

Counting-Based Effective Dimension and Discrete Regularizations

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Fractal-like structures of varying complexity are common in nature, and measure-based dimensions (Minkowski, Hausdorff) supply their basic geometric characterization. However, at the level of fundamental dynamics, which is quantum, structure does not enter via geometry of fixed sets but is encoded in probability distributions on associated spaces. The question then arises whether a robust notion of fractal measure-based dimension exists for structures represented in this way. Starting from effective number theory, we construct all counting-based schemes to select effective supports on collections of objects with probabilities and associate the *effective counting dimension* (ECD) with each. We then show that ECD is scheme-independent and, thus, a well-defined measure-based dimension with meaning analogous to the Minkowski dimension of fixed sets. In physics language, ECD characterizes probabilistic descriptions arising in a theory or model via discrete “regularization”. For example, our analysis makes recent surprising results on effective spatial dimensions in quantum chromodynamics and Anderson models well founded. We discuss how to assess the reliability of regularization removals in practice and perform such analysis in the context of 3d Anderson criticality.

Keywords: Minkowski dimension, effective counting dimension, effective number theory, effective support, effective description, minimal effective description, regularization, Anderson localization, lattice QCD

1. Prologue. Consider the prototypical example of fractal structure, the ternary Cantor set $\mathcal{C} \subset [0, 1]$. Evaluating its Minkowski dimension [1] involves introduction of the regularization parameter $a > 0$, namely the size of elementary interval (“box”), and the use of ordinary counting to determine the number $N(a)$ of such boxes required to cover \mathcal{C} . The scaling of $N(a)$ in the process of “regularization removal”, namely

$$N(a) \propto a^{-d_M} \quad \text{for } a \rightarrow 0 \quad (1)$$

then specifies the Minkowski dimension $d_M[\mathcal{C}] (= \log_3 2)$.

Assume now that, instead of a fixed set such as \mathcal{C} , we are given a probability measure μ over the sample space $[0, 1]$ as a way to introduce structure on this interval. The ensuing probabilities make certain parts of the interval preferred to others, which is the probabilistic analogue of sharply selecting \mathcal{C} in case of fixed sets. Is it possible to characterize the probabilistic case by a robust fractal dimension with meaning akin to Minkowski?

Here we construct such dimension and clarify in what sense it is unique. To convey the idea, consider a schematic analogue of Minkowski prescription in the above example. As a **first step**, introduce a discrete regularization parameter $a = 1/N$, where N now refers to number of equal-size intervals forming a partition $\mathcal{I} = \{\mathcal{I}_i; i = 1, 2, \dots, N\}$ of $[0, 1]$. With each \mathcal{I}_i associate the probability $p_i = \mu[\mathcal{I}_i]$ to obtain the distribution $P(a) = (p_1, \dots, p_{N(a)})$. Thus, for each $a \in \{1, 1/2, 1/3, \dots\}$ we have a collection of Minkowski boxes which, however, come with probabilities. In a **second step**, assume that we modify ordinary counting $N = N[\mathcal{I}]$ of boxes to $\mathcal{N} = \mathcal{N}[\mathcal{I}, P]$ so that probabilities

are properly taken into account. In fact, $\mathcal{N}[\mathcal{I}, P] = \mathcal{N}[P]$ since P already carries the information on $N[\mathcal{I}]$. Scaling of \mathcal{N} upon the regularization removal, namely

$$\mathcal{N}[P(a)] \propto a^{-d_{UV}} \quad \text{for } a \rightarrow 0 \quad (2)$$

would then specify the dimension $d_{UV} = d_{UV}[\mu, \mathcal{N}]$ in analogy to (1). The subscript UV (ultraviolet) conveys that regularization controls the structure at small distances. In this sense, d_M is of course a UV construct as well.

The above plan for d_{UV} becomes a well-founded concept analogous to d_M assuming that: (i) scheme \mathcal{N} is additive like ordinary N ; (ii) there is a notion of well-delineated effective support $\mathcal{S} = \mathcal{S}[\mu, \mathcal{N}]$ induced by μ and specified by \mathcal{N} , namely an analog of \mathcal{C} ; (iii) d_{UV} only depends on μ , i.e. $d_{UV}[\mu, \mathcal{N}] = d_{UV}[\mu]$ for all \mathcal{N} satisfying (i) and (ii). Indeed, (i) makes d_{UV} measure-based in the same sense d_M is, and (ii) allows for a structure induced probabilistically by μ to be described in the same way as structure of fixed sets. Part (iii) then guarantees that d_{UV} is robust (unique).

