A NOTE ON QUANTUM ODOMETERS

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Abstract. We discuss various aspects of noncommutative geometry of smooth subalgebras of Bunce-Deddens-Toeplitz Algebras.

1. Introduction

In noncommutative geometry it is often necessary to consider dense $*$-subalgebras of C*-algebras, in particular, in connection with cyclic cohomology or with the study of unbounded derivations on C*-algebras [5]. Smooth subalgebras of noncommutative spaces are also naturally present in studying spectral triples. If C*-algebras are thought of as generalizations of topological spaces, then dense subalgebras may be regarded as specifying additional structures on the underlying space, like a smooth structure. In analogy with the algebras of smooth functions on a compact manifold, such a smooth subalgebra should be closed under holomorphic functional calculus of all elements and under smooth-functional calculus of self-adjoint elements. It should also be complete with respect to a locally convex algebra topology, see [1].

The purpose of this note is to study smooth subalgebras A_S^∞ of Bunce-Deddens-Toeplitz C*-algebras A_S associated to a supernatural number S, objects that capture their smooth structure. This work is a continuation of, and heavily relies on, our previous papers on the subject of smooth subalgebras, in particular [7], [8] which investigated smooth structures on Bunce-Deddens algebras, the algebras of compact operators, and the Toeplitz algebra.

Bunce-Deddens algebras B_S [3], [4], are crossed-product C*-algebras obtained from odometers and Bunce-Deddence-Toeplitz algebras A_S are their extensions by compact operators $\mathcal{K}$:

$$0 \to \mathcal{K} \to A_S \to B_S \to 0.$$

Due to the topology of odometers [6], which are Cantor sets with a minimal action of a homeomorphism, the smooth subalgebras are naturally equipped with inductive limit Frechet (LF) topology.

Using a version of the Toeplitz map [9], we build smooth subalgebras A_S^∞ from Toeplitz operators with smooth symbols and from smooth compact operators. Smooth compact operators, introduced in [11], were studied in details in [8]. Smooth Bunce-Deddens algebras B_S^∞, the symbols of Toeplitz operators, were introduced in [8]. We explicitly construct appropriate LF structures on A_S^∞ and prove that those algebras are closed under holomorphic calculus so that they have the same K-Theory as their corresponding C*-algebra closures, and we verify that they are closed under smooth functional calculus of self-adjoint elements.

We also focus on describing continuous derivations [14] on smooth subalgebras A_S^∞. In particular, using results from [7], [8], we classify derivations on A_S^∞ and show that, up to

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inner derivations with compact range, they are lifts of derivations on B_S^∞, the factor algebra of A_S^∞ modulo the ideal $\mathcal{K}^\infty$ of smooth compact operators. Since many derivations on B_S^∞ are themselves inner, the factor space of continuous inner derivations on A_S^∞ modulo inner derivations turns out to be one-dimensional. Additionally we shortly describe K-Theory and K-Homology of A_S.

The paper is organized as follows. Preliminary section 2 contains our notation and a short review of relevant results from [9] and [7]. In section 3 we review smooth compact operators and introduce and study smooth Bunce-Deddens-Toeplitz. Section 4 contains a detailed discussion of stability of A_S^∞ under both the holomorphic functional calculus, and the smooth calculus of self-adjoint elements. In sections 5 we investigate the structure and classifications of derivations. Finally, section 6 contains remarks on K-Theory and K-Homology.

2. Preliminaries

2.1. **Supernatural Numbers.** A *supernatural number* S is defined as the formal product:

$$S = \prod_{p-\text{prime}} p^{\varepsilon_p}, \quad \varepsilon_p \in \{0, 1, \cdots, \infty\}.$$

We will assume $\sum \varepsilon_p = \infty$ so that S an infinite supernatural number. We define S-adic ring:

$$\mathbb{Z}/S\mathbb{Z} = \prod_{p-\text{prime}} \mathbb{Z}/p^{\varepsilon_p}\mathbb{Z}.$$

Here if $S = p^\infty$ for a prime p, then $\mathbb{Z}/S\mathbb{Z}$ is equal to $\mathbb{Z}_p$, the ring of p-adic integers.

If the ring $\mathbb{Z}/S\mathbb{Z}$ is equipped with the Tychonoff topology it forms a compact, Abelian topological ring with unity, though only the group structure is relevant for this paper. In addition, if S is an infinite supernatural number then $\mathbb{Z}/S\mathbb{Z}$ is a Cantor set.

The ring $\mathbb{Z}/S\mathbb{Z}$ contains a dense copy of $\mathbb{Z}$ by the following indentification:

$$\mathbb{Z} \ni k \leftrightarrow \{k \pmod{p^{\varepsilon_p}}\} \in \prod_{p-\text{prime}} \mathbb{Z}/p^{\varepsilon_p}\mathbb{Z}. \tag{2.1}$$

2.2. **Hilbert Spaces.** We use two concrete Hilbert spaces for this paper: $H = \ell^2(\mathbb{Z})$ and $H_+ = \ell^2(\mathbb{Z}_{\geq 0})$ Let $\{E_l\}_{l\in\mathbb{Z}}$ and $\{E_k^+ : k \geq 0\}$ be the canonical bases for H and H_+ respectively. We need the following shift operator $V : H \to H$ on H and the unilateral shift operator $U : H_+ \to H_+$ on H_+:

$$VE_l = E_{l+1} \text{ and } UE_k^+ = E_{k+1}^+.$$

Notice that V is a unitary while U is an isometry. We have:

$$[U^*, U] = P_0,$$

where P_0 is the orthogonal projection onto the one-dimensional subspace spanned by E_0^+.

