SOBOLEV REGULARITY THEORY FOR THE NON-LOCAL ELLIPTIC AND PARABOLIC EQUATIONS ON $C^{1,1}$ OPEN SETS

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Abstract. We study the zero exterior problem for the elliptic equation

$$\Delta^{\alpha/2}u - \lambda u = f, \quad x \in D\,; \quad u|_{D^c} = 0$$

as well as for the parabolic equation

$$u_t = \Delta^{\alpha/2}u + f, \quad t > 0,\, x \in D\,; \quad u(0,\cdot)|_D = u_0,\, u|_{[0,T]\times D^c} = 0.$$

Here, $\alpha \in (0,2)$, $\lambda \geq 0$ and D is a $C^{1,1}$ open set. We prove uniqueness and existence of solutions in weighted Sobolev spaces, and obtain global Sobolev and Hölder estimates of solutions and their arbitrary order derivatives. We measure the Sobolev and Hölder regularities of solutions and their arbitrary derivatives using a system of weights consisting of appropriate powers of the distance to the boundary. The range of admissible powers of the distance to the boundary is sharp.

1. Introduction

We study the elliptic equation

$$\begin{cases} \Delta^{\alpha/2}u(x) - \lambda u(x) = f(x), & x \in D, \\ u(x) = 0, & x \in D^c, \end{cases} \tag{1.1}$$

and the parabolic equation

$$\begin{cases} \partial_t u(t,x) = \Delta^{\alpha/2}u(t,x) + f(t,x), & (t,x) \in (0,T)\times D, \\ u(0,x) = u_0(x), & x \in D, \\ u(t,x) = 0, & (t,x) \in [0,T]\times D^c, \end{cases} \tag{1.2}$$

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where $\alpha \in (0,2)$ and D is either a half space or a bounded $C^{1,1}$ open set. The fractional Laplacian $\Delta^{\alpha/2}u$ is defined as

$$\Delta^{\alpha/2}u(x) := c_d \lim_{\varepsilon\downarrow 0}\int_{|y|>\varepsilon}\frac{u(x+y)-u(x)}{|y|^{d+\alpha}}dy, \qquad (c_d := \frac{2^\alpha\Gamma(\frac{d+\alpha}{2})}{\pi^{d/2}|\Gamma(-\alpha/2)|}).$$

In the probabilistic point of view, equations (1.2) and (1.1) are related to a certain pure-jump process which is forced to assume undefined or killed state when it leaves the open set D. The zero exterior condition describes that the influence of the jump process vanishes or is ignored when the process is outside of D. See Section 2 for detail. In fact, the equations are ill-posed if only zero-boundary condition is assigned.

In this article we study equations (1.1) and (1.2) in the weighted Sobolev spaces $H^\gamma_{p,\theta}(D)$ and $L_p((0,T);H^\gamma_{p,\theta}(D))$, respectively. Here $p>1$ and $\theta,\gamma\in\mathbb{R}$. For instance, if $\gamma=0,1,2,\cdots$, then

$$\|u\|_{H^\gamma_{p,\theta}(D)} = \left(\sum_{k=0}^{\gamma}\int_D |\rho^k D^k u|^p \rho^{\theta-d}dx\right)^{1/p},$$

where $\rho(x) := dist(x,\partial D)$. In general, we use a unified way to define the spaces $H^\gamma_{p,\theta}(D)$ for all $\gamma\in\mathbb{R}$. The powers of ρ are used to control the behaviors of functions near the boundary.

The main contribution of this article is to present weighted Sobolev regularity of arbitrary nonnegative real order derivatives of solutions. To be more precise, for elliptic equation (1.1) we prove for any $\gamma\geq 0$ and $p>1$,

$$\|u\|_{H^{\gamma+\alpha}_{p,\theta-\alpha p/2}(D)} \leq C\|f\|_{H^\gamma_{p,\theta+\alpha p/2}(D)} \tag{1.3}$$

provided that $\theta\in(d-1,d-1+p)$. The admissible range of θ is sharp (cf. Remark 2.5). We also prove a parabolic version of (1.3) for parabolic equation (1.2). See Theorems 2.9 and 2.10 for our full Sobolev regularity results of the elliptic and parabolic equations. In particular, if $\gamma=0$ then (1.3) implies

$$\int_D (|\rho^{-\alpha/2}u|^p + |\rho^{\alpha/2}\Delta^{\alpha/2}u|^p)\rho^{\theta-d}dx \leq C\int_D |\rho^{\alpha/2}f|^p\rho^{\theta-d}dx. \tag{1.4}$$

Note that due to the presence of $\rho^{\alpha/2}$ beside f in (1.4), the function f is allowed to blow up near the boundary of D. Indeed, it can behave like $\rho^{-\alpha/2}$ near ∂D.

We also obtain global space-time Hölder estimates of arbitrary derivatives of solutions (see Corollaries 2.16 and 2.17). One advantage of our results is that it gives Hölder estimates of solutions even when the free terms are quite rough. For instance, if $\alpha-\frac{d}{p}\geq\delta\in(0,1)$, then for the elliptic equation we prove

$$|\rho^{\frac{\theta}{p}-\frac{\alpha}{2}}u|_{C(D)} + |\rho^{\delta+\frac{\theta}{p}-\frac{\alpha}{2}}u|_{C^\delta(D)} \leq C\|\psi^{\alpha/2}f\|_{L_{p,\theta}(D)}. \tag{1.5}$$

Now we give a description on the mostly related works below. Our focus lies in the results on domains. Accordingly, regarding the results on the whole space $\mathbb{R}^d$, we only refer e.g. to [4, 6, 18, 29, 40] for Hölder estimates and [17, 28, 30, 31, 44, 45] for L_p estimates.

First, we describe Höder estimates. As for elliptic equation (1.1), it was proved in [47] that

$$f\in L_\infty(D) \quad\Longrightarrow\quad u\in C^{\alpha/2}(\mathbb{R}^d),\ \rho^{-\alpha/2}u\in C^s(D)$$

for some $s>0$. Here, $\rho(x):=dist(x,\partial D)$. Higher order estimate

$$|u|^{(-\alpha/2)}_{\beta+\alpha;D}\leq C\left(|u|_{C^{\alpha/2}(\mathbb{R}^d)}+|f|^{(\alpha/2)}_{\beta;D}\right),\qquad \beta>0$$

was also obtained in [47], where $|\cdot|^{(a)}_b$ denotes the interior Hölder norm (see e.g. [23] or [32]). The result of [47] was generalized for elliptic equations with stable-like operators in [3, 34, 49]. We also refer to [41, 50] for the local result of the type

$$f\in C^{\beta}(D)\quad\implies u\in C^{\beta+\alpha}_{loc}(D),\quad \beta>0$$

proved for non-local elliptic equations with singular kernels or general operators. Also, see [12, 35, 48] for related works on non-linear elliptic equations. Now, we discuss the results on parabolic equation (1.2). In [21], it was proved that if $u_0\in L_2(D)$ and $f\in L_\infty((0,T)\times D)$, then

$$u\in C^{1-\varepsilon,\alpha/2}_{t,x}((t_0,T)\times D),\ \rho^{-\alpha/2}u\in C^{\frac{1}{2}-\frac{\varepsilon}{\alpha},\frac{\alpha}{2}-\varepsilon}_{t,x}((t_0,T)\times D)\tag{1.6}$$

for any $\varepsilon>0$ and $t_0\in(0,T)$. Note that this result is local with respect to the time variable. For a global estimate, we refer to [53], which in particular proved

$$\sup_{t\leq T}\left(|u|^{(-\theta)}_{\alpha+\gamma;D}+|\Delta^{\alpha/2}u|^{(\alpha-\theta)}_{\gamma;D}\right)\leq C\left(|u_0|^{(-\theta)}_{\alpha+\gamma;D}+\sup_{t\leq T}|f|^{(\alpha-\theta)}_{\gamma;D}\right)\tag{1.7}$$

for any $\theta\in(0,\alpha/2)$ and $\gamma\in(0,1)$. This estimate does not give Hölder regularity with respect to the time variable. As compared to (1.6) and (1.7), our results give global Hölder regularity with respect to both time and spacial variables.

Next, we describe results in L_p spaces. The global summability results were studied e.g. in [1, 42]. For instance, for elliptic equation (1.1), the inequality

$$\|u\|_{L_{\frac{dp}{d-\alpha p}}(D)}+\|\Delta^{\alpha/4}u\|_{L_{\frac{dp}{d-\alpha p/2}}(D)}\leq C\|f\|_{L_p(D)},\quad (1<p<2d/(d+\alpha))\tag{1.8}$$

was proved in [42]. We remark that (1.8) does not cover the full regularity of solution, that is, estimate of $\Delta^{\alpha/2}u$ is not covered. There are also several interior regularity results, such as those introduced in [7, 8, 16, 46]. For instance, the results

$$f\in L_p(D)\implies\quad u\in H^{\alpha}_{p,loc}(D),\quad (1<p<\infty)$$

and

$$f\in L_{p_*}(D)\implies\quad u\in H^{\alpha/2}_{p,loc}(D)\qquad (p>2,\, p_*:=\max\{\frac{pd}{d+p\alpha/2},2\})\tag{1.9}$$

were proved in [7] and [46] respectively. Note that $p_*<p$ since $p>2$. We also refer to [10, 20, 27] for results on Hilbert spaces. We finally refer to [24, 25] for the regularity results in the μ-transmission spaces $H^{\mu(s)}_p(D)$, proved for the equations with pseudo-differential operators satisfying the μ-transmission property. In particular, it is proved that if $f\in L_p(D)$, then elliptic problem (1.1) has a unique solution $u\in H^{\alpha/2(\alpha)}_p(D)$.

Our approach is different from those in the above mentioned articles and is based on weighted Sobolev spaces. We summarize our results and their differences from above mentioned results as follow.

- We prove the weighted Sobolev regularity result of solutions to both elliptic and parabolic equations. For instance, for the elliptic equation we prove (1.3), which is

$$\|u\|_{H^{\gamma+\alpha}_{p,\theta-\alpha p/2}(D)}\leq C\|f\|_{H^{\gamma}_{p,\theta+\alpha p/2}(D)},\tag{1.10}$$

for any $p>1, \gamma\geq 0$ and $\theta\in(d-1,d-1+p)$.

- We emphasize that unlike in any of above mentioned articles, our results, e.g. (1.10), are proved for any $\gamma\geq 0$, despite that D is an only $C^{1,1}$ open set. This is possible because we use appropriate weighted Sobolev spaces.
- As can be seen from (1.4), regularity of $\rho^{-\alpha/2}u$ near ∂D changes according to θ. For instance, as $\theta\downarrow d-1$, $\rho^{-\alpha/2}u$ decays faster. Among the results described above, the closest result to (1.4)(or (1.10)) can be found in [24, 25], which in particular show that if D is a bounded C^{∞} domain and $f\in L_p(D)$ then $\rho^{-\alpha/2}u\in H_p^{\alpha/2}(D)$. This result is close to (1.4) if $\theta=d$. Besides that θ can vary in the present article, we require the weaker assumption $\rho^{\alpha/2}f\in L_p(D,\rho^{\theta-d}dx)$.
- We use (1.10) to prove versions of (1.8) and (1.9) under weaker assumption on f. See Remark 2.14 for a comparison of our results with those in [1, 42, 46].
- We also obtain global Hölder estimates for the elliptic and parabolic equations; the free terms can be quite irregular and unbounded. For instance, in (1.5) we only require $\rho^{\alpha/2}f\in L_{p,\theta}(D)$.

Now, we introduce the organization of this article. In Section 2, we introduce our main results, Sobolev space theory and Hölder estimates of solutions. In Section 3, we study the representation of solutions and estimate the zero-th order derivative of solutions. In Section 4, we prove higher regularity of solutions, and we give the proofs of main results in Section 5.

We finish the introduction with notations used in this article. We use " := " or " =: " to denote a definition. $\mathbb{N}$ and $\mathbb{Z}$ denote the natural number system and the integer number system, respectively. We denote $\mathbb{N}_+:=\mathbb{N}\cup\{0\}$, and as usual $\mathbb{R}^d$ stands for the Euclidean space of points $x=(x^1,\dots,x^d)$,

$$B_r(x)=\{y\in\mathbb{R}^d:|x-y|<r\},\quad \mathbb{R}^d_+=\{(x^1,\dots,x^d)\in\mathbb{R}:x^1>0\}.$$

For nonnegative functions f and g, we write $f(x)\approx g(x)$ if there exists a constant $C>0$, independent of x, such that $C^{-1}f(x)\leq g(x)\leq Cf(x)$. For multi-indices $\beta=(\beta_1,\cdots,\beta_d)$, $\beta_i\in\mathbb{N}_+$, and functions $u(x)$ depending on x,

$$D_iu(x):=\frac{\partial u}{\partial x^i},\quad D^{\beta}_xu(x):=D^{\beta_d}_d\cdots D^{\beta_1}_1u(x).$$

We also use D^n_xu to denote the partial derivatives of order $n\in\mathbb{N}_+$ with respect to the space variables. For an open set $U\subset\mathbb{R}^d$, $C(U)$ denotes the space of continuous functions u in U such that $|u|_{C(U)}:=\sup_U|u(x)|<\infty$. $C_0(U)$ is the set of functions in $C(U)$ satisfying $\lim_{|x|\to\infty}u(x)=0$ and $\lim_{x\to\partial U}u(x)=0$. By $C^2_b(U)$ we denote the space of functions whose derivatives of order up to 2 are in $C(U)$. For an open set $V\subset\mathbb{R}^m$, where $m\in\mathbb{N}$, by $C^{\infty}_c(V)$ we denote the space of infinitely differentiable functions with compact support in V. For a Banach space F and $\delta\in(0,1]$, $C^{\delta}(V;F)$ denotes the space of F-valued continuous functions u on V such that

$$\begin{aligned}|u|_{C^{\delta}(V;F)} &:= |u|_{C(V;F)}+[u]_{C^{\delta}(V;F)}\\ &:= \sup_{x\in V}|u(x)|_F+\sup_{x,y\in V}\frac{|u(x)-u(y)|_F}{|x-y|^{\delta}}<\infty.\end{aligned}$$

Also, for $p>1$ and a measure μ on V, $L_p(V,\mu;F)$ denotes the set of F-valued Lebesgue measurable functions u such that

$$\|u\|_{L_p(V,\mu;F)} := \left(\int_V |u|_F^p \, d\mu\right)^{1/p} < \infty.$$

We drop F and μ if $F=\mathbb{R}$ and μ is the Lebesgue measure. By $\mathcal{D}(U)$, where U is an open set in $\mathbb{R}^d$, we denote the space of all distributions on U, and for given $f\in\mathcal{D}(U)$, the action of f on $\phi\in C_c^\infty(U)$ is denoted by

$$(f,\phi)_U := f(\phi).$$

Finally, if we write $C=C(a,b,\cdots)$, then this means that the constant C depends only on $a,b,\cdots$.

2. Main results

In subsection 2.1 we prove the uniqueness and existence results in a wide class of function spaces, and we give the regularity of solutions in subsection 2.2.

Throughout this article, D is either a half space $\mathbb{R}^d_+$ or a bounded $C^{1,1}$ open set.

2.1. **Uniqueness and existence.** For suitable functions f defined on $\mathbb{R}^d$ (e.g. $f\in C_b^2(\mathbb{R}^d)$), we define the fractional Laplacian $\Delta^{\alpha/2}f$ as

$$\Delta^{\alpha/2}f(x) := c_d \lim_{\varepsilon\downarrow 0}\int_{|y|>\varepsilon}\frac{f(x+y)-f(x)}{|y|^{d+\alpha}}dy, \tag{2.1}$$

where $c_d=\frac{2^\alpha\Gamma(\frac{d+\alpha}{2})}{\pi^{d/2}|\Gamma(-\alpha/2)|}$. Also, for functions f,g defined on $E\subset\mathbb{R}^d$, we set

$$\langle f,g\rangle_E := \int_E fg\,dx.$$

Definition 2.1. (i) (Parabolic problem) For given $f\in L_{1,loc}([0,T]\times D)$ and $u_0\in L_{1,loc}(D)$, we say that u is a (weak) solution to the problem

$$\begin{cases}\partial_t u(t,x)=\Delta^{\alpha/2}u(t,x)+f(t,x), & (t,x)\in(0,T)\times D,\\ u(0,x)=u_0(x), & x\in D,\\ u(t,x)=0, & (t,x)\in[0,T]\times D^c,\end{cases} \tag{2.2}$$

if (a) $u=0$ $a.e.$ in $[0,T]\times D^c$, (b) $\langle u(t,\cdot),\phi\rangle_{\mathbb{R}^d}$ and $\langle u(t,\cdot),\Delta^{\alpha/2}\phi\rangle_{\mathbb{R}^d}$ exist for any $t\le T$ and test function $\phi\in C_c^\infty(D)$, and (c) for any $\phi\in C_c^\infty(D)$ the equality

$$\langle u(t,\cdot),\phi\rangle_{\mathbb{R}^d}=\langle u_0,\phi\rangle_D+\int_0^t\langle u(s,\cdot),\Delta^{\alpha/2}\phi\rangle_{\mathbb{R}^d}ds+\int_0^t\langle f(s,\cdot),\phi\rangle_D ds \tag{2.3}$$

holds for all $t\le T$.

(ii) (Elliptic problem) Let $\lambda\in[0,\infty)$. For given $f\in L_{1,loc}(D)$, we say that u is a (weak) solution to

$$\begin{cases}\Delta^{\alpha/2}u(x)-\lambda u(x)=f(x), & x\in D,\\ u(x)=0, & x\in D^c,\end{cases} \tag{2.4}$$

if (a) $u=0$ $a.e.$ in D^c, (b) $\langle u,\phi\rangle_{\mathbb{R}^d}$ and $\langle u,\Delta^{\alpha/2}\phi\rangle_{\mathbb{R}^d}$ exist for any test function $\phi\in C_c^\infty(D)$, and (c) for any $\phi\in C_c^\infty(D)$ we have

$$\langle u,\Delta^{\alpha/2}\phi\rangle_{\mathbb{R}^d}-\lambda\langle u,\phi\rangle_{\mathbb{R}^d}=\langle f,\phi\rangle_D. \tag{2.5}$$

It is clear if $u(t,x)$ is a strong (or point-wise) solution to (2.2) and sufficiently regular, then u becomes a weak solution in the sense of Definition 2.1.

For an explicit representation of weak solutions, we introduce some related stochastic processes. Let $X=(X)_{t\geq 0}$ be a rotationally symmetric α-stable d-dimensional Lévy process defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, that is, X_t is a Lévy process such that

$$\mathbb{E}e^{i\xi\cdot X_t}=e^{-|\xi|^\alpha t},\quad \forall\,\xi\in\mathbb{R}^d.$$

Let

$$\tau_D=\tau_D^x:=\inf\{t\geq 0: x+X_t\notin D\}$$

denote the first exit time of D by X. We add an element, called a cemetery point, $\partial\notin\mathbb{R}^d$ to $\mathbb{R}^d$, and define the killed process of X upon D by

$$X_t^D=X_t^{D,x}:=\begin{cases} x+X_t & t<\tau_D^x,\\ \partial & t\geq \tau_D^x.\end{cases}$$

The cemetery point ∂ is introduced to define $f(\partial):=0$ for any function f so that $f(X_t^D)=0$ if $t\geq\tau_D^x$. Let $p^D(t,x,y)$ denote the transition density of X^D, i.e., for any Borel set $B\subset\mathbb{R}^d$,

$$\mathbb{P}(X_t^{D,x}\in B)=\int_B p^D(t,x,y)dy.$$

Recall $\rho(x)=dist(x,\partial D)$. We denote $L_{p,\theta}(D):=L_p(D,\rho^{\theta-d}dx)$ for any $\theta\in\mathbb{R}$ and $p>1$. In other words, $L_{p,\theta}(D)$ is the set of functions u such that

$$\|u\|_{L_{p,\theta}(D)}:=\left(\int_D |u|^p\rho^{\theta-d}dx\right)^{1/p}<\infty.$$

For $T<\infty$, we also define the space

$$\mathbb{L}_{p,\theta}(D,T):=L_p((0,T);L_{p,\theta}(D))$$

given with the norm

$$\|u\|_{\mathbb{L}_{p,\theta}(D,T)}=\left(\int_0^T\int_D |u|^p\rho^{\theta-d}dxdt\right)^{1/p}.$$

Here are our uniqueness and existence results of the elliptic and parabolic equations. The proofs are given in Section 5.