The notion of effective counting is clearly central to d_{UV} . Its theoretical framework is the effective number theory (ENT) [2] which, among other things, determines all \mathcal{N} satisfying (i). In this work we give an affirmative status to (ii) and (iii) by developing the concept of *effective counting dimension* (ECD), which is more general than d_{UV} .

Before giving a self-contained account of ECD, few points are worth emphasizing. (a) The variety of structures characterized by d_{UV} is much larger than that by d_M . For example, μ may describe Cantor set with non-uniform probability measure on it, for which d_M is not applicable. (b) While probability/stochasticity play important role in

the fractal world [3], they usually enter differently than here. For example, representative of a random Cantor set or an individual Brownian path arise via a random process, but such sample is still a fixed set. Our extension in these situations treats sample itself as a probability measure. (c) The above setup arises e.g. in quantum description of natural world since quantum states encode probabilities of physical events. Here it is common that dynamics is in fact defined by regularization, be it UV or IR (system size $L \rightarrow \infty$). The input for computing d_{UV} or d_{IR} is then directly $P(a)$ or $P(L)$ rather than (usually unknown) μ . Such calculations of d_{IR} for Dirac eigenmodes in quantum chromodynamics [4] and for critical states of Anderson transitions [5] recently led to new geometric insights.

2. Qualitative Outline & Summary. In the study of physical and various model situations we frequently need to describe or analyze some collection $O = \{o_1, o_2, \dots\}$ of objects. The most basic quantitative analysis of O is to count its objects, namely to label it by a natural number N . However, descriptive value of an ordinary count becomes limited when objects in a collection differ substantially. Indeed, consider O_\bullet containing the Sun, the Proxima Centauri and 10^{10} individual grains of sand from Earth's beaches. The ordinary count $N = 2 + 10^{10}$ is clearly a poor characteristic of O_\bullet if the individual importance of its objects is judged by their masses.

The recent works [2, 6, 7] revisited counting with the aim of making it more informative in such situations. The resulting effective number theory studies possible ways to assign counts to collections of objects distinguished by an additive importance weight such as probability or mass. For example, with O_\bullet it associates an *intrinsic (minimal) effective count* by mass $N_\star = 2 + 2.0 \times 10^{-16}$, leading to useful quantitative insight. [The textbook masses for the stars and the average mass of 4.5 mg for a grain of sand went into this calculation using Eq. (4). Note that a specific normalization is involved in the prescription.]

The utility of ordinary counting in analyzing the natural world stems mainly from its *additivity* which guarantees consistent bookkeeping for stable objects and leads to predictability: given disjoint collections O_1 and O_2 with labels N_1 and N_2 we can predict what the label of the combined collection $O_1 \cup O_2$ would be ($N_1 + N_2$) without actually performing the merger. Since merging and partitioning are at the heart of dealing with objects, counting became a theoretical tool: it is usually easier and faster to handle numbers than physical objects.

ENT requires any extension of counting to its effective form to be additive in order to preserve the above features [2]. Generalized additivity arises naturally if one views ordinary counting as a process carried out by a machine. Its "ready state" includes an empty list $I = ()$ and its operation entails receiving objects from O sequentially. Each input causes I to update via $I \rightarrow I \sqcup (1)$, where \sqcup denotes concatenation, resulting in a certain $I = (1, 1, \dots, 1)$ upon exhausting all objects. The machine then calls its

number function $N = N[I]$ (length of a list) and outputs the label. This representation of ordinary counting makes it plain that the scheme is encoded by function $N[I]$ and its additivity is expressed by the functional equation

$$N[I_1 \sqcup I_2] = N[I_1] + N[I_2] \quad , \quad \forall I_1, I_2 \quad (3)$$

In effective counting, each object o comes with a label specifying its own weight w . The associated machine initializes the weight list $W = ()$ and then sequentially inputs the objects. Upon each input, it scans the weight w and updates W via $W \rightarrow W \sqcup (w)$. After finishing the input, it calls its number function $N = N[W]$ to get the ordinary count, and uses it to rescale W into a canonical counting form $W \rightarrow C = (c_1, c_2, \dots, c_N)$ satisfying $\sum_i c_i = N$. This rescaling is allowed since, like its ordinary prototype, effective counting is scale invariant by construction. The machine then calls its effective number function $\mathcal{N} = \mathcal{N}[C]$ and outputs the result. Additivity of the procedure is then expressed by the functional equation [2]

$$\mathcal{N}[C_1 \sqcup C_2] = \mathcal{N}[C_1] + \mathcal{N}[C_2] \quad , \quad \forall C_1, C_2 \quad (A)$$

It ensures that effective counts of disjoint collections with equal average weight per object add up upon merging.

