THE CALDERÓN PROBLEM FOR SPACE-TIME FRACTIONAL PARABOLIC OPERATORS WITH VARIABLE COEFFICIENTS

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Abstract. We study an inverse problem for variable coefficient fractional parabolic operators of the form $(\partial_t - \operatorname{div}(A(x)\nabla_x))^s + q(x,t)$ for $s \in (0,1)$ and show the unique recovery of q from exterior measured data. Similar to the fractional elliptic case, we use Runge type approximation argument which is obtained via a global weak unique continuation property. The proof of such a unique continuation result involves a new Carleman estimate for the associated variable coefficient extension operator. In the latter part of the work, we prove analogous unique determination results for fractional parabolic operators with drift.

Contents

1. Introduction and statement of the main results

Let Ω be a domain in $\mathbb{R}^n$ and let $T > 0$. Let $A(x)$ be a positive definite $n \times n$ matrix on Ω with Lipschitz coefficients. We denote by $\mathcal{H} = \partial_t - \operatorname{div}(A(x)\nabla_x)$ the parabolic operator in $\mathbb{R}^{n+1}$, and for $s \in (0,1)$, by $\mathcal{H}^s$ the fractional parabolic operator. In this article, we study two inverse problems associated to this fractional parabolic operator, which we now proceed to describe precisely.

Let us denote the cylindrical domain $\Omega \times (-T,T)$ by Q and the exterior domain $\Omega_e \times (-T,T)$ by Q_e where $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$.

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Let the potential term $q \in L^\infty(Q)$. We consider the initial-exterior problem

$$\begin{cases} (\mathcal{H}^s + q(x,t))\, u = 0, & \text{in } Q \\ u(x,t) = f(x,t), & \text{in } Q_e \\ u(x,t) = 0, & \text{for } t \leq -T. \end{cases} \tag{1.1}$$

We will assume that

$$0 \text{ is not a Dirichlet eigenvalue for (1.1).} \tag{1.2}$$

We define the nonlocal Dirichlet to Neumann (DN) map as follows

$$\Lambda_q : u|_{Q_e} \to \mathcal{H}^s u|_{Q_e} \tag{1.3}$$

Our first result is that one can recover the potential term q in Q uniquely given the nonlocal DN map.

Next we consider a fractional parabolic problem involving a first order term as well. For $q \in L^\infty(Q)$ and $b \in L^\infty\left((-T,T); W^{1-s,\infty}(\Omega)\right)$, we consider the initial-exterior problem

$$\begin{cases} (\mathcal{H}^s + \langle b(x,t), \nabla_x \rangle + q(x,t))\, u = 0, & \text{in } Q \\ u(x,t) = f(x,t), & \text{in } Q_e \\ u(x,t) = 0, & \text{for } t \leq -T. \end{cases} \tag{1.4}$$

As before, we assume that

$$0 \text{ is not a Dirichlet eigenvalue for (1.4).} \tag{1.5}$$

and define the nonlocal parabolic DN map

$$\Lambda_{b,q} : u|_{Q_e} \to \mathcal{H}^s u|_{Q_e}. \tag{1.6}$$

Our second result is that one can uniquely recover the coefficients b and q from the data $\Lambda_{b,q}$.

We now give a brief survey of local and non-local versions of the Calderón inverse problem in the elliptic and parabolic settings. Calderón initiated the study in this direction in his fundamental article [17], where he asked the question whether one can determine the conductivity of a medium from boundary Dirichlet to Neumann data, and gave some partial answers. This work served as the initial impetus for several deep and insightful works in the context of elliptic inverse problems; see [52, 42, 3, 19, 34]. The problem of unique determination of the conductivity from boundary Dirichlet to Neumann map is typically transformed to an inverse problem for the Schrödinger equation, that is an equation of the type $(-\Delta + q)$, from the corresponding Dirichlet to Neumann map. The method of complex geometric optics (CGO) solutions has served as a crucial ingredient in the proofs of these inverse problems. This has proven versatile to be effective in the solution of several inverse problems involving PDEs. It is not our intention to give a broad survey of exisiting results in inverse problems for PDEs and for this reason we limit ourselves to those problems whose fractional analogues we study in this paper. Analogous to the case of the Schrödinger equation, an inverse problem for the magnetic Schrödinger equation, $\sum_{i=1}^{n} \left(\frac{1}{i}\frac{\partial}{\partial x_j} + W_j\right)^2 + q(x)$, was studied in [51] under a smallness assumption on the first order term W, and removing this assumption later in [43]. However, in this situation, the inverse problem exhibits a phenomenon of gauge invariance, that is, there is an obstruction to uniquely recovering the first order term from boundary Dirichlet to Neumann data. Inverse problems for parabolic equations have been studied extensively as well. We refer to the following initial works in this context; [32, 7].