Theorem 2.2 (Parabolic case). *Let $\alpha\in(0,2)$ and $p\in(1,\infty)$. Assume $\theta\in(d-1,d-1+p)$, $f\in\mathbb{L}_{p,\theta+\alpha p/2}(D,T)$ and $u_0\in L_{p,\theta-\alpha p/2+\alpha}(D)$.*

(i) The function

$$u(t,x):=\int_D p^D(t,x,y)u_0(y)dy+\int_0^t\int_D p^D(t-s,x,y)f(s,y)dyds \tag{2.6}$$

belongs to $\mathbb{L}_{p,\theta-\alpha p/2}(D,T)\cap\{u=0 \text{ on } [0,T]\times D^c\}$ and is the unique weak solution to (2.2) in this function space.

(ii) For the solution u, we have

$$\|u\|_{\mathbb{L}_{p,\theta-\alpha p/2}(D,T)}\leq C(\|f\|_{\mathbb{L}_{p,\theta+\alpha p/2}(D,T)}+\|u_0\|_{L_{p,\theta-\alpha p/2+\alpha}(D)}), \tag{2.7}$$

where C is independent of u and T.

Theorem 2.3 (Elliptic case)**.** *Let $\alpha \in (0,2)$ and $p \in (1,\infty)$. Assume $\theta \in (d-1, d-1+p)$ and $f \in L_{p,\theta+\alpha p/2}(D)$.*

(i) Let $\lambda > 0$ or D be bounded. Then, the function

$$u(x) = u^{(\lambda)}(x) := \int_D \left(\int_0^\infty e^{-\lambda t} p^D(t,x,y)dt \right) f(y)dy$$

belongs to $L_{p,\theta-\alpha p/2}(D) \cap \{u = 0 \text{ on } D^c\}$ and is the unique weak solution to (2.4) in this function space.

(ii) Let $\lambda = 0$ and $D = \mathbb{R}^d_+$. Then, $u^{(1/n)}$ converges weakly in $L_p(\mathbb{R}^d, \rho^{\theta-d-\alpha p/2}dx)$, and the weak limit u is the unique solution to equation (2.4) in the function space $L_{p,\theta-\alpha p/2}(D) \cap \{u = 0 \text{ on } D^c\}$.

(iii) For the solution u, we have

$$\|u\|_{L_{p,\theta-\alpha p/2}(D)} \le C\|f\|_{L_{p,\theta+\alpha p/2}(D)},$$

where C is independent of u and λ.

Remark 2.4. By definition of the norm in $\mathbb{L}_{p,\theta-\alpha p/2}(D,T)$ and (2.7),

$$\|u\|^p_{\mathbb{L}_{p,\theta-\alpha p/2}(D,T)} = \int_0^T \int_D |\rho^{-\alpha/2}u|^p \rho^{\theta-d} dxdt < \infty$$

provided that $-1 < \theta - d < -1 + p$. This suggests that u vanishes at a certain rate near the boundary of D. The detailed behaviors of solutions and their derivatives will be handled in the following subsection.

Remark 2.5. The range $\theta \in (d-1, d-1+p)$ in Theorems 2.2 and 2.3 is sharp. We demonstrate this with a simple example for the elliptic problem. The parabolic problem can be handled similarly.

Let $D = B_1(0)$ and f be a (non-zero) nonnegative function in $C_c^\infty(D)$ so that $f \in L_{p,\theta}(D)$ for any $\theta \in \mathbb{R}$.

1. First, we show $\theta > d-1$ is necessary. Denote

$$G^0_D(x,y) := \int_0^\infty p^D(t,x,y)dt \quad \text{and} \quad u(x) := \int_D G^0_D(x,y)f(y)dy.$$

Due to [13, Corollary 1.2], if $y \in supp(f)$ and $(r+1)/2 < |x| < 1$ where $r := 1 - dist(supp(f), \partial D) > 0$, then $G^0_D(x,y) \approx \rho(x)^{\alpha/2}$. Hence, for $(r+1)/2 < |x| < 1$,

$$u(x) \approx \rho(x)^{\alpha/2} = (1-|x|)^{\alpha/2},$$

and consequently

$$\|u\|^p_{L_{p,\theta-\alpha p/2}(D)} \ge C \int_{(r+1)/2}^1 (1-s)^{\theta-d} s^{d-1} ds.$$

The right-hand side above is finite only if $\theta - d > -1$. Therefore, the condition $\theta - d > -1$ is needed to have $u \in L_{p,\theta-\alpha p/2}(D)$.

2. Next, we show $\theta < d-1+p$ is also necessary. Suppose Theorem 2.3 holds for some $\theta \ge d-1+p$. Then,

$$\left\| \int_D G^0_D(\cdot,y)g(y)dy \right\|_{L_{p,\theta-\alpha p/2}(D)} \le C\|g\|_{L_{p,\theta+\alpha p/2}}, \quad \forall g \in L_{p,\theta+\alpha p/2}(D).$$

Since $G^0_D(x,y)=G^0_D(y,x)$ (see e.g. [15, Theorem 2.4]), by Hölder's inequality,

$$\begin{aligned}\left|\int_D u(x)g(x)dx\right| &= \left|\int_D\left(\int_D G^0_D(x,y)g(x)dx\right)f(y)dy\right|\\ &\leq \left\|\int_D G^0_D(\cdot,y)g(y)dy\right\|_{L_{p,\theta-\alpha p/2}(D)}\|f\|_{L_{p',\theta'+\alpha p'/2}(D)}\\ &\leq C\|g\|_{L_{p,\theta+\alpha p/2}(D)}\|f\|_{L_{p',\theta'+\alpha p'/2}(D)},\end{aligned}$$

where $1/p+1/p'=1$ and $\theta/p+\theta'/p'=d$. Since $L_{p',\theta'-\alpha p'/2}(D)$ is the dual space of $L_{p,\theta+\alpha p/2}(D)$ (cf. Lemma 2.7(iii)), this leads to

$$\|u\|_{L_{p',\theta'-\alpha p'/2}(D)} \leq C\|f\|_{L_{p',\theta'+\alpha p'/2}(D)} < \infty. \tag{2.8}$$

Note that $\theta' \leq d-1$. As shown above, (2.8) is not possible, and therefore we get a contradiction.

Remark 2.6. Since $\Delta^{\alpha/2}\phi$ belongs to the dual space of $L_{p,\theta-\alpha p/2}(D)$ for any $\phi\in C_c^\infty(D)$ (cf. Lemma 4.4) and $C_c^\infty(D)$ is dense in the dual space, we can replace $\langle\cdot,\cdot\rangle_D$ and $\langle\cdot,\cdot\rangle_{\mathbb{R}^d}$ in (2.3) and (2.5) by $(\cdot,\cdot)_D$ for the solutions in Theorems 2.2 and 2.3.

2.2. **Regularity of solutions.** In this subsection, we present Sobolev regularity of solutions. We also obtain Hölder estimates of solutions based on a Sobolev embedding theorem. In particular, we give asymptotic behaviors of solutions and their 'arbitrary' order derivatives near the boundary of D.

To describe such results, we first recall Sobolev and Besov spaces on $\mathbb{R}^d$. For $p\in(1,\infty)$ and $\gamma\in\mathbb{R}$, the Sobolev space $H_p^\gamma=H_p^\gamma(\mathbb{R}^d)$ is defined as the space of all tempered distributions f on $\mathbb{R}^d$ satisfying

$$\|f\|_{H_p^\gamma} := \|(1-\Delta)^{\gamma/2}f\|_{L_p}<\infty,$$

where

$$(1-\Delta)^{\gamma/2}f(x) := \mathcal{F}^{-1}\left[(1+|\cdot|^2)^{\gamma/2}\mathcal{F}[f]\right](x).$$

Here, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the d-dimensional Fourier transform and the inverse Fourier transform respectively, i.e.,

$$\mathcal{F}[f](\xi) := \int_{\mathbb{R}^d} e^{-i\xi\cdot x}f(x)dx, \quad \mathcal{F}^{-1}[f](x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\xi\cdot x}f(\xi)d\xi.$$

As is well known, if $\gamma\in\mathbb{N}_+$, then we have

$$H_p^\gamma = W_p^\gamma := \{f : D_x^\beta u\in L_p(\mathbb{R}^d), |\beta|\leq\gamma\}.$$

For $T\in(0,\infty)$, define

$$\mathbb{H}_p^\gamma(T) := L_p((0,T);H_p^\gamma), \quad \mathbb{L}_p(T) := \mathbb{H}_p^0(T) = L_p((0,T);L_p).$$

Now we take a function Ψ whose Fourier transform $\mathcal{F}[\Psi]$ is infinitely differentiable, supported in an annulus $\{\xi\in\mathbb{R}^d : \frac{1}{2}\leq|\xi|\leq 2\}$, $\mathcal{F}[\Psi]\geq 0$ and

$$\sum_{j\in\mathbb{Z}}\mathcal{F}[\Psi](2^{-j}\xi)=1, \qquad \forall\xi\neq 0.$$

For a tempered distribution f and $j\in\mathbb{Z}$, define

$$\Delta_j f(x) := \mathcal{F}^{-1}\left[\mathcal{F}[\Psi](2^{-j}\cdot)\mathcal{F}[f]\right](x), \qquad S_0 f(x) := \sum_{j=-\infty}^{0}\Delta_j f(x).$$

The Besov space $B_p^\gamma = B_p^\gamma(\mathbb{R}^d)$, where $p > 1, \gamma \in \mathbb{R}$, is defined as the space of all tempered distributions f satisfying

$$\|f\|_{B_p^\gamma} := \|S_0 f\|_{L_p} + \left(\sum_{j=1}^{\infty} 2^{\gamma p j} \|\Delta_j f\|_{L_p}^p \right)^{1/p} < \infty.$$

It is well known (see e.g. [51, Remark 2.5.12/2]) that if $\gamma = n + \delta$, where $n \in \mathbb{N}_+$ and $\delta \in (0,1)$, then

$$\|f\|_{B_p^\gamma} \approx \|f\|_{H_p^n} + \left(\sum_{|\beta|=n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|D_x^\beta f(x+y) - D_x^\beta f(x)|^p}{|y|^{d+\delta p}} dy dx \right)^{1/p}. \tag{2.9}$$

Moreover, for any $p > 1$, we have

$$H_p^{\gamma_2} \subset B_p^{\gamma_1} \quad \text{if} \quad \gamma_1 < \gamma_2. \tag{2.10}$$

Next, we introduce weighted Sobolev and Besov spaces on $D \subset \mathbb{R}^d$. Recall $\rho(x) = dist\,(x, \partial D)$ and $L_{p,\theta}(D) := L_p(D, \rho^{\theta-d} dx)$. For any $\theta \in \mathbb{R}$ and $n \in \mathbb{N}_+$, define

$$H_{p,\theta}^n(D) = \{u : u, \rho D_x u, \cdots, \rho^n D_x^n u \in L_{p,\theta}(D)\}.$$

The norm in this space is defined as

$$\|u\|_{H_{p,\theta}^n(D)} = \sum_{|\beta| \le n} \left(\int_D |\rho^{|\beta|} D_x^\beta u(x)|^p \rho^{\theta-d} dx \right)^{1/p}. \tag{2.11}$$

To generalize this space and define $H_{p,\theta}^\gamma(D)$ for any $\gamma \in \mathbb{R}$, we proceed as follows. We choose a sequence of nonnegative functions $\zeta_n \in C^\infty(D), n \in \mathbb{Z}$, having the following properties:

$$(i)\ supp(\zeta_n) \subset \{x \in D : k_1 e^{-n} < \rho(x) < k_2 e^{-n}\}, \quad k_2 > k_1 > 0, \tag{2.12}$$

$$(ii)\ \sup_{x \in \mathbb{R}^d} |D_x^m \zeta_n(x)| \le C(m) e^{mn}, \quad \forall m \in \mathbb{N}_+ \tag{2.13}$$

$$(iii)\ \sum_{n \in \mathbb{Z}} \zeta_n(x) > c > 0, \quad \forall x \in D. \tag{2.14}$$

Such functions can be easily constructed by considering mollifications of indicator functions of the sets of the type $\{x \in D : k_3 e^{-n} < \rho(x) < k_4 e^{-n}\}$. If the set $\{x \in D : k_1 e^{-n} < \rho(x) < k_2 e^{-n}\}$ is empty, we just take $\zeta_n = 0$.

Now we define weighted Sobolev spaces $H_{p,\theta}^\gamma(D)$ and weighed Besov spaces $B_{p,\theta}^\gamma(D)$ for any $\gamma, \theta \in \mathbb{R}$ and $p > 1$. To understand these spaces, one needs to notice that for any distribution u on D, $\zeta_{-n} u$ becomes a distribution on $\mathbb{R}^d$. Obviously, the action of $\zeta_{-n} u$ on $C_c^\infty(\mathbb{R}^d)$ is defined as

$$(\zeta_{-n} u, \phi)_{\mathbb{R}^d} = (u, \zeta_{-n} \phi)_D, \quad \phi \in C_c^\infty(\mathbb{R}^d). \tag{2.15}$$

By $H_{p,\theta}^\gamma(D)$ and $B_{p,\theta}^\gamma(D)$ we denote the sets of distributions u on D such that

$$\|u\|_{H_{p,\theta}^\gamma(D)}^p := \sum_{n \in \mathbb{Z}} e^{n\theta} \|\zeta_{-n}(e^n \cdot) u(e^n \cdot)\|_{H_p^\gamma}^p < \infty, \tag{2.16}$$

and

$$\|u\|_{B_{p,\theta}^\gamma(D)}^p := \sum_{n \in \mathbb{Z}} e^{n\theta} \|\zeta_{-n}(e^n \cdot) u(e^n \cdot)\|_{B_p^\gamma}^p < \infty,$$

respectively. The spaces $H^\gamma_{p,\theta}(D)$ and $B^\gamma_{p,\theta}(D)$ are independent of choice of $\{\zeta_n\}$ (see e.g. [43, Proposition 2.2]). More precisely, if $\{\xi_n \in C^\infty(D) : n \in \mathbb{Z}\}$ satisfies (2.12) and (2.13), then

$$\sum_{n\in\mathbb{Z}} e^{n\theta}\|\xi_{-n}(e^n\cdot)u(e^n\cdot)\|^p_{H^\gamma_p} \leq C\|u\|^p_{H^\gamma_{p,\theta}(D)}, \tag{2.17}$$

and the reverse inequality of (2.17) also holds if $\{\xi_n\}$ satisfies (2.14). The similar statements hold in the space $B^\gamma_{p,\theta}(D)$ as well. Furthermore, if $\gamma = n \in \mathbb{N}_+$, then the norms defined in (2.11) and (2.16) are equivalent (cf. [43, Proposition 2.2]).

Obviously, by (2.10), we have for any $p > 1$ and $\theta \in \mathbb{R}$,

$$H^{\gamma_2}_{p,\theta}(D) \subset B^{\gamma_1}_{p,\theta}(D) \quad \text{if} \quad \gamma_1 < \gamma_2. \tag{2.18}$$

Furthermore, for an equivalent norm in $B^\gamma_{p,\theta}(D)$, we can apply (2.9) and prove the following: if $\gamma = n + \delta > 0$, where $n \in \mathbb{N}_+$, $\delta \in (0, 1)$, and $\theta - d + \gamma p > -1$, then

$$\|u\|_{B^\gamma_{p,\theta}(D)} \approx \|u\|_{H^n_{p,\theta}(D)} + \left(\sum_{|\beta|=n} \int_D \int_D \rho^{\theta-d+\gamma p}_{x,y} \frac{|D^\beta u(x) - D^\beta u(y)|^p}{|x-y|^{d+\delta p}} dydx\right)^{1/p}, \tag{2.19}$$

where $\rho_{x,y} = \rho(x) \wedge \rho(y)$. The proof of (2.19) is left to the reader. Relation (2.19) will not be used elsewhere in this article.

Next, we choose (cf. [32]) an infinitely differentiable function ψ in D such that $\psi \approx \rho$ on D, and for any $m \in \mathbb{N}_+$

$$\sup_D |\rho^m(x) D^{m+1}_x \psi(x)| \leq C(m) < \infty.$$

For instance, one can take $\psi(x) := \sum_{n\in\mathbb{Z}} e^{-n}\zeta_n(x)$.

Below we collect some other properties of the spaces $H^\gamma_{p,\theta}(D)$ and $B^\gamma_{p,\theta}(D)$. For $\nu \in \mathbb{R}$, we write $u \in \psi^{-\nu} H^\gamma_{p,\theta}(D)$ (resp. $u \in \psi^{-\nu} B^\gamma_{p,\theta}(D)$) if $\psi^\nu u \in H^\gamma_{p,\theta}(D)$ (resp. $\psi^\nu u \in B^\gamma_{p,\theta}(D)$).

Lemma 2.7. *Let $\gamma, \theta \in \mathbb{R}$ and $p \in (1, \infty)$.*

(i) The space $C^\infty_c(D)$ is dense in $H^\gamma_{p,\theta}(D)$ and $B^\gamma_{p,\theta}(D)$.

(ii) For $\delta \in \mathbb{R}$, $H^\gamma_{p,\theta}(D) = \psi^\delta H^\gamma_{p,\theta+\delta p}(D)$ and $B^\gamma_{p,\theta}(D) = \psi^\delta B^\gamma_{p,\theta+\delta p}(D)$. Moreover,

$$\|u\|_{H^\gamma_{p,\theta}(D)} \approx \|\psi^{-\delta}u\|_{H^\gamma_{p,\theta+\delta p}(D)}, \quad \|u\|_{B^\gamma_{p,\theta}(D)} \approx \|\psi^{-\delta}u\|_{B^\gamma_{p,\theta+\delta p}(D)}.$$

(iii) (Duality) Let

$$1/p + 1/p' = 1, \quad \theta/p + \theta'/p' = d.$$

Then, the dual spaces of $H^\gamma_{p,\theta}(D)$ and $B^\gamma_{p,\theta}(D)$ are $H^{-\gamma}_{p',\theta'}(D)$ and $B^{-\gamma}_{p',\theta'}(D)$, respectively.

(iv) (Sobolev embedding) Let $\mu \leq \gamma$, $1 < p \leq q$ and $\theta \leq \tau$ such that

$$\mu - d/q \leq \gamma - d/p, \quad \tau/q = \theta/p.$$

Then, we have

$$\|u\|_{H^\mu_{q,\tau}(D)} \leq C\|u\|_{H^\gamma_{p,\theta}(D)}.$$

(v) (Sobolev-Hölder embedding)
Let $\gamma - \frac{d}{p} \geq n + \delta$ for some $n \in \mathbb{N}_+$ and $\delta \in (0, 1)$. Then, for any $k \leq n$,

$$|\psi^{k+\frac{\theta}{p}} D^k_x u|_{C(D)} + [\psi^{n+\frac{\theta}{p}+\delta} D^n_x u]_{C^\delta(D)} \leq C(d, \gamma, p, \theta)\|u\|_{H^\gamma_{p,\theta}(D)}.$$

Proof. The proofs for $B^{\gamma}_{p,\theta}(D)$ are similar to those for $H^{\gamma}_{p,\theta}(D)$, and we only consider the claims for $H^{\gamma}_{p,\theta}(D)$. When D is a half space, the claims are proved by Krylov in [37, Lemma 2.2, Theorem 2.5], and those are generalized by Lototsky in [43] for arbitrary domains. Here, we remark that the results in [43] are still valid for bounded $C^{1,1}$ open sets. The lemma is proved. □

Now we define solution spaces for the parabolic equation. For $T \in (0,\infty)$, denote

$$\mathbb{H}^{\gamma}_{p,\theta}(D,T) := L_p((0,T);H^{\gamma}_{p,\theta}(D)).$$

We write $u \in \mathfrak{H}^{\gamma}_{p,\theta}(D,T)$ if $u \in \psi^{\alpha/2}\mathbb{H}^{\gamma}_{p,\theta}(D,T)$, $u(0,\cdot) \in \psi^{\alpha/2-\alpha/p}B^{\gamma-\alpha/p}_{p,\theta}(D)$, and there exists $f \in \psi^{-\alpha/2}\mathbb{H}^{\gamma-\alpha}_{p,\theta}(D,T)$ such that for any $\phi \in C^{\infty}_c(D)$

$$(u(t,\cdot),\phi)_D = (u(0,\cdot),\phi)_D + \int_0^t (f(s,\cdot),\phi)_D ds, \quad \forall\, t \leq T.$$

In this case, we write $u_t := \partial_t u := f$. The norm in $\mathfrak{H}^{\gamma}_{p,\theta}(D,T)$ is defined as

$$\begin{aligned}\|u\|_{\mathfrak{H}^{\gamma}_{p,\theta}(D,T)} :=& \|\psi^{-\alpha/2}u\|_{\mathbb{H}^{\gamma}_{p,\theta}(D,T)} + \|\psi^{\alpha/2}u_t\|_{\mathbb{H}^{\gamma-\alpha}_{p,\theta}(D,T)} \\ &+ \|\psi^{-\alpha/2+\alpha/p}u(0,\cdot)\|_{B^{\gamma-\alpha/p}_{p,\theta}(D)}.\end{aligned} \tag{2.20}$$

Remark 2.8. (i) The Banach space $\mathfrak{H}^{\gamma}_{p,\theta}(D,T)$ is a modification of the corresponding space defined for $\alpha = 2$ (see e.g. [32] for C^1 domains and [39] for a half space). The completeness of this space for $\alpha \in (0,2)$ can be proved by repeating the argument in [39, Remark 3.8].

