arXiv:2205.12500v2 [math.CV] 17 Oct 2022

INTERPOLATION AND DUALITY IN SPACES OF PSEUDOCONTINUABLE FUNCTIONS

KONSTANTIN M. DYAKONOV

Abstract. Given an inner function θ on the unit disk, let $K^p_\theta := H^p \cap \theta\overline{z}\overline{H^p}$ be the associated star-invariant subspace of the Hardy space H^p. Also, we put $K_{*\theta} := K^2_\theta \cap \mathrm{BMO}$. Assuming that $B = B_{\mathcal{Z}}$ is an interpolating Blaschke product with zeros $\mathcal{Z} = \{z_j\}$, we characterize, for a number of smoothness classes X, the sequences of values $\mathcal{W} = \{w_j\}$ such that the interpolation problem $f\big|_{\mathcal{Z}} = \mathcal{W}$ has a solution f in $K^2_B \cap X$. Turning to the case of a general inner function θ, we further establish a non-duality relation between K^1_θ and $K_{*\theta}$. Namely, we prove that the latter space is properly contained in the dual of the former, unless θ is a finite Blaschke product. From this we derive an amusing non-interpolation result for functions in K_{*B}, with $B = B_{\mathcal{Z}}$ as above.

1. Introduction and results

We write $\mathbb{T}$ for the unit circle $\{\zeta \in \mathbb{C} : |\zeta| = 1\}$ and m for the normalized arc length measure on $\mathbb{T}$; thus, $dm(\zeta) = |d\zeta|/(2\pi)$. We then define the spaces $L^p := L^p(\mathbb{T}, m)$ in the usual way and let $\|\cdot\|_p$ denote the standard norm on L^p. Also, for $1 \le p \le \infty$, we introduce the *Hardy space* H^p by putting

$$H^p := \{f \in L^p : \widehat{f}(n) = 0 \text{ for } n = -1, -2, \dots\},$$

where $\widehat{f}(n)$ is the nth *Fourier coefficient* of f given by

$$\widehat{f}(n) := \int_{\mathbb{T}} \overline{\zeta}^n f(\zeta)\, dm(\zeta), \qquad n \in \mathbb{Z}.$$

The Poisson integral (i.e., harmonic extension) of an H^p function being holomorphic on the disk

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$$

(see [15, Chapter II]), we may use this extension to view elements of H^p as holomorphic functions on $\mathbb{D}$ when convenient.

Furthermore, we write P_+ (resp., P_-) for the orthogonal projection from L^2 onto H^2 (resp., onto $\overline{z}\overline{H^2} = L^2 \ominus H^2$). By a classical theorem of M. Riesz (see [15, Chapter III]), each of these projections admits a bounded extension—or restriction—to L^p, with $1 < p < \infty$, and maps L^p onto H^p (resp., onto $\overline{z}\overline{H^p}$).

2010 *Mathematics Subject Classification.* 30H05, 30H10, 30J05, 46J15.

Key words and phrases. Hardy space, smoothness class, BMO, inner function, interpolating Blaschke product, star-invariant subspace, duality.

Supported in part by grants MTM2017-83499-P and PID2021-123405NB-I00 from El Ministerio de Ciencia e Innovación (Spain).

Now suppose θ is an *inner function*, meaning that $\theta \in H^\infty$ and $|\theta| = 1$ a.e. on $\mathbb{T}$. The corresponding *star-invariant* (or *model*) *subspace* K_θ^p is then defined by

$$K_\theta^p := \{f \in H^p : \overline{z}\overline{f}\theta \in H^p\}, \qquad 1 \le p \le \infty, \tag{1.1}$$

so that $K_\theta^p = H^p \cap \theta\overline{z}\overline{H^p}$. (When $p = 2$, yet another equivalent definition is $K_\theta^2 = H^2 \ominus \theta H^2$.) It is clear from (1.1) that the antilinear isometry

$$f \mapsto \overline{z}\overline{f}\theta =: \widetilde{f} \tag{1.2}$$

leaves K_θ^p invariant. Also, it is well known (see [6, 20]) that each K_θ^p is invariant under the *backward shift operator*

$$\mathfrak{B} : f \mapsto \frac{f - f(0)}{z}, \qquad f \in H^p,$$

and conversely, that every closed and nontrivial $\mathfrak{B}$-invariant subspace of H^p, with $1 \le p < \infty$, arises in this way.

The functions belonging to some K_θ^p space (i.e., noncyclic vectors of $\mathfrak{B}$) are known as *pseudocontinuable* functions. In fact, they are characterized by the property of having a meromorphic pseudocontinuation to $\mathbb{D}_- := \mathbb{C}\setminus(\mathbb{D}\cup\mathbb{T})$; that is, the function in question should agree a.e. on $\mathbb{T}$ with the boundary values of some meromorphic function of bounded characteristic in $\mathbb{D}_-$ (see [6] for details).

The orthogonal projection from H^2 onto K_θ^2 is given by $f \mapsto \theta P_-(\overline{\theta} f)$, and the M. Riesz theorem shows that the same formula provides, for $1 < p < \infty$, a bounded projection from H^p onto K_θ^p parallel to θH^p. This yields the direct sum decomposition

$$H^p = K_\theta^p \oplus \theta H^p, \qquad 1 < p < \infty, \tag{1.3}$$

with orthogonality for $p = 2$.

