + i\gamma} \mathcal{J} \quad \text{(say)},\tag{3.5}
$$

where

$$
f(u) := -2\pi \xi u + \frac{\gamma \log u}{X}, \qquad g(u) := \frac{\mathcal{B}(u)}{u^{1/2}}.
$$

Note that $\gamma_{\star} = \gamma/(2\pi \xi X)$ is the only real number for which $f'(\gamma_{\star}) = 0$. For any given $\Delta>0,$ we write $\mathcal{J}=\mathcal{J}_\infty+\mathcal{J}_\star$ with

$$
\mathcal{J}_{\infty} := \int_{\mathbb{R}^+ \atop (|u - \gamma_{\star}| > \Delta)} e^{iX f(u)} g(u) \, du, \qquad \mathcal{J}_{\star} := \int_{\mathbb{R}^+ \atop (|u - \gamma_{\star}| \leq \Delta)} e^{iX f(u)} g(u) \, du.
$$

We study \mathcal{J}_{∞} first. Let $\mathcal L$ and $\mathcal D$ be the operators defined in Lemma [3.2](#page-3-1) with $u_0 := \gamma_\star$, and put

$$
g_k := [\mathcal{DL}]^k g
$$
 and $\tilde{g}_k := \mathcal{L}[\mathcal{DL}]^k g$ $(k \ge 0).$

According to Lemma [3.2,](#page-3-1)

$$
g_k(u) \underset{\mathcal{B},k}{\ll} \max\left\{ |u - \gamma_\star|^{-k}, |u - \gamma_\star|^{-2k} \right\} \qquad (u \neq \gamma_\star). \tag{3.6}
$$

Taking into account that $f'(u) = 2\pi \xi u^{-1}(\gamma_{\star} - u)$, we have

$$
\mathcal{J}_{\infty} = \frac{1}{2\pi\xi} \int_{\mathbb{R}^+ \atop (|u - \gamma_{\star}| > \Delta)} e^{iXf(u)} f'(u) \cdot \tilde{g}_0(u) \, du = -\frac{1}{2\pi i\xi X} \int_{\mathbb{R}^+} e^{iXf(u)} g_1(u) \, du,
$$

where we have used integration by parts in the last step. Similarly, by induction on k , we see that

$$
\mathcal{J}_{\infty} = \frac{1}{(-2\pi i \xi X)^k} \int_{\mathbb{R}^+ \atop (|u - \gamma_*| > \Delta)} e^{iX f(u)} g_k(u) \, du. \tag{3.7}
$$

The numbers a_{\star} and b_{\star} defined in [\(3.4\)](#page-5-1) have the property that if $\gamma_{\star} \notin [a_{\star}, b_{\star}],$ then $|u - \gamma_{\star}| > |\gamma_{\star}|^{1/2}$ for all $u \in [a, b]$. Hence, choosing $\Delta := |\gamma_{\star}|^{1/2}$ in the case that $\gamma_{\star} \notin [a_{\star}, b_{\star}],$ it follows that $\mathcal{J}_{\infty} = \mathcal{J}$ and $\mathcal{J}_{\star} = 0$. Setting $k := 4$ and combining (3.6) and (3.7) , we derive the bound

$$
\mathcal{J} \ll (\xi X)^{-4} \max \left\{ |\gamma_{\star}|^{-2}, |\gamma_{\star}|^{-4} \right\} \ll \max \left\{ (\xi X)^{-2} |\gamma|^{-2}, |\gamma|^{-4} \right\}
$$

In view of (3.5) , we obtain the first statement of the lemma.

From now on, we assume that $\gamma_{\star} \in [a_{\star}, b_{\star}]$. We further assume that X is large enough (depending on B) so that

$$
\Delta := X^{-2/5} < \min\{1, 2^{-1/2} a_\star\}. \tag{3.8}
$$

Write $\mathcal{J} = \mathcal{J}_{\infty} + \mathcal{J}_{\star}$ as before with this Δ . Setting $k := 5$ (say) and combining (3.6) and (3.7) , we derive the bound

$$
\mathcal{J}_{\infty} \underset{\xi, \mathcal{B}}{\ll} X^{-1}.\tag{3.9}
$$

Turning to the estimate of \mathcal{J}_{\star} , observe that the upper bound [\(3.8\)](#page-6-2) implies that the interval $\Omega_{\Delta} := [\gamma_{\star} - \Delta, \gamma_{\star} + \Delta]$ lies entirely inside \mathbb{R}^+ ; in particular,