(ii) The same argument in [38, Remark 5.5] shows that $C^{\infty}_c([0,T]\times D)$ is dense in $\mathfrak{H}^{\gamma}_{p,\theta}(D,T)$.

The following two theorems address our Sobolev regularity results. The proofs are given in Section 5.

Theorem 2.9 (Parabolic case)**.** *Let $\gamma \in [0,\infty)$, and assume $f \in \psi^{-\alpha/2}\mathbb{H}^{\gamma}_{p,\theta}(D,T)$ and $u_0 \in \psi^{\alpha/2-\alpha/p}B^{\gamma+\alpha-\alpha/p}_{p,\theta}(D)$. The unique solution u in Theorem 2.2 belongs to $\mathfrak{H}^{\gamma+\alpha}_{p,\theta}(D,T)$, and for this solution we have*

$$\|u\|_{\mathfrak{H}^{\gamma+\alpha}_{p,\theta}(D,T)} \leq C\left(\|\psi^{\alpha/2}f\|_{\mathbb{H}^{\gamma}_{p,\theta}(D,T)} + \|\psi^{-\alpha/2+\alpha/p}u_0\|_{B^{\gamma+\alpha-\alpha/p}_{p,\theta}(D)}\right), \tag{2.21}$$

where C depends only on d,p,α,γ,θ and D.

Theorem 2.10 (Elliptic case)**.** *Let $\gamma,\lambda \in [0,\infty)$ and assume $f \in \psi^{-\alpha/2}H^{\gamma}_{p,\theta}(D)$. Then, the unique solution u in Theorem 2.3 belongs to $\psi^{\alpha/2}H^{\gamma+\alpha}_{p,\theta}(D)$, and for this solution we have*

$$\lambda\|\psi^{\alpha/2}u\|_{H^{\gamma}_{p,\theta}(D)} + \|\psi^{-\alpha/2}u\|_{H^{\gamma+\alpha}_{p,\theta}(D)} \leq C\|\psi^{\alpha/2}f\|_{H^{\gamma}_{p,\theta}(D)}, \tag{2.22}$$

where C depends only on d,p,α,γ,θ and D. In particular, it is independent of λ.

Remark 2.11. (i) Let $\gamma+\alpha \geq n$, where $n \in \mathbb{N}_+$. Then, (2.11) and (2.22) certainly yield

$$\int_D \left(|\rho^{-\alpha/2}u|^p + |\rho^{1-\alpha/2}Du|^p + \cdots + |\rho^{n-\alpha/2}D^n u|^p\right)\rho^{\theta-d}dx < \infty.$$

(ii) The parabolic version of (i) also holds.

(*iii*) Let $H^{\gamma}_{p,loc}(D)$ denote the space of all distributions on D such that $u\eta \in H^{\gamma}_{p}$ for any $\eta \in C^{\infty}_{c}(D)$. Then, due to the definition of $H^{\gamma}_{p,\theta}(D)$ (see (2.16)), one can easily find that $H^{\gamma}_{p,\theta}(D) \subset H^{\gamma}_{p,loc}(D)$. Note that if D is bounded, then $L_p(D) \subset L_{p,d+\alpha p/2}(D)$. Thus, our result directly implies the ones in [7, 8, 16] when D is a bounded $C^{1,1}$ open set, while the latter ones cover a more general class of open sets.

The following estimates are consequences of Lemma 2.7(*iv*).

Corollary 2.12. *Let u be taken from Theorem 2.9 and*

$$\mu - d/q \leq \gamma + \alpha - d/p, \quad \tau/q = \theta/p.$$

Then, we have

$$\|\psi^{-\alpha/2}u\|_{L_p((0,T);H^{\mu}_{q,\tau}(D))} \leq C\left(\|\psi^{\alpha/2}f\|_{\mathbb{H}^{\gamma}_{p,\theta}(D,T)} + \|\psi^{-\alpha/2+\alpha/p}u_0\|_{B^{\gamma+\alpha-\alpha/p}_{p,\theta}(D)}\right).$$

Corollary 2.13. *Let u be taken from Theorem 2.10 and*

$$\mu - d/q \leq \gamma + \alpha - d/p, \quad \tau/q = \theta/p.$$

Then, we have

$$\|\psi^{-\alpha/2}u\|_{H^{\mu}_{q,\tau}(D)} \leq C\|\psi^{\alpha/2}f\|_{H^{\gamma}_{p,\theta}(D)}.$$

Remark 2.14. We compare Corollary 2.13 with the results in [1, 42, 46]. Below we assume D is bounded.

(*i*) Let $q \in (1,\infty)$, $p = \max\{2, \frac{dq}{d+\alpha q/2}\}$ and $f \in L_{p,d+\alpha p/2}(D)$. Then,

$$\frac{\alpha}{2} - \frac{d}{q} \leq \alpha - \frac{d}{p}.$$

Thus, by Corollary 2.13 and Theorem 2.10,

$$\|\psi^{-\alpha/2}u\|_{H^{\alpha/2}_{q,\tau}(D)} \leq C\|\psi^{\alpha/2}f\|_{L_{p,d}(D)},$$

where $\tau/q = d/p$, which implies $u \in H^{\alpha/2}_{q,loc}(D)$. This is proved in [46] given that $f \in L_p(D)$ and $q > 2$ (instead of $q > 1$). Since $L_p(D) \subseteq L_{p,d+\alpha p/2}(D)$, our result extends the one in [46], although [46] considered more general domains and non-local equations.

(*ii*) Let $\alpha/2 \leq \beta \leq \alpha$ and $q \geq p$ such that

$$\beta - \frac{d}{q} \leq \alpha - \frac{d}{p}, \quad q < \frac{dp}{d-1}. \tag{2.23}$$

In this case, $pd/q \in (d-1, d-1+p)$, which allows us to apply Corollaries 2.13 and 4.5(*i*) to get

$$\begin{aligned}\|\psi^{\beta-\alpha/2}\Delta^{\beta/2}u\|_{L_q(D)} &= \|\psi^{\beta-\alpha/2}\Delta^{\beta/2}u\|_{L_{q,d}(D)}\\ &\leq C\|\psi^{-\alpha/2}u\|_{H^{\beta}_{q,d}(D)}\\ &\leq C\|\psi^{\alpha/2}f\|_{L_{p,pd/q}(D)}.\end{aligned} \tag{2.24}$$

According to [1, Theorem 1.4], if $\alpha/2 \leq \beta < (1 \wedge \alpha)$ and

$$\beta - \frac{d}{q} < \alpha - \frac{d}{p} < \beta, \tag{2.25}$$

then

$$\|\psi^{\beta-\alpha/2}\Delta^{\beta/2}u\|_{L_q(D)} \leq C\|f\|_{L_p(D)}.$$

Also, by [42, Theorem 24], if

$$\frac{\alpha}{2}-\frac{d}{q}=\alpha-\frac{d}{p},\quad 1<p<\frac{2d}{d+\alpha}, \tag{2.26}$$

then it holds that

$$\|\Delta^{\alpha/4}u\|_{L_q(D)}\leq C\|f\|_{L_p(D)}.$$

One can note that (2.23), (2.25), and (2.26) are distinct conditions. Note that given that $\alpha/2\leq\beta$, we have $L_p(D)\subset L_{p,d+p(\alpha/2+d/q)}(D)$ for $p,q\in(1,\infty)$ satisfying (2.23). Consequently, (2.24) allows a broader class of data f.

For Hölder regularity of the solution to the parabolic equation, we use the following parabolic embedding.

Proposition 2.15. *Let $\alpha\in(0,2)$, $p\in(1,\infty)$, and $\gamma,\theta\in\mathbb{R}$. Then, for any $1/p<\nu\leq 1$,*

$$\left|\psi^{\alpha(\nu-1/2)}\left(u-u(0,\cdot)\right)\right|_{C^{\nu-1/p}([0,T];H^{\gamma+\alpha-\nu\alpha}_{p,\theta}(D))}\leq C\|u\|_{\mathfrak{H}^{\gamma+\alpha}_{p,\theta}(D,T)}, \tag{2.27}$$

where C depends only on d, ν, p, θ, α and T.

Proof. We repeat the argument in [39] which treats the case $\alpha=2$. Considering $u-u_0$ in place of u, we may assume $u_0=0$. Let $u_t=f$. By (2.16) and Lemma 2.7(ii),

$$\begin{aligned}&\left|\psi^{\alpha(\nu-1/2)}u\right|^p_{C^{\nu-1/p}([0,T];H^{\gamma+\alpha-\nu\alpha}_{p,\theta}(D))}\\&\leq C\sum_{n\in\mathbb{Z}}e^{n(\theta+p\alpha(\nu-1/2))}|u(\cdot,e^n\cdot)\zeta_{-n}(e^n\cdot)|^p_{C^{\nu-1/p}([0,T];H^{\gamma+\alpha-\nu\alpha}_p)}.\end{aligned} \tag{2.28}$$

Denote $v_n(t,x)=u(t,e^nx)\zeta_{-n}(e^nx)$. Then, $\partial_tv_n(t,x)=f(t,e^nx)\zeta_{-n}(e^nx)$. Thus, by Lemma A.5 with $a=e^{-np\alpha/2}$,

$$\begin{aligned}&e^{np\alpha(\nu-1/2)}|u(\cdot,e^n\cdot)\zeta_{-n}(e^n\cdot)|^p_{C^{\nu-1/p}([0,T];H^{\gamma+\alpha-\nu\alpha}_p)}\\&\leq Ce^{-np\alpha/2}\|u(\cdot,e^n\cdot)\zeta_{-n}(e^n\cdot)\|^p_{\mathbb{H}^{\gamma+\alpha}_p(T)}+Ce^{np\alpha/2}\|f(\cdot,e^n\cdot)\zeta_{-n}(e^n\cdot)\|^p_{\mathbb{H}^{\gamma}_p(T)}.\end{aligned}$$

Coming back to (2.28) and using (2.16),

$$\begin{aligned}&\left|\psi^{\alpha(\nu-1/2)}u\right|^p_{C^{\nu-1/p}([0,T];H^{\gamma+\alpha-\nu\alpha}_{p,\theta}(D))}\\&\leq C\|\psi^{-\alpha/2}u\|^p_{\mathbb{H}^{\gamma+\alpha}_{p,\theta}(D,T)}+C\|\psi^{\alpha/2}f\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(D,T)}.\end{aligned}$$

This and Lemma 2.7(ii) prove (2.27). □

Proposition 2.15 and Lemma 2.7(v) yield the following results.

Corollary 2.16. *(Hölder regularity for parabolic equation) Let u be taken from Theorem 2.9, $1/p<\nu\leq 1$, and*

$$\gamma+\alpha-\nu\alpha-\frac{d}{p}\geq n+\delta,\quad n\in\mathbb{N}_+,\ \delta\in(0,1).$$

Then,

$$\sum_{k=0}^{n}|\psi^{k+\frac{\theta}{p}+\alpha\left(\nu-\frac{1}{2}\right)}D_x^k(u-u(0,\cdot))|_{C^{\nu-1/p}([0,T];C(D))}$$
$$+\sup_{t,s\in[0,T]}\frac{[\psi^{n+\delta+\frac{\theta}{p}+\alpha\left(\nu-\frac{1}{2}\right)}D_x^n(u(t,\cdot)-u(s,\cdot))]_{C^{\delta}(D)}}{|t-s|^{\nu-1/p}}\leq C\|u\|_{\mathfrak{H}^{\gamma+\alpha}_{p,\theta}(D,T)}.$$

Corollary 2.17. *(Hölder regularity for elliptic equation) Let u be taken from Theorem 2.10 and*

$$\gamma+\alpha-\frac{d}{p}\geq n+\delta,\quad n\in\mathbb{N}_+,\ \delta\in(0,1).$$

Then,

$$\sum_{k=0}^{n}|\psi^{k+\frac{\theta}{p}-\frac{\alpha}{2}}D_x^k u|_{C(D)}+[\psi^{n+\delta+\frac{\theta}{p}-\frac{\alpha}{2}}D_x^n u]_{C^{\delta}(D)}\leq C\|\psi^{-\alpha/2}u\|_{H^{\gamma+\alpha}_{p,\theta}(D)}.$$

Remark 2.18. Corollaries 2.16 and 2.17 give various Hölder estimates of solutions and their arbitrary order derivatives. Below we elaborate some special cases. We only consider $\gamma=0,1,2,\cdots$ and $\theta=d$. Note $L_{p,d}(D)=L_p(D)$.

(i) Parabolic Hölder estimates when $\gamma=0$. Let $u_0=0$ for simplicity, and assume $\psi^{\alpha/2}f\in\cap_{p>d/\alpha}\mathbb{L}_{p,d}(D,T)$. Obviously this holds e.g. if D is bounded and $\psi^{\alpha/2}f\in L_\infty([0,T]\times D)$. Taking $\nu\uparrow 1$ and $p\uparrow\infty$, from Corollary 2.16 we get

$$\sup_{x\in D}|\psi^{\alpha/2-\delta}(x)u(\cdot,x)|_{C^{1-\varepsilon}([0,T])}<\infty$$

for any small $\delta,\varepsilon>0$. This gives maximal regularity with respect to time variable. Now, we take p sufficiently large and ν sufficiently close to $1/p$ to get

$$\sup_{x\in D}|\psi^{-\alpha/2+\delta'}(x)u(\cdot,x)|_{C^{\varepsilon'}([0,T])}+\sup_{t\in[0,T]}|\psi^{\alpha/2-\delta'}u(t,\cdot)|_{C^{\alpha-\varepsilon'}(D)}<\infty$$

for any small $\delta',\varepsilon'>0$. The second term above gives the maximal interior regularity with respect to space variable, and the first one gives a decay rate near the boundary of D. In particular,

$$\sup_{t\in[0,T]}|u(t,x)|\leq C(\delta')\psi^{\alpha/2-\delta'}(x),\quad\forall\delta'>0.$$

(ii) Elliptic Hölder estimates when $\gamma=0$. Let $\psi^{\alpha/2}f\in\cap_{p>d/\alpha}L_{p,d}(D)$. Taking p sufficiently large, from Corollary 2.17 we get

$$|\psi^{\alpha/2}u|_{C^{\alpha-\varepsilon}(D)}+|u|_{C^{\alpha/2-\varepsilon}(D)}+|\psi^{-\alpha/2+\delta}u|_{C^{\varepsilon}(D)}<\infty \tag{2.29}$$

for any small $\delta,\varepsilon>0$. In [47], it is proved that if $\lambda=0$ and $f\in L_\infty(D)$, then

$$|u|_{C^{\alpha/2}(D)}+|\psi^{-\alpha/2}u|_{C^{\beta}(D)}<\infty \tag{2.30}$$

for some $\beta>0$. Thus, there is a slight gap between (2.29) and (2.30). However, our result holds even when f blows up near the boundary since we assume (at most) $\psi^{\alpha/2}f$ is bounded.

(iii) Higher order estimates. Let $\gamma=n\in\mathbb{N}$. Then, the same arguments above show that all the claims in (i)-(ii) also hold for $\psi D_x u,\psi^2D_x^2u,\cdots,\psi^nD_x^nu$. That is, the estimates hold if one replaces u by any of these functions. In particular, if $\psi^{\alpha/2}f,\psi^{\alpha/2+1}D_xf\in\cap_{p>d/\alpha}L_{p,d}(D)$, then, together with (2.29), we also have

$$|\psi^{1+\alpha/2}D_xu|_{C^{\alpha-\varepsilon}(D)}+|\psi D_xu|_{C^{\alpha/2-\varepsilon}(D)}+|\psi^{1-\alpha/2+\delta}D_xu|_{C^{\varepsilon}(D)}<\infty$$

for any small $\delta, \varepsilon > 0$.

3. The zero-th order derivative estimates

In this section, we estimate the zero-th order derivative of the solutions to the parabolic equation

$$\begin{cases} \partial_t u(t,x) = \Delta^{\alpha/2} u(t,x) + f(t,x), & (t,x) \in (0,T) \times D, \\ u(0,x) = u_0(x), & x \in D, \\ u(t,x) = 0, & (t,x) \in [0,T] \times D^c. \end{cases} \tag{3.1}$$

as well as to the elliptic equation

$$\begin{cases} \Delta^{\alpha/2} u(x) - \lambda u(x) = f(x), & x \in D, \\ u(x) = 0, & x \in D^c. \end{cases} \tag{3.2}$$

3.1. **Weak solutions for smooth data.** Recall that $X = (X)_{t \geq 0}$ is a rotationally symmetric α-stable d-dimensional Lévy process. Let $p(t,x) = p_d(t,x)$ denote the transition density function of X. Then, it is well known (e.g. [33, (3.6)]) that

$$\begin{aligned} p_d(t,x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-t|\xi|^\alpha} d\xi \\ &\approx t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x|^{d+\alpha}} \approx \frac{t}{(t^{1/\alpha} + |x|)^{d+\alpha}}, \quad \forall (t,x) \in (0,\infty) \times \mathbb{R}^d. \end{aligned}$$

The equality above also implies that $p_d(t,\cdot)$ is a radial function and

$$p_d(t,x) = t^{-\frac{d}{\alpha}} p_d(1, t^{-\frac{1}{\alpha}} x).$$

Denote

$$d_x = d_{D,x} := \begin{cases} \rho(x) & : x \in D, \\ 0 & : x \notin D. \end{cases}$$

The following lemma gives an upper bound of $p^D(t,x,y)$.

Lemma 3.1. *For any $x, y \in \mathbb{R}^d$,*

$$p^D(t,x,y) \leq \begin{cases} C\left(1 \wedge \frac{d_x^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{d_y^{\alpha/2}}{\sqrt{t}}\right) p(t,x-y) & \text{if } D \text{ is a half space}, \\ Ce^{-ct}\left(1 \wedge \frac{d_x^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{d_y^{\alpha/2}}{\sqrt{t}}\right) p(t,x-y) & \text{if } D \text{ is bounded}. \end{cases}$$

Here, $C, c > 0$ depend only on d, α and D.

Proof. See [9, Theorem 5.8] for the case $D = \mathbb{R}^d_+$. Let D be bounded. Then, by [9, Theorem 4.5], there exist $C, c, r > 0$, depending only on α, d and D, such that for any $x, y \in D$

$$p^D(t,x,y) \leq Ce^{-2ct}\left(\frac{d_x^{\alpha/2}}{\sqrt{t} \wedge r^{\alpha/2}} \wedge 1\right)\left(\frac{d_y^{\alpha/2}}{\sqrt{t} \wedge r^{\alpha/2}} \wedge 1\right) p(t \wedge r^\alpha, x-y).$$

This actually implies the claim of the lemma. Indeed, the case $t < r^\alpha$ is obvious, and if $t > r^\alpha$ then

$$p(r^\alpha, x-y) = r^{-d} p(1, r^{-1}(x-y)) \leq r^{-d} p(1, t^{-\frac{1}{\alpha}}(x-y)) = r^{-d} t^{d/\alpha} p(t, x-y).$$

This certainly proves the claim. The lemma is proved. □

For $x \in \mathbb{R}^d$, we use $\mathbb{E}_x$ and $\mathbb{P}_x$ to denote the expectation and distribution of $x+X$. For instance, $\mathbb{P}_x(X_t \in A) := \mathbb{P}(x+X_t \in A)$. Recall that $f(\partial) := 0$ for any function f, where ∂ is the cemetery point.

Now, we introduce the probabilistic representation of equation (3.1) for smooth data.

Lemma 3.2. *(i) Suppose $u_0 \in C_c^\infty(D)$ and $f \in C_c^\infty((0,T)\times D)$. Then,*

$$\begin{aligned} u(t,x) &:= \mathbb{E}_x[u_0(X_t^D)] + \int_0^t \mathbb{E}_x[f(s, X_{t-s}^D)]ds \\ &= \int_D p^D(t,x,y)u_0(y)dy + \int_0^t \int_D p^D(t-s,x,y)f(s,y)dyds \end{aligned}$$

is a weak solution to (3.1) in the sense of Definition 2.1(i).