Among the inner functions θ, of special relevance to us are Blaschke products. Recall that, for a sequence $\mathcal{Z} = \{z_j\} \subset \mathbb{D}$ with

$$\sum_j (1 - |z_j|) < \infty, \tag{1.4}$$

the associated *Blaschke product* is given by

$$B(z) = B_{\mathcal{Z}}(z) := \prod_j \frac{|z_j|}{z_j}\frac{z_j - z}{1 - \overline{z}_j z}$$

(if $z_j = 0$, then we set $|z_j|/z_j = -1$). The product converges uniformly on compact subsets of $\mathbb{D}$ and defines an inner function that vanishes precisely at the z_j's; see [15, Chapter II]. If, in addition,

$$\inf_j |B'(z_j)|\,(1 - |z_j|) > 0, \tag{1.5}$$

then we say that B is an *interpolating Blaschke product*. Accordingly, the sequences $\mathcal{Z} = \{z_j\}$ in $\mathbb{D}$ that satisfy (1.4) and (1.5), with $B = B_{\mathcal{Z}}$, are called *interpolating* (or

H^∞*-interpolating) sequences.* By a celebrated theorem of Carleson (see [3] or [15, Chapter VII]), these are precisely the sequences $\mathcal{Z}$ with the property that

$$H^\infty\big|_{\mathcal{Z}} = \ell^\infty.$$

Here and below, the following standard notation (and terminology) is used. Given a sequence $\mathcal{Z} = \{z_j\}$ of pairwise distinct points in $\mathbb{D}$, the *trace* $f\big|_{\mathcal{Z}}$ of a function $f : \mathbb{D} \to \mathbb{C}$ is defined to be the sequence $\{f(z_j)\}$; and if $\mathcal{X}$ is a certain function space on $\mathbb{D}$, then the corresponding *trace space* is

$$\mathcal{X}\big|_{\mathcal{Z}} := \left\{ f\big|_{\mathcal{Z}} : f \in \mathcal{X} \right\}.$$

We shall be concerned with interpolation problems for functions in star-invariant subspaces—specifically, for those in K^p_B, where B is an interpolating Blaschke product. Some of the earlier results in this area can be found in [1, 7, 16, 18], while others, more relevant to our current topic, will be recalled presently.

First, we need yet another piece of notation. Given numbers $p > 0$, $\gamma \in \mathbb{R}$ and a sequence $\mathcal{Z} = \{z_j\} \subset \mathbb{D}$, we write $\ell^p_\gamma(\mathcal{Z})$ for the set of all sequences $\{w_j\} \subset \mathbb{C}$ satisfying

$$\sum_j |w_j|^p (1 - |z_j|)^\gamma < \infty.$$

Now, if $1 < p < \infty$ and if $B = B_{\mathcal{Z}}$ is an interpolating Blaschke product with zero sequence $\mathcal{Z}$, then we have

$$K^p_B\big|_{\mathcal{Z}} = H^p\big|_{\mathcal{Z}} = \ell^p_1(\mathcal{Z}).$$

Indeed, the left-hand equality follows from (1.3) with $\theta = B$, while the other holds by a well-known theorem of Shapiro and Shields [21]. In addition, for each sequence $\mathcal{W} = \{w_j\}$ in $\ell^p_1(\mathcal{Z})$, there is a *unique* function $f \in K^p_B$ with $f\big|_{\mathcal{Z}} = \mathcal{W}$; the uniqueness is due to the fact that $K^p_B \cap BH^p = \{0\}$.

The case of K^∞_B is subtler, as the next result shows.

Theorem A. *Suppose that* $\mathcal{Z} = \{z_j\}$ *is an interpolating sequence in* $\mathbb{D}$ *and* $B = B_{\mathcal{Z}}$ *is the associated Blaschke product. Then we have*

$$K^\infty_B|_{\mathcal{Z}} = \ell^\infty \tag{1.6}$$

if and only if

$$\sup\left\{ \sum_j \frac{1 - |z_j|}{|\zeta - z_j|} : \zeta \in \mathbb{T} \right\} < \infty. \tag{1.7}$$

This theorem is essentially a consequence of Hruščev and Vinogradov's work in [19]; see also [5, Section 3] for details.

Condition (1.7) above is known as the *uniform Frostman condition*, and the sequences $\mathcal{Z} = \{z_j\}$ in $\mathbb{D}$ that obey it are called *Frostman sequences.* While a Frostman sequence need not be interpolating (in fact, its points are not even supposed to be pairwise distinct), it does necessarily split into finitely many interpolating sequences; see [19] for a proof. Finally, a Blaschke product whose zeros form a Frostman sequence will be referred to as a *Frostman Blaschke product.*

We mention in passing that, by a theorem of Vinogradov [22], the identity

$$K^\infty_{B^2}|_{\mathcal Z}=\ell^\infty$$

is valid whenever $\mathcal Z$ is an interpolating sequence and $B=B_{\mathcal Z}$. It should be noted, however, that $K^\infty_{B^2}$ is strictly larger than K^∞_B.