$$
\mathcal{J}_{\star} := \int_{\Omega_{\Delta}} e^{iXf(u)} g(u) \, du.
$$

Since $\gamma_{\star} \in [a_{\star}, b_{\star}]$ (and thus, $\gamma_{\star} \approx_{\mathcal{B}} 1$), the estimate

$$
f'''(u) = \frac{4\pi\xi\gamma_\star}{u^3} \underset{\mathcal{B}}{\simeq} \xi
$$

holds uniformly for all $u \in \Omega_{\Delta}$, hence by Taylor's approximation we have

$$
e^{iXf(u)} = e^{iX\{f(\gamma_\star) + \frac{1}{2}f''(\gamma_\star)(u - \gamma_\star)^2\}}(1 + r_1(u))
$$

for some complex function r_1 such that

$$
r_1(u) \underset{\xi, \mathcal{B}}{\ll} X|u - \gamma_\star|^3 \qquad (u \in \Omega_\Delta).
$$

We can also write

$$
g(u) = g(\gamma_\star) + r_2(u),
$$

where r_2 satisfies the bound

$$
r_2(u) \underset{\xi, \mathcal{B}}{\ll} |u - \gamma_\star| \qquad (u \in \Omega_\Delta).
$$

Therefore, since

$$
\int_{\Omega_{\Delta}} |r_1(u)| du \underset{\xi, \mathcal{B}}{\ll} X \int_{\gamma_{\star} - \Delta}^{\gamma_{\star} + \Delta} |u - \gamma_{\star}|^3 du \ll X\Delta^4,
$$

$$
\int_{\Omega_{\Delta}} |r_2(u)| du \underset{\xi, \mathcal{B}}{\ll} \int_{\gamma_{\star} - \Delta}^{\gamma_{\star} + \Delta} |u - \gamma_{\star}| du \ll \Delta^2,
$$

and $\Delta^2 \ll X \Delta^4 = X^{-3/5}$ by [\(3.8\)](#page-6-2), we derive the estimate

$$
\mathcal{J}_{\star} = \int_{\Omega_{\Delta}} e^{iX \{f(\gamma_{\star}) + \frac{1}{2}f''(\gamma_{\star})(u - \gamma_{\star})^2\}} (1 + r_1(u)) (g(\gamma_{\star}) + r_2(u)) du
$$
\n
$$
= e^{-i\gamma} \gamma_{\star}^{-1/2 + i\gamma} \mathcal{B}(\gamma_{\star}) \int_{\Omega_{\Delta}} e^{-\pi i \xi X \gamma_{\star}^{-1} (u - \gamma_{\star})^2} du + O_{\xi, \mathcal{B}}(X^{-3/5}),
$$
\n(3.10)

where in the last step we used the identities

$$
Xf(\gamma_\star) = -\gamma + \gamma \log \gamma_\star, \qquad f''(\gamma_\star) = -2\pi \xi \gamma_\star^{-1}, \qquad g(\gamma_\star) := \gamma_\star^{-1/2} \mathcal{B}(\gamma_\star).
$$

Next, we extend the range of integration in the preceding integral to all of $\mathbb R$ (with an acceptable error). Consider the integral

$$
\mathcal{K} := \int_{\mathbb{R}} e^{-\pi i \xi X \gamma_{\star}^{-1} (u - \gamma_{\star})^2} du.
$$

Making the change of variables $u \mapsto cu + \gamma_{\star}$, where $c := \sqrt{\gamma_{\star}/(\pi \xi X)}$, and applying Lemma [3.3,](#page-5-3) we have

$$
\mathcal{K} = c \int_{\mathbb{R}} e^{-iu^2} du \ll c^2 \Delta^{-1} \underset{\xi, \mathcal{B}}{\ll} X^{-3/5}.
$$

Using this bound together with (3.10) , it follows that

$$
\mathcal{J}_{\star} = e^{-i\gamma} \gamma_{\star}^{-1/2 + i\gamma} \mathcal{B}(\gamma_{\star}) \int_{\mathbb{R}} e^{-\pi i \xi X \gamma_{\star}^{-1} (u - \gamma_{\star})^2} du + O_{\xi, \mathcal{B}}(X^{-3/5}).
$$