In recent years, study of inverse problems involving fractional powers of local operators has been gaining significant attention. The work in this direction for the fractional Laplacian involving a zeroth order term was initiated by [29]. The results in [29] were subsequently extended to variable coefficient operators with smooth principal part in [28]. An inverse problem for the fractional Laplacian with both zeroth and first order term was recently considered in [18]; see also [14] for a related work. We should mention the important feature that unlike the local case, the phenomenon of gauge invariance disappears in the nonlocal framework. Moving on to fractional analogoues of the parabolic operator, an inverse problem for a fractional parabolic operator of the form $(\partial_t-\Delta)^s+q$ was recently considered in [36]. Two related works with slightly different fractional parabolic operators are [48, 38].

In this work, our main focus is the unique determination of the potential and the drift term from the nonlocal DN map for more general operators of the type $(\partial_t - \mathrm{div}(A(x)\nabla))^s$ where A is assumed to be Lipschitz continuous. Our results therefore generalize those in [36] as well as those in the elliptic case in [28] where, instead, smooth coefficients are considered.

1.1. **Main results.** We now proceed to give the main results of the article. Our first main result concerns the unique determination of the potential q.

Theorem 1.1. *Let $T>0$ and $\Omega\subset\mathbb{R}^n, n\geq 1$ be an open bounded set. Consider $q_1,q_2\in L^\infty(Q)$ and any two nonempty open sets in Ω_e say W_1 and W_2 such that*

$$\Lambda_{q_1}(f)|_{W_2\times(-T,T)} = \Lambda_{q_2}(f)|_{W_2\times(-T,T)}, \quad \text{for all } f\in C_0^\infty(W_1\times(-T,T))$$

then $q_1=q_2$ in Q.

We also uniquely recover the coefficients b,q for (1.4) given the nonlocal DN map. The following result below is the parabolic generalization of Theorem 1.1 in [18].

Theorem 1.2. *Let $T>0$ and $\Omega\subset\mathbb{R}^n, n\geq 1$ be an open bounded Lipschitz set. Consider $q_1,q_2\in L^\infty(Q)$ and $b_1,b_2\in L^2\left((-T,T);W^{1-s,\infty}(\Omega)\right)$. We further choose two nonempty open sets from Ω_e say W_1 and W_2 such that*

$$\Lambda_{b_1,q_1}(f)|_{W_2\times(-T,T)} = \Lambda_{b_2,q_2}(f)|_{W_2\times(-T,T)}, \quad \text{for all } f\in C_0^\infty(W_1\times(-T,T))$$

then $b_1=b_2,\ q_1=q_2$ in Q.

The proofs of our main results, Theorem 1.1 and Theorem 1.2, crucially rely on a global weak unique continuation property for $\mathcal{H}^s$ (see Theorem 1.3 below), and substantial parts of this article rest on proving this result. In the following result, for the fractional parabolic space $\mathbb{H}^s(\mathbb{R}^{n+1})$, we refer to the definition in (2.12).

Theorem 1.3. *Let $T>0$ and U be a nontrivial open set in $\mathbb{R}^n, n\geq 1$. For some $u\in\mathbb{H}^s(\mathbb{R}^{n+1})$, if*

$$u=\mathcal{H}^s u = 0 \text{ in } U\times(-T,T),$$

then $u=0$ in $\mathbb{R}^n\times(-T,T)$.

In an exactly analogous way as in [29], Theorem 1.3 is used to prove the Runge approximation properties in Theorem 5.3 and Theorem 5.6 that is tailored for Theorem 1.1 and Theorem 1.2 respectively. This allows bypassing the method of complex geometric optics (CGO) solutions which is a crucial ingredient in the local case (see for instance [52]). Over here, it is worthwhile to mention that that Runge approximation results were first obtained for $(-\Delta)^s$ in [20]; see also [21].

Now regarding Theorem 1.3, we mention that in the case when $A = \mathbb{I}_n$, such a result has been established in [36, Proposition 5.6] as a consequence of the following weak unique continuation property for the corresponding extension problem. We refer to Section 4 for the precise notations.

Before proceeding further, let us declare that we will denote $1-2s$ by a from now on. Notice that $a \in (-1,1)$.