To describe the trace class $K^\infty_B|_{\mathcal Z}$ in the general case (i.e., when (1.7) no longer holds), we first introduce a bit of notation. Once the interpolating sequence $\mathcal Z=\{z_j\}$ is fixed, we associate with each sequence $\mathcal W=\{w_j\}$ from $\ell^1_1(\mathcal Z)$ the *conjugate sequence* $\widetilde{\mathcal W}=\{\widetilde w_k\}$ whose elements are

$$\widetilde w_k:=\sum_j\frac{w_j}{B'(z_j)\cdot(1-z_j\overline z_k)}\qquad(k=1,2,\dots).\tag{1.8}$$

The absolute convergence of the series in (1.8) is ensured, for any $k\in\mathbb N$, by the fact that $\mathcal W\in\ell^1_1(\mathcal Z)$ in conjunction with (1.5). Because $\ell^1_1(\mathcal Z)$ contains ℓ^∞, as well as every $\ell^p_1(\mathcal Z)$ with $1<p<\infty$, the sequence $\widetilde{\mathcal W}$ is well defined whenever $\mathcal W$ belongs to one of these spaces.

The following result was established in [13].

Theorem B. *Suppose that $\mathcal Z=\{z_j\}$ is an interpolating sequence in $\mathbb D$ and $B=B_{\mathcal Z}$ is the associated Blaschke product. Given a sequence $\mathcal W\in\ell^\infty$, one has $\mathcal W\in K^\infty_B\big|_{\mathcal Z}$ if and only if $\widetilde{\mathcal W}\in\ell^\infty$.*

It was further conjectured in [13, 14] that the trace space $K^1_B\big|_{\mathcal Z}$ is describable in similar terms, i.e., that the necessary conditions $\mathcal W\in\ell^1_1(\mathcal Z)$ and $\widetilde{\mathcal W}\in\ell^1_1(\mathcal Z)$ are also sufficient for $\mathcal W$ to be in $K^1_B\big|_{\mathcal Z}$. To the best of our knowledge, the conjecture is still open.

Here, our purpose is to supplement Theorem B by characterizing the values of *smooth*, not just bounded, functions in K^2_B on the (interpolating) sequence $\mathcal Z=B^{-1}(0)$. To be more precise, of concern are trace spaces of the form $(K^2_B\cap X)\big|_{\mathcal Z}$, where X is a certain smoothness class on $\mathbb T$. Specifically, X will be one of the following spaces.

• The *Lipschitz–Zygmund space* $\Lambda^\alpha=\Lambda^\alpha(\mathbb T)$ with $\alpha>0$. This is the set of functions $f\in C(\mathbb T)$ satisfying

$$\|\Delta^n_h f\|_\infty=O(|h|^\alpha),\qquad h\in\mathbb R,$$

where n is some (any) integer with $n>\alpha$, and Δ^n_h denotes the nth order difference operator with step h. (As usual, the difference operators Δ^k_h are defined inductively: we put

$$(\Delta^1_h f)(\zeta):=f(e^{ih}\zeta)-f(\zeta),\qquad\zeta\in\mathbb T,$$

and $\Delta^k_h f:=\Delta^1_h\Delta^{k-1}_h f$ for $k\ge 2$.)

• BMO = BMO($\mathbb T$), the space of functions of *bounded mean oscillation* on $\mathbb T$. Recall that an integrable function f on $\mathbb T$ belongs to BMO if and only if

$$\|f\|_*:=\left|\int_{\mathbb T}f\,dm\right|+\sup_I\frac{1}{m(I)}\int_I|f-f_I|\,dm<\infty,$$

where $f_I := m(I)^{-1}\int_I f\,dm$; the supremum is taken over the open arcs $I \subset \mathbb{T}$. Even though BMO contains discontinuous and unbounded functions, there are reasons for viewing it as a smoothness class. In a sense, it corresponds to the endpoint as $\alpha \to 0$ of the Λ^α scale. We also need the analytic subspace $\mathrm{BMOA} := \mathrm{BMO} \cap H^2$.

• The *Gevrey class* $G_\alpha = G_\alpha(\mathbb{T})$ with $\alpha > 0$. This is the set of functions $f \in C^\infty(\mathbb{T})$ satisfying

$$\|f^{(n)}\|_\infty \le Q_f^{n+1}(n!)^{1+1/\alpha}, \qquad n = 0, 1, 2, \dots,$$

with some constant $Q_f > 0$. Here, we write $f^{(n)}(e^{it})$ for the nth order derivative of the function $t \mapsto f(e^{it})$, which is assumed to be C^∞-smooth on $\mathbb{R}$.

• The *Sobolev space* $\mathcal{L}^p_s = \mathcal{L}^p_s(\mathbb{T})$ with $1 < p < \infty$ and $s > 0$, defined by

$$\mathcal{L}^p_s := \{f \in L^p : f^{(s)} \in L^p\},$$

with the appropriate interpretation of the (possibly fractional) derivative $f^{(s)}$. Precisely speaking, we write $f^{(s)} \in L^p$ to mean that there is a function $g \in L^p$ satisfying $\widehat{g}(n) = (in)^s \widehat{f}(n)$ for all $n \in \mathbb{Z}$.