Combining the previous bound with [\(3.9\)](#page-6-3), we have

$$
\mathcal{J} = e^{-i\gamma} \gamma_\star^{-1/2 + i\gamma} \mathcal{B}(\gamma_\star) \int_{\mathbb{R}} e^{-\pi i \xi X \gamma_\star^{-1} (u - \gamma_\star)^2} du + O_{\xi, \mathbb{B}}(X^{-3/5}). \tag{3.11}
$$

The integral here can be explicitly evaluated:

$$
\int_{\mathbb{R}} e^{-\pi i \xi X \gamma_{\star}^{-1} (u - \gamma_{\star})^2} du = e^{-i\pi/4} \sqrt{\frac{\gamma_{\star}}{\xi X}}.
$$

Inserting this result into (3.11) and recalling (3.5) , after some simplification we find that

$$
\mathcal{I}(\gamma) = \xi^{-1/2 - i\gamma} \mathbf{e} \left(\frac{\gamma}{2\pi} \log \frac{\gamma}{2\pi e} - \frac{1}{8} \right) \mathcal{B}(\gamma_\star) + O_{\xi, \mathcal{B}}(X^{-1/10}),
$$

and the proof is complete.

4. Twisting the von Mangoldt function

THEOREM 4.1. *Assume RH. Let* $\xi \in \mathbb{R}^+$, $\mathcal{B} \in C_c^{\infty}(\mathbb{R}^+)$, and $X > X_{\mathcal{B}}$. Then

$$
\sum_{n\geqslant 1}\Lambda(n)\mathbf{e}(-n\xi)\mathcal{B}\left(\frac{n}{X}\right)=-\sum_{\rho=\frac{1}{2}+i\gamma}\xi^{-1/2-i\gamma}\mathcal{Z}(\rho)\mathcal{B}\left(\frac{\gamma}{2\pi\xi X}\right)+O_{\xi,\mathcal{B}}(X^{9/10}).
$$

Proof. Our goal is to estimate

$$
\sum_{n\geqslant 1} \Lambda(n) \mathbf{e}(-n\xi) \mathcal{B}\left(\frac{n}{X}\right) = \sum_{n\geqslant 1} \Lambda(n) \varphi(n),
$$

where $\varphi(u) := \mathbf{e}(-u\xi) \mathcal{B}(u/X)$. By the explicit formula (see, e.g., Iwaniec and Kowalski [[4](#page-11-6), Exercise 5, p. 109]) we have

$$
\sum_{n\geqslant 1} \Lambda(n)\varphi(n) = \int_{\mathbb{R}^+} \left(1 - \frac{1}{(u-1)u(u+1)}\right) \varphi(u) \, du - \sum_{\rho} \hat{\varphi}(\rho),\tag{4.1}
$$

where $\hat{\varphi}$ is the Mellin transform of φ given by

$$
\hat{\varphi}(s) := \int_{\mathbb{R}^+} \varphi(u) u^{s-1} \, du.
$$

Using Lemma [3.1](#page-3-2) to bound the integral in (4.1) , we get that

$$
\sum_{n\geqslant 1} \Lambda(n)\varphi(n) = -\sum_{\rho} \hat{\varphi}(\rho) + O_{\xi, \mathcal{B}}(X^{-1}). \tag{4.2}
$$

Next, for any complex zero $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$ we have

$$
\hat{\varphi}(\rho) := \int_{\mathbb{R}^+} \mathbf{e}(-u\xi) \mathcal{B}(u/X) u^{-1/2 + i\gamma} du,
$$

which is the integral $\mathcal{I}(\gamma)$ considered in Lemma [3.4.](#page-5-0) Defining $a, a_{\star}, a, b_{\star}, \gamma_{\star}$ as in Lemma [3.4,](#page-5-0) it follows that

$$
\sum_{\rho=\frac{1}{2}+i\gamma} \hat{\varphi}(\rho) = \Sigma_1 + O_{\xi,\mathcal{B}}(X^{9/10}),
$$
\n(4.3)

where

$$
\Sigma_1 := \sum_{\substack{\rho = \frac{1}{2} + i\gamma \\ \gamma_{\star} \in [a_{\star}, b_{\star}]}} \xi^{-1/2 - i\gamma} \mathbf{e} \Big(\frac{\gamma}{2\pi} \log \frac{\gamma}{2\pi e} + \frac{7}{8} \Big) \mathcal{B}(\gamma_{\star}).
$$