Hence, in the same way that $N = N[I]$ encodes ordinary counting, each $\mathcal{N} = \mathcal{N}[C]$ obeying (A) and other necessary conditions [2] specifies a valid effective counting scheme. When used consistently in all situations, each offers bookkeeping and predictability features analogous to those of ordinary counting. ENT identifies all such effective schemes \mathcal{N} . A key property of the resulting concept is that the scheme specified by [2]

$$N_\star[C] = \sum_{i=1}^N \mathbf{n}_\star(c_i) \quad , \quad \mathbf{n}_\star(c) = \min\{c, 1\} \quad (4)$$

satisfies $N_\star[C] \leq N[C] \leq N[C]$ for all C and all N . Hence, the effective total prescribed by N_\star cannot be lowered by a change of counting scheme and is intrinsic to a collection. In fact, each collection of objects with additive weights is characterized by two key counting characteristics: the ordinary count N and the intrinsic effective count N_\star .

In this work we show that effective counting entails a unique notion of dimension. The associated setting involves an infinite sequence of collections O_k with N_k objects, and an associated sequence $C_k = (c_{k,1}, c_{k,2}, \dots, c_{k,N_k})$ of counting weights. Here N_k is strictly increasing and hence $\lim_{k \rightarrow \infty} N_k = \infty$. The pair O_k, C_k may specify e.g. an increasingly refined representation of a complex composite object or of a physical system with infinitely many parts. Following the standard physics language, we refer to it as "regularization" of the target $k \rightarrow \infty$ situation.

Assume that we fix a counting scheme \mathcal{N} and associate with each weighted collection O, C its effective description $O_s = O_s[C, \mathcal{N}]$, containing only $\mathcal{N}[C]$ highest-weighted objects of O . To any regularization sequence O_k, C_k this

assigns a sequence of effective descriptions $O_{s,k}$ yielding the effective description of the target. Then the idea of *effective counting dimension* (ECD) is to convey how the abundance of objects in the effective description of the target scales with that in its full representation. In other words, ECD corresponds to Δ in

$$N[C_k] \propto N[C_k]^\Delta \quad \text{for } k \rightarrow \infty \quad , \quad 0 \leq \Delta \leq 1 \quad (5)$$

However, it turns out that the above notion of effective description (effective support) O_s is only consistent for certain schemes \mathcal{N} . Indeed, for \mathcal{N} to delineate the support properly, a separation property formulated in Sec. 4 below has to be imposed. Formally, if \mathfrak{N} is the set of all schemes \mathcal{N} , then only elements of its subset $\mathfrak{N}_s \subset \mathfrak{N}$ assign effective supports. We will show in Sec. 4 that \mathfrak{N}_s is spanned by

$$N_{(u)}[C] = \sum_{i=1}^N n_{(u)}(c_i) \quad , \quad n_{(u)}(c) = \min\{c/u, 1\} \quad (6)$$

where $u \in (0, 1]$. Note that $N_{(1)} = \mathcal{N}_* \in \mathfrak{N}_s$.

The above leads us to consider $\Delta = \Delta[\{C\}, \mathcal{N}]$ with $\{C\}$ a shorthand for regularization sequence and $\mathcal{N} \in \mathfrak{N}_s$. The minimal nature of \mathcal{N}_* implies that $\Delta_*[\{C\}] \equiv \Delta[\{C\}, \mathcal{N}_*]$ is the smallest possible ECD. However, ECD is in fact fully robust and doesn't depend on \mathcal{N} at all. Indeed, in Sec. 5 we will show that

$$\Delta_*[\{C\}] = \Delta[\{C\}, \mathcal{N}] \quad , \quad \forall \mathcal{N} \in \mathfrak{N}_s \quad (7)$$

Hence, ECD is a well-defined property of the target specified by regularization pair $\{O\}, \{C\}$. The use of additive counting makes it the measure-based effective dimension.