For a continuous function $f \in C(\mathbb{Z}/S\mathbb{Z})$ we define two operators $m_f : H \to H$ and $M_f : H_+ \to H_+$ via formulas:

$$m_f E_l = f(l)E_l \text{ and } M_f E_k^+ = f(k)E_k^+.$$

In those formulas we considered integers k, l as elements of $\mathbb{Z}/S\mathbb{Z}$ using identification (2.1). Since $\mathbb{Z}$ is a dense subgroup of $\mathbb{Z}/S\mathbb{Z}$ we obtain immediately that

$$\|m_f\| = \|M_f\| = \sup_{l\in\mathbb{Z}} |f(l)| = \sup_{k\in\mathbb{Z}_{\geq 0}} |f(k)| = \sup_{x\in\mathbb{Z}/S\mathbb{Z}} |f(x)| = \|f\|_\infty .$$

The algebras of operators generated by the m_f's or by the M_f's are thus isomorphic to $C(\mathbb{Z}/S\mathbb{Z})$ and so they carry all the information about the space $\mathbb{Z}/S\mathbb{Z}$, while operators U and V reflect the odometer dynamics φ on $\mathbb{Z}/S\mathbb{Z}$ given by:

$$\varphi(x) = x + 1. \tag{2.2}$$

The relation between those operators is:

$$V^{-1} m_f V = m_{f\circ\varphi}. \tag{2.3}$$

Similarly we have:

$$M_f U = U M_{f\circ\varphi}. \tag{2.4}$$

There is also another, less obvious relation between U and the M_f's, namely:

$$M_f P_0 = P_0 M_f = f(0) P_0. \tag{2.5}$$

2.3. **Algebras.** Following [9], we define the Bunce-Deddens and Bunce-Deddens-Toeplitz algebras, B_S and A_S respectively, to be the following C*-algebras: B_S is generated by the operators V and m_f:

$$B_S = C^*\{V, m_f : f \in C(\mathbb{Z}/S\mathbb{Z})\}$$

while A_S is generated by the operators U and M_f:

$$A_S = C^*\{U, M_f : f \in C(\mathbb{Z}/S\mathbb{Z})\}.$$

The algebra A_S contains the projection P_0 and in fact all compact operators $\mathcal{K}$ and the quotient $A_S/\mathcal{K}$ can be naturally identified with B_S, see [7]. Let τ be the natural homomorphism $\tau : A_S \to B_S$.

The algebra B_S is isomorphic with the crossed product algebra:

$$B_S \cong C(\mathbb{Z}/S\mathbb{Z}) \rtimes_\varphi \mathbb{Z}\,.$$

and is simple [7]. Consequently it is isomorphic the universal C*-algebra with generators v and f, where v is unitary, $f \in C(\mathbb{Z}/S\mathbb{Z})$, with relations (compare with (2.3)):

$$v^{-1} f v = f \circ \varphi.$$

Interestingly, algebras A_S can also be described in terms of generators and relations as follows.

Proposition 2.1. *The universal C*-algebra A with generators u and f, such that u is an isometry, $f \in C(\mathbb{Z}/S\mathbb{Z})$, with relations (compare with* (2.3) *and* (2.5)*):*

$$f u = u\,(f \circ \varphi) \quad \textit{and} \quad f p_0 = f(0) p_0,$$

where $[u^*, u] = p_0$, is isomorphic with A_S.

Proof. We will show that any irreducible representation of A either factors through B_S or is isomorphic to the defining representation of A_S. Since $B_S \cong A_S/\mathcal{K}$ is a factor algebra, the defining representation of A_S dominates the factor representation and so, by universality, A is isomorphic to A_S.

Consider an irreducible representation of A and let U represents u and M_f represent f. Notice that $P_0 := I - UU^*$ is the orthogonal projection onto the kernel of U^*. If that kernel is zero then U is unitary and U, M_f give a representation of B_S by universality, since they satisfy the crossed-product relations.

If the kernel of U^* is not zero, pick a unit vector E_0^+ such that $U^*E_0^+ = 0$. Since U is an isometry, the set $\{E_k^+\}$, $k = 0, 1, \ldots$ is orthonormal, where $E_k^+ := U^k E_0^+$. Moreover, we have by using relations:

$$M_f E_0^+ = M_f P_0 E_0^+ = f(0)E_0^+,$$

and similarly:

$$M_f E_k^+ = M_f U^k E_0^+ = U^k M_{f\circ\varphi^k} E_0^+ = f(k)U^k E_0^+ = f(k)E_k^+.$$

It follows that vectors $\{E_k^+\}$ span an invariant subspace and so, by irreducibility, $\{E_k^+\}$ is an orthonormal basis. Since U is the unilateral shift in this basis, we reproduced the defining representation of A_S, finishing the proof. □

2.4. **Toeplitz Map.** Next we discuss the key relation between the two algebras A_S and B_S. Let $P_{\geq 0} : H \to H_+$ be the following map from H onto H_+ given by

$$P_{\geq 0}E_k = \begin{cases} E_k^+ & \text{if } k \geq 0 \\ 0 & \text{if } k < 0. \end{cases}$$

We also need another map $s : H_+ \to H$ given by:

$$sE_k^+ = E_k.$$

Define the map $T : B(H) \to B(H_+)$, between the spaces of bounded operators on H and H_+, in the following way: given $b \in B(H)$ we set

$$T(b) = P_{\geq 0} b s.$$

T is known as a Toeplitz map. It has the following properties [10]:

(1) $T(I_H) = I_{H_+}$.
(2) $T(bV^n) = T(b)U^n$ and $T(V^{-n}b) = (U^*)^n T(b)$ for $n \geq 0$ and all $b \in B(H)$.
(3) $T(bm_f) = T(b)M_f$ and $T(m_f b) = M_f T(b)$ for all $f \in C(\mathbb{Z}/S\mathbb{Z})$ and all $b \in B(H)$
(4) $T(b^*) = T(b)^*$ for all $b \in B(H)$.

Consequently, it follows that T is a $*$-preserving map from B_S to A_S. If τ is the natural homomorphism from A_S to B_S then we have

$$\tau T(b) = b$$

for all $b \in B_S$. It follows that for any a in A_S there is a compact operator c such that we have a decomposition:

$$a = T(b) + c, \tag{2.6}$$

where $b = \tau(a) \in B_S$. One can verify that if b is an element in B_S then $T(b)$ is compact if and only if $b = 0$. This implies the uniqueness of the above decomposition (2.6).