Theorem 1.4 (Weak unique continuation property). *Let U_0 be a weak solution to the following extension problem*

$$\begin{cases} \operatorname{div}_X(x_{n+1}^a \nabla U_0) = x_{n+1}^a \partial_t U_0 \ \ in\ \mathbb{B}_1^+, \ \ X=(x,x_{n+1}) \\ \lim_{x_{n+1}\to 0^+} x_{n+1}^a \partial_{n+1} U_0(x,0,t) = U_0(x,0,t) = 0 \ \ in\ B_1 \times (0,1). \end{cases} \tag{1.7}$$

Then $U_0 \equiv 0$ in $\mathbb{B}_1^+ \times (0,1)$.

Such a result has been derived in [36] by the following two steps.

Step 1: By means of repeated differentiation and a bootstrap argument, it is first shown that the zero Cauchy data in (1.7) implies that U_0 vanishes to infinite order in space and time at every point $(x,0,t) \in B_1 \times (0,1)$.

Step 2: Then by means of a Carleman estimate for the constant coefficient operator in (1.7) that the authors establish, it is shown that $U_0 \equiv 0$ in $\mathbb{B}_1^+ \times (0,1)$.

Finally once Theorem 1.4 is proven in [36], it is applied to the solution U of the extension problem (3.1) corresponding to Dirichlet data u (when $A = \mathbb{I}_n$) using which one can assert that U vanishes in $\mathbb{B}_1^+ \times (0,1)$. Then noting that U solves a uniformly parabolic PDE with real analytic coefficients away from $\{x_{n+1}=0\}$, one can thus spread the zero set using the classical theory and thus conclude that $U \equiv 0$ in $\mathbb{R}_+^{n+1} \times (0,1)$. Theorem 1.3 now follows since $U=u$ at $\{x_{n+1}=0\}$.

It turns out that more recently in [12], it has been shown that solutions to a more general class of equations modelled on (1.7) are real analytic in the space variable x which includes the extension variable and therefore the use of the Carleman estimate in *Step 2* above can be avoided. We also mention that a certain variant of the weak unique continuation property in Theorem 1.4 is also used to characterize singular points in the fractional heat obstacle problem in [10].

Similar to that in [36], in this work we derive Theorem 1.3 by obtaining an analogous weak unique continuation property for extension problems of the type (4.45) where the matrix valued function A satisfies the uniform ellipticity condition in (2.2) and the Lipschitz growth condition in (2.3). This constitutes one of the key novelties of this work. This is done by means of a new Carleman estimate that we establish for degenerate operators of the type (4.3). The estimate in Lemma 4.5 below can be regarded as a generalization of the Carleman estimate for uniformly parabolic operators with Lipschitz coefficients as in the fundamental works of Escauriaza, Fernandez and Vessella in [23, 22]. Such a generalization has required some very subtle adaptations of the ideas in [23, 22] for which we refer to the discussion in Section 4 below. Inspired by ideas in the recent work [2], we combine such an estimate with a monotonicity in time result as in Lemma 4.9 that we derive using which we show the validity of a conditional doubling property for solutions to the extension problem. This facilitates the use of blowup arguments which then reduces the weak unique continuation property for (4.45) to that of the constant coefficient extension problem as in Theorem 1.4 above. We would also like to mention that Theorem 3.1 is another central result in our work where we show that for $u \in \mathbb{H}^s(\mathbb{R}^{n+1})$, the solution U to the corresponding extension problem (3.1) belongs to the right energy space and moreover the weighted Neumann derivative can be interpreted as a limit in an appropriate norm. Therefore weak type methods can be applied and this is finally extremely crucial for the proof of the unique continuation result, Theorem 1.3 in Section 4. As the reader will see, the proof of our main results rely on several non-trivial facts from analysis and PDE that in our context beautifully combine.

In closing, we provide a brief account of unique continuation results in the nonlocal setting. For nonlocal elliptic equations of the type $(-\Delta)^s+V$ a strong unique continuation theorem was obtained by Fall and Felli, see Theorem 1.3 in [24]. Their analysis combines the approach in [26], [27] with the Caffarelli-Silvestre extension method in [16]. We also mention the interesting work of Ruland [46], [47], where the Carleman method has been used, together with [16], to obtain results similar to those in [24] but with weaker assumptions on the potential V. In the time dependent case, for global solutions of

$$(\partial_t - \Delta)^s u = Vu, \tag{1.8}$$

a backward space-time strong unique continuation theorem was previously established in [11] with appropriate assumptions on V. We also refer to [8] for some interesting results on the regularity of the nodal sets of such solutions. The result in [11] represents the nonlocal counterpart of the one first obtained by Poon in [45] for the local case $s = 1$. More recently, a space like strong unique continuation result for local solutions to (1.8) has been obtained in [2] which constitutes the nonlocal counterpart of the space-like strong unique continuation results in the aforementioned works [23, 22]. It is to be noted that both the works [2] and [11] uses the extension problem for the fractional heat operator that has been developed in [44] and [50] independently. When $s = 1/2$ the extension problem was first introduced in [33] by Jones.