For each of these choices of X, we now characterize the sequences $\mathcal{W}$ from the trace space $(K^2_B \cap X)\big|_{\mathcal{Z}}$ in terms of the conjugate sequence $\widetilde{\mathcal{W}}$, as defined by (1.8) above. The description always involves a certain decay condition (or growth restriction) on $\widetilde{\mathcal{W}}$, as we shall presently see.

Theorem 1.1. *Let $\alpha > 0$, $1 < p < \infty$ and $s > 0$. Also, let X be one of the following spaces: Λ^α, BMO, G_α or $\mathcal{L}^p_s$. Given an interpolating Blaschke product $B = B_{\mathcal{Z}}$ with zeros $\mathcal{Z} = \{z_k\}$ and a sequence $\mathcal{W} = \{w_k\} \in \ell^2_1(\mathcal{Z})$, we have*

$$\mathcal{W} \in \left(K^2_B \cap X\right)\big|_{\mathcal{Z}}$$

if and only if

(a) $|\widetilde{w}_k| = O\left((1-|z_k|)^\alpha\right)$ *when* $X = \Lambda^\alpha$,

(b) $\widetilde{\mathcal{W}} \in \ell^\infty$ *when* $X = \mathrm{BMO}$,

(c) *there is a constant $c > 0$ such that*

$$|\widetilde{w}_k| = O\left(\exp\left(-\frac{c}{(1-|z_k|)^\alpha}\right)\right)$$

when $X = G_\alpha$,

(d) $\widetilde{\mathcal{W}} \in \ell^p_{1-sp}(\mathcal{Z})$ *when* $X = \mathcal{L}^p_s$.

The intersection $K^2_B \cap \mathrm{BMO}$, which corresponds to case (b) above, will be henceforth denoted by K_{*B}. Similarly, for a general inner function θ, we define

$$K_{*\theta} := K^2_\theta \cap \mathrm{BMO}.$$

Comparing Theorem B with the BMO part of Theorem 1.1, we see that the structure of the trace space $K^\infty_B\big|_{\mathcal{Z}}$ is remarkably similar to that of $K_{*B}\big|_{\mathcal{Z}}$. In light of this observation, we may wonder what the BMO counterpart of Theorem A could look like. Specifically, we may ask if there exist infinite Blaschke products $B = B_{\mathcal{Z}}$ for which the trace space $K_{*B}\big|_{\mathcal{Z}}$ is completely determined by the natural (and necessary) logarithmic growth condition on the values.

To be more precise, suppose that $\mathcal{Z} = \{z_k\}$ is a sequence of pairwise distinct points in $\mathbb{D}$, and write $\ell^\infty_{\log}(\mathcal{Z})$ for the space of sequences $\mathcal{W} = \{w_k\} \subset \mathbb{C}$ with

$$|w_k| = O\left(\log \frac{2}{1-|z_k|}\right).$$

It is well known (and easily shown) that every $f \in \mathrm{BMOA}$ satisfies

$$|f(z)| = O\left(\log \frac{2}{1-|z|}\right), \qquad z \in \mathbb{D},$$

so $\mathrm{BMOA}\big|_{\mathcal{Z}}$ is always contained in $\ell^\infty_{\log}(\mathcal{Z})$. The equality

$$\mathrm{BMOA}\big|_{\mathcal{Z}} = \ell^\infty_{\log}(\mathcal{Z}) \tag{1.9}$$

obviously need not hold in general, but it does actually occur for some infinite sequences $\mathcal{Z} = \{z_k\}$ (which form a tiny subfamily among the H^∞-interpolating sequences). For instance, (1.9) will be valid provided that

$$|z_j - z_k| \ge c(1-|z_j|)^s, \qquad j \ne k,$$

for some constants $c > 0$ and $s \in (0, \frac{1}{2})$; see [10, Theorem 11].

The question is what happens to (1.9) when BMOA gets replaced by its subspace K_{*B}, with $B = B_{\mathcal{Z}}$. The property that arises is thus

$$K_{*B}\big|_{\mathcal{Z}} = \ell^\infty_{\log}(\mathcal{Z}), \tag{1.10}$$

and we regard it as an analogue of (1.6) in the BMO setting. In contrast to (1.6), however, (1.10) does not lead to any nontrivial class of sequences. Indeed, our next result shows that (1.10) is only possible when $\mathcal{Z} = B^{-1}(0)$ is a finite set.

Proposition 1.2. *Whenever $B = B_{\mathcal{Z}}$ is an infinite Blaschke product with simple zeros, the trace space $K_{*B}\big|_{\mathcal{Z}}$ is properly contained in $\ell^\infty_{\log}(\mathcal{Z})$.*

This will be deduced from another result, which deals with the case of a general inner function θ and asserts an amusing lack of duality between the star-invariant subspaces K^1_θ and $K_{*\theta}$.

It is well known that, for $1 < p < \infty$, the dual of the Hardy space H^p (under the pairing $\langle f, g\rangle = \int_{\mathbb{T}} f\overline{g}\,dm$) is H^q with $q = p/(p-1)$, while the dual of H^1 is BMOA; see, e.g., [15, Chapter VI]. The former duality relation has a natural counterpart in the K^p_θ setting, namely $(K^p_\theta)^* = K^q_\theta$ for p and q as above (see [4, Lemma 4.2]), and one may wonder if the identity $(K^1_\theta)^* = K_{*\theta}$ has any chance of being true, at least for some inner functions θ. Our last theorem says that this is never the case, except when θ is a finite Blaschke product.