Before demonstrating the results (6) and (7) we wish to make few remarks. (i) The fact that ECD doesn't require metric allows for large range of applications. Indeed, models in some areas (e.g. ecosystems and social sciences) often do not involve distances. (ii) Setups with metric frequently entail UV cutoff a (\propto shortest distance) and IR cutoff L (\propto longest distance). Sequence $\{C\}$ can facilitate their removals: C_k may be associated e.g. with $a_k \rightarrow 0$ at fixed L , or with $L_k \rightarrow \infty$ at fixed a . Defining the nominal dimensions via $N[C_k] \propto a_k^{-D_{UV}(L)}$ and $N[C_k] \propto L_k^{D_{IR}(a)}$ for UV and IR cases, their effective counterparts are [4]

$$N_*[C_k] \propto a_k^{-d_{UV}(L)} \quad , \quad N_*[C_k] \propto L_k^{d_{IR}(a)} \quad (8)$$

If Δ_{UV}, Δ_{IR} denote their associated ECDs then

$$d_{UV} = \Delta_{UV} D_{UV} \quad , \quad d_{IR} = \Delta_{IR} D_{IR} \quad (9)$$

Dimension d_{IR} was recently calculated in QCD [4] and in Anderson models [5]. (iii) The meaning of d_{UV} is fully analogous to Minkowski (box-counting) dimension of fixed sets. In fact, for $\{O\}$ that UV-regularizes a bounded region in \mathbb{R}^D , d_{UV} is exactly the Minkowski dimension of $\{O_s\}$ treated as a fixed set. Uniqueness (7) of ECD

suggests that measure-based dimensions are meaningful even for geometric figures emerging effectively, e.g. from probabilities. (iv) We refer to $O_s = O_s[O, C, \mathcal{N}]$, $\mathcal{N} \in \mathfrak{N}_s$, as both the support and the description of O . The latter is more suitable in situations involving information and complexity. Our analysis implies the existence of well-defined *minimal effective description* $O_*[O, C]$, namely $O_s[O, C, \mathcal{N}_*]$, which may find uses in these contexts.

3. Effective Counting Schemes. Our starting point is ENT [2] which determines the set \mathfrak{N} of all effective counting schemes $\mathcal{N} = \mathcal{N}[C] = \mathcal{N}(c_1, \dots, c_N)$. Apart from additivity (A), symmetry, continuity and boundary conditions, the axiomatic definition of \mathfrak{N} also ensures that increasing the cumulation of weights in C doesn't increase the effective number. This monotonicity is expressed by

$$N(\dots c_i + \epsilon \dots c_j - \epsilon \dots) \leq N(\dots c_i \dots c_j \dots) \quad (M)$$

for each $c_i \geq c_j$ and $0 \leq \epsilon \leq c_j$. The resulting \mathfrak{N} consists of additively separable functions $\mathcal{N}(c_1, \dots, c_N) = \sum_{i=1}^N n(c_i)$, such that the *counting function* $\mathbf{n} = \mathbf{n}(c)$, $c \in [0, \infty)$, is

$$\begin{aligned} (i) \text{ continuous} & & (iii) \mathbf{n}(0) = 0 \\ (ii) \text{ concave} & & (iv) \mathbf{n}(c) = 1 \text{ for } c \geq 1 \end{aligned} \quad (10)$$

Representation of $\mathcal{N} \in \mathfrak{N}$ by \mathbf{n} satisfying (10) is unique.

4. Effective Supports. We will assume from now on that the order of objects in $O = \{o_1, \dots, o_N\}$ is set by their relevance in C , i.e. that $c_1 \geq c_2 \geq \dots \geq c_N$. Given a counting scheme \mathcal{N} , we collect the first $N[C]$ objects to form the intended effective support (effective description) O_s of O . Since \mathcal{N} is real-valued, we represent O_s as

$$O = \{o_1, \dots, o_N\} \longrightarrow O_s[O, C] = \{o_1, \dots, o_J, f\} \quad (11)$$

where J is the ceiling of $N[C]$, and $N[C] = (J-1) + f$. Hence, $0 < f \leq 1$ is the fraction of o_J included in O_s .

The rationale for effective support so conceived is clear: O_s is a subcollection of most relevant elements from O that behaves under the ordinary counting measure in the same way as O under the effective one. Indeed, additivity (A) translates into (dependence on \mathcal{N}, C_1, C_2 is implicit)

$$N[O_s[O_1 \sqcup O_2]] = N[O_s[O_1]] + N[O_s[O_2]] \quad (12)$$

where an obvious real-valued extension of $N[\dots]$ to collections with fractional last element was made.