Note that the error term in [\(4.3\)](#page-8-3) is a consequence of the following bounds on the sums that arise naturally in our application of Lemma [3.4:](#page-5-0)

$$
\sum_{\substack{\rho=\frac{1}{2}+i\gamma \\ \gamma_{\star}\notin[a_{\star},b_{\star}]}} X^{1/2} \max\left\{X^{-2}|\gamma|^{-2},|\gamma|^{-4}\right\} \underset{\xi,\mathcal{B}}{\ll} X^{1/2}, \qquad \sum_{\substack{\rho=\frac{1}{2}+i\gamma \\ \gamma_{\star}\in[a_{\star},b_{\star}]}} X^{-1/10} \underset{\xi,\mathcal{B}}{\ll} X^{9/10}.
$$

The condition " $\gamma_{\star} \in [a_{\star}, b_{\star}]$ " in the above definition of Σ_1 is redundant (indeed, we have $a_{\star} < a < b < b_{\star}$ by [\(3.4\)](#page-5-1), hence $\mathcal{B}(\gamma_{\star}) = 0$ if $\gamma_{\star} \notin [a_{\star}, b_{\star}]$), and so we can simply write

$$
\Sigma_1 = \sum_{\rho = \frac{1}{2} + i\gamma} \xi^{-1/2 - i\gamma} \mathbf{e} \left(\frac{\gamma}{2\pi} \log \frac{\gamma}{2\pi e} + \frac{7}{8} \right) \mathcal{B} \left(\frac{\gamma}{2\pi \xi X} \right).
$$

Next, observe that [\(1.1\)](#page-1-1) implies

$$
\mathbf{e}\Big(\frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{7}{8}\Big) = \mathbf{e}\big(N(T) - S(T) + O(T^{-1})\big) = \overline{\mathbf{e}\big(S(T)\big)} + O(T^{-1})
$$

provided that $T > 0$ is not the ordinate of a zero of $\zeta(s)$ (since $N(T) \in \mathbb{Z}$). Taking the limit as $T \to \gamma$, we get that

$$
\mathbf{e}\left(\frac{\gamma}{2\pi}\log\frac{\gamma}{2\pi\mathrm{e}}+\frac{7}{8}\right)=\mathcal{Z}(\rho)+O(\gamma^{-1}),
$$

where (as in §[1\)](#page-0-0)

$$
\mathcal{Z}(\rho) := \lim_{T \to \gamma} \overline{\mathbf{e}(S(T))}.
$$

Thus, up to an acceptable error, we can replace Σ_1 in (4.3) with the quantity

$$
\Sigma_1 := \sum_{\rho = \frac{1}{2} + i\gamma} \xi^{-1/2 - i\gamma} \mathcal{Z}(\rho) \mathcal{B}\left(\frac{\gamma}{2\pi \xi X}\right).
$$

The theorem now follows by combining (4.2) and (4.3) .

5. Proofs of Theorems [1.1](#page-2-1) and [1.2](#page-2-2)

We begin by considering the GRH for individual Dirichlet L-functions. For convenience, we formulate the following hypothesis.

HYPOTHESIS GRH $[\beta_0, \chi]$: *Given* $\beta_0 \in [\frac{1}{2}]$ $(\frac{1}{2}, 1)$ *and a Dirichlet character* χ *, the inequality* $\beta \leq \beta_0$ *holds for all zeros* $\rho = \beta + i\gamma$ *of the L-function* $L(s, \chi)$ *.*

Note that $GRH[\beta_0]$ is true if and only if $GRH[\beta_0, \chi]$ holds for all characters χ .