The article is organized as follows. We outline the background to define the nonlocal operator $\mathcal{H}^s$ and its domain in section 2. Also we discuss the mapping property of $\mathcal{H}^s$ there. In section 3, we derive some results on the extension problem for $\mathcal{H}^s$ which will be followed by the well-posedness of the initial-exterior problems for $\mathcal{H}^s + q$ and $\mathcal{H}^s + \langle b, \nabla_x\rangle + q$. In section 4, we present the unique continuation part where we first establish a Carleman estimate for the extended operator and combine it with a blow-up analysis to deduce the weak unique continuation result mentioned in Theorem 1.3. In section 5, we prove Runge appoximation properties and provide the unique determination results for the inverse problems, Theorem 1.1 and Theorem 1.2.

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2. Preliminaries

In this section we introduce the relevant notation and gather some auxiliary results that will be useful in the rest of the paper. Generic points in $\mathbb{R}^n \times \mathbb{R}$ will be denoted by $(x_0, t_0), (x, t)$, etc. For an open set $\Omega \subset \mathbb{R}^n_x \times \mathbb{R}_t$, by $C_0^\infty(\Omega)$ we mean the set of compactly supported smooth functions in Ω. We will assume that the uniformly parabolic operator $\partial_t - \operatorname{div}(A(x)\nabla_x)$ in $\mathbb{R}^n \times \mathbb{R}$ has a globally defined fundamental solution $p(x, x', t)$ that satisfies for every $x \in \mathbb{R}^n$ and $t > 0$

$$P_t 1(x,t) = \int_{\mathbb{R}^n} p(x, x', t)\mathrm{d}x' = 1. \tag{2.1}$$

We also assume that the matrix valued function A is uniformly elliptic, i.e.

$$\Lambda^{-1}\mathbb{I} \le A \le \Lambda\mathbb{I}, \tag{2.2}$$

for some $\Lambda > 1$ and satisfies the following Lipschitz boundedness assumption

$$|A(x) - A(y)| \leq K|x-y|. \tag{2.3}$$

We start by introducing the following notion of evolutive semigroup

$$P^{\mathcal{H}}_{\tau} u(x,t) = \int_{\mathbb{R}^n} p(x,y,\tau) u(y, t-\tau)\, \mathrm{d}y, \quad \text{for } u \in \mathcal{S}(\mathbb{R}^{n+1}) \tag{2.4}$$

where $p(x,z,t)$ is the heat kernel associated to the elliptic operator

$$\mathcal{L} \stackrel{def}{=} \operatorname{div}(A(x)\nabla). \tag{2.5}$$

Note that, $\{P^{\mathcal{H}}_{\tau}\}_{\tau\geq 0}$ is a strongly continuous contractive semigroup satisfying $\|P^{\mathcal{H}}_{\tau} u - u\|_{L^2(\mathbb{R}^{n+1})} = O(\tau)$.

Definition 2.1. *For $s \in (0,1)$ and $u \in \mathcal{S}(\mathbb{R}^{n+1})$, we define $\mathcal{H}^s$ based on the Balakrishnan formula* [49, Eq. (9.63) on pp. 285]*) in the following way*

$$\mathcal{H}^s u(x,t) = -\frac{s}{\Gamma(1-s)} \int_0^\infty \left(P^{\mathcal{H}}_{\tau} u(x,t) - u(x,t)\right) \frac{\mathrm{d}\tau}{\tau^{1+s}}. \tag{2.6}$$

Next we denote by $\{E_\lambda\}$ the spectral measures associated to $\mathcal{L}$. More precisely, we let

$$\mathcal{L} = -\int_0^\infty \lambda \; dE_\lambda. \tag{2.7}$$

Invoking such a spectral decomposition for the operator $\mathcal{L}$ and by using Fourier transform in t variable, we alternatively express $\mathcal{H}^s u$ in Fourier terms. To do so, we first observe the following representation of the heat semigroup $\{P_t\}_{t\geq 0}$ in terms of spectral measures as well as an important identity for gamma functions

$$P_t = \int_0^\infty e^{-\lambda t} \mathrm{d}E_\lambda, \text{ and } \quad -\frac{s}{\Gamma(1-s)} \int_0^\infty \frac{e^{-(\lambda+i\sigma)t} - 1}{\tau^{1+s}} \mathrm{d}\tau = (\lambda + i\sigma)^s, \; \lambda > 0, \sigma \in \mathbb{R}. \tag{2.8}$$