(ii) Let $u \in C_c^\infty([0,T]\times D)$. Then,

$$u(t,x) = \mathbb{E}_x[u(0,X_t^D)] + \int_0^t \mathbb{E}_x[f(s,X_{t-s}^D)]ds, \tag{3.3}$$

where $f := \partial_t u - \Delta^{\alpha/2}u$.

Proof. (i) If $u_0 = 0$, then it follows from [53, Lemma 8.4]. The general case is handled similarly.

(ii) This follows from [53, Theorem 5.5]. We remark that [53, Theorem 5.5] is proved only on bounded open sets, but the result holds even on a half space. Indeed, let $D_n \subset D = \mathbb{R}^d_+$ be a sequence of bounded $C^{1,1}$ open sets such that $D_n \uparrow D$ and $supp(u(t,\cdot)) \subset D_n$ for all $t \in [0,T]$. Since D_n is bounded, by [53, Theorem 5.5],

$$u(t,x) = \mathbb{E}_x[u(0,X_t^{D_n})] + \mathbb{E}_x\left[\int_0^{t\wedge\tau_{D_n}} f(t-s,X_s)ds\right],$$

where τ_{D_n} is the first exit time of D_n by X, and X^{D_n} is the killed process of X upon D_n (see Section 2). By following the proof of [53, Lemma 5.4], we have $\tau_{D_n} \uparrow \tau_D$. This, since both $u(0)$ and f are bounded, certainly yields (3.3). The lemma is proved. □

Let $\{T_t\}_{t\geq 0}$ and $\{T_t^D\}_{t\geq 0}$ be the transition semigroups of X and X^D defined by

$$T_t f(x) := \mathbb{E}_x[f(X_t)], \qquad T_t^D f(x) := \mathbb{E}_x[f(X_t^D)],$$

respectively. It is known (see e.g. [11, Example 1.3] and page 68 of [14]) that $\{T_t\}_{t\geq 0}$ and $\{T_t^D\}_{t\geq 0}$ are Feller semigroups. For instance, $\{T_t^D\}_{t\geq 0}$ is a family of linear operators on $L_\infty(D)$ such that

(i) for any $f \in L_\infty(D)$,

$$T_t^D T_s^D f = T_{t+s}^D f,$$

(ii) for any $f \in C_0(D)$, $T_t^D f \in C_0(D)$ and

$$\lim_{t\to 0}\|T_t^D f - f\|_{L_\infty(D)} = 0.$$

We also define infinitesimal generators A and A_D by

$$Af(x) := \lim_{t\downarrow 0}\frac{T_t f(x) - f(x)}{t}, \qquad A_D f(x) := \lim_{t\downarrow 0}\frac{T_t^D f(x) - f(x)}{t}$$

provided that the limits exist. It is well known (e.g. [5, Theorem 2.3]) if $f \in C_0(D)$ and one of $Af(x)$ and $A_D f(x)$ exists, then the other also exists and $Af(x) = A_D f(x)$. Moreover if $f \in C_b^2(\mathbb{R}^d)$, then $Af(x) = \Delta^{\alpha/2}f(x)$ (e.g. [5, Lemma 2.6]).

Lemma 3.3. *Assume $u \in C_0(D)$ and $Au(x)$ exists for all $x \in D$. If u satisfies*

$$Au - \lambda u = 0 \text{ in } D, \quad \lambda > 0, \tag{3.4}$$

then, $u \equiv 0$.

Proof. Assume $\sup_{x\in D} u > 0$. Since $u \in C_0(D)$, there exists $x_0 \in D$ such that

$$u(x_0) = \sup_{x\in D} u(x).$$

By the definition of the infinitesimal generator,

$$Au(x_0) = \lim_{t\downarrow 0} \frac{\mathbb{E}_{x_0}[u(X_t)] - u(x_0)}{t} \le 0.$$

Hence, (3.4) yields a contradiction. Using the similar argument for $-u$, we conclude that $u \equiv 0$. The lemma is proved. □

For $\lambda \ge 0$, we define the Green function

$$G_D^\lambda(x,y) := \int_0^\infty e^{-\lambda t} p^D(t,x,y)dt.$$

By Lemma 3.1, $G_D^\lambda(x,y)$ is well defined if $x \ne y$.

Lemma 3.4. *Let D be a half space (resp. a bounded $C^{1,1}$ open set) and $\lambda > 0$ (resp. $\lambda \ge 0$). For $f \in C(D)$, define*

$$v(x) := \int_D G_D^\lambda(x,y)f(y)dy. \tag{3.5}$$

(i) $v \in C_0(D)$, $Av(x)$ exists for all $x \in D$, and v is a strong(point-wise) solution to

$$\begin{cases} Av(x) - \lambda v(x) = f(x), & x \in D, \\ v(x) = 0, & x \in D^c. \end{cases} \tag{3.6}$$

(ii) v is a weak solution to (3.2) in the sense of Definition 2.1(ii).

(iii) Let $u \in C_c^\infty(D)$ and $g := \Delta^{\alpha/2}u - \lambda u$. Then,

$$u(x) = \int_D G_D^\lambda(x,y)g(y)dy. \tag{3.7}$$

Proof. (i) The claim follows from [34, Lemma 3.6] if D is bounded and $\lambda = 0$. We repeat its proof for the case $\lambda > 0$. First, we show $v \in C_0(D)$. By Lemma 3.1,

$$\begin{aligned}\int_0^\infty \int_D e^{-\lambda t} p^D(t,x,y)|f(y)|dydt &\le C\|f\|_{L_\infty(D)} \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda t} p(t,x-y)dydt \\ &= C\|f\|_{L_\infty(D)} \int_0^\infty e^{-\lambda t}dt < \infty.\end{aligned}$$

Thus, by Fubini's theorem,

$$v(x) = \int_0^\infty e^{-\lambda t} T_t^D f(x)dt.$$

Since $T_t^D f \in C_0(D)$ and $\|T_t^D f\|_{L_\infty(D)} \le \|f\|_{L_\infty(D)}$, the dominated convergence theorem easily yields $v \in C_0(D)$.

Since $\{T_t^D\}_{t\geq 0}$ is a Feller semigroup, for any $x \in D$,

$$\begin{aligned}
A_D v(x) &:= \lim_{t\downarrow 0} \frac{T_t^D v(x) - v(x)}{t} \quad \text{(provided that the limit exists)} \\
&= \lim_{t\downarrow 0} \frac{1}{t}\left(T_t^D \int_0^\infty e^{-\lambda s} T_s^D f(x) ds - \int_0^\infty e^{-\lambda s} T_s^D f(x) ds \right) \\
&= \lim_{t\downarrow 0} \frac{1}{t}\left(\int_0^\infty e^{-\lambda s} T_{t+s}^D f(x) ds - \int_0^\infty e^{-\lambda s} T_s^D f(x) ds \right) \\
&= \lim_{t\downarrow 0} \frac{1}{t}\left(e^{\lambda t} \int_t^\infty e^{-\lambda s} T_s^D f(x) ds - \int_0^\infty e^{-\lambda s} T_s^D f(x) ds \right) \\
&= \lim_{t\downarrow 0} \frac{e^{\lambda t}-1}{t} \int_t^\infty e^{-\lambda s} T_s^D f(x) ds + \lim_{t\to 0} \frac{1}{t}\left(\int_0^t e^{-\lambda s} T_s^D f(x) ds \right) \\
&= \lambda v(x) + f(x).
\end{aligned}$$

Since the limits exist, we conclude that $A_D v$ exists, $A_D v = Av$, and v satisfies (3.6).

(ii) Let $\varphi \in C_c^\infty(D)$. Since $\|T_t^D \varphi\|_{L_\infty(D)} \leq \|\varphi\|_{L_\infty(D)}$,

$$\lim_{t\to\infty} e^{-\lambda t} T_t^D \varphi = 0.$$

Since $\varphi \in C_c^\infty(D)$, we can use the relation $\partial_t T_t^D \varphi = T_t^D \Delta^{\alpha/2}\varphi$ (see [53, Lemma 8.4]) to get

$$\begin{aligned}
(v, \Delta^{\alpha/2}\varphi)_{\mathbb{R}^d} &= \int_0^\infty (e^{-\lambda t} T_t^D f, \Delta^{\alpha/2}\varphi)_D dt \\
&= \int_0^\infty (f, e^{-\lambda t} T_t^D \Delta^{\alpha/2}\varphi)_D dt \\
&= \int_0^\infty (f, e^{-\lambda t} \partial_t T_t^D \varphi)_D dt \\
&= -\lim_{t\to\infty} (f, e^{-\lambda t} T_t^D \varphi)_D + (f,\varphi)_D + \int_0^\infty (f, \lambda e^{-\lambda t} T_t^D \varphi)_D dt \\
&= (f,\varphi)_D + \lambda (v,\varphi)_D.
\end{aligned}$$

(iii) Note that $f := \Delta^{\alpha/2} u - \lambda u \in C(D)$. Assume $\lambda > 0$ for the moment. Take $v(x)$ from (3.5). Then, since $u \in C_b^2(\mathbb{R}^d)$, we have $Au = \Delta^{\alpha/2} u$ (e.g. [5, Lemma 2.6]), and therefore both u and v satisfy the equation $Aw(x) - \lambda w(x) = f(x)$ for each $x \in D$. We conclude $u = v$ due to Lemma 3.3. If $\lambda = 0$ and D is a bounded $C^{1,1}$ open set, then the uniqueness result in [34, Theorem 3.10] easily yields (3.7). The lemma is proved. □

3.2. **Estimates of zero-th order of solutions.** Denote

$$\mathcal{T}_D^0 u_0(t,x) := \int_D p^D(t,x,y) u_0(y) dy,$$

$$\mathcal{T}_D f(t,x) := \int_0^t \int_D p^D(t-s,x,y) f(s,y) dy ds,$$

$$\mathcal{G}_D^\lambda f(x) := \int_D G_D^\lambda(x,y) f(y) dy.$$

In this subsection, we prove the operators

$$\begin{aligned}\mathcal{T}_D^0 &: \psi^{\alpha/2-\alpha/p}L_{p,\theta}(D) \to \psi^{\alpha/2}\mathbb{L}_{p,\theta}(D,T),\\ \mathcal{T}_D &: \psi^{-\alpha/2}\mathbb{L}_{p,\theta}(D,T) \to \psi^{\alpha/2}\mathbb{L}_{p,\theta}(D,T),\\ \mathcal{G}_D^\lambda &: \psi^{-\alpha/2}L_{p,\theta}(D) \to \psi^{\alpha/2}L_{p,\theta}(D)\end{aligned}$$

are bounded. Our proofs highly depend on the following lemma, which is proved in Lemma A.3.

Lemma 3.5. *Let $\alpha \in (0,2)$, γ_0, $\gamma_1 \in \mathbb{R}$. Suppose that*

$$-\frac{2}{\alpha} < \gamma_0, \quad -2 < \gamma_1 - \gamma_0 \leq 2 + \frac{2}{\alpha}.$$

Then, for any $(t,x) \in (0,\infty) \times \mathbb{R}^d$,

$$\int_D p(t,x-y)\frac{d_y^{\gamma_0\alpha/2}}{(\sqrt{t}+d_y^{\alpha/2})^{\gamma_1}}dy \leq C(\sqrt{t}+d_x^{\alpha/2})^{\gamma_0-\gamma_1},$$

where $C = C(d,\alpha,\gamma_0,\gamma_1,D)$.

We first consider the operator $\mathcal{T}_D$.

Lemma 3.6. *Let $\alpha \in (0,2)$ and $p \in (1,\infty)$. Suppose that*

$$d-1 < \theta < d-1+p.$$

Then, there exists $C = C(d,\alpha,\theta,p,D)$ such that for any $f \in \psi^{-\alpha/2}\mathbb{L}_{p,\theta}(D,T)$,

$$\|\psi^{-\alpha/2}\mathcal{T}_D f\|_{\mathbb{L}_{p,\theta}(D,T)} \leq C\|\psi^{\alpha/2}f\|_{\mathbb{L}_{p,\theta}(D,T)}.$$

Proof. By Lemma 2.7(ii) and (2.11), it suffices to show

$$\int_0^T\int_D d_x^{\mu-\alpha p/2}|\mathcal{T}_D f(t,x)|^p dxdt \leq C\int_0^T\int_D d_x^{\mu+\alpha p/2}|f(t,x)|^p dxdt, \tag{3.8}$$

where $\mu := \theta - d$. For $p' = p/(p-1)$, since $\mu \in (-1,p-1)$, we can take β_0 satisfying

$$\frac{2\mu}{p\alpha}+1-\frac{4}{p} < \beta_0 < \frac{2\mu}{p\alpha}+1+\frac{2}{p\alpha} \tag{3.9}$$

and

$$-\frac{2(p-1)}{p} = -\frac{2}{p'} < \beta_0 < \left(2+\frac{2}{\alpha}\right)\frac{1}{p'} = \left(2+\frac{2}{\alpha}\right)\frac{p-1}{p}. \tag{3.10}$$

Since $1-\frac{2}{p} < \frac{2\mu}{p\alpha}+1+\frac{2}{p\alpha}-\frac{2}{p}$ and $\frac{2\mu}{p\alpha}+1 < \frac{2(p-1)}{p\alpha}+1$, we can take constants β_1 and β_2 such that

$$1-\frac{2}{p} < \beta_0-\beta_1 < \frac{2\mu}{p\alpha}+1+\frac{2}{p\alpha}-\frac{2}{p} \tag{3.11}$$

and

$$\frac{2\mu}{p\alpha}+1 < \beta_0+\beta_2 < \frac{2(p-1)}{p\alpha}+1. \tag{3.12}$$

Let $R_{t,x} := \frac{d_x^{\alpha/2}}{\sqrt{t}+d_x^{\alpha/2}}$. By Lemma 3.1 and Hölder's inequality,

$$\begin{aligned}|\mathcal{T}_D f(t,x)| \leq & C\left(\int_0^t\int_D p(t-s,x-y)d_y^{-\alpha\beta_0 p'/2}R_{t-s,x}^{(1-\beta_1)p'}R_{t-s,y}^{(1-\beta_2)p'}dyds\right)^{1/p'} \\ & \times\left(\int_0^t\int_D p(t-s,x-y)d_y^{\alpha\beta_0 p/2}R_{t-s,x}^{\beta_1 p}R_{t-s,y}^{\beta_2 p}|f(s,y)|^p dyds\right)^{1/p} \\ =: & C\times I(t,x)\times II(t,x). \end{aligned} \tag{3.13}$$

By Lemma 3.5 with $\gamma_0 = (1-\beta_2)p' - \beta_0 p'$ and $\gamma_1 = (1-\beta_2)p'$, we have

$$\int_D p(t-s,x-y)d_y^{-\alpha\beta_0 p'/2}R_{t-s,y}^{(1-\beta_2)p'}dy \leq C(\sqrt{t-s}+d_x^{\alpha/2})^{-\beta_0 p'}.$$

Using this inequality and changing variables,

$$\begin{aligned} I(t,x)^{p'} &\leq Cd_x^{\alpha(1-\beta_1)p'/2}\int_0^t(\sqrt{t-s}+d_x^{\alpha/2})^{-\beta_0 p'-(1-\beta_1)p'}ds \\ &\leq Cd_x^{-\alpha\beta_0 p'/2}\int_0^\infty d_x^\alpha(\sqrt{s}+1)^{-\beta_0 p'-(1-\beta_1)p'}ds = Cd_x^{\alpha-\alpha\beta_0 p'/2}. \end{aligned} \tag{3.14}$$

Therefore, due to (3.13), (3.14) and Fubini's theorem,

$$\begin{aligned} &\int_0^T\int_D d_x^\mu|d_x^{-\alpha/2}\mathcal{T}_D f(t,x)|^p dxdt \leq C\int_0^T\int_D d_x^{\mu+\alpha p/2-\alpha-\alpha\beta_0 p/2}II(t,x)^p dxdt \\ &= C\int_0^T\int_D |f(s,y)|^p d_y^{\alpha\beta_0 p/2} \\ &\qquad\times\left(\int_s^T\int_D d_x^{\mu+\alpha p/2-\alpha-\alpha\beta_0 p/2}p(t-s,x-y)R_{t-s,x}^{\beta_1 p}R_{t-s,y}^{\beta_2 p}dxdt\right)dyds. \end{aligned} \tag{3.15}$$

Now, again by Lemma 3.5 with $\gamma_0 = 2\mu/\alpha + p - 2 - \beta_0 p + \beta_1 p$ and $\gamma_1 = \beta_1 p$,

$$\begin{aligned} &\int_s^T\int_D d_x^{\mu+\alpha p/2-\alpha-\alpha\beta_0 p/2}p(t-s,x-y)R_{t-s,x}^{\beta_1 p}R_{t-s,y}^{\beta_2 p}dxdt \\ &\leq Cd_y^{\alpha\beta_2 p/2}\int_s^T(\sqrt{t-s}+d_y^{\alpha/2})^{2\mu/\alpha+p-2-\beta_0 p-\beta_2 p}dt \\ &\leq Cd_y^{\mu+\alpha p/2-\alpha\beta_0 p/2}\int_0^\infty(\sqrt{t}+1)^{2\mu/\alpha+p-2-\beta_0 p-\beta_2 p}dt \leq Cd_y^{\mu+\alpha p/2-\alpha\beta_0 p/2}. \end{aligned} \tag{3.16}$$

This and (3.15) yield (3.8), and the lemma is proved. □

Next, we consider the operator $\mathcal{T}_D^0$ defined for initial data.

Lemma 3.7. *Let $\alpha \in (0,2)$ and $p \in (1,\infty)$. Suppose that*

$$d-1 < \theta < d-1+p+\left(\alpha(p-1)\wedge\frac{3}{2}\alpha p\right).$$

Then, there exists $C = C(d,\alpha,\theta,p,D)$ such that for any $u_0 \in \psi^{-\alpha/2+\alpha/p}L_{p,\theta}(D)$,

$$\|\psi^{-\alpha/2}\mathcal{T}_D^0 u_0\|_{\mathbb{L}_{p,\theta}(D,T)} \leq C\|\psi^{-\alpha/2+\alpha/p}u_0\|_{L_{p,\theta}(D)}.$$

Proof. As in the proof of Lemma 3.6, it is enough to prove

$$\int_0^T \int_D d_x^{\mu-\alpha p/2} |\mathcal{T}_D^0 u_0(t,x)|^p dxdt \le C \int_D d_x^{\mu+\alpha-\alpha p/2} |u_0(x)|^p dx,$$

where $\mu := \theta - d$. Since $\mu \in (-1, p-1+\frac{3}{2}\alpha p)$, we can choose β_0 satisfying

$$\frac{2\mu}{p\alpha} - 1 - \frac{2}{p} < \beta_0 < \frac{2\mu}{p\alpha} - 1 + \frac{2}{p} + \frac{2}{p\alpha}$$

and

$$-\frac{2(p-1)}{p} = -\frac{2}{p'} < \beta_0 < \left(2 + \frac{2}{\alpha}\right)\frac{1}{p'} = \left(2 + \frac{2}{\alpha}\right)\frac{p-1}{p},$$

where $p' = p/(p-1)$. Also, since $\frac{2\mu}{p\alpha} - 1 + \frac{2}{p} < \frac{2}{p'\alpha} + 1$, we can choose β_1 satisfying

$$\frac{2\mu}{p\alpha} - 1 + \frac{2}{p} < \beta_0 + \beta_1 < \frac{2}{p'\alpha} + 1.$$

Let $R_{t,x} := \frac{d_x^{\alpha/2}}{\sqrt{t} + d_x^{\alpha/2}}$. By Lemma 3.1 and Hölder's inequality,

$$\begin{aligned}
|\mathcal{T}_D^0 u_0(t,x)| \le & C \left(\int_D p(t,x-y) d_y^{-\alpha\beta_0 p'/2} R_{t,y}^{(1-\beta_1)p'} dy \right)^{1/p'} \\
& \times \left(\int_D p(t,x-y) d_y^{\alpha\beta_0 p/2} R_{t,x}^p R_{t,y}^{\beta_1 p} |u_0(y)|^p dy \right)^{1/p} \\
=: & C \times I(t,x) \times II(t,x),
\end{aligned}$$

By Lemma 3.5 with $\gamma_0 = (1-\beta_1)p' - \beta_0 p'$ and $\gamma_1 = (1-\beta_1)p'$, we have

$$I(t,x)^{p'} = \int_D p(t,x-y) d_y^{-\alpha\beta_0 p'/2} R_{t,y}^{(1-\beta_1)p'} dy \le C(\sqrt{t} + d_x^{\alpha/2})^{-\beta_0 p'}$$

Therefore, applying Fubini's theorem,

$$\begin{aligned}
& \int_0^T \int_D d_x^{\mu} |d_x^{-\alpha/2} \mathcal{T}_D^0 u_0(t,x)|^p dxdt \\
& \le C \int_0^T \int_D d_x^{\mu-\alpha p/2} (\sqrt{t} + d_x^{\alpha/2})^{-\beta_0 p} II(t,x)^p dxdt \\
& \le C \int_D |u_0(y)|^p d_y^{\alpha\beta_0 p/2} K(T,y) dy, \qquad (3.17)
\end{aligned}$$

where

$$\begin{aligned}
K(T,y) :=& \int_0^T R_{t,y}^{\beta_1 p} \int_D p(t,x-y) d_x^{\mu-\alpha p/2} (\sqrt{t} + d_x^{\alpha/2})^{-\beta_0 p} R_{t,x}^p dxdt \\
=& \int_0^T R_{t,y}^{\beta_1 p} \int_D p(t,x-y) d_x^{\mu} (\sqrt{t} + d_x^{\alpha/2})^{-\beta_0 p - p} dxdt.
\end{aligned}$$

By Lemma 3.5 with $\gamma_0 = 2\mu/\alpha$ and $\gamma_1 = \beta_0 p + p$,

$$\begin{aligned}
K(T,y) & \le C \int_0^T R_{t,y}^{\beta_1 p} (\sqrt{t} + d_y^{\alpha/2})^{2\mu/\alpha - \beta_0 p - p} dt \\
& \le C d_y^{\mu - \alpha\beta_0 p/2 - \alpha p/2 + \alpha} \int_0^\infty (\sqrt{t} + 1)^{2\mu/\alpha - \beta_0 p - p - \beta_1 p} dt \\
& \le C d_y^{\mu - \alpha\beta_0 p/2 - \alpha p/2 + \alpha}.
\end{aligned}$$

This with (3.17) proves the lemma. □

Finally, we consider the operator $\mathcal{G}_D^\lambda$ for elliptic equation (3.2).