However, ENT axioms only deal with counting, and their compatibility with the above notion of effective support needs to be examined. Effective numbers are crucially shaped by additivity (A) and monotonicity (M). With (A) being the basis for O_s via (12), it is (M) that requires attention. To that end, consider the operation on left-hand side of (M), involving the last object included in O_s (i.e. o_J) and first object fully left out (o_{J+1}). Since o_J gains relevance at the expense of o_{J+1} , its presence

in O_s (measured by f) cannot decrease. This *separation property* (SP) is expressed by

$$\mathcal{N}(\dots c_J + \epsilon, c_{J+1} - \epsilon \dots) \geq \mathcal{N}(\dots c_J, c_{J+1} \dots) \quad (\text{SP})$$

for all C such that $c_J > c_{J+1} > 0$ and all sufficiently small $\epsilon > 0$. Ensuring a meaningful split of effective support from the rest, (SP) has to hold in order to define O_s consistently. Note that $J = \text{ceil}(\mathcal{N}[C])$ depends on both \mathcal{N} and C .

We now show that the only counting schemes \mathcal{N} compatible with (SP) are specified by Eq. (6). To start, note that in order to make the separation property compatible with monotonicity (M), we have to impose the equality sign in (SP). In terms of counting function \mathbf{n} of \mathcal{N} we have

$$\mathbf{n}(c_J + \epsilon) + \mathbf{n}(c_{J+1} - \epsilon) = \mathbf{n}(c_J) + \mathbf{n}(c_{J+1}) \quad (13)$$

for all C with $c_J > c_{J+1} > 0$ and all sufficiently small $\epsilon > 0$.

Note next that properties (10) of \mathbf{n} imply the existence of $0 < u \leq 1$ such that $\mathbf{n}(c) = 1$ for all $c \geq u$, and $0 < \mathbf{n}(c) < 1$ for all $0 < c < u$. Given this $u = u[\mathbf{n}]$, the separation operation in (13) cannot be performed when $c_J \geq u$. Indeed, since each $\mathbf{n}(c_j)$ with $j \leq J$ contributes unity to $\mathcal{N}[C]$, we have $J = \text{ceil}(J + \sum_{i=J+1}^N c_i)$, leading to $c_{J+1} = 0$. Hence, it is sufficient to consider (13) for C with $u > c_J > c_{J+1} > 0$. In this form it is readily satisfied by $\mathbf{n}_{(u)}$ of Eq. (6) due to its linearity on $[0, u]$. Consequently, $\mathcal{N}_{(u)} \in \mathfrak{N}_s$.

However, all other \mathbf{n} featuring the same u violate (13). To show that, consider $N=3$ vectors $C = (3-y-x, y, x)$ with $0 < x < y < u$ and $J=2$. Definition of J , namely $J-1 < \mathcal{N}[C] \leq J$, demands that $\mathbf{n}(x) + \mathbf{n}(y) \leq 1$. We will specify x, y that satisfy this, as well as $\epsilon_0 > 0$ such that $\mathbf{n}(x - \epsilon) + \mathbf{n}(y + \epsilon) < \mathbf{n}(x) + \mathbf{n}(y)$ for all $0 < \epsilon < \epsilon_0$, thus failing (13). Since $\mathbf{n}_{(u)}$ produces equality in this relation, we can proceed using $\mathbf{g}(x) = \mathbf{n}(x) - \mathbf{n}_{(u)}(x)$ and

$$\mathbf{g}(x - \epsilon) + \mathbf{g}(y + \epsilon) < \mathbf{g}(x) + \mathbf{g}(y) \quad , \quad 0 < \epsilon < \epsilon_0 \quad (14)$$