2.5. **Fourier Series.** There are natural one-parameter groups of automorphisms of B_S and A_S respectively. They are given by the formulas:

$$\rho_\theta^{\mathbb{L}}(b) = e^{2\pi i\theta\mathbb{L}} b e^{-2\pi i\theta\mathbb{L}} \text{ for } b \in B_S \text{ and } \rho_\theta^{\mathbb{K}}(a) = e^{2\pi i\theta\mathbb{K}} a e^{-2\pi i\theta\mathbb{K}} \text{ for } a \in A_S,$$

where $\theta \in \mathbb{R}/\mathbb{Z}$. Here we using the following diagonal label operators on H and H_+ respectively:

$$\mathbb{L}E_l = lE_l \text{ and } \mathbb{K}E_k^+ = kE_k^+.$$

We have the following relations:

$$\rho_\theta^{\mathbb{L}}(V) = e^{2\pi i\theta}V \text{ and } \rho_\theta^{\mathbb{L}}(m_f) = m_f\,.$$

Automorphisms $\rho_\theta^{\mathbb{K}}$ satisfy analogous relations and the extra relation on U^*, namely

$$\rho_\theta^{\mathbb{K}}(U^*) = e^{-2\pi i\theta}U^*\,.$$

Define $E : B_S \to C^*\{m_f : f \in C(\mathbb{Z}/S\mathbb{Z})\} \cong C(\mathbb{Z}/S\mathbb{Z})$ via

$$E(b) = \int_0^1 \rho_\theta^{\mathbb{L}}(b)\, d\theta\,.$$

It's easily checked that E is an expectation on B_S. For a $b \in B_S$ we define the *n-th Fourier coefficient* b_n by the following:

$$b_n = e(V^{-n}b) = \int_0^1 \rho_\theta(V^{-n}b)\, d\theta = \int_0^1 e^{-2\pi i n\theta}V^{-n}\rho_\theta(b)\, d\theta.$$

From this definition, it's clear that $b_n \in C^*\{M_f : f \in C(\mathbb{Z}/S\mathbb{Z})\}$ so we can write $b_n = m_{f_n}$ for some $f_n \in C(\mathbb{Z}/S\mathbb{Z})$. We define an expectation, E on A_S, in a similar fashion:

$$E : A_S \to C^*\{M_f : f \in C(\mathbb{Z}/S\mathbb{Z})\} \cong C(\mathbb{Z}/S\mathbb{Z})\,.$$

For an $a \in A_S$, its n-th Fourier coefficient a_n is also defined similarly and also $a_n = M_{f_n}$ for some $f_n \in C(\mathbb{Z}/S\mathbb{Z})$. Additionally, notice that we have the following relation with the Toeplitz map:

$$(T(b))_n = T(b_n) \text{ for all } n\,.$$

3. Smooth Subalgebras

3.1. **Smooth Compact Operators.** We begin by reviewing properties of smooth compact operators from [8]. Let $\mathcal{K}$ be the algebra of compact operators on H_+. The orthonormal basis $\{E_k^+\}_{k\geq 0}$ of H_+ determines a system of units $\{P_{ks}\}_{k,s\geq 0}$ in $\mathcal{K}$ that satisfy the following relations:

$$P_{ks}^* = P_{sk} \quad\text{and}\quad P_{ks}P_{rt} = \delta_{sr}P_{kt}\,,$$

where $\delta_{sr} = 1$ for $s = r$ and is equal to zero otherwise. The set of *smooth compact operators* with respect to $\{E_k^+\}$ is the set of operators of the form

$$c = \sum_{k,s\geq 0} c_{ks}P_{ks}\,,$$

so that the coefficients $\{c_{ks}\}_{k,s\geq 0}$ are rapidly decaying (RD). We denote the set of smooth compact operators by $\mathcal{K}^\infty$.

We now introduce norms on $\mathcal{K}^\infty$. They are constructed using the following useful derivation on $\mathcal{K}^\infty$:

$$d_{\mathbb{K}}(c) = [\mathbb{K}, c]\,.$$

Clearly $d_{\mathbb{K}}$ is linear and satisfies the the Leibniz rule as $d_{\mathbb{K}}$ is a commutator. We define $\|\cdot\|_{M,N}$ norms on $\mathcal{K}^\infty$ by the following formulas:

$$\|c\|_{M,N} = \sum_{j=0}^{M} \binom{M}{j} \|d_{\mathbb{K}}^j(c)(I+\mathbb{K})^N\|\,,$$

with $\delta_{\mathbb{K}}^0(c) := c$. The following proposition from [8] summarizes the basic properties of $\|\cdot\|_{M,N}$ norms.

Proposition 3.1. *Let a and b be bounded operators in H, then*

1. $a \in \mathcal{K}^\infty$ *if and only if* $\|a\|_{M,N} < \infty$ *for all nonnegative integers* M *and* N.
2. $\|a\|_{M+1,N} = \|a\|_{M,N} + \|d_{\mathbb{K}}(a)\|_{M,N}$.
3. $\|a\|_{M,N} \le \|a\|_{M,N+1}$.
4. $\|ab\|_{M,N} \le \|a\|_{M,0}\|b\|_{M,N} \le \|a\|_{M,N}\|b\|_{M,N}$.
5. $\|d_{\mathbb{K}}(a)\|_{M,N} \le \|a\|_{M+1,N}$.
6. $\|a^*\|_{M,N} \le \|a\|_{M+N,N}$.
7. $\mathcal{K}^\infty$ *is a complete topological vector space.*

This proposition implies that $\mathcal{K}^\infty$ is a Fréchet $*$-algebra with respect to the norms, $\|\cdot\|_{M,N}$.

3.2. **Smooth Bunce-Deddens Algebras.** Next we review smooth Bunce-Deddens algebras B_S^∞ from [7]. We need the following terminology. We say a family of locally constant functions on $\mathbb{Z}/S\mathbb{Z}$ is *Uniformly Locally Constant*, ULC, if there exists a divisor l of S such that for every f in the family we have

$$f(x+l) = f(x)$$

for all $x \in \mathbb{Z}/S\mathbb{Z}$.

We define the space of smooth elements of the Bunce-Deddens algebra, B_S^∞, to be the space of elements in B_S whose Fourier coefficients are ULC and whose norms are RD. Using Fourier series those conditions can be written as:

$$B_S^\infty = \left\{ b = \sum_{n\in\mathbb{Z}} V^n m_{f_n} : \{\|f_n\|\} \text{ is } RD,\ \text{there is an } l|S,\ V^l b V^{-l} = b \right\}.$$

It's immediate that B_S^∞ is indeed a nonempty subset of B_S and it was proved in [7] that B_S^∞ is a $*$-subalgebra of B_S.

Let $\delta_{\mathbb{L}} : B_S^\infty \to B_S^\infty$ be given by

$$\delta_{\mathbb{L}}(b) = [\mathbb{L}, b]$$

This derivation is very fundamental below. We have the following simple relations:

$$\delta_{\mathbb{L}}(v^n) = nV^n \quad\text{and}\quad \delta_{\mathbb{L}}(m_f) = 0.$$

This derivative is in particular used to define the following norms on B_S^∞ that capture the RD property of the Fourier coefficients of elements of B_S^∞. They are defined by:

$$\|b\|_P = \sum_{j=0}^{P} \binom{P}{j} \|\delta_{\mathbb{L}}^j(b)\|\,.$$

The following proposition from [7] states the basic properties the P-norms.