Lemma 3.8. *Let $\alpha \in (0,2)$, $p \in (1,\infty)$ and $\theta \in (d-1, d-1+p)$. Suppose that D is a half space (resp. a bounded $C^{1,1}$ open set) and $\lambda > 0$ (resp. $\lambda \geq 0$). Then, for any $f \in \psi^{-\alpha/2} L_{p,\theta}(D)$,*

$$\|\psi^{-\alpha/2}\mathcal{G}_D^\lambda f\|_{L_{p,\theta}(D)} \leq C\|\psi^{\alpha/2} f\|_{L_{p,\theta}(D)},$$

where $C = C(d,p,\alpha,\theta,D)$ is independent of λ.

Proof. As before, we need to show

$$\int_D d_x^{\mu-\alpha p/2}|\mathcal{G}_D^\lambda f(x)|^p dx \leq C\int_D d_x^{\mu+\alpha p/2}|f(x)|^p dx, \tag{3.18}$$

where $\mu := \theta - d$. Take β_0, β_1 and β_2 satisfying (3.9)-(3.12). Let $R_{t,x} := \frac{d_x^{\alpha/2}}{\sqrt{t}+d_x^{\alpha/2}}$. By Lemma 3.1 and Hölder's inequality,

$$\begin{aligned}|\mathcal{G}_D^\lambda f(x)| \leq & C\left(\int_0^\infty \int_D p(t,x-y) d_y^{-\alpha\beta_0 p'/2} R_{t,x}^{(1-\beta_1)p'} R_{t,y}^{(1-\beta_2)p'} dydt\right)^{1/p'} \\ & \times \left(\int_0^\infty \int_D p(t,x-y) d_y^{\alpha\beta_0 p/2} R_{t,x}^{\beta_1 p} R_{t,y}^{\beta_2 p}|f(y)|^p dydt\right)^{1/p} \\ =: & C \times I(t,x) \times II(t,x). \end{aligned} \tag{3.19}$$

Similar argument used to prove (3.14) yields

$$I(t,x)^{p'} \leq C d_x^{\alpha(1-\beta_1)p'/2} \int_0^\infty (\sqrt{t}+d_x^{\alpha/2})^{-\beta_0 p' - (1-\beta_1)p'} dt \leq C d_x^{\alpha - \alpha\beta_0 p'/2}. \tag{3.20}$$

Therefore, by (3.19), (3.20) and Fubini's theorem,

$$\begin{aligned}&\int_D d_x^\mu |d_x^{-\alpha/2}\mathcal{G}_D^\lambda f(x)|^p dx \\ &\leq C\int_D d_x^{\mu+\alpha p/2-\alpha-\alpha\beta_0 p/2} II(t,x)^p dxdt \\ &= C\int_D |f(y)|^p d_y^{\alpha\beta_0 p/2}\Big(\int_0^\infty \int_D d_x^{\mu+\alpha p/2-\alpha-\alpha\beta_0 p/2} p(t,x-y) R_{t,x}^{\beta_1 p} R_{t,y}^{\beta_2 p} dxdt\Big) dy.\end{aligned}$$

As in (3.16), we get

$$\begin{aligned}&\int_0^\infty \int_D d_x^{\mu+\alpha p/2-\alpha-\alpha\beta_0 p/2} p(t,x-y) R_{t,x}^{\beta_1 p} R_{t,y}^{\beta_2 p} dxdt \\ &\leq C d_y^{\alpha\beta_2 p/2}\int_0^\infty (\sqrt{t}+d_y^{\alpha/2})^{2\mu/\alpha + p - 2 - \beta_0 p - \beta_2 p} dt \leq C d_y^{\mu+\alpha p/2 - \alpha\beta_0 p/2}.\end{aligned}$$

Thus, we prove (3.18) and the lemma. □

4. Higher order estimates

In this section, we prove that one can raise regularity of solutions as long as the free terms are in appropriate function spaces.

We first prepare some auxiliary results below. Let $\{\zeta_n : n \in \mathbb{Z}\}$ be a collection of functions satisfying (2.12)-(2.14) with $(k_1, k_2) = (1, e^2)$. We also take $\{\eta_n : n \in \mathbb{Z}\}$ satisfying (2.12)-(2.14) with $(k_1, k_2) = (e^{-2}, e^4)$ and

$$\eta_n = 1 \text{ on } \{x \in D : e^{-n-1} < \rho(x) < e^{-n+3}\}.$$

Consequently, $\eta_n = 1$ on the support of ζ_n and $\zeta_n\eta_n = \zeta_n$.

Lemma 4.1. *For any $\gamma \in \mathbb{R}$, there exists a constant $C = C(d, \alpha, \gamma)$ such that for $u \in C_c^\infty(D)$ and $n \in \mathbb{Z}$,*

$$\begin{aligned}&\left\|\Delta^{\alpha/2}\Big((u\zeta_{-n}\eta_{-n})(e^n\cdot)\Big) - \zeta_{-n}(e^n\cdot)\Delta^{\alpha/2}\Big((u\eta_{-n})(e^n\cdot)\Big)\right\|_{H_p^\gamma}\\ &\leq C\left(\left\|\Delta^{\alpha/4}\Big((u\eta_{-n})(e^n\cdot)\Big)\right\|_{H_p^\gamma} + \|u(e^n\cdot)\eta_{-n}(e^n\cdot)\|_{H_p^\gamma}\right).\end{aligned}$$

Proof. By (2.1),

$$\begin{aligned}&\Delta^{\alpha/2}\big((u\zeta_{-n}\eta_{-n})(e^n\cdot)\big)(x) - \zeta_{-n}(e^n x)\Delta^{\alpha/2}\big((u\eta_{-n})(e^n\cdot)\big)(x)\\ &- u(e^n x)\eta_{-n}(e^n x)\Delta^{\alpha/2}\zeta_{-n}(e^n\cdot)(x) = C\int_{\mathbb{R}^d} H_n(x,y)|y|^{-d-\alpha}dy, \qquad (4.1)\end{aligned}$$

where

$$H_n(x,y) := [(u\eta_{-n})(e^n(x+y)) - (u\eta_{-n})(e^n x)][\zeta_{-n}(e^n(x+y)) - \zeta_{-n}(e^n x)].$$

In the virtue of (2.13), for any $m \in \mathbb{N}_+$,

$$\left|D_x^m\big(\zeta_{-n}(e^n(x+y)) - \zeta_{-n}(e^n x)\big)\right| \leq C(m)(1 \wedge |y|).$$

Thus, $\zeta_{-n}(e^n(x+y)) - \zeta_{-n}(e^n x)$ becomes a point-wise multiplier in H_p^γ (see e.g. [36, Lemma 5.2]), and therefore

$$\|H_n(\cdot,y)\|_{H_p^\gamma} \leq C(1 \wedge |y|)\|(u\eta_{-n})(e^n(\cdot + y)) - (u\eta_{-n})(e^n\cdot)\|_{H_p^\gamma}.$$

By [52, Lemma 2.1], the above is bounded by

$$C\left(\|u(e^n\cdot)\eta_{-n}(e^n\cdot)\|_{H_p^\gamma} \wedge |y|^{\alpha/2+1}\|\Delta^{\alpha/4}\big(u(e^n\cdot)\eta_{-n}(e^n\cdot)\big)\|_{H_p^\gamma}\right). \qquad (4.2)$$

By Minkowski's inequality and (4.2),

$$\begin{aligned}&\left\|\int_{\mathbb{R}^d} H_n(\cdot,y)|y|^{-d-\alpha}dy\right\|_{H_p^\gamma}\\ &\leq C\|\Delta^{\alpha/4}\big(u(e^n\cdot)\eta_{-n}(e^n\cdot)\big)\|_{H_p^\gamma}\int_{|y|\leq 1}|y|^{-d-\alpha/2+1}dy\\ &\quad + C\|u(e^n\cdot)\eta_{-n}(e^n\cdot)\|_{H_p^\gamma}\int_{|y|>1}|y|^{-d-\alpha}dy\\ &\leq C\left(\|\Delta^{\alpha/4}\big((u\eta_{-n})(e^n\cdot)\big)\|_{H_p^\gamma} + \|u(e^n\cdot)\eta_{-n}(e^n\cdot)\|_{H_p^\gamma}\right). \qquad (4.3)\end{aligned}$$

On the other hand, by (2.1) and (2.13),

$$\begin{aligned}|D_x^m\Delta^{\alpha/2}(\zeta_{-n}(e^n\cdot))(x)| &\leq C\|D_x^{m+2}\zeta_{-n}(e^n\cdot)\|_{L_\infty}\int_{|y|\leq 1}|y|^{-d-\alpha+2}dy\\ &\quad + C\|D_x^m\zeta_{-n}(e^n\cdot)\|_{L_\infty}\int_{|y|>1}|y|^{-d-\alpha}dy \leq C.\end{aligned}$$

Thus, again by [36, Lemma 5.2], we have

$$\|u(e^n\cdot)\eta_{-n}(e^n\cdot)\Delta^{\alpha/2}(\zeta_{-n}(e^n\cdot))\|_{H_p^\gamma} \leq C\|u(e^n\cdot)\eta_{-n}(e^n\cdot)\|_{H_p^\gamma}. \tag{4.4}$$

Combining (4.1), (4.3) and (4.4), we prove the lemma. □

Lemma 4.2. *Let $d-1-\alpha p/2 < \theta < d-1+p+\alpha p/2$. Then, for any $\gamma \in \mathbb{R}$ and $u \in C_c^\infty(D)$,*

$$\sum_{n\in\mathbb{Z}} e^{n(\theta-\alpha p/2)} \left\|\zeta_{-n}(e^n\cdot)\Delta^{\alpha/2}\Big([1-\eta_{-n}(e^n\cdot)]u(e^n\cdot)\Big)\right\|^p_{H_p^\gamma} \leq C\|\psi^{-\alpha/2}u\|^p_{L_{p,\theta}(D)}. \tag{4.5}$$

where $C = C(d,p,\gamma,\alpha,\theta,D)$.

Proof. It is certainly enough to prove (4.5) for only $\gamma = m \in \mathbb{N}_+$.

By the choice of $\{\eta_{-n} : n \in \mathbb{Z}\}$, we have $\zeta_n(x) = \zeta_n(x)\eta_n(x)$ for all x, and

$$\zeta_{-n}(e^n x)(1-\eta_{-n}(e^n(x+y))) = 0 \quad \text{if} \quad |y| < \delta_0,$$

where $\delta_0 := 1-e^{-1}$. Thus, by (2.1),

$$\begin{aligned} F_n(x) &:= \zeta_{-n}(e^n x)\Delta^{\alpha/2}\Big([1-\eta_{-n}(e^n\cdot)]u(e^n\cdot)\Big)(x) \\ &= C\int_{|y|\geq\delta_0} u(e^n(x+y))\zeta_{-n}(e^n x)[1-\eta_{-n}(e^n(x+y))]|y|^{-d-\alpha}dy \\ &= C\int_{|x-y|\geq\delta_0} u(e^n y)\Big(\zeta_{-n}(e^n x)[1-\eta_{-n}(e^n y)]|x-y|^{-d-\alpha}\Big)dy. \end{aligned} \tag{4.6}$$

Denote

$$B_n := supp(\zeta_{-n}).$$

Then, since $\zeta_{-n}(e^n x)(1-\eta_{-n}(e^n y)) = 0$ for $|x-y| < \delta_0$, by (2.13), we have

$$\begin{aligned} &\Big|D_x\Big(\zeta_{-n}(e^n x)(1-\eta_{-n}(e^n y))|x-y|^{-d-\alpha}\Big)\Big| \\ &\leq C1_{B_n}(e^n x)|1-\eta_{-n}(e^n y)||x-y|^{-d-\alpha} \\ &\quad + C|\zeta_{-n}(e^n x)(1-\eta_{-n}(e^n y))||x-y|^{-d-\alpha-1} \\ &\leq C1_{B_n}(e^n x)|1-\eta_{-n}(e^n y)||x-y|^{-d-\alpha}. \end{aligned}$$

Similarly, for $k \in \mathbb{N}_+$,

$$\begin{aligned} &\Big|D^k_x\Big(\zeta_{-n}(e^n x)(1-\eta_{-n}(e^n y))|x-y|^{-d-\alpha}\Big)\Big| \\ &\leq C(k)1_{B_n}(e^n x)|1-\eta_{-n}(e^n y)||x-y|^{-d-\alpha}. \end{aligned}$$

It follows from (4.6) for each $k \in \mathbb{N}_+$,

$$|D^k_x F_n(x)| \leq C(k)H_n(x), \tag{4.7}$$

where

$$H_n(x) := 1_{B_n}(e^n x)\int_{|x-y|\geq\delta_0} |u(e^n y)|\,|1-\eta_{-n}(e^n y)|\,|x-y|^{-d-\alpha}dy.$$

Since $\|F_n\|_{H_p^m} \approx \sum_{k\le m}\|D_x^k F_n\|_{L_p}$, from (4.7) we get

$$\sum_{n\in\mathbb{Z}} e^{n(\theta-\alpha p/2)}\|\zeta_{-n}(e^n\cdot)\Delta^{\alpha/2}\big([1-\eta_{-n}(e^n\cdot)]u(e^n\cdot)\big)\|^p_{H_p^m}$$
$$\le C\sum_{n\in\mathbb{Z}} e^{n(\theta-\alpha p/2)}\|H_n\|^p_{L_p}.$$

Therefore, to finish the proof of (4.5), we only need to show

$$\sum_{n\in\mathbb{Z}} e^{n(\theta-\alpha p/2)}\|H_n\|^p_{L_p} \le C\|\psi^{-\alpha/2}u\|^p_{L_{p,\theta}(D)}. \tag{4.8}$$

Case 1. Let $d-1+\alpha p/2<\theta<d-1+p+\alpha p/2$. Observe that

$$\int_{|y|\ge\delta_0}|u(e^n(x+y))(1-\eta_{-n}(e^n(x+y)))||y|^{-d-\alpha}dy$$
$$\le \sum_{k=0}^{\infty}\int_{2^k\delta_0\le|y|<2^{k+1}\delta_0}|u(e^n(x+y))||y|^{-d-\alpha}dy$$
$$=C(d)e^{n\alpha}\sum_{k=0}^{\infty}\int_{2^ke^n\delta_0\le|y|<2^{k+1}e^n\delta_0}|u(e^nx+y)||y|^{-d-\alpha}dy$$
$$\le C\sum_{k=0}^{\infty}2^{-k\alpha}\frac{1}{e^{nd}2^{kd}}\int_{2^ke^n\delta_0\le|y|<2^{k+1}e^n\delta_0}|u(e^nx+y)|dy$$
$$\le C\sum_{k=0}^{\infty}2^{-k\alpha}\mathbb{M}u(e^nx)=C\mathbb{M}u(e^nx),$$

where $\mathbb{M}u$ is the maximal function of u defined by

$$\mathbb{M}u(x)=\sup_{x\in B_r(z)}\frac{1}{|B_r(z)|}\int_{B_r(z)}|u(y)|dy.$$

Therefore, $H_n(x)\le C(1_{B_n}\mathbb{M}u)(e^nx)$. Since $e^n\approx\rho$ on B_n, by the change of variables,

$$\sum_{n\in\mathbb{Z}} e^{n(\theta-\alpha p/2)}\|H_n\|^p_{L_p}\le C\int_D|\mathbb{M}u(x)|^p\rho(x)^{\theta-d-\alpha p/2}dx.$$

Due to [19, Theorem 1.1], the function $\rho^{\theta-\alpha p/2-d}$ belongs to the class of Muckenhoupt A_p-weights, and therefore we can apply the Hardy-Littlewood Maximal inequality ([22, Theorem 7.1.9]) to get

$$\sum_{n\in\mathbb{Z}} e^{n(\theta-\alpha p/2)}\|H_n\|^p_{L_p}\le C\int_{\mathbb{R}^d}|u(x)|^p\rho(x)^{\theta-d-\alpha p/2}dx.$$

This proves (4.8) if $d-1+\alpha p/2<\theta<d-1+p+\alpha p/2$.

Case 2. Let $d-1-\alpha p/2<\theta<d+\alpha p/2$. Then, we can choose $\beta\in(0,\alpha)$ such that

$$-1<\theta-d-\alpha p/2+\beta p\le 0.$$

By (4.6) and Hölder's inequality, for $p' := p/(p-1)$,

$$\begin{aligned} H_n(x) &\leq 1_{B_n}(e^n x)\left(\int_{|x-y|\geq\delta_0}\frac{|u(e^ny)(1-\eta_{-n}(e^ny))|^p}{|x-y|^{d+\beta p}}dy\right)^{1/p} \\ &\qquad\times\left(\int_{|x-y|\geq\delta_0}|x-y|^{-d-(\alpha-\beta)p'}dy\right)^{1/p'} \\ &\leq C1_{B_n}(e^nx)\left(\int_{|x-y|\geq\delta_0}\frac{|u(e^ny)|^p}{|x-y|^{d+\beta p}}dy\right)^{1/p}. \end{aligned} \tag{4.9}$$

By the change of variables and Fubini's theorem,

$$\begin{aligned} &\sum_{n\in\mathbb{Z}} e^{n(\theta-\alpha p/2)}\|H_n\|^p_{L_p} \\ &\leq \sum_{n\in\mathbb{Z}} e^{n(\theta-\alpha p/2)}\int_{\mathbb{R}^d}\int_{|x-y|\geq\delta_0} 1_{B_n}(e^nx)\frac{|u(e^ny)|^p}{|x-y|^{d+\beta p}}dydx \\ &= \int_{\mathbb{R}^d}\left[\sum_{n\in\mathbb{Z}} e^{n(\theta-d-\alpha p/2+\beta p)}\int_{|x-y|\geq e^n\delta_0} 1_{B_n}(x)|x-y|^{-d-\beta p}dx\right]|u(y)|^p dy. \end{aligned} \tag{4.10}$$

In the virtue of (4.9) and (4.10), to prove (4.8), it suffices to show that for $y\in D$,

$$\sum_{n\in\mathbb{Z}} e^{n(\theta-d-\alpha p/2+\beta p)}\int_{|x-y|\geq e^n\delta_0} 1_{B_n}(x)|x-y|^{-d-\beta p}dx \leq C d_y^{\theta-d-\alpha p/2}. \tag{4.11}$$

For fixed $y\in D$, we take $n_0 = n_0(y)\in\mathbb{Z}$ such that

$$e^{n_0+3}\leq d_y < e^{n_0+4}.$$

If $n\leq n_0$ and $x\in B_n$, then $e^n < d_x < e^{n+2}\leq e^{n_0+2} < e^{n_0+3}\leq d_y$, and consequently $|x-y|\geq d_y - d_x \geq Ce^{n_0}$. Thus,

$$\begin{aligned} &\sum_{n\leq n_0} e^{n(\theta-d-\alpha p/2+\beta p)}\int_{|x-y|\geq Ce^n} 1_{B_n}(x)|x-y|^{-d-\beta p}dx \\ &\leq C\int_{|x-y|\geq Ce^{n_0}}\sum_{n\leq n_0} 1_{B_n}(x)\frac{d_x^{\theta-d-\alpha p/2+\beta p}}{|x-y|^{d+\beta p}}dx \\ &\leq C\int_{|x-y|\geq Ce^{n_0}, d_x<d_y}\frac{d_x^{\theta-d-\alpha p/2+\beta p}}{|x-y|^{d+\beta p}}dx \\ &\leq Ce^{-n_0\beta p}d_y^{\theta-d-\alpha p/2+\beta p}\leq Cd_y^{\theta-d-\alpha p/2}, \end{aligned} \tag{4.12}$$

where C is independent of y. For the second inequality above, we used $\sum_{n\leq n_0} 1_{B_n}(x)\leq C1_{d_x<d_y}$, and for the third inequality, we used Lemma A.4(ii) with $\rho = Ce^{n_0}$ and $r = d_y$.