We refer to Section 2 in [13] for a detailed account on this aspect. Now we consider the Fourier transform in time variable to obtain from (2.4)

$$\mathcal{F}_t \left(P^{\mathcal{H}}_{\tau} u\right)(\xi, \sigma) = e^{-i\sigma\tau} P_\tau \left(\mathcal{F}_t u(\cdot,\sigma)\right)(\xi)$$

which results into the Fourier analogue of the definition (2.6)

$$\begin{aligned}\mathcal{F}_t\left(\mathcal{H}^s u\right)(\cdot,\sigma) &= -\frac{s}{\Gamma(1-s)} \int_0^\infty \frac{1}{\tau^{1+s}} \int_0^\infty \left(e^{-(\lambda+i\sigma)\tau} - 1\right) \mathrm{d}E_\lambda(\mathcal{F}_t u(\cdot,\sigma))\, \mathrm{d}\tau \\ &= \int_0^\infty (\lambda + i\sigma)^s \, \mathrm{d}E_\lambda(\mathcal{F}_t u(\cdot,\sigma))\end{aligned}$$

Here we have crucially used the relations in (2.8). Consequently, we can write for $u \in \mathcal{S}(\mathbb{R}^{n+1})$

$$\|\mathcal{F}_t\left(\mathcal{H}^s u\right)(\cdot,\sigma)\|_{L^2(\mathbb{R}^n)} = \int_0^\infty |\lambda + i\sigma|^{2s} \mathrm{d}\|E_\lambda(\mathcal{F}_t u(\cdot,\sigma))\|^2, \quad \sigma \in \mathbb{R}.$$

Keeping this in mind, we now define the appropriate function space which constitutes the domain of $\mathcal{H}^s$ and associated norm.

Definition 2.2. *For $s \in (0,1)$, we define the space $\mathcal{H}^{2s}(\mathbb{R}^{n+1})$ to be the completion of $\mathcal{S}(\mathbb{R}^{n+1})$ with respect to the norm*

$$\|u\|_{\mathcal{H}^{2s}(\mathbb{R}^{n+1})} = \left(\int_{\mathbb{R}}\int_0^\infty \left(1+|\lambda+i\sigma|^2\right)^s \mathrm{d}\|E_\lambda(\mathcal{F}_t u(\cdot,\sigma))\|^2 \mathrm{d}\sigma\right)^{\frac{1}{2}}. \tag{2.9}$$

It is worth noting that $\mathrm{Dom}(\mathcal{H}) \subseteq \mathrm{Dom}(\mathcal{H}^s)$.

More generally, we introduce the various function spaces that are needed in this set-up. Let $\mathcal{O}$ be an open set in $\mathbb{R}^{n+1}$ and $r \in \mathbb{R}$. We define

$$\mathcal{H}^r\left(\mathbb{R}^{n+1}\right) = \Bigg\{\text{Completion of } \mathcal{S}(\mathbb{R}^{n+1}) \text{ w.r.t the norm} :$$
$$\int_{\mathbb{R}}\int_0^\infty \left(\left(1+|\lambda+i\sigma|^2\right)^{r/2} d||E_\lambda(\mathcal{F}_t u(\cdot,\sigma))||^2 \,\mathrm{d}\sigma\right)^{1/2}\Bigg\},$$
$$\mathcal{H}^r(\mathcal{O}) = \left\{u|_{\mathcal{O}}; u \in \mathcal{H}^r\left(\mathbb{R}^{n+1}\right)\right\}, \quad \widetilde{\mathcal{H}}^r(\mathcal{O}) = \text{closure of } C_0^\infty(\mathcal{O}) \text{ in } H^r\left(\mathbb{R}^{n+1}\right).$$

Also we define

$$||u||_{\mathcal{H}^r(\mathcal{O})} = \inf\{||v||_{\mathcal{H}^r(\mathbb{R}^{n+1})} : v|_{\mathcal{O}} = u\}. \tag{2.10}$$

Now from resolution of the parabolic version of the Kato square root problem as in [4] and interpolation type arguments, we note that the following equivalence holds

$$\mathbb{H}^s(\mathbb{R}^{n+1}) = \mathcal{H}^s(\mathbb{R}^{n+1}), \quad s \in (0,1), \tag{2.11}$$

where $\mathbb{H}^s(\mathbb{R}^{n+1})$ is the parabolic fractional Sobolev space defined as

$$\mathbb{H}^s(\mathbb{R}^{n+1}) \overset{def}{=} \{u \in L^2 : (|\xi|^2 + i\rho)^{s/2}\mathcal{F}_{x,t}u(\xi,\rho) \in L^2\}. \tag{2.12}$$

We refer to [41, pages 6-7] for relevant details. Sometimes when the context is clear, the space-time fourier transform $\mathcal{F}_{x,t}u$ will be denoted by $\hat{u}$. From now on in view of (2.11), we will identify both the spaces $\mathbb{H}^s$ and $\mathcal{H}^s$ and furthermore for a closed set E in $\mathbb{R}^{n+1}$ we let

$$\mathbb{H}^s_E = \{u \in \mathbb{H}^s(\mathbb{R}^{n+1}) : \mathrm{supp}(u) \subset E\}. \tag{2.13}$$

It is easily seen that $\mathbb{H}^s_E$ is a Hilbert space.