Proposition 3.2. *Let b_1 and b_2 be in B_S^∞, then*

1. $\|b_1\|_{P+1} = \|b_1\|_P + \|\delta_{\mathbb{L}}(b_1)\|_P$ *with* $\|b_1\|_0 := \|b_1\|$.
2. $\|b_1 b_2\|_P \le \|b_1\|_P \|b_2\|_P$.
3. $\|\delta_{\mathbb{L}}(b_1)\|_P \le \|b_1\|_{P+1}$.

It follows that we have the following useful way to describe elements in B_S^∞:

$$B_S^\infty = \{b \in B_S : \|b\|_M < \infty\,,\text{ for every } M,\text{ there is an } l|S,\ V^l b V^{-l} = b\}\,.$$

3.3. **Smooth Bunce-Deddens-Toeplitz Algebras.** Finally, following similar considerations for the Toeplitz algebra in [8], we define the smooth Bunce-Deddens-Toeplitz algebra A_S^∞ by

$$A_S^\infty = \{a = T(b) + c : b \in B_S^\infty,\ c \in \mathcal{K}^\infty\} \subseteq A_S\,.$$

Much like with the short exact sequence for A_S and B_S, these smooth subalgebras have the following related short exact sequence:

$$0 \longrightarrow \mathcal{K}^\infty \longrightarrow A_S^\infty \longrightarrow B_S^\infty \longrightarrow 0\,.$$

Thus, we can view the topology on A_S^∞, as a vector space, in the usual way:

$$A_S^\infty \cong B_S^\infty \oplus \mathcal{K}^\infty\,.$$

This gives A_S its LF topology.

The Toeplitz map $T : B_S \to A_S$ can naturally be restricted to B_S^∞ and considered as a map $T : B_S^\infty \to A_S^\infty$. In addition, the homomorphism τ can be restricted to A_S^∞ and we have a homomorphism $\tau : A_S^\infty \to B_S^\infty$.

It is easy to verify on generators that we have

$$d_{\mathbb{K}}(T(b)) = T(\delta_{\mathbb{L}}(b)).$$

As a consequence of continuity of T this formula is true for all $b \in B_S^\infty$.

It remains to verify that A_S^∞ is indeed a subalgebra of A_S. This follows from the following two propositions.

Proposition 3.3. *Let b be in B_S^∞ and c be in $\mathcal{K}^\infty$. Then $T(b)c$ and $cT(b)$ are in $\mathcal{K}^\infty$.*

Proof. Because $T(b^*) = T(b)^*$, we only need to prove $T(b)c$ is in $\mathcal{K}^\infty$. Proceeding as in [8] we prove by induction on M that we have the following estimate:

$$\|T(b)c\|_{M,N} \le \|b\|_M \|c\|_{M,N}\,. \tag{3.1}$$

The $M = 0$ case is immediate from the definition of the norms. The inductive step is:

$$\begin{aligned}\|T(b)c\|_{M+1,N} &= \|T(b)c\|_{M,N} + \|d_{\mathbb{K}}(T(b))c + T(b)d_{\mathbb{K}}(c)\|_{M,N} \le \\ &\le (\|b\|_M + \|\delta_{\mathbb{L}}(b)\|_M)\,(\|c\|_{M,N} + \|d_{\mathbb{K}}(c)\|_{M,N}) = \|b\|_{M+1}\|c\|_{M+1,N}\,.\end{aligned}$$

Notice also that, again proceeding as in [8], we can obtain the following inequality:

$$\|cT(b)\|_{M,N} \leq \|b\|_{M+N}\|c\|_{M,N}\,. \tag{3.2}$$

□

Proposition 3.4. *Let b_1 and b_2 be smooth Bunce-Deddens elements, then the following expression is a smooth compact element:*

$$T(b_1)T(b_2) - T(b_1b_2)\,.$$

Proof. We follow [8]. Let b_1 and b_2 be in B_S^∞ with the following decompositions:

$$b_1 = b_1^+ + b_1^- = \sum_{n\geq 0} V^n m_{f_n} + \sum_{n<0} m_{f_n} V^n \quad \text{and} \quad b_2 = b_2^+ + b_2^- = \sum_{n\geq 0} V^n m_{g_n} + \sum_{n<0} m_{g_n} V^n$$

where $\{\|f_n\|\}$ and $\{\|g_n\|\}$ are RD sequences and $\{f_n\}$ and $\{g_n\}$ are ULC. Since T is linear we only need to study the following differences:

$$\begin{aligned} &T(b_1^+)T(b_2^+) - T(b_1^+b_2^+), \quad T(b_1^-)T(b_2^-) - T(b_1^-b_2^-) \\ &T(b_1^-)T(b_2^+) - T(b_1^-b_2^+), \quad T(b_1^+)T(b_2^-) - T(b_1^-b_2^+)\,. \end{aligned}$$

First consider the following:

$$\begin{aligned} T(b_1^+)T(b_2^+) - T(b_1^+b_2^+) &= \sum_{m,n\geq 0} U^n M_{f_n} U^m M_{g_m} - \sum_{m,n\geq 0} T\left(V^n m f_n V^m m_{g_m}\right) \\ &= \sum_{m,n\geq 0} U^{n+m} M_{f_n\circ\varphi^m} M_{g_m} - \sum_{m,n\geq 0} T\left(V^{n+m} m_{f_n\circ\varphi^m} m_{g_m}\right) \\ &= \sum_{m,n\geq 0} U^{n+m} M_{f_n\circ\varphi^m} M_{g_m} - \sum_{m,n\geq 0} T\left(V^{n+m}\right) M_{f_n\circ\varphi^m} M_{g_m}. \end{aligned}$$

Since $T(V^{n+m}) = U^{n+m}$, so the above is zero. A similar argument can be made for $T(b_1^-)T(b_2^-) - T(b_1^-b_2^-)$. For the next difference we have

$$T(b_1^-)T(b_2^+) - T(b_1^-b_2^+) = \sum_{m\geq 0, n<0} M_{f_n}(U^*)^{-n}U^m M_{g_m} - \sum_{m\geq 0, n<0} M_{f_n} T(V^nV^m) M_{g_m}.$$