Theorem 1.3. *Given an inner function θ, other than a finite Blaschke product, there exists a non-*BMO *function $g \in K^2_\theta$ such that the functional*

$$f \mapsto \int_{\mathbb{T}} f\overline{g}\,dm,$$

defined initially for $f \in K^2_\theta$, extends continuously to K^1_θ.

In other words, whenever θ is an “interesting” (i.e., nonrational) inner function, $K_{*\theta}$ is properly contained in $(K^1_\theta)^*$. One might compare this non-duality result with Bessonov's duality theorem for K^1_θ that appears in [2]. There, θ was assumed to be a one-component inner function, meaning that the set $\{z \in \mathbb{D} : |\theta(z)| < \varepsilon\}$ is connected for some $\varepsilon \in (0,1)$, and the dual of $K^1_\theta \cap zH^1$ was identified with a certain discrete BMO space on $\mathbb{T}$.

In the remaining part of the paper, we first list a number of auxiliary facts (these are collected in Section 2) and then use them to prove our current results. The proofs are in Sections 3 and 4.

Acknowledgement. I thank Carlo Bellavita for a helpful conversation. In particular, Theorem 1.3 of this paper arose in response to a question he asked me.

2. Preliminaries

Several background results will be needed. When stating the first of these, we shall assume that X is one of our smoothness spaces (namely, Λ^α, BMO, G_α or $\mathcal{L}^p_s$), the admissible values of the parameters α, p and s being as above.

Lemma 2.1. *Let $f \in H^2$ and let $B = B_{\mathcal{Z}}$ be an interpolating Blaschke product with zeros $\mathcal{Z} = \{z_k\}$. In order that $P_-(\overline{B}f) \in X$, it is necessary and sufficient that*

(a) $|f(z_k)| = O\left((1-|z_k|)^\alpha\right)$ *when* $X = \Lambda^\alpha$,

(b) $\{f(z_k)\} \in \ell^\infty$ *when* $X = \mathrm{BMO}$,

(c) *for some* $c > 0$,

$$|f(z_k)| = O\left(\exp\left(-\frac{c}{(1-|z_k|)^\alpha}\right)\right)$$

when $X = G_\alpha$,

(d) $\{f(z_k)\} \in \ell^p_{1-sp}(\mathcal{Z})$ *when* $X = \mathcal{L}^p_s$.

The statements corresponding to parts (a) and (b) were proved in [8] as Theorems 4.1 and 5.2. For parts (c) and (d), we refer to [11]; specifically, see Theorems 1 and 7 therein.

Another (well-known) fact to be used below is that the space BMOA enjoys the so-called *K-property* of Havin, as defined in [17]. The precise meaning of this assertion is as follows.

Lemma 2.2. *For every $\psi \in H^\infty$, the Toeplitz operator $T_{\overline{\psi}}$ given by*

$$T_{\overline{\psi}} f := P_+(\overline{\psi} f), \qquad f \in \mathrm{BMOA},$$

maps BMOA *boundedly into itself.*

To prove this, it suffices to observe (in the spirit of [17]) that $T_{\overline{\psi}}$ is the adjoint of the multiplication operator $g \mapsto \psi g$, which is obviously bounded on H^1.

Before proceeding, we need to introduce a bit of notation. Namely, with an inner function θ and a number $\varepsilon \in (0,1)$ we associate the sublevel set

$$\Omega(\theta, \varepsilon) := \{z \in \mathbb{D} : |\theta(z)| < \varepsilon\}.$$

The following result is a restricted version of [9, Theorem 1].

Lemma 2.3. *Suppose that $f \in \mathrm{BMOA}$ and θ is an inner function. Then $f\overline{\theta} \in \mathrm{BMO}$ if and only if*

$$\sup\{|f(z)| : z \in \Omega(\theta, \varepsilon)\} < \infty \tag{2.1}$$

for some (or every) ε with $0 < \varepsilon < 1$.

Next, we recall a remarkable maximum principle for K^2_θ functions that was established by Cohn in [5].

Lemma 2.4. *Let θ be inner, and suppose $f \in K^2_\theta$ is a function that satisfies* (2.1) *for some $\varepsilon \in (0, 1)$. Then $f \in H^\infty$.*

Our last lemma reproduces yet another result of Cohn (see [4, p. 737]), which characterizes the inner functions θ with the property that $K_{*\theta}$ contains only bounded functions. This characterization is, in turn, a consequence of Hruščev and Vinogradov's earlier work from [19] on the multipliers of Cauchy type integrals.

Lemma 2.5. *Let θ be an inner function. Then $K_{*\theta} = K^\infty_\theta$ if and only if θ is a Frostman Blaschke product.*

We also refer to [12, Theorem 1.7] for a refinement of this result in terms of $\mathrm{inn}(K_{*\theta})$, the set of inner factors for functions from $K_{*\theta}$.