We now note that the adjoint operator $\mathcal{H}^s_*$ is defined in terms of the spectral resolution in the following manner

$$\mathcal{F}_t\left(\mathcal{H}^s_* u\right)(\cdot,\sigma) = \int_0^\infty (\lambda - i\sigma)^s \mathrm{d}E_\lambda(\mathcal{F}_t u(\cdot,\sigma)), \quad \text{for } u \in \mathcal{S}(\mathbb{R}^{n+1}).$$

For $f, g \in \mathcal{S}(\mathbb{R}^{n+1})$, we observe from the properties of the spectral family of projection operators $\{E_\lambda\}_{\lambda>0}$ that

$$\begin{aligned}\langle \mathcal{H}^s f, g\rangle = \langle \mathcal{H}^{\frac{s}{2}} f, \mathcal{H}^{\frac{s}{2}}_* g\rangle = \langle f, \mathcal{H}^s_* g\rangle &= \int_{\mathbb{R}}\int_0^\infty (\lambda+i\sigma)^s \,\mathrm{d}\langle E_\lambda \mathcal{F}_t u, \overline{\mathcal{F}_t v}\rangle(\cdot,\sigma)\mathrm{d}\sigma \\ &\preceq \|f\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}\|g\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}.\end{aligned} \tag{2.14}$$

As an outcome of the inequality (2.14), we have the mapping property $\mathcal{H}^s : \mathbb{H}^s(\mathbb{R}^{n+1}) \to \mathbb{H}^{-s}(\mathbb{R}^{n+1})$ where $\mathbb{H}^{-s}$ denotes the dual space.

3. Some Direct problems

In this section, we study some direct problems related to the fractional operator $\mathcal{H}^s$. We start with the discussion on the extension problem for $\mathcal{H}^s$. Then the well-posedness results for (1.1) and (1.4) will be presented which mainly relies on the Lax-Milgram type arguments.

3.1. **The extension problem for $\mathcal{H}^s$.** Our objective here is to solve the extension problem for $\mathcal{H}^s$ with prescribed Dirichlet data $u \in \mathbb{H}^s(\mathbb{R}^{n+1})$. More specifically, we consider solution to the following Dirichlet problem in $\mathbb{R}^{n+2}_+$

$$\begin{cases} \mathcal{L}_a U = z^a \partial_t U - \operatorname{div}(z^a A(x) \nabla_{x,z} U) = 0, & \text{in } \mathbb{R}^{n+1} \times \mathbb{R}_+,\ a = 1-2s \\ U(x,t,0) = u(x,t), & \text{on } \mathbb{R}^{n+1}. \end{cases} \tag{3.1}$$

by introducing a new variable $z \in \mathbb{R}_+$. As it is well known by now, (3.1) represents the parabolic counterpart of the Caffarelli-Silvestre extension problem as in [16] for $\mathcal{H}^s$. See [13, 15, 44, 50]. More precisely, it has been shown in the cited works that if $u \in \mathbb{H}^{2s}$, then we have in $L^2(\mathbb{R}^{n+1})$,

$$\lim_{z\to 0^+} z^a \partial_z U = \mathcal{H}^s. \tag{3.2}$$

In our setting of the Calderon problem, it turns out that we need to deal with $u \in \mathbb{H}^s$. Therefore, this requires generalizing the convergence in (3.2) with respect to a weaker norm for functions in $\mathbb{H}^s$ and this is the one of the main contents of Theorem 3.1 below. Such a result has already been established in the case when $A = \mathbb{I}$ in [18, Proposition 4.2]. Moreover, we also show that the extended function belongs to the right energy space so that weak type methods as in [11] can be subsequently applied. This is finally relevant to the weak unique continuation result Theorem 1.3 that we prove in Section 4. In this regard, we now introduce the relevant energy space.