Next, we handle the summation for $n > n_0$. Since $\theta - \alpha p/2 - d < 0$,

$$\begin{aligned}
&\sum_{n>n_0} e^{n(\theta-d-\alpha p/2+\beta p)} \int_{|x-y|\geq \delta_0 e^n} 1_{B_n}(x)|x-y|^{-d-\beta p}dx \\
&\leq C \sum_{n>n_0} e^{n(\theta-d-\alpha p/2+\beta p)} \int_{|x-y|\geq \delta_0 e^n} |x-y|^{-d-\beta p}dx \\
&= C \sum_{n>n_0} e^{n(\theta-d-\alpha p/2)} = Ce^{n_0(\theta-d-\alpha p/2)} \leq C d_y^{\theta-d-\alpha p/2}. \qquad (4.13)
\end{aligned}$$

Combining (4.12) and (4.13), we obtain (4.11). Thus, (4.8) and the lemma are proved. □

Lemma 4.3. *Let $d-1-\alpha p/2 < \theta < d-1+p+\alpha p/2$ and $\gamma \in \mathbb{R}$. Then, for any $u \in C_c^\infty(D)$,*

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} e^{n(\theta-\alpha p/2)} \left\| \Delta^{\alpha/2}\Big(u(e^n\cdot)\zeta_{-n}(e^n\cdot)\Big) - \zeta_{-n}(e^n\cdot)\Delta^{\alpha/2}(u(e^n\cdot)) \right\|_{H_p^\gamma}^p \\
&\leq C\|\psi^{-\alpha/2}u\|^p_{H^{0\vee(\gamma+\alpha/2)}_{p,\theta}(D)}, \qquad (4.14)
\end{aligned}$$

where $C = C(d,p,\alpha,\theta,\gamma,D)$.

Proof. Recall $\eta_{-n}\zeta_{-n} = \zeta_{-n}$. Thus, by the triangle inequality,

$$\begin{aligned}
&\left\| \Delta^{\alpha/2}\Big(u(e^n\cdot)\zeta_{-n}(e^n\cdot)\Big) - \zeta_{-n}(e^n\cdot)\Delta^{\alpha/2}(u(e^n\cdot)) \right\|_{H_p^\gamma} \\
&\leq \left\| \Delta^{\alpha/2}\Big((u\zeta_{-n}\eta_{-n})(e^n\cdot)\Big) - \zeta_{-n}(e^n\cdot)\Delta^{\alpha/2}\Big((u\eta_{-n})(e^n\cdot)\Big) \right\|_{H_p^\gamma} \\
&\quad + \left\| \zeta_{-n}(e^n\cdot)\Delta^{\alpha/2}\Big([1-\eta_{-n}(e^n\cdot)]u(e^n\cdot)\Big) \right\|_{H_p^\gamma}.
\end{aligned}$$

Also, note

$$\|u\|_{H_p^{\gamma+\alpha/2}} \approx \Big(\|u\|_{H_p^\gamma} + \|\Delta^{\alpha/4}u\|_{H_p^\gamma} \Big).$$

Therefore, Lemma 4.1 and Lemma 4.2 easily lead to the claim of the lemma. □

Lemma 4.4. *Let $d-1-\alpha p/2 < \theta < d-1+p+\alpha p/2$, and $\gamma \geq -\alpha$. Then, for any $u \in C_c^\infty(D)$, we have $\Delta^{\alpha/2}u \in \psi^{-\alpha/2}H^\gamma_{p,\theta}(D)$ and*

$$\|\psi^{\alpha/2}\Delta^{\alpha/2}u\|_{H^\gamma_{p,\theta}(D)} \leq C\|\psi^{-\alpha/2}u\|_{H^{\gamma+\alpha}_{p,\theta}(D)}. \qquad (4.15)$$

Proof. By Lemma 2.7(*ii*) and the relation $\Delta^{\alpha/2}u(e^nx) = e^{-n\alpha}(\Delta^{\alpha/2}u(e^n\cdot))(x)$,

$$\begin{aligned}
\|\psi^{\alpha/2}\Delta^{\alpha/2}u\|^p_{H^\gamma_{p,\theta}(D)} &\leq C\sum_n e^{n(\theta+\alpha p/2)}\|\zeta_{-n}(e^n\cdot)\Delta^{\alpha/2}u(e^n\cdot)\|^p_{H_p^\gamma} \\
&= C\sum_n e^{n(\theta-\alpha p/2)}\|\zeta_{-n}(e^n\cdot)\Delta^{\alpha/2}(u(e^n\cdot))\|^p_{H_p^\gamma}.
\end{aligned}$$

By (4.14), the last term above is bounded by

$$\begin{aligned}
&C\sum_{n\in\mathbb{Z}} e^{n(\theta-\alpha p/2)} \left\| \Delta^{\alpha/2}\Big(u(e^n\cdot)\zeta_{-n}(e^n\cdot)\Big) \right\|^p_{H_p^\gamma} + C\|\psi^{-\alpha/2}u\|^p_{H^{0\vee(\gamma+\alpha/2)}_{p,\theta}(D)} \\
&\leq C\|\psi^{-\alpha/2}u\|^p_{H^{\gamma+\alpha}_{p,\theta}(D)}.
\end{aligned}$$

The lemma is proved. □

By Lemma 4.4, for any $\gamma_0 \in \mathbb{R}$ and $\phi \in C_c^\infty(D)$, $\Delta^{\alpha/2}\phi$ belongs to the dual space of $H^{\gamma_0+\alpha}_{p,\theta-\alpha p/2}(D)$ (see Lemma 2.7(iii)). Therefore, for $u \in \psi^{\alpha/2}H^{\gamma_0+\alpha}_{p,\theta}(D)$, we can define $\Delta^{\alpha/2}u$ as a distribution on D by

$$(\Delta^{\alpha/2}u,\phi)_D := (u,\Delta^{\alpha/2}\phi)_D, \quad \phi \in C_c^\infty(D). \tag{4.16}$$

Corollary 4.5. *Let $d-1-\alpha p/2 < \theta < d-1+p+\alpha p/2$.*

(i) Let $\gamma \in \mathbb{R}$, $u \in \psi^{\alpha/2}H^{\gamma+\alpha}_{p,\theta}(D)$, and $\Delta^{\alpha/2}u$ be defined as in (4.16)*. Then, $\Delta^{\alpha/2}u \in \psi^{-\alpha/2}H^{\gamma}_{p,\theta}(D)$, and* (4.15) *holds.*

(ii) If $\gamma \geq -\alpha/2$ and $u \in H^{\gamma+\alpha/2}_{p,\theta-\alpha p/2}(D)$, then the left-hand side of (4.14) *makes sense, and inequality* (4.14) *holds.*

Proof. If $\gamma \geq 0$, then (i) is a consequence of Lemma 4.4 and Lemma 2.7(i). If $\gamma < 0$, then by Lemmas 4.4 and 2.7(iii),

$$\begin{aligned} |(\Delta^{\alpha/2}u,\phi)_D| &\leq C\|\psi^{-\alpha/2}u\|_{H^{\gamma+\alpha}_{p,\theta}(D)}\|\psi^{\alpha/2}\Delta^{\alpha/2}\phi\|_{H^{-\gamma-\alpha}_{p',\theta'}(D)} \\ &\leq C\|\psi^{-\alpha/2}u\|_{H^{\gamma+\alpha}_{p,\theta}(D)}\|\psi^{-\alpha/2}\phi\|_{H^{-\gamma}_{p',\theta'}(D)}, \end{aligned}$$

where $1/p+1/p'=1$ and $\theta/p+\theta'/p'=d$. This implies $\Delta^{\alpha/2}$ is a bounded linear operator from $\psi^{\alpha/2}H^{\gamma+\alpha}_{p,\theta}(D)$ to $\psi^{-\alpha/2}H^{\gamma}_{p,\theta}(D)$. Thus, (i) is proved.

Next, we show (ii). The left-hand side of (4.14) makes sense due to (i) and (2.15). Now, the claim of (ii) follows from Lemma 4.3 and Lemma 2.7(i). The corollary is proved. □

Theorem 4.6 (Higher regularity for parabolic equation)**.** *Let $0 \leq \mu \leq \gamma$, and $\theta \in (d-1-\frac{\alpha p}{2}, d-1+p+\frac{\alpha p}{2})$. Suppose that $f \in \psi^{-\alpha/2}\mathbb{H}^{\gamma-\alpha}_{p,\theta}(D,T)$, $u_0 \in \psi^{\alpha/2-\alpha/p}B^{\gamma-\alpha/p}_{p,\theta}(D)$, and $u \in \psi^{\alpha/2}\mathbb{H}^{\mu}_{p,\theta}(D,T) \cap \{u = 0 \text{ on } [0,T]\times D^c\}$ is a weak solution to* (2.2)*. Then, $u \in \psi^{\alpha/2}\mathbb{H}^{\gamma}_{p,\theta}(D,T)$, and for this solution*

$$\begin{aligned} \|\psi^{-\alpha/2}u\|_{\mathbb{H}^{\gamma}_{p,\theta}(D,T)} \leq C\Big(&\|\psi^{-\alpha/2+\alpha/p}u_0\|_{B^{\gamma-\alpha/p}_{p,\theta}(D)} \\ &+ \|\psi^{\alpha/2}f\|_{\mathbb{H}^{\gamma-\alpha}_{p,\theta}(D,T)} + \|\psi^{-\alpha/2}u\|_{\mathbb{H}^{\mu}_{p,\theta}(D,T)}\Big), \end{aligned} \tag{4.17}$$

where $C = C(d,p,\alpha,\gamma,\mu,\theta,D)$.

Proof. **1**. We first note that it is enough to consider the case $\gamma \leq \mu+\alpha/2$. Indeed, if the claim holds for the case $\gamma \leq \mu+\alpha/2$, then repeating the result with $\mu' = \mu+\alpha/2, \mu+2\alpha/2, \cdots$ in order, we prove the lemma when $\gamma = \mu+k\alpha/2$, $k \in \mathbb{N}_+$. Now let $\gamma = \mu+k\alpha/2+c$, where $k \in \mathbb{N}_+$ and $c \in (0,\alpha/2)$. Then, applying the previous result with $\mu' = \mu+k\alpha/2$, we prove the general case.

2. For each $n \in \mathbb{Z}$, denote

$$u_n(t,x) := u(e^{n\alpha}t, e^n x), \quad f_n(t,x) := f(e^{n\alpha}t, e^n x), \quad u_{0n}(x) := u_0(e^n x).$$

Then, $u_n(\cdot)\zeta_{-n}(e^n\cdot) \in \mathbb{H}^{\mu}_p(e^{-n\alpha}T)$ and it is a weak solution (or solution in the sense of distribution) to the equation

$$\begin{cases} \partial_t v_n(t,x) = \Delta^{\alpha/2}v_n(t,x) + F_n(t,x), & (t,x) \in (0,e^{-n\alpha}T)\times\mathbb{R}^d \\ v_n(0,x) = (u_{0n}(\cdot)\zeta_{-n}(e^n\cdot))(x), & x \in \mathbb{R}^d \end{cases}$$

where

$$\begin{aligned}F_n(t,x) &= e^{n\alpha}(f_n(\cdot,\cdot)\zeta_{-n}(e^n\cdot))(t,x)\\ &\quad - \Big(\Delta^{\alpha/2}(u_n(\cdot,\cdot)\zeta_{-n}(e^n\cdot))(t,x) - \zeta_{-n}(e^nx)\Delta^{\alpha/2}u_n(t,x)\Big)\\ &=: e^{n\alpha}(f_n(\cdot,\cdot)\zeta_{-n}(e^n\cdot))(t,x) - G_n(t,x).\end{aligned}$$

3. By Corollary 4.5(ii) with $\gamma' = \mu - \alpha/2$, we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} e^{n(\theta-\alpha p/2)}\|G_n(e^{-n\alpha}t,\cdot)\|^p_{H_p^{\gamma-\alpha}} &\leq C\sum_{n\in\mathbb{Z}} e^{n(\theta-\alpha p/2)}\|G_n(e^{-n\alpha}t,\cdot)\|^p_{H_p^{\mu-\alpha/2}}\\ &\leq C\|\psi^{-\alpha/2}u(t,\cdot)\|^p_{H^\mu_{p,\theta}(D)}. \end{aligned}\tag{4.18}$$

Therefore, due to $f\in\psi^{-\alpha/2}\mathbb{H}^{\gamma-\alpha}_{p,\theta}(D,T)$,

$$F_n \in \mathbb{H}_p^{\gamma-\alpha}(e^{-n\alpha}T).$$

Thus, we apply [45, Theorem 1] to conclude $u_n\zeta_{-n}(e^n\cdot)\in\mathbb{H}^\gamma_p(e^{-n\alpha}T)$, and

$$\begin{aligned}&\|\Delta^{\alpha/2}(u(\cdot,e^n\cdot)\zeta_{-n}(e^n\cdot))\|^p_{\mathbb{H}_p^{\gamma-\alpha}(T)}\\ &= e^{n\alpha}\|\Delta^{\alpha/2}(u_n(\cdot,\cdot)\zeta_{-n}(e^n\cdot))\|^p_{\mathbb{H}_p^{\gamma-\alpha}(e^{-n\alpha}T)}\\ &\leq Ce^{n\alpha}\|\zeta_{-n}(e^n\cdot)u_{0n}(\cdot)\|^p_{B_p^{\gamma-\alpha/p}} + Ce^{n\alpha}\|G_n(\cdot,\cdot)\|^p_{\mathbb{H}_p^{\gamma-\alpha}(e^{-n\alpha}T)}\\ &\quad + Ce^{n\alpha}\|e^{n\alpha}\zeta_{-n}(e^n\cdot)f_n(\cdot,\cdot)\|^p_{\mathbb{H}_p^{\gamma-\alpha}(e^{-n\alpha}T)}\\ &= Ce^{n\alpha}\|\zeta_{-n}(e^n\cdot)u_{0n}(\cdot)\|^p_{B_p^{\gamma-\alpha/p}} + C\|G_n(e^{-n\alpha}\cdot,\cdot)\|^p_{\mathbb{H}_p^{\gamma-\alpha}(T)}\\ &\quad + C\|e^{n\alpha}\zeta_{-n}(e^n\cdot)f(\cdot,e^n\cdot)\|^p_{\mathbb{H}_p^{\gamma-\alpha}(T)}. \end{aligned}\tag{4.19}$$

By (4.18) and (4.19) (also see Lemma 2.7(ii)),

$$\begin{aligned}&\sum_{n\in\mathbb{Z}} e^{n(\theta-\alpha p/2)}\|\Delta^{\alpha/2}(u(\cdot,e^n\cdot)\zeta_{-n}(e^n\cdot))\|^p_{\mathbb{H}_p^{\gamma-\alpha}(T)}\\ &\leq C\Big(\|\psi^{-\alpha/2+\alpha/p}u_0\|^p_{B^{\gamma-\alpha/p}_{p,\theta}(D)}\\ &\quad + \|\psi^{\alpha/2}f\|^p_{\mathbb{H}^{\gamma-\alpha}_{p,\theta}(D,T)} + \|\psi^{-\alpha/2}u\|^p_{\mathbb{H}^\mu_{p,\theta}(D,T)}\Big). \end{aligned}\tag{4.20}$$

Therefore, (4.20), Lemma 2.7(ii), and the relation

$$\|u\|_{H^\gamma_p} \approx \Big(\|u\|_{H_p^{\gamma-\alpha}} + \|\Delta^{\alpha/2}u\|_{H_p^{\gamma-\alpha}}\Big)$$

yield (4.17) for $\gamma\leq\mu+\alpha/2$. The theorem is proved. □

Theorem 4.7 (Higher regularity for elliptic equation)**.** *Let $\lambda\geq 0$, $0\leq\mu\leq\gamma$, and $\theta\in(d-1-\frac{\alpha p}{2}, d-1+p+\frac{\alpha p}{2})$. Suppose that $f\in\psi^{-\alpha/2}H^{\gamma-\alpha}_{p,\theta}(D)$, and $u\in\psi^{\alpha/2}H^\mu_{p,\theta}(D)\cap\{u=0 \;on\; D^c\}$ is a solution to (2.4), then, $u\in\psi^{\alpha/2}H^\gamma_{p,\theta}(D)$, and moreover*

$$\begin{aligned}&\lambda\|\psi^{\alpha/2}u\|_{H^{\gamma-\alpha}_{p,\theta}(D)} + \|\psi^{-\alpha/2}u\|_{H^\gamma_{p,\theta}(D)}\\ &\leq C\left(\|\psi^{\alpha/2}f\|_{H^{\gamma-\alpha}_{p,\theta}(D)} + \|\psi^{-\alpha/2}u\|_{H^\mu_{p,\theta}(D)}\right),\end{aligned}$$

where $C = C(d,p,\alpha,\gamma,\mu,\theta,D)$. In particular, C is independent of λ.

Proof. We repeat the argument of the proof of Theorem 4.6. As before, we may assume $\gamma \leq \mu + \alpha/2$.

Let $n \in \mathbb{Z}$. Since u is a weak solution to (2.4), $u_n(x) := u(e^n x)$ and $f_n(x) := f(e^n x)$ satisfy the following equation in weak sense;

$$\Delta^{\alpha/2}(u_n(\cdot)\zeta_{-n}(e^n\cdot))(x) - e^{n\alpha}\lambda(u_n(\cdot)\zeta_{-n}(e^n\cdot))(x) = F_n(x), \quad x \in \mathbb{R}^d, \tag{4.21}$$

where

$$\begin{aligned} F_n(x) &= e^{n\alpha} f_n(x)\zeta_{-n}(e^n x) - G_n(x) \\ &:= e^{n\alpha} f_n(x)\zeta_{-n}(e^n x) - \left(\Delta^{\alpha/2}(u_n(\cdot)\zeta_{-n}(e^n\cdot))(x) - \zeta_{-n}(e^n x)\Delta^{\alpha/2}u_n(x)\right). \end{aligned}$$

By Corollary 4.5(ii), we get $G_n \in H_p^{\gamma-\alpha}$ and

$$\begin{aligned} \sum_{n\in\mathbb{Z}} e^{n(\theta-\alpha p/2)}\|G_n\|^p_{H_p^{\gamma-\alpha}} &\leq C\sum_{n\in\mathbb{Z}} e^{n(\theta-\alpha p/2)}\|G_n\|^p_{H_p^{\mu-\alpha/2}} \\ &\leq C\|\psi^{-\alpha/2}u\|^p_{H^\mu_{p,\theta}(D)}. \end{aligned} \tag{4.22}$$

This implies that $F_n \in H_p^{\gamma-\alpha}$.