From properties (10) of \mathbf{n} , $\mathbf{n}_{(u)}$ and the explicit form of $\mathbf{n}_{(u)}$ it follows that $\mathbf{g}(x)$ is a continuous function satisfying: (a) $\mathbf{g}(0) = \mathbf{g}(u) = 0$; (b) $\mathbf{g}(x) > 0$ for $0 < x < u$; (c) there are $0 < x_0 \leq y_0 < u$ such that $\mathbf{g}(x)$ is increasing on $[0, x_0]$ and decreasing on $[y_0, u]$. Hence, any $0 < x \leq x_0, y_0 < y < u$ and $0 < \epsilon_0 \leq \min\{x, u - y\}$ form a triple satisfying (14). Finally, for any y chosen as above, we can select sufficiently small $x > 0$ such that $\mathbf{n}(x) < 1 - \mathbf{n}(y)$ due to $\mathbf{n}(0) = 0$ and continuity. Hence, all schemes \mathcal{N} based on $\mathbf{n} \neq \mathbf{n}_{(u)}$ fail to satisfy (SP), and \mathfrak{N}_s is spanned by $\mathcal{N}_{(u)}$.

5. Uniqueness of ECD. It is now straightforward to show (7). Consider a pair $\mathcal{N}_\star, \mathcal{N}_{(u)}$ for arbitrary but fixed $0 < u \leq 1$. We compare their counting functions $\mathbf{n}_\star(c)$ and $\mathbf{n}_{(u)}(c)$ on the following partition of their domain. (i) $0 \leq c \leq u$. Here $\mathbf{n}_\star(c) = c$ and $\mathbf{n}_{(u)}(c) = c/u$ and hence $\mathbf{n}_{(u)}(c) = \mathbf{n}_\star(c)/u$. (ii) $u < c < 1$. Here $\mathbf{n}_\star(c) = c$ and $\mathbf{n}_{(u)}(c) = 1$ and so $\mathbf{n}_{(u)}(c) = \mathbf{n}_\star(c)/c < \mathbf{n}_\star(c)/u$. (iii)

$c \geq 1$. Here $\mathbf{n}_\star(c) = \mathbf{n}_{(u)}(c) = 1$ and so $\mathbf{n}_{(u)}(c) \leq \mathbf{n}_\star(c)/u$.

Taken together, this yields $\mathbf{n}_\star(c) \leq \mathbf{n}_{(u)}(c) \leq \mathbf{n}_\star(c)/u$ for any $c \geq 0$, as well as

$$\mathcal{N}_\star[C] \leq \mathcal{N}_{(u)}[C] \leq \frac{1}{u} \mathcal{N}_\star[C] \quad , \quad \forall C \quad , \quad \forall u \in (0, 1] \quad (15)$$

Now, consider a regularization sequence $\{C\}$ such that its ECD (5) associated with \mathcal{N}_\star (i.e. Δ_\star) exists. From (15) it follows that the power governing the growth of $\mathcal{N}_{(u)}[C_k]$ with N_k also exists and is equal to Δ_\star as claimed in (7). Hence, the concept of effective counting dimension associated with $\{C\}$ is well-defined (unique).

6. Generality of ECD. The general context we associated with ECD, namely that of arbitrary collections O of objects, may raise questions about using the term ‘‘dimension’’. Indeed, its intuitive notion is frequently reserved for less general setups and, in particular, for those involving metric (see comment (i) in Sec. 2). We thus elaborate more on the underlying rationale.

In line with the usual practice, we aimed at minimal conditions under which the notion of ECD is applicable. Such minimal setup turns out to be a sequence (O_k, C_k) of objects and associated additive weights. This arises due to the fact that, while ordinary counting doesn’t give any structure to this most bare of settings, the effective counting does. In fact, the most relevant consequence of present analysis is that ECD provides for the robust and well-founded quantitative characteristic of this structure.

To illustrate the reasoning, consider an extreme example of a sequence where (O_1, C_1) involves apples, (O_2, C_2) potatoes, (O_3, C_3) apples again, (O_4, C_4) peanuts and so on. While a casual observer presented with the sequence may be puzzled by its meaning, it may have a clear rationale for a fertilizer company which generated it as part of their efficiency analysis. The associated ECD has the same nominal meaning for both (it specifies how effective number of objects scales with their ordinary number), but the fertilizer company will find it natural to call it dimension. After all, at the heart of ordinary measure-based dimensions is the scaling of measure, and they are dealing with scaling of their own measure represented by the effective count. The casual observer, from whom the meaning of effective count is hidden, may object.

In the next section we will discuss an example of ECD stability study using critical wave functions of 3d Anderson transitions. This involves sequences labeled by size L of the system which, similarly to the example above, may seem like sequences of independent distinct objects. However, they have common origin in probability distributions of wave functions generated by quantum dynamics of the Anderson model. This makes the ensuing sequences meaningful. Moreover, the objects are actually elements of physical space in this case, which allows us to interpret ECD as an effective dimension of space occupied by the Anderson electron.