LEMMA 5.1. *Fix* $\beta_0 \in \left[\frac{1}{2}\right]$ $\frac{1}{2}$, 1), and let χ be a nonprincipal character of modulus q. *Then the following are equivalent:*

- (*i*) *Hypothesis* GRH[β_0, χ] *is true;*
- (*ii*) For any function $B \in C_c^{\infty}(\mathbb{R}^+)$, we have

$$
\sum_{n\geqslant 1}\Lambda(n)\chi(n)\mathcal{B}(n/X)\underset{q,\mathcal{B},\varepsilon}{\ll}X^{\beta_0+\varepsilon}.
$$

Proof. Consider the intermediate bound

$$
\sum_{n \leq X} \Lambda(n) \chi(n) \ll X^{\beta_0 + \varepsilon}.
$$
\n(5.1)

The equivalence $(i) \iff (5.1)$ $(i) \iff (5.1)$ is standard and well known, and the implication $(5.1) \Longrightarrow (ii)$ $(5.1) \Longrightarrow (ii)$ is immediate using partial summation. To prove $(ii) \Longrightarrow (5.1)$ one uses a <u>fixed</u> test function $\mathcal{B} \in C_c^{\infty}(\mathbb{R}^+)$ such that $0 \le \mathcal{B}(u) \le 1$ for all $u \in \mathbb{R}$, and $\mathcal{B}(u) = 1$ for all $u \in \left[\frac{1}{2}\right]$ $\frac{1}{2}$, 1]. By (ii) we have

$$
\sum_{X/2 < n \leqslant X} \Lambda(n)\chi(n) = \sum_{X/2 < n \leqslant X} \Lambda(n)\chi(n)\mathcal{B}(n/X) \leqslant \sum_{n \geqslant 1} \Lambda(n)\chi(n)\mathcal{B}(n/X) \lll X^{\beta_0 + \varepsilon},
$$

and then (5.1) follows by a standard dyadic argument.

LEMMA 5.2. *Fix* $\beta_0 \in \left[\frac{1}{2}\right]$ $\frac{1}{2}$, 1)*.* Let $\xi := m/q$ with $0 < m < q$ and $(m, q) = 1$. If $GRH[\beta_0, \chi]$ *holds for all nonprincipal characters* χ mod q, then for any $\varepsilon > 0$ we *have*

$$
\sum_{n\geqslant 1} \Lambda(n) \mathbf{e}(-nm/q) \mathcal{B}(n/X) = \frac{\mu(q)}{\phi(q)} \sum_{n\geqslant 1} \Lambda(n) \mathcal{B}(n/X) + O_{\xi, \mathcal{B}, \varepsilon}(X^{\beta_0 + \varepsilon}) \tag{5.2}
$$

Proof. Introducing an error of at most $O_{\mathcal{B}}(X^{1/2})$, the sum on the left side of [\(5.2\)](#page-10-0) can be replaced by

$$
\sum_{(n,q)=1} \Lambda(n) \mathbf{e}(-nm/q) \mathcal{B}(n/X) = \sum_{(a,q)=1} \mathbf{e}(-am/q) \sum_{n \equiv a \bmod q} \Lambda(n) \mathcal{B}(n/X)
$$

$$
= \frac{1}{\phi(q)} \sum_{(a,q)=1} \mathbf{e}(-am/q) \sum_{\chi \bmod q} \overline{\chi}(a) \sum_{n \ge 1} \Lambda(n) \chi(n) \mathcal{B}(n/X).
$$

The contribution from the principal character χ_0 is

$$
\frac{1}{\phi(q)} \sum_{(a,q)=1} \mathbf{e}(-am/q) \sum_{n\geq 1} \Lambda(n) \mathcal{B}(n/X) = \frac{\mu(q)}{\phi(q)} \sum_{n\geq 1} \Lambda(n) \mathcal{B}(n/X)
$$

(the first sum is a Ramanujan sum), and by Lemma 5.1 (ii) the total contribution from all nonprincipal characters is at most $O_{q,\mathfrak{B},\varepsilon}(X^{\beta_0+\varepsilon})$.). $\qquad \qquad \Box$

We are now ready to prove Theorems [1.1](#page-2-1) and [1.2.](#page-2-2)

Proof of Theorem [1.1.](#page-2-1) Assume RH. Since RH is equivalent to GRH in the case of principal characters, we can assume χ is nonprincipal. Moreover, if χ is induced from a primitive character χ_{\star} of conductor q_{*} , then $L(s, \chi)$ and $L(s, \chi_{\star})$ have the same zeros in the critical strip since

$$
L(s, \chi) = L(s, \chi_*) \prod_{\substack{p \mid q \\ p \nmid q_*}} (1 - \chi_*(p) p^{-s}).
$$