For an open set $\Sigma \subseteq \mathbb{R}^{n+1} \times \mathbb{R}_+$, we define the energy space $\mathcal{L}^{1,2}(\Sigma; z^a \mathrm{d}x\mathrm{d}t\mathrm{d}z)$ which consists of all those $U \in L^2(\Sigma; z^a \mathrm{d}x\mathrm{d}t\mathrm{d}z)$ such that $\nabla_x U$ and $\partial_z U \in L^2(\Sigma; z^a \mathrm{d}x\mathrm{d}t\mathrm{d}z)$, endowed with the norm

$$\|U\|_{\mathcal{L}^{1,2}(\Sigma; z^a \mathrm{d}x\mathrm{d}t\mathrm{d}z)} \overset{def}{=} \int_{\Sigma\times\mathbb{R}_+} z^a \left(|U|^2 + |\nabla_x U|^2 + |\partial_z U|^2\right) \mathrm{d}x\mathrm{d}t\mathrm{d}z.$$

As mentioned above, we now state the central result of this subsection which concerns the various convergence properties of the extended function in (3.1) (and its weighted Neumann derivative) corresponding to $u \in \mathbb{H}^s$. Theorem 3.1 below can be regarded as the variable coefficient analogue of Proposition 4.2 in [18]. We crucially adapt some ideas from [13] in our proof of this result. However unlike that in [13], since the convergence results are established in different norms, therefore our proof has required some delicate reworking of the ideas in [13].

Theorem 3.1. *Let $s \in (0,1)$ and $u \in \mathbb{H}^s(\mathbb{R}^{n+1})$. There exists a solution to (3.1) satisfying*

i) $\lim_{z\to 0+} U(\cdot,\cdot,z) = u$ in $\mathbb{H}^s(\mathbb{R}^{n+1})$,

ii) $\lim_{z\to 0+} \dfrac{2^{-a}\Gamma(s)}{\Gamma(1-s)} z^a \partial_z U(\cdot,\cdot,z) = \mathcal{H}^s u$ in $\mathbb{H}^{-s}(\mathbb{R}^{n+1})$,

iii) $\|U\|_{\mathcal{L}^{1,2}(\mathbb{R}^{n+1}\times(0,M); z^a \mathrm{d}x\mathrm{d}t\mathrm{d}z)} \preceq_M \|u\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}$.

Proof. We first note that the solution to (3.1) is given by

$$U(x,t,z) = \int_0^\infty \int_{\mathbb{R}^n} P^a_z(x,y,\tau) u(y,t-\tau)\, \mathrm{d}y\mathrm{d}\tau \tag{3.3}$$

where

$$P_z^a(x,y,t) := \frac{1}{2^{1-a}\Gamma(\frac{1-a}{2})}\frac{z^{1-a}}{t^{\frac{3-a}{2}}}e^{-\frac{z^2}{4t}}p(x,y,t).$$

By taking the Fourier transform in time variable in (3.3), we have the expression

$$\begin{aligned}\mathcal{F}_tU(x,\sigma,z) &= \frac{z^{1-a}}{2^{1-a}\Gamma(\frac{1-a}{2})}\int_0^\infty \frac{e^{-\frac{z^2}{4\tau}}}{\tau^{\frac{3-a}{2}}}e^{-i\tau\sigma}P_\tau(\mathcal{F}_tu(\cdot,\sigma))(x)\ \mathrm{d}\tau.\\ &= \frac{z^{1-a}}{2^{1-a}\Gamma(\frac{1-a}{2})}\int_0^\infty\int_0^\infty \frac{e^{-\frac{z^2}{4\tau}}}{\tau^{\frac{3-a}{2}}}e^{-(\lambda+i\sigma)\tau}\mathrm{d}\tau\ \mathrm{d}E_\lambda(\mathcal{F}_tu)(\cdot,\sigma)(x) \end{aligned} \tag{3.4}$$

where we used the spectral representation of P_τ as in (2.8) in the last line. The relation (3.4) readily implies

$$\mathcal{F}_t\left(U(\cdot,\sigma,z)-u(\cdot,\sigma)\right) = \frac{z^{2s}}{4^s\Gamma(s)}\int_0^\infty\int_0^\infty \frac{e^{-\frac{z^2}{4\tau}}}{\tau^{1+s}}\left(e^{-(\lambda+i\sigma)\tau}-1\right)\mathrm{d}\tau\ \mathrm{d}E_\lambda(\mathcal{F}_tu)(\cdot,\sigma) \tag{3.5}$$