7. Anderson Criticality. The results of Ref. [2] and the present work suggest that using \mathcal{N}_* alone suffices for many effective counting analyses. But even then, additional input from other schemes in \mathfrak{N} may be informative. For example, it can be used to assess the reliability of regularization removals. Indeed, \mathfrak{N}_s is spanned by schemes $\mathcal{N}_{(u)}$ and we showed that the associated $\Delta(u)$ is constant. However, carrying out the $k \rightarrow \infty$ extrapolation in practice can be affected by large systematic errors if available collections are not sufficiently large to achieve scaling in (5). The computed $\Delta(u)$ is then expected to vary significantly. On the other hand, when approximate scaling is in place, the degree of non-constant behavior can be used to judge the level of systematic errors.

To explain, note first that possible effective supports $\mathcal{O}_s = \mathcal{O}_s(u)$ of \mathcal{O} contain populations of $\mathcal{N}_{(u)}$ objects that decrease with increasing u . Hence, for given $\mathcal{O} = \{O_{k_1}, O_{k_2}, \dots, O_{k_M}\}$ used in regularization removal, function $\Delta(u, \mathcal{O})$ is also expected to be decreasing. At the same time, u can be lowered to make the fraction of objects in effective support arbitrarily close to one and there is a guaranteed over-representation of the scaling population at sufficiently small u . In fact, it is expected that $\lim_{u \rightarrow 0} \Delta(u, \mathcal{O}) \approx 1$ for generic \mathcal{O} , regardless of true ECD. The signature of \mathcal{O} suitable for regularization removal is the existence of a “scaling window” in u , where $\Delta(u, \mathcal{O})$ changes slowly. The value $\Delta(u_0, \mathcal{O})$ at point u_0 of slowest change is expected to produce the most reliable estimate of ECD from \mathcal{O} . The change of $\Delta(u, \mathcal{O})$ within the window sets an approximate scale of systematic error.

We now apply this general strategy to the recent calculation [5] of spatial effective dimensions d_{IR} at Anderson transitions [8–10] in three dimensions ($d_{\text{IR}} = 3\Delta_{\text{IR}}$). We will focus on 3d Anderson model in the orthogonal class, defined on $(L/a)^3$ cubic lattice with sites labeled by $r = (x_1, x_2, x_3)$ and periodic boundary conditions. The model is diagonal in spin and it is thus sufficient to consider 1-component fermionic operators c_r . Denoting by ϵ_r the on-site random energies chosen from a box distribution in the range $[-W/2, +W/2]$, the Hamiltonian is [8]

$$\mathcal{H} = \sum_r \epsilon_r c_r^\dagger c_r + \sum_{r,j} c_r^\dagger c_{r-e_j} + h.c. \quad (16)$$

Here e_j ($j = 1, 2, 3$) are unit lattice vectors. For energy $E = 0$, there is a critical point at $W = W_c = 16.543(2)$ [11] separating extended states at $W < W_c$ from exponentially localized ones at $W > W_c$.

Objects o_i involved in the calculation of $d_{\text{IR}} = d_{\text{IR}}(E, W)$ are elementary cubes of space at positions r_i , with weights specified by wave function ψ via $w_i = p_i = \psi^\dagger \psi(r_i)$. Collection \mathcal{O} forms the space occupied by the system with volume $V = N[\mathcal{O}]a^3 = L^3$. Electron in state ψ is effectively present in a subregion $\mathcal{O}_*[\psi]$ of volume $V_{\text{eff}} = \mathcal{N}_*[\psi]a^3$. Dimension d_{IR} gauges the asymptotic response of V_{eff} to increasing L . The model involves averaging over disor-