If $\lambda = 0$, then the equality (4.21) easily yields

$$\begin{aligned} \|\Delta^{\alpha/2}(u_n(\cdot)\zeta_{-n}(e^n\cdot))\|^p_{H_p^{\gamma-\alpha}} &= \|F_n\|^p_{H_p^{\gamma-\alpha}} \\ &\leq \|e^{n\alpha} f_n(\cdot)\zeta_{-n}(e^n\cdot)\|^p_{H_p^{\gamma-\alpha}} + \|G_n\|^p_{H_p^{\gamma-\alpha}}. \end{aligned} \tag{4.23}$$

Next, let $\lambda > 0$. Then, by [44, Theorem 1] (or [17, Theorem 2.1]), we have $u_n(\cdot)\zeta_{-n}(e^n\cdot) \in H_p^{\gamma-\alpha}$ and

$$\begin{aligned} & e^{n\alpha p}\lambda^p\|u_n(\cdot)\zeta_{-n}(e^n\cdot)\|^p_{H_p^{\gamma-\alpha}} + \|\Delta^{\alpha/2}(u_n(\cdot)\zeta_{-n}(e^n\cdot))\|^p_{H_p^{\gamma-\alpha}} \\ &\leq C\|F_n\|^p_{H_p^{\gamma-\alpha}} \leq C\left(\|e^{n\alpha} f_n(\cdot)\zeta_{-n}(e^n\cdot)\|^p_{H_p^{\gamma-\alpha}} + \|G_n\|^p_{H_p^{\gamma-\alpha}}\right). \end{aligned} \tag{4.24}$$

We multiply by $e^{n(\theta-\alpha p/2)}$ to (4.23) and (4.24), then take sum over $n \in \mathbb{Z}$. Finally, we use (4.22) and Lemma 2.7(ii) to finish the proof of the theorem. □

5. Proof of Theorems 2.2, 2.3, 2.9 and 2.10

We only need to prove Theorems 2.2 and 2.3. This is because Theorem 2.9 is a consequence of Theorems 2.2 and 4.6, and Theorem 2.10 is a consequence of Theorems 2.3 and 4.7.

Proof of Theorem 2.2

1. Existence and estimate of solution.

First, assume $u_0 \in C_c^\infty(D)$ and $f \in C_c^\infty((0,T)\times D)$. Then, by Lemma 3.2, the function u defined in (2.6) becomes a weak solution to (2.2). Also, by Lemmas 3.6 and 3.7,

$$\|\psi^{-\alpha/2}u\|_{\mathbb{L}_{p,\theta}(D,T)} \leq C\left(\|\psi^{-\alpha/2+\alpha/p}u_0\|_{L_{p,\theta}(D)} + \|\psi^{\alpha/2}f\|_{\mathbb{L}_{p,\theta}(D,T)}\right).$$

Now we fix $\gamma \in (0,\alpha/p)$. By (2.18), we have $L_{p,\theta'}(D) \subset B^{\gamma-\alpha/p}_{p,\theta'}(D)$ for any $\theta' \in \mathbb{R}$, and therefore applying Theorem 4.6 with $\mu = 0$, we conclude $u \in \psi^{\alpha/2}\mathbb{H}^\gamma_{p,\theta}(D,T)$ and

$$\|\psi^{-\alpha/2}u\|_{\mathbb{H}^\gamma_{p,\theta}(D,T)} \leq C\left(\|\psi^{-\alpha/2+\alpha/p}u_0\|_{L_{p,\theta}(D)} + \|\psi^{\alpha/2}f\|_{\mathbb{L}_{p,\theta}(D,T)}\right).$$

Using this and Corollary 4.5(i), we have $u_t = \Delta^{\alpha/2}u + f \in \psi^{-\alpha/2}\mathbb{H}^{\gamma-\alpha}_{p,\theta}(D,T)$, $u \in \mathfrak{H}^{\gamma}_{p,\theta}(D,T)$, and

$$\|u\|_{\mathfrak{H}^{\gamma}_{p,\theta}(D,T)} \leq C\left(\|\psi^{-\alpha/2+\alpha/p}u_0\|_{L_{p,\theta}(D)} + \|\psi^{\alpha/2}f\|_{\mathbb{L}_{p,\theta}(D,T)}\right). \tag{5.1}$$

For general data, we take $\{u_{0n}\}_{n\in\mathbb{N}} \subset C^{\infty}_c(D)$ and $\{f_n\}_{n\in\mathbb{N}} \subset C^{\infty}_c((0,T)\times D)$ such that

$$u_{0n} \to u_0 \quad \text{in} \quad \psi^{\alpha/2-\alpha/p}B^{\gamma-\alpha/p}_{p,\theta}(D),$$
$$f_n \to f \quad \text{in} \quad \psi^{-\alpha/2}\mathbb{H}^{\gamma}_{p,\theta}(D,T).$$

Define u_n (resp. u) by (2.6) with u_{0n} (resp. u_0) and f_n (resp. f). Then, by Lemmas 3.6 and 3.7, u_n converges to u in the space $\mathbb{L}_{p,\theta-\alpha p/2}(D,T)$. Also, considering the estimate (5.1) corresponding to $u_n - u_m$, we conclude u_n is a Cauchy sequence in $\mathfrak{H}^{\gamma}_{p,\theta}(D,T)$. Let v denote the limit of u_n in $\mathfrak{H}^{\gamma}_{p,\theta}(D,T)$. Then, $v = u$ (a.e.) and therefore u (or its version) is in $\mathfrak{H}^{\gamma}_{p,\theta}(D,T)$.

Now we prove that (2.3) holds for all $t \leq T$. Since u_n is a solution to (2.2) in the sense of Definition 2.1, taking $n \to \infty$ and using

$$\|\psi^{-\alpha/2}(u_n - u)\|_{\mathbb{H}^{\gamma}_{p,\theta}(D,T)} + \|\psi^{\alpha/2}(\Delta^{\alpha/2}u_n - \Delta^{\alpha/2}u)\|_{\mathbb{H}^{\gamma-\alpha}_{p,\theta}(D,T)} \to 0,$$

we find that (2.3) holds for u almost everywhere on $[0,T]$. By Theorem 2.15, we know that $(u(t)-u_0,\phi)_D$ is a continuous in t, and therefore we conclude that (2.3) holds for all $t \leq T$. Thus, u becomes a weak solution. (2.7) also follows from the estimates of u_n.

2. Uniqueness.

Let $u \in \psi^{\alpha/2}\mathbb{L}_{p,\theta}(D,T) \cap \{u = 0 \text{ on } [0,T]\times D^c\}$ be a weak solution to

$$\begin{cases} \partial_t u(t,x) = \Delta^{\alpha/2}u(t,x), & (t,x) \in (0,T)\times D,\\ u(0,x) = 0, & x \in D,\\ u(t,x) = 0, & (t,x) \in [0,T]\times D^c. \end{cases}$$

Then, by Theorem 4.6 with $\mu = 0$ and $\gamma > 0$, we have $u \in \psi^{\alpha/2}\mathbb{H}^{\gamma}_{p,\theta}(D,T)$ for any $\gamma > 0$. This and Corollary 4.5(i) imply $u \in \mathfrak{H}^{\gamma+\alpha}_{p,\theta}(D,T)$ for any $\gamma \in \mathbb{R}$.

Now we take a sequence $u_n \in C^{\infty}_c([0,T]\times D)$ (cf. Remark 2.8(ii)) such that $u_n \to u$ in $\mathfrak{H}^{\gamma+\alpha}_{p,\theta}(D,T)$. In particular, $u_n \to u$, $u_n(0,\cdot) \to 0$, and $\partial_t u_n \to \partial_t u$ in their corresponding spaces. Define $f_n := \partial_t u_n - \Delta^{\alpha/2}u_n$, then u_n trivially satisfies

$$\partial_t u_n - \Delta^{\alpha/2}u_n = f_n.$$

By Lemma 3.2(ii),

$$u_n(t,x) = \mathbb{E}_x[u_n(0,X^D_t)] + \int_0^t \mathbb{E}_x[f_n(s,X^D_{t-s})]ds.$$

Also, by Lemmas 3.6 and 3.7 and Theorem 4.6,

$$\begin{aligned} &\|\psi^{-\alpha/2}u_n\|_{\mathbb{H}^{\gamma+\alpha}_{p,\theta}(D,T)} \\ &\leq C\left(\|\psi^{\alpha/p-\alpha/2}u_n(0,\cdot)\|_{B^{\gamma+\alpha-\alpha/p}_{p,\theta}(D)} + \|\psi^{\alpha/2}f_n\|_{\mathbb{H}^{\gamma}_{p,\theta}(D,T)}\right). \end{aligned} \tag{5.2}$$

By Corollary 4.5(i), we have $\Delta^{\alpha/2}u_n \to \Delta^{\alpha/2}u$ in $\psi^{-\alpha/2}\mathbb{H}^{\gamma}_{p,\theta}(D,T)$, and therefore

$$f_n = \partial_t u_n - \Delta^{\alpha/2}u_n \to \partial_t u - \Delta^{\alpha/2}u = 0$$

as $n\to\infty$ in the space $\psi^{-\alpha/2}\mathbb{H}^{\gamma}_{p,\theta}(D,T)$. From (5.2), we conclude that $u=0$. The uniqueness is also proved.

Proof of Theorem 2.3

If $\lambda>0$ or D is bounded, then it is enough to repeat the proof of Theorem 2.2. During the proof, one only needs to replace results for parabolic equations by their corresponding elliptic versions.

Therefore, we only consider the case when $\lambda=0$ and D is a half space.

1. A priori estimate and uniqueness.

We first prove the a priori estimate

$$\|\psi^{-\alpha/2}u\|_{H^{\alpha}_{p,\theta}(D)}\leq C\|\psi^{\alpha/2}f\|_{L_{p,\theta}(D)} \tag{5.3}$$

holds given that $u\in L_{p,\theta-\alpha p/2}(D)\cap\{u=0\,\text{on}\,D^c\}$ is a weak solution to (2.4).

Note that by Theorem 4.7, we have $u\in\psi^{\alpha/2}H^{\alpha}_{p,\theta}(D)$. Assume $u\in C_c^{\infty}(D)$ for a moment. Then, for any $\lambda>0$,

$$\Delta^{\alpha/2}u-\lambda u=f-\lambda u \text{ on } D,$$

where $f:=\Delta^{\alpha/2}u$. Thus, applying (2.22) for $\lambda>0$ and letting $\lambda\downarrow 0$, we get estimate (5.3) for $\lambda=0$. For general case, we take $u_n\in C_c^{\infty}(D)$ such that $u_n\to u$ in $\psi^{\alpha/2}H^{\alpha}_{p,\theta}(D)$. Then, by Corollary 4.5(i), $\Delta^{\alpha/2}u_n\to\Delta^{\alpha/2}u=f$ in $\psi^{-\alpha/2}L_{p,\theta}(D)$. Consequently, this leads to (5.3), which certainly implies

$$\|\psi^{-\alpha/2}u\|_{L_{p,\theta}(D)}\leq C\|\psi^{\alpha/2}f\|_{L_{p,\theta}(D)}.$$

The uniqueness result of solution easily follows from this.

2. Weak convergence and existence.

Let $u_n\in\psi^{\alpha/2}L_{p,\theta}(D)\cap\{u=0\text{ on}\,D^c\}$ denote the solution to equation (2.4) corresponding to $\lambda=\frac{1}{n}$. Then, by (2.22), $\{u_n\}$ is a bounded sequence in the space $L_p(\mathbb{R}^d,\rho^{\theta-d-\alpha p/2}dx)$, and therefore there exists a subsequence $\{u_{n_i}\}$ which converges weakly to some $u\in L_p(\mathbb{R}^d,\rho^{\theta-d-\alpha p/2}dx)$. Obviously, we have $u=0$ (a.e.) on D^c. By Lemma 4.4, for any $\phi\in C_c^{\infty}(D)$, $\Delta^{\alpha/2}\phi$ belongs to the dual space of $\psi^{\alpha/2}H^{\alpha}_{p,\theta}(D)$. Therefore,

$$(u_{n_i},\phi)_{\mathbb{R}^d}=(u_{n_i},\phi)_D\to(u,\phi)_D=(u,\phi)_{\mathbb{R}^d}$$

and

$$(u_{n_i},\Delta^{\alpha/2}\phi)_{\mathbb{R}^d}=(u_{n_i},\Delta^{\alpha/2}\phi)_D\to(u,\Delta^{\alpha/2}\phi)_D=(u,\Delta^{\alpha/2}\phi)_{\mathbb{R}^d},$$

as $n_i\to\infty$. Thus, we conclude u is a weak solution to (2.4) in $L_{p,\theta-\alpha p/2}(D)\cap\{u=0\,\text{on}\,D^c\}$. Now we prove the weak convergence. The above argument shows that any subsequence of u_n has a further subsequence which converges weakly in $L_p(\mathbb{R}^d,\rho^{\theta-d-\alpha p/2}dx)$, and the limit becomes a solution to (2.4) in $L_{p,\theta-\alpha p/2}(D)\cap\{u=0\,\text{on}\,D^c\}$. Due to the uniqueness of solution proved above, we conclude that this limit coincides with u. This proves the weak convergence, and the theorem is proved.

Appendix A. Auxiliary results

Recall that $p(t,x)=p_d(t,x)$ is the transition density function of a rotationally symmetric α-stable d-dimensional Lévy process. It is a radial function and

$$p_d(t,x)\approx t^{-\frac{d}{\alpha}}\wedge\frac{t}{|x|^{d+\alpha}}\approx\frac{t}{(t^{1/\alpha}+|x|)^{d+\alpha}}, \tag{A.1}$$

and

$$p_d(t,x) = t^{-\frac{d}{\alpha}} p_d(1, t^{-\frac{1}{\alpha}}x). \tag{A.2}$$

If f is a radial function, then we put $f(r) := f(x)$ if $r = |x|$.

Lemma A.1. *(i) Let $d \geq 2$ and f be a nonnegative radial function on $\mathbb{R}^d$. Then, for $x^1 \neq 0$,*

$$\int_{\mathbb{R}^{d-1}} f(x^1, x')dx' = C(d)|x^1|^{d-1} \int_0^\infty f(|x^1|(1+s^2)^{1/2})s^{d-2}ds. \tag{A.3}$$

(ii) Let $d \geq 2$. For any $t > 0$ and $x^1 \neq 0$,

$$\int_{\mathbb{R}^{d-1}} p_d(t, x^1, x')dx' \approx p_1(t, x^1), \tag{A.4}$$

where the comparability relation depends only on d and α.

Proof. (i) By the change of variables,

$$\begin{aligned}\int_{\mathbb{R}^{d-1}} f(x^1, x')dx' &= \int_{\mathbb{R}^{d-1}} f(x^1, |x^1|x')|x^1|^{d-1}dx' \\ &= |x^1|^{d-1} \int_{\mathbb{R}^{d-1}} f(|x^1|(1+|x'|^2)^{1/2})dx' \\ &= C(d)|x^1|^{d-1} \int_0^\infty f(|x^1|(1+s^2)^{1/2})s^{d-2}ds.\end{aligned}$$

(ii) By (A.1) and (A.3), it suffices to prove that

$$\int_0^\infty \frac{t|x^1|^{d-1}s^{d-2}}{(t^{1/\alpha} + |x^1|(1+s^2)^{1/2})^{d+\alpha}}ds \approx \left(t^{-\frac{1}{\alpha}} \wedge \frac{t}{|x^1|^{1+\alpha}}\right). \tag{A.5}$$

Let $t|x^1|^{-\alpha} \leq 1$, then

$$\begin{aligned}\int_0^\infty \frac{t|x^1|^{d-1}s^{d-2}}{(t^{1/\alpha} + |x^1|(1+s^2)^{1/2})^{d+\alpha}}ds &\approx \int_0^\infty \frac{t|x^1|^{d-1}s^{d-2}}{\left(|x^1|(1+s^2)^{1/2}\right)^{d+\alpha}}ds \\ &= C(d,\alpha)\frac{t}{|x^1|^{1+\alpha}}. \end{aligned}\tag{A.6}$$

Now let $t|x^1|^{-\alpha} \geq 1$. We put

$$\int_0^\infty \frac{t|x^1|^{d-1}s^{d-2}}{(t^{1/\alpha} + |x^1|(1+s^2)^{1/2})^{d+\alpha}}ds = \int_0^{t^{1/\alpha}|x^1|^{-1}} \cdots + \int_{t^{1/\alpha}|x^1|^{-1}}^\infty \cdots =: I + II.$$

Then,

$$\begin{aligned}I &\leq t^{-\frac{d}{\alpha}}|x^1|^{d-1} \int_0^{t^{1/\alpha}|x^1|^{-1}} s^{d-2}ds = C(d,\alpha)t^{-\frac{1}{\alpha}}, \\ II &\leq t|x^1|^{-1-\alpha} \int_{t^{1/\alpha}|x^1|^{-1}}^\infty \frac{s^{d-2}}{(1+s^2)^{\frac{d+\alpha}{2}}}ds = C(\alpha)t^{-\frac{1}{\alpha}}.\end{aligned}$$

Therefore, the left-hand side of (A.5) is controlled by the right-hand side. Due to (A.6), to prove (A.5), we only need a proper lower bound of I. Let $t|x^1|^{-\alpha} \geq 1$.

By the changing variables $s = t^{1/\alpha}|x^1|^{-1}l$,

$$I \geq \int_0^{t^{1/\alpha}|x^1|^{-1}} \frac{t|x^1|^{d-1}s^{d-2}}{(t^{1/\alpha} + (t^{2/\alpha} + |x^1|^{-2}s^2)^{1/2})^{d+\alpha}} ds$$
$$= t^{-\frac{1}{\alpha}} \int_0^1 \frac{l^{d-2}}{(1 + (1 + l^2)^{1/2})^{d+\alpha}} dl = C(d, \alpha) t^{-\frac{1}{\alpha}}.$$

The lemma is proved. □

Lemma A.2. *Let $\alpha \in (0, 2)$ and γ_0, $\gamma_1 \in \mathbb{R}$. Suppose that*

$$-\frac{2}{\alpha} < \gamma_0, \quad -2 < \gamma_1 - \gamma_0 \leq 2 + \frac{2}{\alpha}. \tag{A.7}$$

Then, for any $(t, x) \in (0, \infty) \times \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p(t, x - y) \frac{|y^1|^{\gamma_0 \alpha/2}}{(\sqrt{t} + |y^1|^{\alpha/2})^{\gamma_1}} dy \leq C(\sqrt{t} + |x^1|^{\alpha/2})^{\gamma_0 - \gamma_1}. \tag{A.8}$$

where $C = C(d, \alpha, \gamma_0, \gamma_1)$.

Proof. It suffices to prove (A.8) when $t = 1$. Indeed, if it holds for $t = 1$, then by (A.2),

$$\int_{\mathbb{R}^d} p(t, x - y) \frac{|y^1|^{\gamma_0 \alpha/2}}{(\sqrt{t} + |y^1|^{\alpha/2})^{\gamma_1}} dy$$
$$= Ct^{\frac{\gamma_0 - \gamma_1}{2}} \int_{\mathbb{R}^d} p(1, t^{-\frac{1}{\alpha}}x - y) \frac{|y^1|^{\gamma_0 \alpha/2}}{(1 + |y^1|^{\alpha/2})^{\gamma_1}} dy$$
$$\leq Ct^{\frac{\gamma_0 - \gamma_1}{2}} (1 + t^{-\frac{1}{2}}|x^1|^{\alpha/2})^{\gamma_0 - \gamma_1}$$
$$= C(\sqrt{t} + |x^1|^{\alpha/2})^{\gamma_0 - \gamma_1}.$$

Thus, we may assume $t = 1$. By (A.4) and (A.1),

$$\int_{\mathbb{R}^d} p_d(1, x - y) \frac{|y^1|^{\gamma_0 \alpha/2}}{(1 + |y^1|^{\alpha/2})^{\gamma_1}} dy$$
$$\approx \int_{\mathbb{R}} \left(1 \wedge \frac{1}{|x^1 - y^1|^{1+\alpha}}\right) \frac{|y^1|^{\gamma_0 \alpha/2}}{(1 + |y^1|^{\alpha/2})^{\gamma_1}} dy^1 =: I(x^1).$$

Thus, it only remains to show for $x^1 \in \mathbb{R}$,

$$I(x^1) \leq C(1 + |x^1|^{\alpha/2})^{\gamma_0 - \gamma_1}. \tag{A.9}$$

Case 1. Let $|x^1| \leq 1$. Put

$$I(x^1) = \int_{|y^1| \leq 2} \cdots dy^1 + \int_{|y^1| > 2} \cdots dy^1 =: I_1(x^1) + I_2(x^1).$$

If $|y^1| \leq 2$, then by (A.7),

$$I_1(x^1) \leq C \int_{|y^1| \leq 2} |y^1|^{\alpha \gamma_0 / 2} dy^1 = C.$$

If $|y^1| > 2$, then $|x^1 - y^1| \geq |y^1|/2$. Thus, by (A.7),

$$\begin{aligned} I_2(x^1) &\leq C \int_{|y^1|>2} \frac{1}{|x^1 - y^1|^{1+\alpha}} \left(\frac{|y^1|^{\alpha/2}}{1 + |y^1|^{\alpha/2}} \right)^{\gamma_1} |y^1|^{\alpha(\gamma_0 - \gamma_1)/2} dy^1 \\ &\leq C \int_{|y^1|>2} |y^1|^{\frac{\alpha(\gamma_0 - \gamma_1 - 2)}{2} - 1} dy^1 = C. \end{aligned}$$

Therefore, I is bounded and (A.9) is proved for $|x^1| \leq 1$.