However, since $T(V^{n+m}) = (U^*)^{-n}U^m$ since $n < 0$, this difference is also zero. Finally, for the last difference, we have

$$\begin{aligned} C := T(b_1^+)T(b_2^-) - T(b_1^+b_2^-) &= T(b_1^+)\sum_{m<0} M_{g_m}(U^*)^{-m} - \sum_{m<0} T(b_1^+ m_{g_m} V^m) \\ &= \sum_{m<0}\left(T(b_1^+ m_{g_m})(U^*)^{-m} - T(b_1^+ m_{g_m} V^m)\right) \\ &= -\sum_{m<0} T(b_1^+ m_{g_m} V^m) P_{<-m} \end{aligned}$$

where we used the following formula for $m < 0$:

$$U^{-m}(U^*)^{-m} - I = -P_{<-m}\,.$$

Clearly, C is compact but we still need to prove it's smooth compact. To this end, we prove the M, N-norms of C are finite. A straightforward calculation gives:

$$d_{\mathbb{K}}^j(C) = -\sum_{m<0} d_{\mathbb{K}}^j\left(T(b_1^+ m_{g_m} V^m) P_{<-m}\right) = -\sum_{m<0} T\left(d_{\mathbb{K}}^j(b_1^+ V^m)\right) P_{<-m}$$

Next we estimate norms of C using $\|P_{<-m}\|_{0,N} = |m|^N$ to obtain:

$$\begin{aligned}\|d_{\mathbb{K}}^j(C)\|_{0,N} &\leq \sum_{m<0}\sum_{l=0}^{j}\binom{j}{l}|m|^{j-l+N}\|d_{\mathbb{K}}^l(b_1^+)\|\|g_m\| \\ &\leq \sum_{m<0}(1+|m|)^{N+j}\left(\sum_{l=0}^{j}\binom{j}{l}\|d_{\mathbb{K}}^l(b_1^+)\|\right)\|g_m\| \\ &= \sum_{m<0}\|b_1^+\|_j(1+|m|)^{N+j}\|g_m\| \leq \text{const}\|b_1^+\|_j\|b_2^-\|_{N+j+2}\,.\end{aligned}$$

Consequently, since b_1 and b_2 are in B_S^∞ we get $||C||_{M,N} < \infty$. This shows $T(b_1)T(b_2) - T(b_1b_2)$ is smooth compact. A more careful analysis following [8] yields the following estimate:

$$\|T(b_1)T(b_2) - T(b_1b_2)\|_{M,N} \leq \text{const}\|b_1\|_j\|b_2\|_{N+j+2},\,. \tag{3.3}$$

□

4. Stability of Smooth Bunce-Deddens-Toeplitz Algebra

The purpose of this section is to establish stability of A_S^∞ under both the holomorphic functional calculus, and the smooth calculus of self-adjoint elements. It is well known that showing the former automatically implies that the K-Theories of A_S^∞ and A_S coincide [2].

Proposition 4.1. *The smooth Bunce-Deddens-Toeplitz algebra A_S^∞ is closed under the holomorphic functional calculus.*

Proof. Since A_S^∞ is a complete locally convex topological vector space, it is enough to check that if $a \in A_S^\infty$ and invertible in A_S, then $a^{-1} \in A_S^\infty$. Consequently, the Cauchy integral representation finishes the proof. To this end, let $a \in A_S^\infty$ and thus $a = T(b) + c$ with $b \in B_S^\infty$ and $c \in \mathcal{K}^\infty$ and suppose a is invertible in A_S. Since τ is a homomorphism, $\tau(a) = b$ is invertible in B_S^∞. It is proved in [7] that if $b \in B_S^\infty$ and invertible, then $b^{-1} \in B_S^\infty$. Since $\mathcal{K}$ is an ideal of A_S and τT is the identity map, it follows that

$$a^{-1} = T(b^{-1}) + c'$$

for some $c' \in \mathcal{K}$. The proof will be complete if we can show that $c' \in \mathcal{K}^\infty$. Notice that

$$c' = a^{-1} - T(b^{-1}) = a^{-1}(I - aT(b^{-1}) = a^{-1}(I - T(b)T(b^{-1}) + cT(b^{-1}))\,.$$

From Propositions 3.3 and 3.4, we have that both $I - T(b)T(b^{-1})$ and $cT(b^{-1})$ are in $\mathcal{K}^\infty$. Consequently, there is a $\tilde{c} \in \mathcal{K}^\infty$ such that $c' = a^{-1}\tilde{c}$. It follows from the properties of norms on $\mathcal{K}^\infty$ that

$$\|c'\|_{0,N} \leq \|a^{-1}\|\|\tilde{c}\|_{0,N} < \infty\,. \tag{4.1}$$

Computing $\delta_{\mathbb{K}}$ on c we have

$$\delta_{\mathbb{K}}(c') = \delta_{\mathbb{K}}(a^{-1})\tilde{c}) = -a^{-1}\delta_{\mathbb{K}}(a)a^{-1}\tilde{c} + a^{-1}\delta_{\mathbb{K}}(\tilde{c})\,.$$

Similarly to the proof of Proposition 3.3, we have, inductively, for any j that

$$\delta_{\mathbb{K}}^j(b) = \sum_i a_i b_i \quad \text{finite sum},$$

with a_i bounded and b_i are smooth compact. Using this and the estimate in equation (4.1), we see that $\|c'\|_{M,N}$ is finite for all M and N. Thus $c' \in \mathcal{K}^\infty$, completing the proof. □

To prove closure under the calculus of self-adjoint elements, the approach used in [7] works in this setting as well. Hence, we need results regarding the growth of exponentials of elements of B_S^∞ and $\mathcal{K}^\infty$. For $\mathcal{K}^\infty$, the exact result needed was proved in [7]. We state it here for convenience.

Proposition 4.2. *Suppose that $c \in \mathcal{K}^\infty$ is a self-adjoint smooth compact operator. Then we have an estimate:*

$$\|e^{ic}\|_{M,0} \le \prod_{j=1}^{M} (1 + \|c\|_{j,0})^{2^{M-j}} .$$

The second result needed is a minor adaptation of Proposition 3.4 in [7].