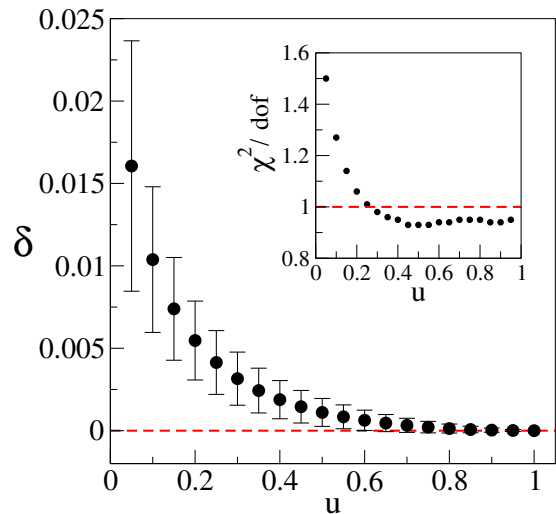


FIG. 1. Deviation $\delta(u, \mathcal{O}) = d_{\text{IR}}(u, \mathcal{O}) - d_{\text{IR}}(1, \mathcal{O})$ obtained via 2-power fit as described in the text. The inset shows χ^2 per degree of freedom (dof) for the fits involved.

der $\{\epsilon_r\}$, and hence $\mathcal{N}_* \rightarrow \langle \mathcal{N}_* \rangle$ in definition (8). The critical effective dimension $d_{\text{IR}}(0, W_c) \approx 8/3$ was found for (16) and models in three other universality classes [5]. This commonality was expressed by super-universal value $d_{\text{IR}}^{\text{su}} = 2.665(3)$ with the quoted uncertainty including the spread over classes.

We have generated a new set of data for system (16) at the critical point $(0, W_c)$, producing \mathcal{O} including 26 systems with sizes in the range $22 \leq L/a \leq 160$. JADAMILU library [12] was used for matrix diagonalization. We then strictly followed the analysis procedure of Ref. [5] and obtained $d_{\text{IR}}(\mathcal{O}) = 2.6654(11)$, consistently with the original estimate. However, in the present calculation of critical eigenmodes we recorded $\mathcal{N}_{(u)}$ for 20 equidistant values of u starting with $u = 0.05$, rather than just $\mathcal{N}_* = \mathcal{N}_{(1)}$. This allows us to perform the proposed stability analysis utilizing \mathfrak{N}_s .

The latter is most efficiently performed by computing $\delta(u, \mathcal{O}) = d_{\text{IR}}(u, \mathcal{O}) - d_{\text{IR}}(1, \mathcal{O})$ which can be extracted directly from the raw data without any intermediate steps. Indeed, the large L behavior of $\langle \mathcal{N}_{(u)} \rangle_L / \langle \mathcal{N}_{(1)} \rangle_L$ is governed by power $\delta(u)$. Moreover, relation $d_{\text{IR}}(u) = d_{\text{IR}}$ is replaced by the definite $\delta(u) = 0$. While still featuring the expected decreasing behavior and slow variation in the scaling window, the size of $\delta(u, \mathcal{O})$ directly conveys the scale of systematic errors. Note that since the above ratio defining $\delta(u)$ involves correlated data in the way we performed the calculation, Jackknife procedure was used to estimate its error in the analysis described below.

To extract $\delta(u, \mathcal{O})$ from the data in an unbiased way, we included it as a parameter in general 2-power fits of the above ratio in the form $c_1 L^\delta + c_2 L^{-\beta}$. The role of the second power is to absorb finite-volume effects, and its presence resulted in very stable results. Unconstrained 2-power fits were mainly afforded by our extensive statistics

(30K–100K of disorder realizations). We proceeded by finding the smallest size L_{\min} in \mathcal{O} , such that the fit in the range $L_{\min}/a \leq L/a \leq 160$ yielded $\chi^2/\text{dof} < 1$ for $u = 0.95$ data. The resulting $L_{\min} = 30a$ was then fixed for fits at all u , leading to $\delta(u, \mathcal{O})$ shown in Fig. 1. The respective χ^2/dof are shown in the inset.

The resulting $\delta(u, \mathcal{O})$ is indicative of \mathcal{O} suitable for regularization removal. Indeed, populations associated with the window $0.75 \leq u \leq 1$ scale essentially in sync. The slowest change occurs at $u_0 = 1$, suggesting that the quoted result for d_{IR} , which is based on \mathcal{N}_* , is nominally the most reliable for this \mathcal{O} . Note that, according to Eq. (15), effective support at $u = 1/2$ is up to twice as abundant as the minimal one. The associated $\delta(1/2, \mathcal{O}) \approx 0.002$ offers a convenient canonical benchmark for the level of systematic error. Given the position of the scaling window and its degree of stability, it is likely an upper bound in this case. These findings suggest that $\approx 10^{-3}$ is the scale of statistical as well as systematic error associated with calculation of d_{IR} in Ref. [5].

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