Hence, to prove Theorem [1.1](#page-2-1) it suffices to show that $\text{GRH}[\frac{9}{10}, \chi]$ holds for any Dirichlet L-function attached to a *primitive* Dirichlet character χ of modulus $q > 1$. In this situation, the following identity is well known:

$$
\chi(n) = \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi}(m) \mathbf{e}(-nm/q) \qquad (n \in \mathbb{Z}), \tag{5.3}
$$

where the sum runs over any complete set of residue classes m mod q, and $\tau(\chi)$ is the Gauss sum given by

$$
\tau(\chi) := \sum_{n \bmod q} \chi(n) \mathbf{e}(n/q);
$$

see, e.g., Bump [[2](#page-11-3), Chapter 1]. In particular, for every $\mathcal{B} \in C_c^{\infty}(\mathbb{R}^+)$ we have

$$
\sum_{n\geqslant 1} \Lambda(n)\chi(n)\mathcal{B}(n/X) = \frac{\chi(-1)\tau(\chi)}{q} \sum_{\substack{0 < m < q \\ (m,q)=1}} \overline{\chi}(m) \sum_{n\geqslant 1} \Lambda(n) \mathbf{e}(-nm/q) \mathcal{B}(n/X). \tag{5.4}
$$

12 W. BANKS

Combining [\(1.3\)](#page-2-3) and Theorem [4.1,](#page-8-1) the estimate

$$
\sum_{n\geqslant 1} \Lambda(n) \mathbf{e}(-nm/q) \mathcal{B}\left(\frac{n}{X}\right) = -F_{q,\mathcal{B}}(X) + O_{q,\mathcal{B},\varepsilon}(X^{9/10+\varepsilon})\tag{5.5}
$$

holds uniformly for $0 < m < q$ with $(m, q) = 1$, where

$$
F_{q,\mathfrak{B}}(X) := \frac{\mu(q)}{\phi(q)} \sum_{n \geqslant 1} \Lambda(n) \mathfrak{B}(n/X).
$$

We insert [\(5.5\)](#page-11-7) into [\(5.4\)](#page-10-1). Since $F_{q,\mathcal{B}}(X)$ is independent of m, and

$$
\sum_{\substack{0 < m < q \\ (m,q)=1}} \overline{\chi}(m) = 0
$$

for the nonprincipal character χ , we derive the bound

$$
\sum_{n\geqslant 1} \Lambda(n)\chi(n)\mathcal{B}(n/X) \underset{q,\mathcal{B},\varepsilon}{\ll} X^{9/10+\varepsilon}.
$$

Since $\mathcal{B} \in C_c^{\infty}(\mathbb{R}^+)$ is arbitrary, Lemma [5.1](#page-9-2) shows that $\text{GRH}[\frac{9}{10}, \chi]$ is true, and we are done.

Proof of Theorem [1.2.](#page-2-2) Assume RH and GRH $\left[\frac{9}{10}\right]$. For any $\mathcal{B} \in C_c^{\infty}(\mathbb{R}^+)$ and $\xi := m/q$ with $0 < m < q$ and $(m, q) = 1$, we have by Theorem [4.1:](#page-8-1)

$$
\sum_{n\geqslant 1} \Lambda(n) \mathbf{e}(-n\xi) \mathcal{B}\left(\frac{n}{X}\right) = -\sum_{\rho=\frac{1}{2}+i\gamma} \xi^{-1/2-i\gamma} \mathcal{Z}(\rho) \mathcal{B}\left(\frac{\gamma}{2\pi\xi X}\right) + O_{\xi,\mathcal{B}}(X^{9/10}),
$$

and by Lemma [5.2:](#page-10-2)

$$
\sum_{n\geqslant 1} \Lambda(n) \mathbf{e}(-n\xi) \mathcal{B}(n/X) = \frac{\mu(q)}{\phi(q)} \sum_{n\geqslant 1} \Lambda(n) \mathcal{B}(n/X) + O_{\xi, \mathcal{B}, \varepsilon}(X^{9/10+\varepsilon}).
$$

Combining these results, we obtain [\(1.3\)](#page-2-3).

$$
\qquad \qquad \Box
$$

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