Proposition 4.3. *If $b \in B_S^\infty$ is self-adjoint, then we have an estimate:*

$$\|e^{ib}\|_{M} \le \prod_{j=1}^{M} (1 + \|b\|_{j})^{2^{M-j}} .$$

Proof. For $M = 0$, notice that $\|e^{ib}\|_0 = 1$. We continue by induction, utilizing part (1) of Proposition 3.2:

$$\|e^{ib}\|_{M+1} = \|e^{ib}\|_M + \|\delta_{\mathbb{L}}(e^{ib})\|_M.$$

Using that

$$\delta_{\mathbb{L}}(e^{ib}) = i\int_0^1 e^{i(1-t)b}\delta_{\mathbb{L}}(b)e^{itb}\, dt\,,$$

we have the following estimate for the inductive step:

$$\|e^{ib}\|_{M+1} \le \|e^{ib}\|_M + i\int_0^1 \|e^{i(1-t)b}\|_M \|\delta_{\mathbb{L}}(b)\|_M \|e^{itb}\|_M\, dt \le$$
$$\le \prod_{j=1}^{M} (1 + \|b\|_{j})^{2^{M-j}} + \left[\prod_{j=1}^{M} (1 + \|b\|_{j})^{2^{M-j}}\right]^2 \|\delta_{\mathbb{L}}(b)\|_M\,.$$

Since $\|\delta_{\mathbb{L}}(b)\|_M \le \|b\|_{M+1}$, we have:

$$\|e^{ib}\|_{M+1} \le \prod_{j=1}^{M} (1 + \|b\|_{j})^{2^{M-j}} (1 + \prod_{j=1}^{M} (1 + \|b\|_{j})^{2^{M-j}} \|b\|_{M+1}) \le$$
$$\le \prod_{j=1}^{M} (1 + \|b\|_{j})^{2^{M-j}} \prod_{j=1}^{M} (1 + \|b\|_{j})^{2^{M-j}} (1 + \|b\|_{M+1}) = \prod_{j=1}^{M+1} (1 + \|b\|_{j})^{2^{M+1-j}} .$$

This establishes the inductive step and finishes the proof. □

Theorem 4.4. *The smooth Bunce-Deddens-Toeplitz algebra A_S^∞ is closed under the smooth functional calculus of self-adjoint elements.*

Proof. We need to prove that, given a self-adjoint element a of A_S^∞ and a smooth function $f(x)$ defined on an open neighborhood of the spectrum $\sigma(a)$ of a we have $f(a)$ is in A_S^∞. It is without loss of generality to assume that $f(x)$ is smooth on $\mathbb{R}$ and is L-periodic: $f(x+L) = f(x)$ for some L.. Then $f(x)$ admits a Fourier series representation with rapid decay coefficients $\{f_n\}$, and hence

$$f(a) = \sum_{n\in\mathbb{Z}} f_n e^{2\pi i n a/L}$$

for a self-adjoint $a = T(b) + c \in A_S^\infty$. Thus, it remains to establish at most polynomial growth in n of norms $\|e^{2\pi i n a/L}\|_{M,N}$.

Notice that $\tau\left(e^{2\pi i n a/L}\right)$ in B_S^∞ is $e^{2\pi i n b/L}$, which indeed grows at most polynomially in n, by Proposition 4.3. Thus, we only need to show that the $\|\cdot\|_{M,N}$ of the difference

$$e^{2\pi i n (T(b)+c)/L} - T\left(e^{2\pi i n b/L}\right) \in \mathcal{K}^\infty$$

are at most polynomially growing in n.

To analyze the above, we use a version of the Duhamel's formula:

$$e^{i(T(b)+c)} - T\left(e^{ib}\right) = \int_0^1 \frac{d}{dt}\left(e^{it(T(b)+c)}T\left(e^{i(1-t)b}\right)\right) dt =$$

$$= \int_0^1 e^{it(T(b)+c)} c\, T\left(e^{i(1-t)b}\right) dt + \int_0^1 e^{it(T(b)+c)} \left[T(b)T\left(e^{i(1-t)b}\right) - T\left(be^{i(1-t)b}\right)\right] dt\,.$$

Employing Proposition 3.1 we can estimate the norms as follows:

$$\|e^{i(T(b)+c)} - T\left(e^{ib}\right)\|_{M,N} \le \int_0^1 \|e^{it(T(b)+c)}\|_{M,0}\|c\,T\left(e^{i(1-t)b}\right)\|_{M,N}\, dt +$$

$$+ \int_0^1 \|e^{it(T(b)+c)}\|_{M,0}\|T(b)T\left(e^{i(1-t)b}\right) - T\left(be^{i(1-t)b}\right)\|_{M,N}\, dt\,.$$

All terms above can now be estimated using (3.2), as well as Propositions 4.2 and 4.3. We obtain the following bounds:

$$\|e^{i(T(b)+c)} - T\left(e^{ib}\right)\|_{M,N} \le \prod_{j=1}^{M}(1+\|b\|_j+\|c\|_{j,0})^{2^{M-j}}\,\|c\|_{M,N}\prod_{j=1}^{M+N}(1+\|b\|_j)^{2^{M+N-j}} +$$

$$+\,\mathrm{const}\prod_{j=1}^{M}(1+\|b\|_j+\|c\|_{j,0})^{2^{M-j}}\,\|b\|_M\prod_{j=1}^{M+N+2}(1+\|b\|_j)^{2^{M+N+2-j}}\,.$$

Clearly those estimates establish the desired at most polynomial growth, finishing the proof. □

5. Classification of Derivations

We begin with recalling the basic concepts from [9]. Let A be a complete locally compact topological algebra and let $d : A \to A$ be continuous derivation on A. Suppose that there is a continuous one-parameter family of automorphisms $\rho_\theta : A \to A$ of A, $\theta \in \mathbb{R}/\mathbb{Z}$.

Given $n \in \mathbb{Z}$, a continuous derivation $d : A \to A$ is said to be a *n-covariant derivation* if the relation

$$\rho_\theta^{-1} d\rho_\theta(a) = e^{-2\pi i n\theta} d(a)$$

holds for all θ. When $n = 0$ we say the derivation is invariant. In this definition A could be any of the following algebras: A_S^∞, B_S^∞, or $\mathcal{K}^\infty$ and with the appropriate one-parameter family of automorphisms $\rho_\theta^{\mathbb{K}}$ or $\rho_\theta^{\mathbb{L}}$. With this definition, we point out that $\delta_{\mathbb{L}} : B_S^\infty \to B_S^\infty$ is an invariant continuous derivation as is $d_{\mathbb{K}} : A_S^\infty \to A_S^\infty$ and $d_{\mathbb{K}} : \mathcal{K}^\infty \to \mathcal{K}^\infty$.