3. Proof of Theorem 1.1

We shall only give a detailed proof of part (a), the other cases being similar. Since $\mathcal{W} = \{w_k\} \in \ell^2_1(\mathcal{Z})$, we know that there exists a unique $f \in K^2_B$ such that $f\big|_{\mathcal{Z}} = \mathcal{W}$. Therefore, in order that

$$\mathcal{W} \in \left(K^2_B \cap \Lambda^\alpha\right)\big|_{\mathcal{Z}} \tag{3.1}$$

it is necessary and sufficient that

$$f \in \Lambda^\alpha. \tag{3.2}$$

To find out when the latter condition holds, we apply Lemma 2.1, part (a), to the function $g := \overline{z}\overline{f}B$ in place of f. (Note that $g \in H^2$ because $f \in K^2_B$.) This tells us that $P_-(\overline{B}g) \in \Lambda^\alpha$ if and only if

$$|g(z_k)| = O\left((1 - |z_k|)^\alpha\right), \qquad k \in \mathbb{N}. \tag{3.3}$$

On the other hand,

$$P_-(\overline{B}g) = P_-(\overline{z}\overline{f}) = \overline{z}\overline{f},$$

and it is clear that the function $\overline{z}\overline{f}$ belongs to Λ^α if and only if f does. Thus, we may rephrase (3.2) as (3.3).

To arrive at a further—and definitive—restatement of (3.3), we need to express the numbers $g(z_k)$ in terms of $\mathcal{W}$. For $z \in \mathbb{D}$, Cauchy's formula yields

$$g(z) = \int_{\mathbb{T}} \frac{g(\zeta)}{1 - \overline{\zeta}z}\, dm(\zeta).$$

Consequently,

$$\begin{aligned}\overline{g(z_k)} &= \int_{\mathbb{T}} \frac{\overline{g(\zeta)}}{1-\zeta\overline{z}_k}\,dm(\zeta) = \int_{\mathbb{T}} \frac{\zeta f(\zeta)\overline{B(\zeta)}}{1-\zeta\overline{z}_k}\,dm(\zeta) \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\zeta)}{B(\zeta)\cdot(1-\zeta\overline{z}_k)}\,d\zeta.\end{aligned}$$

Computing the last integral by residues, while recalling that $f(z_j)=w_j$, we find that

$$\overline{g(z_k)} = \sum_j \frac{w_j}{B'(z_j)\cdot(1-z_j\overline{z}_k)} = \widetilde{w}_k \tag{3.4}$$

for each $k\in\mathbb{N}$. (To justify the application of the residue theorem, one may begin by evaluating the integral over the circle $r_n\mathbb{T}$, where $\{r_n\}\subset(0,1)$ is a suitable sequence tending to 1, and then pass to the limit as $n\to\infty$.)

Finally, we use (3.4) to rewrite (3.3) in the form

$$|\widetilde{w}_k| = O\left((1-|z_k|)^\alpha\right), \qquad k\in\mathbb{N}. \tag{3.5}$$

The equivalence of (3.1) and (3.5) is thereby established, proving the Λ^α part of the theorem.

The remaining statements (i.e., those involving BMO, G_α and $\mathcal{L}^p_s$) are proved similarly, by combining the appropriate parts of Lemma 2.1 with identity (3.4).

4. Proofs of Proposition 1.2 and Theorem 1.3

We begin by proving Theorem 1.3. Once this is done, Proposition 1.2 will be derived as a corollary.

Proof of Theorem 1.3. Given an inner function θ distinct from a finite Blaschke product, we want to find a function $g\in K^2_\theta\setminus\text{BMO}$ that induces a bounded linear functional on K^1_θ. We shall distinguish two cases.

Case 1. Assume that θ is an infinite Frostman Blaschke product. Its zero sequence, say $\mathcal{Z}=\{z_j\}$, must then have a limit point on $\mathbb{T}$. Of course, nothing is lost by assuming that $\mathcal{Z}$ clusters at 1. Now let

$$\varphi(z) := \log(1-z),$$

where "log" stands for the holomorphic branch of the logarithm that lives on the right half-plane and satisfies $\log 1 = 0$. We have $\varphi\in\text{BMOA}$ (because $\operatorname{Im}\varphi\in L^\infty$), so the corresponding linear functional acts boundedly on H^1 and hence on K^1_θ. Clearly, the same functional on K^1_θ is also induced, in a similar manner, by the function

$$g := \theta P_-(\overline{\theta}\varphi),$$

which is the orthogonal projection (in H^2) of φ onto K^2_θ. Precisely speaking, the functional

$$f\mapsto \int_{\mathbb{T}} f\overline{g}\,dm \left(=\int_{\mathbb{T}} f\overline{\varphi}\,dm\right), \qquad f\in K^2_\theta,$$

extends continuously to K^1_θ.

We know that $g \in K^2_\theta$, and to conclude that g does the job, we only need to check that

$$g \notin \mathrm{BMO}. \tag{4.1}$$

To this end, observe first that $\sup_j |\varphi(z_j)| = \infty$ and hence, *a fortiori*,

$$\sup\{|\varphi(z)| : z \in \Omega(\theta, \varepsilon)\} = \infty$$

for every $\varepsilon \in (0,1)$. By Lemma 2.3, this implies that $\overline{\theta}\varphi \notin \mathrm{BMO}$. On the other hand,

$$\overline{\theta}\varphi = P_-(\overline{\theta}\varphi) + P_+(\overline{\theta}\varphi),$$

where the last term, $P_+(\overline{\theta}\varphi)$, is in $\mathrm{BMOA}(\subset \mathrm{BMO})$ thanks to Lemma 2.2. It follows readily that $P_-(\overline{\theta}\varphi) \notin \mathrm{BMO}$. In particular, $P_-(\overline{\theta}\varphi) \notin L^\infty$ (just note that $L^\infty \subset \mathrm{BMO}$). Equivalently, the function $\theta P_-(\overline{\theta}\varphi) = g$ is not in L^∞.