Case 2. Let $|x^1| > 1$. Put

$$\begin{aligned} I(x^1) = &\int_{|y^1| \geq 2|x^1|} \cdots + \int_{|x^1|/2 < |y^1| < 2|x^1|} \cdots + \int_{1/2 < |y^1| \leq |x^1|/2} \cdots + \int_{|y^1| \leq 1/2} \cdots \\ &=: J_1(x^1) + J_2(x^1) + J_3(x^1) + J_4(x^1). \end{aligned}$$

First, we estimate J_1. Note that if $r > 1$, then

$$\frac{1}{2} \leq \frac{r^{\alpha/2}}{1 + r^{\alpha/2}} \leq 1. \tag{A.10}$$

For $|y^1| > 2|x^1|$, we have $|y^1| > 2$. Thus, (A.7) and (A.10) yield

$$\begin{aligned} J_1(x^1) &\leq \int_{|y^1| \geq 2|x^1|} \frac{1}{|x^1 - y^1|^{1+\alpha}} \left(\frac{|y^1|^{\alpha/2}}{1 + |y^1|^{\alpha/2}} \right)^{\gamma_1} |y^1|^{\frac{\alpha(\gamma_0 - \gamma_1)}{2}} dy^1 \\ &\leq C \int_{|y^1| \geq 2|x^1|} |y^1|^{\frac{\alpha(\gamma_0 - \gamma_1 - 2)}{2} - 1} dy^1 \\ &= C|x^1|^{\frac{\alpha(\gamma_0 - \gamma_1 - 2)}{2}} \leq C|x^1|^{\frac{\alpha(\gamma_0 - \gamma_1)}{2}} \leq C(1 + |x^1|^{\alpha/2})^{\gamma_0 - \gamma_1}. \end{aligned} \tag{A.11}$$

Secondly, we estimate J_2. If $|x^1|/2 < |y^1| < 2|x^1|$, then

$$\frac{1}{2} \leq \frac{1 + |y^1|^{\alpha/2}}{1 + |x^1|^{\alpha/2}} \leq 2, \quad \frac{1}{3} \leq \frac{|y^1|^{\alpha/2}}{1 + |y^1|^{\alpha/2}} \leq 1.$$

Therefore, we have

$$\begin{aligned} J_2(x^1) &\leq C(1 + |x^1|^{\alpha/2})^{\gamma_0 - \gamma_1} \int_{\mathbb{R}} p_1(1, x^1 - y^1) dy^1 \\ &= C(1 + |x^1|^{\alpha/2})^{\gamma_0 - \gamma_1}. \end{aligned} \tag{A.12}$$

Next, we estimate J_3. If $1/2 \leq |y^1| \leq |x^1|/2$, then

$$\frac{1}{3} \leq \frac{|y^1|^{\alpha/2}}{1 + |y^1|^{\alpha/2}} \leq 1, \quad |x^1 - y^1| \geq \frac{|x^1|}{2}.$$

Hence, by (A.7) and (A.10),

$$\begin{aligned}
J_3(x^1) &\le \int_{1/2\le|y^1|\le|x^1|/2} \frac{1}{|x^1-y^1|^{1+\alpha}} \left(\frac{|y^1|^{\alpha/2}}{1+|y^1|^{\alpha/2}}\right)^{\gamma_1} |y^1|^{\frac{\alpha(\gamma_0-\gamma_1)}{2}} dy^1 \\
&\le C|x^1|^{-1-\alpha} \int_{1/2\le|y^1|\le|x^1|/2} |y^1|^{\frac{\alpha(\gamma_0-\gamma_1)}{2}} dy^1 \\
&\le C|x^1|^{-1-\alpha} \int_{1/2\le|y^1|\le|x^1|/2} |y^1|^{\frac{\alpha(\gamma_0-\gamma_1+2)}{2}} dy^1 \\
&\le C|x^1|^{-1-\alpha} \int_{|y^1|\le|x^1|/2} |y^1|^{\frac{\alpha(\gamma_0-\gamma_1+2)}{2}} dy^1 \\
&= C|x^1|^{\frac{\alpha(\gamma_0-\gamma_1)}{2}} \le C(1+|x^1|^{\alpha/2})^{\gamma_0-\gamma_1}. \qquad \text{(A.13)}
\end{aligned}$$

Lastly, we estimate J_4. If $|y^1| \le 1/2$, then

$$\frac{2}{3} \le \frac{1}{1+|y^1|^{\alpha/2}} \le 1, \quad |x^1-y^1| \ge \frac{|x^1|}{2}.$$

Therefore, by (A.7) and (A.10),

$$\begin{aligned}
J_4(x^1) &\le \int_{|y^1|\le 1/2} \frac{1}{|x^1-y^1|^{1+\alpha}} \left(\frac{1}{1+|y^1|^{\alpha/2}}\right)^{\gamma_1} |y^1|^{\alpha\gamma_0/2} dy^1 \\
&\le C|x^1|^{-1-\alpha} \int_{|y^1|\le 1} |y^1|^{\alpha\gamma_0/2} dy^1 \\
&\le C|x^1|^{-1-\alpha} \le C(1+|x^1|^{\alpha/2})^{-2/\alpha-2} \le C(1+|x^1|^{\alpha/2})^{\gamma_0-\gamma_1}.
\end{aligned}$$

Combining this with (A.11), (A.12) and (A.13), we prove (A.9) for $|x^1| > 1$. The lemma is proved. □

Lemma A.3. *Let* (A.7) *hold for* $\gamma_0, \gamma_1 \in \mathbb{R}$. *Then, for* $(t,x) \in (0,\infty)\times\mathbb{R}^d$,

$$\int_D p(t,x-y) \frac{d_y^{\gamma_0\alpha/2}}{(\sqrt{t}+d_y^{\alpha/2})^{\gamma_1}} dy \le C(\sqrt{t}+d_x^{\alpha/2})^{\gamma_0-\gamma_1},$$

where C *depends only on* $d, \alpha, \gamma_0, \gamma_1$ *and* D.

Proof. Note that it is enough to assume D is bounded. This is because if D is a half space, the result follows from Lemma A.2.

For $R > 0$, denote $D_R := \{x \in D : d_x \ge R\}$. Since D is bounded, one can find $x_1, \dots, x_n \in \partial D$ such that

$$D \subset \left(\bigcup_{i=1}^n (D \cap B_{R/3}(x_i))\right) \cup D_{R/6}.$$

Therefore,

$$\begin{aligned}&\int_D p(t,x-y)\frac{d_y^{\alpha\gamma_0/2}}{(\sqrt{t}+d_y^{\alpha/2})^{\gamma_1}}dy\\&\leq \sum_{i=1}^n \int_{D\cap B_{R/3}(x_i)} p(t,x-y)\frac{d_y^{\alpha\gamma_0/2}}{(\sqrt{t}+d_y^{\alpha/2})^{\gamma_1}}dy\\&\quad+\int_{D_{R/6}} p(t,x-y)\frac{d_y^{\alpha\gamma_0/2}}{(\sqrt{t}+d_y^{\alpha/2})^{\gamma_1}}dy\\&=:\sum_{k=1}^n I_k(t,x)+II(t,x).\end{aligned}$$

1. We estimate $I_k(t,x)$ for fixed $k\in\{1,2,\cdots,n\}$.

First, assume $x\in B_R(x_k)\cap D$. Then, (by reducing R if necessary) we can consider a $C^{1,1}$-bijective (flattening boundary) map $\Phi=(\Phi^1,\cdots,\Phi^d)$ defined on $B_R(x_k)$ such that $\Phi(B_R(x_k)\cap D)\subset\mathbb{R}^d_+$ and $d_z\approx\Phi^1(z)$ on $B_R(x_k)\cap D$. Then, one can easily handle I_k using Lemma A.2.

Second, assume $x\in D\setminus B_R(x_k)$. Since $r\to p(t,r)$ is nonincreasing, for any $y,z\in B_{R/3}(x_k)$, we have $|z-y|\leq 2R/3<|x-y|$, which implies

$$p(t,x-y)\leq p(t,z-y).$$

If $\gamma_1-\gamma_0\geq 0$, choosing $z\in B_{R/3}(x_k)\cap D$ such that $d_x\leq C(D,R)d_z$ and using the result for the first case,

$$\begin{aligned}I_k(t,x)&\leq\int_{D\cap B_{R/3}(x_k)} p(t,z-y)\frac{d_y^{\alpha\gamma_0/2}}{(\sqrt{t}+d_y^{\alpha/2})^{\gamma_1}}dy\\&\leq C(\sqrt{t}+d_z^{\alpha/2})^{\gamma_0-\gamma_1}\leq C(\sqrt{t}+d_x^{\alpha/2})^{\gamma_0-\gamma_1}.\end{aligned}\tag{A.14}$$

If $\gamma_1-\gamma_0<0$, by taking $z\in B_{R/3}(x_k)\cap D$ such that $d_z\leq d_x$, we also have (A.14).

2. We estimate $II(t,x)$.

We first consider the case $x\in D_{R/12}$. For $y\in D_{R/6}$, we have $d_x\approx d_y\approx 1$ and

$$\left(\frac{\sqrt{t}+d_y^{\alpha/2}}{\sqrt{t}+d_x^{\alpha/2}}\right)^{\gamma_0-\gamma_1}\leq C(diam(D),\gamma_0,\gamma_1,R,\alpha),$$

Using this, we get

$$II\leq C(\sqrt{t}+d_x^{\alpha/2})^{\gamma_0-\gamma_1}\int_{D_{R/6}} p(t,x-y)\left(\frac{d_y^{\alpha/2}}{\sqrt{t}+d_y^{\alpha/2}}\right)^{\gamma_0}dy.$$

Also, since $d_y\approx 1$ on $y\in D_{R/6}$, it suffices to show that

$$\int_{D_{R/6}} p(t,x-y)\left(\frac{1}{\sqrt{t}+1}\right)^{\gamma_0}dy\leq C.\tag{A.15}$$

Since (A.15) is obvious if $t\leq 1$ or $\gamma_0\geq 0$. If $t>1$ and $\gamma_0<0$, then by (A.1),

$$\int_{D_{R/6}} p(t,x-y)\left(\frac{1}{\sqrt{t}+1}\right)^{\gamma_0}dy\leq C\int_D t^{-d/\alpha-\gamma_0/2}dy\leq C.$$

Therefore, (A.15) is proved.

Next, we consider the case $x\in D\setminus D_{R/12}$. Since $d_y\approx 1$, we have

$$\frac{d_y^{\alpha\gamma_0/2}}{(\sqrt{t}+d_y^{\alpha/2})^{\gamma_1}}\approx\frac{1}{(\sqrt{t}+1)^{\gamma_1}}.$$

Also note that $|x-y|>R/12$ for $y\in D_{R/6}$. Thus, by (A.1),

$$\begin{aligned} II &\leq C1_{t<1}\int_{|x-y|\geq R/12}\frac{t}{|x-y|^{d+\alpha}}dy+C1_{t\geq 1}t^{-d/\alpha-\gamma_1/2}\\ &\leq C1_{t<1}+C1_{t\geq 1}t^{-\gamma_0/2-d/\alpha}t^{(\gamma_0-\gamma_1)/2}\\ &\leq C1_{t<1}+C1_{t\geq 1}t^{(\gamma_0-\gamma_1)/2}.\end{aligned}$$

Thus if $\gamma_0\geq\gamma_1$, then by (A.7),

$$II\leq Ct^{(\gamma_0-\gamma_1)/2}\leq C(\sqrt{t}+d_x^{\alpha/2})^{\gamma_0-\gamma_1}.$$

Now let $\gamma_0<\gamma_1$. Then, $1_{t<1}(\sqrt{t}+d_x^{\alpha/2})$ is bounded above and $t\approx(t+d_x^{\alpha/2})$ if $t>1$, we get

$$II\leq C1_{t<1}+C1_{t\geq 1}t^{(\gamma_0-\gamma_1)/2}\leq C(\sqrt{t}+d_x^{\alpha/2})^{\gamma_0-\gamma_1}$$

provided that $\gamma_0<\gamma_1$. The lemma is proved. □

Next, we provide some results for the distance function d_x.

Lemma A.4. *Let D be a half space or a bounded $C^{1,1}$ open set.*

(i) Let $x_0\in\partial D$ and $r>0$. Then, for any $\lambda>-1$,

$$⨍_{B_r(x_0)} d_x^{\lambda}dx\leq C(d,\lambda,D)r^{\lambda}. \tag{A.16}$$

(ii) Let $y\in D$, $r,\rho,\kappa_1>0$ and $-1<\kappa_0\leq 0$. Suppose that $r\leq c\rho$ for some $c>0$. Then, there exists a constant $C=C(d,\kappa_1,\kappa_0,c,D)$ such that

$$\int_{D_\rho(y)\cap D^r}\frac{d_x^{\kappa_0}}{|x-y|^{d+\kappa_1}}dx\leq C\rho^{-\kappa_1}r^{\kappa_0},$$

where $D_\rho(y):=\{x\in D:|x-y|>\rho\}$ and $D^r:=\{x\in D:d_x\leq r\}$.

Proof. (i) The result is trivial if D is a half space. If D is a bounded $C^{1,1}$ open set, then ∂D is a $(d-1)$-dimensional compact Lipschitz manifold. Thus, we have (A.16) due to e.g. page 16 of [2].

(ii) **1**. Let D be a half space.

Assume first $d\geq 2$. By the change of variables and Fubini's theorem,

$$\begin{aligned} &\int_{|x-y|>\rho,|x^1|\leq r}\frac{|x^1|^{\kappa_0}}{|x-y|^{d+\kappa_1}}dx\\ &=\int_{|x^1+y^1|\leq r}|x^1+y^1|^{\kappa_0}\int_{\mathbb{R}^{d-1}}|x|^{-d-\kappa_1}1_{|x|>\rho}dx'dx^1\\ &=C\int_{|x^1+y^1|\leq r}\frac{|x^1+y^1|^{\kappa_0}}{|x^1|^{1+\kappa_1}}\int_0^\infty\frac{s^{d-2}}{(1+s^2)^{(d+\kappa_1)/2}}1_{|x^1|(1+s^2)^{1/2}>\rho}dsdx^1\\ &=C\int_0^\infty\frac{s^{d-2}}{(1+s^2)^{(d+\kappa_1)/2}}I(\rho,s,y^1,r)ds, \end{aligned}\tag{A.17}$$

where

$$I(\rho,s,y^1,r):=\int_{\mathbb{R}}\frac{|x^1|^{\kappa_0}}{|x^1-y^1|^{1+\kappa_1}}1_{|x^1-y^1|>(1+s^2)^{-1/2}\rho}1_{|x^1|\leq r}dx^1.$$

Take $p_0=p_0(\kappa_0)>1$ satisfying $-1<p_0\kappa_0$. Since $-1<\kappa_0\leq 0<\kappa_1$, by Hölder's inequality,

$$\begin{aligned}
&I(\rho,s,y^1,r)\\
&\leq\left(\int_{\mathbb{R}}|x^1|^{p_0\kappa_0}1_{|x^1|\leq r}dx^1\right)^{1/p_0}\left(\int_{\mathbb{R}}|x^1|^{-p_0'-p_0'\kappa_1}1_{|x^1|>(1+s^2)^{-1/2}\rho}dx^1\right)^{1/p_0'}\\
&\leq Cr^{\kappa_0+\frac{1}{p_0}}\rho^{-1-\kappa_1+\frac{1}{p_0'}}(1+s^2)^{\frac{(1+\kappa_1-1/p_0')}{2}},
\end{aligned}\tag{A.18}$$

where $p_0'=p_0/(p_0-1)$. Combining (A.17) and (A.18), we have

$$\begin{aligned}
&\int_{|x-y|>\rho,|x^1|\leq r}\frac{|x^1|^{\kappa_0}}{|x-y|^{d+\kappa_1}}dx\\
&\leq C\rho^{-1-\kappa_1+\frac{1}{p_0'}}r^{\kappa_0+\frac{1}{p_0}}\int_0^\infty\frac{s^{d-2}}{(1+s^2)^{(d-1+1/p_0')/2}}ds\\
&=C\rho^{-1-\kappa_1+\frac{1}{p_0'}}r^{\kappa_0+\frac{1}{p_0}}\leq C\rho^{-\kappa_1}r^{\kappa_0}.
\end{aligned}$$

For $d=1$, using (A.18), we get

$$\begin{aligned}
\int_{|x-y|>\rho,|x|\leq r}\frac{|x|^{\kappa_0}}{|x-y|^{1+\kappa_1}}dx&=I(\rho,0,y,r)\\
&\leq Cr^{\kappa_0+\frac{1}{p_0}}\rho^{-1-\kappa_1+\frac{1}{p_0'}}\leq C\rho^{-\kappa_1}r^{\kappa_0}.
\end{aligned}$$

2. Let D be a bounded open set. We take $x_1,\dots,x_n\in\partial D$ such that

$$D^r\subset\bigcup_{i=1}^n B_{2r}(x_i).$$

Therefore, by (i),

$$\begin{aligned}
\int_{D_\rho(y)\cap D^r}\frac{d_x^{\kappa_0}}{|x-y|^{d+\kappa_1}}dx&\leq\sum_{i=1}^n\int_{D_\rho(y)\cap B_{2r}(x_i)}\frac{d_x^{\kappa_0}}{|x-y|^{d+\kappa_1}}dx\\
&\leq C\rho^{-d-\kappa_1}r^{d+\kappa_0}\leq C\rho^{-\kappa_1}r^{\kappa_0}.
\end{aligned}$$

The lemma is proved. □

We write $u\in\mathcal{H}_p^{\gamma+\alpha}(T)$ if $u\in\mathbb{H}_p^{\gamma+\alpha}(T)$, $u(0,\cdot)\in B_p^{\gamma+\alpha-\alpha/p}$ and there exists $f\in\mathbb{H}_p^{\gamma}(T)$ such that for any $\phi\in C_c^\infty(\mathbb{R}^d)$,

$$(u(t,\cdot),\phi)_{\mathbb{R}^d}=(u(0,\cdot),\phi)_{\mathbb{R}^d}+\int_0^t(f(s,\cdot),\phi)_{\mathbb{R}^d}ds,\quad\forall\,t\leq T.$$

In this case, we write $f=u_t$. The norm in $\mathcal{H}_p^{\gamma+\alpha}(T)$ is defined as

$$\|u\|_{\mathcal{H}_p^{\gamma+\alpha}(T)}:=\|u\|_{\mathbb{H}_p^{\gamma+\alpha}(T)}+\|u_t\|_{\mathbb{H}_p^{\gamma}(T)}+\|u(0,\cdot)\|_{B_p^{\gamma+\alpha-\alpha/p}}.$$

Lemma A.5. *Let* $p \in (1,\infty)$, $\alpha \in (0,2)$, $\gamma \in \mathbb{R}$ *and* $1/p < \nu \leq 1$. *For* $a > 0$, $0 \leq s \leq t \leq T$ *and* $u \in \mathcal{H}_p^{\gamma+\alpha}(T)$,

$$\begin{aligned} &\|u(t) - u(s)\|_{H_p^{\gamma+\alpha-\nu\alpha}} \\ &\leq C|t-s|^{\nu-1/p} a^{2\nu-1} \left(a\|u\|_{\mathbb{H}_p^{\gamma+\alpha}(T)} + a^{-1}\|u_t\|_{\mathbb{H}_p^{\gamma}(T)} \right), \end{aligned} \tag{A.19}$$

where $C = C(\alpha, p, \nu)$. *In particular,* C *is independent of* T *and* a.

Proof. One can prove the lemma by following the proof of [39, Theorem 7.3], which treats the case $\alpha = 2$. First, we note that due to the isometry $(1-\Delta)^{\sigma/2} : H_p^{\gamma} \to H_p^{\gamma-\sigma}$, we only need to prove for any particular $\gamma \in \mathbb{R}$, and therefore we assume $\gamma = \nu\alpha - \alpha$. Second, since $C_c^{\infty}([0,T] \times \mathbb{R}^d)$ is dense in $\mathcal{H}_p^{\gamma+\alpha}(T)$, we may further assume $u \in C_c^{\infty}([0,T] \times \mathbb{R}^d)$. Third, due to the scaling argument used at the beginning of the proof of [39, Theorem 7.3], it is enough to consider the case $a = T = 1$.

Finally, to prove (A.19) for the case $a = T = 1$, we just need to repeat the proof of [36, Theorem 7.2] word for word. Although [36, Theorem 7.2] handles the case $\alpha = 2$, its proof works also for $\alpha \in (0,2)$ thanks to [26, Lemma A.2]. The lemma is proved. □

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