If d is a continuous derivation on A, the *n-th Fourier component* of d is defined as:

$$d_n(a) = \int_0^1 e^{2\pi i n\theta} \rho_\theta^{-1} d\rho_\theta(a)\, d\theta\, .$$

We have the following simple observation [9].

Proposition 5.1. *With the above notation the n-th Fourier component $d_n : A_S^\infty \to A_S^\infty$ is a continuous n-covariant derivation.*

To classify continuous derivations on A_S^∞ we follow the strategy from [9]. We use the classification of derivations on B_S^∞ from [7] and show how to lift derivations from B_S to A_S. We handle the remaining derivations, those with range in $\mathcal{K}^\infty$, by using the Fourier decomposition components. This is the heart of the argument and will be described next.

Let $\mathcal{A}_S \subseteq A_S^\infty$ be the subspace of A_S^∞ consisting of elements $a = T(b) + c$ such that b has only finitely many non-zero Fourier components and c has only finitely many non-zero matrix coefficients (in the standard basis). It was observed in [9] that $\mathcal{A}_S$ is a dense subalgebra of A_S. In turn, we note that it is also a dense subalgebra of A_S^∞.

Theorem 5.2. *If $d : A_S^\infty \to \mathcal{K}^\infty$ is a continuous derivation, then there is $c \in \mathcal{K}^\infty$ such that $d(a) = [c, a]$ for every $a \in A_S^\infty$. In particular, d is an inner derivation.*

Proof. Let $d : A_S^\infty \to \mathcal{K}^\infty$ be a continuous derivation. Let d_n be the nth-Fourier component of d. From Proposition 5.1, d_n are n-covariant derivations and $d_n : A_S^\infty \to \mathcal{K}^\infty$. We only consider the case $n \geq 0$ as $n < 0$ can be treated similarly. All n-covariant derivations $d_n : \mathcal{A}_S \to A_S$ were classified in [9] . Thus, we know there exists a sequence, $\{\beta_n(k)\}$, possibly unbounded in k, such that

$$d_n(a) = [U^n \beta_n(\mathbb{K}), a] \tag{5.1}$$

for any $a \in \mathcal{A}_S$. We are requiring here the range of d to belong to $\mathcal{K}^\infty$, which places restrictions on $\{\beta_n(k)\}$.

Let χ be a character on $\mathbb{Z}/S\mathbb{Z}$ and since $d_n(a) \in \mathcal{K}^\infty$ for any $a \in A_S^\infty$ we have

$$\begin{cases} d_n(U) = U^{n+1}(\beta_n(\mathbb{K} + I) - \beta_n(\mathbb{K})) := U^{n+1}\alpha_n(\mathbb{K}) \in \mathcal{K}^\infty & \text{for } n \geq 0 \\ d_n(M_\chi) = U^n \beta_n(\mathbb{K})(1 - \chi(n)) \in \mathcal{K}^\infty & \text{for } n \geq 0\, . \end{cases}$$

Since for each $n > 0$ we can choose χ such that $\chi(n) \neq 1$, and thus we have $\{\alpha_n(k)\}$ and $\{\beta_n(k)\}$ are RD in k for every $n > 0$.

For $n = 0$, the above equation only implies that $\{\alpha_0(k)\}$ is RD in k. We have the following difference equation:

$$\alpha_n(k) = \beta_n(k + 1) - \beta_n(k)\, .$$

This equation has a solution of the form

$$\beta_n(k) = -\sum_{r=k}^{\infty} \alpha_n(r)\,. \tag{5.2}$$

It follows, since $\{\alpha_0(k)\}$ is RD in k, so is $\{\beta_0(k)\}$. Thus $\{\beta_n(k)\}$ is RD for any n and the formula (5.1) extends by continuity to any $a \in A_S^\infty$.

We want to establish that $\{\beta_n(k)\}$ is RD in both n and k. Since $d_n(U) \in \mathcal{K}^\infty$ we have that $\|d_n(U)\|_{M,N}$ are finite for all M and N. So, for any N and j there exists a constant $C_{j,N}$ such that

$$\|d_{\mathbb{K}}^j(d_n(U))(I+\mathbb{K})^N\| \le C_{j,N}$$

On the other hand, consider the following calculation for $n \ge 0$:

$$d_{\mathbb{K}}^j(d_n(U)) = d_{\mathbb{K}}^j(U^{n+1}\alpha_n(\mathbb{K})) = (n+1)^j U^{n+1}\alpha_n(\mathbb{K})$$

since $\alpha_n(\mathbb{K})$ is diagonal. Therefore, we have that

$$(n+1)^j\|\alpha_n(\mathbb{K})(I+\mathbb{K})^N\| \le C_{j,N}\,.$$

However,

$$(n+1)^j\|\alpha_n(\mathbb{K})(I+\mathbb{K})^N\| = (n+1)^j \sup_k \left\{(1+k)^N|\alpha_n(k)|\right\}\,.$$

It follows that

$$(1+n)^j(1+k)^N|\alpha_n(k)| \le C_{j,N}$$

and thus $\{\alpha_n(k)\}$ is RD in both n and k. Consequently, by (5.2), $\{\beta_n(k)\}$ is RD in both n and k. Therefore

$$\begin{aligned} d(a) &= \sum_{n\in\mathbb{Z}} d_n(a) = \sum_{n\ge 0}[U^n\beta_n(\mathbb{K}), a] + \sum_{n<0}[\beta_n(\mathbb{K})(U^*)^{-n}, a] \\ &= \left[\sum_{n\ge 0} U^n\beta_n(\mathbb{K}) + \sum_{n<0}\beta_n(\mathbb{K})(U^*)^{-n}, a\right] = [c, a] \end{aligned}$$

where all the sums converge and $c \in \mathcal{K}^\infty$. Thus d is inner, completing the proof. □

To analyze general derivations $d : A_S^\infty \to A_S^\infty$ we first notice the following.

Proposition 5.3. *Let $d : A_S^\infty \to A_S^\infty$ be a continuous derivation, then $d(\mathcal{K}^\infty) \subseteq \mathcal{K}^\infty$.*

Proof. Since $\mathcal{K}^\infty$ is generated by the system of units $\{P_{ks}\}$ and since d is continuous we only need to verify that $d(P_{ks})$ is in $\mathcal{K}^\infty$. Since $P_{ks} = P_{kr}P_{rs}$, by the Leibniz rule we have that

$$d(P_{ks}) = P_{kr}d(P_{rs}) + d(P_{kr})P_{rs}\,.$$

Since the right-hand side is clearly in $\mathcal{K}^\infty$, the claim follows. □

It follows from this proposition that any continuous derivation $d : A_S^\infty \to A_S^\infty$ defines a continuous derivation on B_S^∞, which is isomorphic to the factor algebra $A_S^\infty/\mathcal{K}^\infty$. We use this observation in the proof of the following main result of this section.