Now, if g were in BMO, then we would have $g \in K_{*\theta}$; and since our current assumption on θ yields $K_{*\theta} = K^\infty_\theta$ (in accordance with Lemma 2.5), g would have to be bounded, which it is not. This proves (4.1).

Case 2. Assume that θ is not a Frostman Blaschke product. This time, using Lemma 2.5 again, we can find an unbounded function $h \in K_{*\theta}$. We have then $\widetilde{h} := \overline{z}\overline{h}\theta \in K^2_\theta$, and we go on to claim that $\widetilde{h} \notin \mathrm{BMO}$. (Here and below, the "tilde operation" (1.2) is being used repeatedly.) Indeed, if $\widetilde{h}$ were in BMO, then so would be $h\overline{\theta}$, and Lemma 2.3 would tell us that

$$\sup\{|h(z)| : z \in \Omega(\theta, \varepsilon)\} < \infty$$

for some (any) $\varepsilon \in (0,1)$. This, however, would imply that $h \in H^\infty$ by virtue of Lemma 2.4, whereas h is actually unbounded by assumption.

Now we know that $\widetilde{h} \in K^2_\theta \setminus \mathrm{BMO}$, and we proceed by showing that $\widetilde{h}$ generates a continuous linear functional on K^1_θ. This will allow us to conclude that $\widetilde{h}$ is eligible as g (the function we are looking for), and the proof will be complete.

Given $f \in K^2_\theta$, we have the elementary identity $\overline{f}\widetilde{h} = \widetilde{f}\overline{h}$. Recalling also the facts that $\widetilde{f} \in H^2(\subset H^1)$ and $h \in \mathrm{BMOA}$, we use the duality relation $(H^1)^* = \mathrm{BMOA}$ to infer that

$$\begin{aligned}\left|\int_{\mathbb{T}} f\,\overline{\widetilde{h}}\,dm\right| &= \left|\int_{\mathbb{T}} \overline{f}\,\widetilde{h}\,dm\right| = \left|\int_{\mathbb{T}} \widetilde{f}\,\overline{h}\,dm\right| \\ &\le C\|\widetilde{f}\|_1\|h\|_* = C\|f\|_1\|h\|_*\end{aligned}$$

with some absolute constant $C > 0$. Consequently, for $g = \widetilde{h}$, the (densely defined) functional

$$f \mapsto \int_{\mathbb{T}} f\overline{g}\,dm \tag{4.2}$$

is indeed continuous on K^1_θ, and we are done. □

Proof of Proposition 1.2. Assuming that $B = B_{\mathcal{Z}}$ is an infinite Blaschke product with zeros $\mathcal{Z} = \{z_j\}$, where the z_j's are pairwise distinct, we want to find a sequence of values $\mathcal{W} = \{w_j\}$ in $\ell^\infty_{\log}(\mathcal{Z})$ that is not the trace of any K_{*B} function on $\mathcal{Z}$. Consider, for each $j \in \mathbb{N}$, the function

$$f_j(z) := \frac{1}{1 - \overline{z}_j z}$$

and note that $f_j \in K^2_B$. Observe also that

$$\|f_j\|_1 \le M \log \frac{2}{1 - |z_j|}, \qquad j \in \mathbb{N}, \tag{4.3}$$

for some fixed constant $M > 0$.

Now, Theorem 1.3 provides us with a function $g \in K^2_B \setminus \mathrm{BMO}$ such that the associated functional (4.2), defined initially for $f \in K^2_B$, acts boundedly on K^1_B, say with norm N_g. When applied to $f = f_j$, this functional takes the value $\overline{g(z_j)}$; indeed, Cauchy's formula gives

$$\int_{\mathbb{T}} \overline{f}_j g \, dm = g(z_j)$$

for each j. In conjunction with (4.3), this yields

$$|g(z_j)| \le N_g \|f_j\|_1 \le M N_g \log \frac{2}{1 - |z_j|}, \qquad j \in \mathbb{N}. \tag{4.4}$$

Finally, we put

$$w_j := g(z_j), \qquad j \in \mathbb{N}.$$

The sequence $\mathcal{W} = \{w_j\}$ is then in $\ell^\infty_{\log}(\mathcal{Z})$, as (4.4) shows, while

$$\mathcal{W} \notin K_{*B}\big|_{\mathcal{Z}} \tag{4.5}$$

as required. To verify (4.5), it suffices to note that g is the *only* function in K^2_B that interpolates $\mathcal{W}$ on $\mathcal{Z}$ (indeed, a K^2_B function is uniquely determined by its trace on $\mathcal{Z} = B^{-1}(0)$), whereas $g \notin \mathrm{BMO}$. The proof is complete. □

Remark. We have seen above that if $g \in K^2_B$, with $B = B_{\mathcal{Z}}$, and if the functional (4.2) is continuous on K^1_B, then $g\big|_{\mathcal{Z}} \in \ell^\infty_{\log}(\mathcal{Z})$. Now, if $\mathcal{Z}$ has the BMOA-interpolating property (1.9), then *every* sequence $\mathcal{W}$ in $\ell^\infty_{\log}(\mathcal{Z})$ is actually writable as $g\big|_{\mathcal{Z}}$ for some $g \in K^2_B$ that induces a continuous linear functional on K^1_B. (To see why, take $G \in \mathrm{BMOA}$ with $G\big|_{\mathcal{Z}} = \mathcal{W}$ and then put $g = BP_-(\overline{B}G)$, so that g is the orthogonal projection of G onto K^2_B.) Of course, things become different if condition (1.9) is dropped. For instance, there are interpolating sequences $\mathcal{Z}$ for which $\ell^\infty_{\log}(\mathcal{Z}) \not\subset \ell^2_1(\mathcal{Z})$; and if this is the case, then no sequence in $\ell^\infty_{\log}(\mathcal{Z}) \setminus \ell^2_1(\mathcal{Z})$ is the trace of any H^2 function on $\mathcal{Z}$.

References

[1] E. Amar, A. Hartmann, *Uniform minimality, unconditionality and interpolation in backward shift invariant subspaces*, Ann. Inst. Fourier (Grenoble) **60** (2010), 1871–1903.

[2] R. V. Bessonov, *Duality theorems for coinvariant subspaces of H^1*, Adv. Math. **271** (2015), 62–90.

[3] L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math. **80** (1958), 921–930.
[4] W. S. Cohn, *Radial limits and star invariant subspaces of bounded mean oscillation*, Amer. J. Math. **108** (1986), 719–749.
[5] W. S. Cohn, *A maximum principle for star invariant subspaces*, Houston J. Math. **14** (1988), 23–37.
[6] R. G. Douglas, H. S. Shapiro, A. L. Shields, *Cyclic vectors and invariant subspaces for the backward shift operator*, Ann. Inst. Fourier (Grenoble) **20** (1970), 37–76.
[7] K. M. Dyakonov, *Interpolating functions of minimal norm, star-invariant subspaces and kernels of Toeplitz operators*, Proc. Amer. Math. Soc. **116** (1992), 1007–1013.
[8] K. M. Dyakonov, *Smooth functions and coinvariant subspaces of the shift operator*, Algebra i Analiz **4** (1992), no. 5, 117–147; translation in St. Petersburg Math. J. **4** (1993), 933–959.
[9] K. M. Dyakonov, *Division and multiplication by inner functions and embedding theorems for star-invariant subspaces*, Amer. J. Math. **115** (1993), 881–902.
[10] K. M. Dyakonov, *Moment problems for bounded functions*, Comm. Anal. Geom. **2** (1994), 533–562.
[11] K. M. Dyakonov, *Smooth functions in the range of a Hankel operator*, Indiana Univ. Math. J. **43** (1994), 805–838.
[12] K. M. Dyakonov, *Two theorems on star-invariant subspaces of* BMOA, Indiana Univ. Math. J. **56** (2007), 643–658.
[13] K. M. Dyakonov, *A free interpolation problem for a subspace of* H^∞, Bull. Lond. Math. Soc. **50** (2018), 477–486.
[14] K. M. Dyakonov, *Interpolating by functions from model subspaces in* H^1, Integral Equations Operator Theory **90** (2018), Paper No. 42, 7 pp.
[15] J. B. Garnett, *Bounded analytic functions*, Revised first edition, Springer, New York, 2007.
[16] P. Gorkin, B. D. Wick, *Interpolation in model spaces*, Proc. Amer. Math. Soc. Ser. B **7** (2020), 170–182.
[17] V. P. Havin, *The factorization of analytic functions that are smooth up to the boundary*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **22** (1971), 202–205.
[18] S. V. Hruščëv, N. K. Nikol'skiĭ, B. S. Pavlov, *Unconditional bases of exponentials and of reproducing kernels*, in: Complex analysis and spectral theory (Leningrad, 1979/1980), pp. 214–335, Lecture Notes in Math., 864, Springer, Berlin and New York, 1981.
[19] S. V. Hruščev, S. A. Vinogradov, *Inner functions and multipliers of Cauchy type integrals*, Ark. Mat. **19** (1981), 23–42.
[20] N. K. Nikolski, *Operators, Functions, and Systems: An Easy Reading, Vol. 2: Model operators and systems*, Mathematical Surveys and Monographs, 93, Amer. Math. Soc., Providence, RI, 2002.
[21] H. S. Shapiro, A. L. Shields, *On some interpolation problems for analytic functions*, Amer. J. Math. **83** (1961), 513–532.
[22] S. A. Vinogradov, *Some remarks on free interpolation by bounded and slowly growing analytic functions*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **126** (1983), 35–46.

Departament de Matemàtiques i Informàtica, IMUB, BGSMath, Universitat de Barcelona, Gran Via 585, E-08007 Barcelona, Spain

ICREA, Pg. Lluís Companys 23, E-08010 Barcelona, Spain
Email address: konstantin.dyakonov@icrea.cat