Theorem 5.4. *Let $d : A_S^\infty \to A_S^\infty$ be any continuous derivation. Then there exist: a constant γ, $b \in B_S^\infty$ and $c \in \mathcal{K}^\infty$ such that:*

$$d = \gamma d_{\mathbb{K}} + [T(b) + c, \cdot].$$

Proof. Let $d : A_S^\infty \to A_S^\infty$ be a continuous derivation and define a derivation $\delta : B_S^\infty \to B_S^\infty$ by

$$\delta(a + \mathcal{K}^\infty) = d(a) + \mathcal{K}^\infty\,.$$

In other words, δ is the class of d in the factor algebra $A_S^\infty/\mathcal{K}^\infty \cong B_S^\infty$. The continuity of d implies the continuity of δ. But all continuous derivations $\delta : B_S^\infty \to B_S^\infty$ were classified in [7]. Therefore, by that paper, there exists a constant γ such that

$$\delta = \gamma\delta_{\mathbb{L}} + \tilde{\delta}$$

where $\tilde{\delta}$ is inner. Thus there exists a $b \in B_S^\infty$ such that $\tilde{\delta} = [b, \cdot]$.

Next notice that $[T(b), \cdot]$ is an inner derivation on A_S^∞ whose class in B_S^∞ is precisely $[b, \cdot]$. Define a derivation $\tilde{d} : A_S^\infty \to A_S^\infty$ by

$$\tilde{d} = d - cd_{\mathbb{K}} - [T(b), \cdot]\,.$$

Since the class of $d_{\mathbb{K}}$ in is $\delta_{\mathbb{L}}$, we have that $\tilde{d} : A_S^\infty \to \mathcal{K}^\infty$ and hence by Theorem 5.2, $\tilde{d} = [c, \cdot]$ for some $c \in \mathcal{K}^\infty$. This concludes the proof. □

6. K-Theory and K-Homology

Since $\mathcal{K}^\infty, A_S^\infty, B_S^\infty$ are closed under the holomorphic functional calculus, each inclusion induces an isomorphism in K-Theory. Using this fact, along with the 6-term exact sequence [12] induced by the short exact sequence of smooth subalgebras, we compute the K-Theory of A_S^∞. We then make use of the Universal Coefficient Theorem [13] to compute the K-Homology of A_S.

6.1. **K Theory.** Recall the short exact sequence

$$0 \longrightarrow \mathcal{K}^\infty \longrightarrow A_S^\infty \longrightarrow B_S^\infty \longrightarrow 0$$

of smooth subalgebras. This induces the following 6-term exact sequence in K-Theory:

$$\begin{array}{ccccc} K_0(\mathcal{K}^\infty) & \longrightarrow & K_0(A_S^\infty) & \xrightarrow{K_0(\tau)} & K_0(B_S^\infty) \\ {\scriptstyle \text{ind}}\uparrow & & & & \downarrow{\scriptstyle \exp} \\ K_1(B_S^\infty) & \xleftarrow[K_1(\tau)]{} & K_1(A_S^\infty) & \longleftarrow & K_1(\mathcal{K}^\infty) \end{array}$$

For details regarding the K-Theory of B_S^∞, see [7]. Since the generating unitary V in B_S^∞ lifts to the partial isometry U, it follows that

$$\text{ind}([V]_1) = [I - U^*U]_0 - [I - UU^*]_0 = -[P_{00}],$$

which generates $K_0(\mathcal{K}^\infty)$. Hence, the index map is an isomorphism. By exactness, it follows that $K_1(\tau)$ is the trivial map. Since $K_1(\mathcal{K}^\infty) = 0$, by exactness $K_1(\tau)$ is also injective, and hence $K_1(A_S^\infty) = 0$. Since exp is trivial, by exactness $K_0(\tau)$ is surjective. But again, since ind is an isomorphism, it follows that the map $K_0(\mathcal{K}^\infty) \to K_0(A_S^\infty)$ is trivial. Hence, $K_0(\tau)$ is injective as well. Using the computation done in [7], it follows that we have:

$$K_0(A_S^\infty) \cong G_S \quad \text{where} \quad G_S = \{k/l \in \mathbb{Q} : k \in \mathbb{Z}, l|S\}.$$

Let us summarize the results in the following proposition.

Proposition 6.1. *The K-Theory of A_S is given by*

$$K_0(A_S) \cong G_S \quad and \quad K_1(A_S) \cong 0.$$

6.2. **K-Homology.** The Universal Coefficient Theorem of Rosenberg and Schochet [13] states that we have two exact sequences:

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_1(A_S), \mathbb{Z}) \longrightarrow K^0(A_S) \longrightarrow \mathrm{Hom}(K_0(A_S), \mathbb{Z}) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A_S), \mathbb{Z}) \longrightarrow K^1(A_S) \longrightarrow \mathrm{Hom}(K_1(A_S), \mathbb{Z}) \longrightarrow 0,$$

where in the above, we have used the identification $KK^i(A_S, \mathbb{C}) = K^i(A_S)$. From the first sequence, it is clear that

$$\mathrm{Ext}^1_{\mathbb{Z}}(K_1(A_S), \mathbb{Z}) \cong 0.$$

In [7] it was shown that

$$\mathrm{Hom}(K_0(A_S), \mathbb{Z}) \cong 0.$$

Hence, we have $K^0(A_S) = 0$. From the second sequence, it is immediate that

$$K^1(A_S) \cong \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A_S), \mathbb{Z}) \cong K^1(B_S),$$

where the last isomorphism is derived in [7]. This group was computed in [7] to be isomorphic to $(\mathbb{Z}/S\mathbb{Z})/\mathbb{Z}$. This reference also contains an explicit description of the precise subgroup being modded out. In fact, this subgroup turns out to be the natural dense copy of $\mathbb{Z} \subseteq \mathbb{Z}/S\mathbb{Z}$. We summarize the above computations in the following proposition.

Proposition 6.2. *The K-Homology of A_S is given by*

$$K^0(A_S) \cong 0 \quad and \quad K^1(A_S) \cong (\mathbb{Z}/S\mathbb{Z})/\mathbb{Z}.$$

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