To show that U attains the prescribed data u at $z=0$ in the $\mathbb{H}^s$ sense, we recall the important identity which can be found in [30, page 369]

$$\int_0^\infty t^{\nu-1}e^{-\left(\frac{\beta}{t}+\gamma t\right)}\ \mathrm{d}t = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}}K_\nu(2\sqrt{\beta\gamma}) \tag{3.6}$$

where $\Re(\beta),\Re(\gamma)>0$ and K_ν is the Bessel function of third kind. We notice that

$$\begin{aligned}\int_0^\infty \frac{e^{-(\lambda+i\sigma)\frac{z^2}{4\theta}}}{\theta^{1+s}}\left[e^{-\theta}-1\right]\mathrm{d}\theta &= 2\left(\frac{z\sqrt{\lambda+i\sigma}}{2}\right)^{-s}K_s\left(z\sqrt{\lambda+i\sigma}\right)-\Gamma(s)\left(\frac{z\sqrt{\lambda+i\sigma}}{2}\right)^{-2s}\\ &= 2^{1+s}\left(z\sqrt{\lambda+i\sigma}\right)^{-2s}\left(\left(z\sqrt{\lambda+i\sigma}\right)^{s}K_s\left(z\sqrt{\lambda+i\sigma}\right)-2^{s-1}\Gamma(s)\right)\end{aligned} \tag{3.7}$$

where we took $\beta=\frac{(\lambda+i\sigma)z^2}{4}, \gamma=1, \nu=-s$ in (3.6) and used analytic continuation to the following identity

$$\int_0^\infty e^{-\frac{\zeta^2}{\theta}}\frac{\mathrm{d}\theta}{\theta^{1+s}} = \int_0^\infty \frac{e^{-p}}{\left(\frac{\zeta^2}{p}\right)^{1+s}}\frac{\zeta^2}{p^2}\mathrm{d}p = \zeta^{-2s}\int_0^\infty p^{s-1}e^{-p}\mathrm{d}p = \Gamma(s)\zeta^{-2s},\ \text{for } \zeta>0. \tag{3.8}$$

Alternatively, extension of the identity (3.8) to complex parameters can also be justified by a contour type integration in the complex plane. See for instance the proof of Theorem 1.1 in [13]. Using (3.5) we have,

$$\begin{aligned}\|U(\cdot,\cdot,z)-u(\cdot,\cdot)\|^2_{\mathbb{H}^s(\mathbb{R}^{n+1})} &= \int_{\mathbb{R}}\int_0^\infty \left(1+|\lambda+i\sigma|^{\frac{s}{2}}\right)^2\mathrm{d}\|E_\lambda(\mathcal{F}_t\left(U(\cdot,\sigma,z)-u(\cdot,\sigma)\right))\|^2\mathrm{d}\sigma\\ &\preceq z^{4s}\int_{\mathbb{R}}\int_0^\infty \left(1+|\lambda+i\sigma|^{\frac{s}{2}}\right)^2\left|\int_0^\infty \frac{e^{-\frac{z^2}{4\tau}}}{\tau^{1+s}}\left(e^{-(\lambda+i\sigma)\tau}-1\right)\mathrm{d}\tau\right|^2\mathrm{d}\|E_\lambda(\mathcal{F}_tu)(\cdot,\sigma)\|^2\mathrm{d}\sigma\\ &\preceq z^{4s}\int_{\mathbb{R}}\int_0^\infty \left(1+|\lambda+i\sigma|^{\frac{s}{2}}\right)^2|\lambda+i\sigma|^{2s}\left|\int_0^\infty \frac{e^{-(\lambda+i\sigma)\frac{z^2}{4\theta}}}{\theta^{1+s}}\left[e^{-\theta}-1\right]\ \mathrm{d}\theta\right|^2\mathrm{d}\|E_\lambda(\mathcal{F}_tu)(\cdot,\sigma)\|^2\mathrm{d}\sigma,\end{aligned}$$

where in the last line, we used a complex change of variable which can be justified by a contour type integration. Now using (3.7) we find from above

$$\begin{aligned}
&\|U(\cdot,\cdot,z)-u(\cdot,\cdot)\|^2_{\mathbb{H}^s(\mathbb{R}^{n+1})}\\
&\preceq \int_{\mathbb{R}}\int_0^\infty \left(1+|\lambda+i\sigma|^{\frac{s}{2}}\right)^2 \left|\left(z\sqrt{\lambda+i\sigma}\right)^s K_s\left(z\sqrt{\lambda+i\sigma}\right)-2^{s-1}\Gamma(s)\right|^2 \mathrm{d}\|E_\lambda(\mathcal{F}_t u)(\cdot,\sigma)\|^2\mathrm{d}\sigma\\
&\preceq \sup_{|\xi|\le\epsilon}\left|\xi^s K_s(\xi)-2^{s-1}\Gamma(s)\right|^2\int_{\mathbb{R}}\int_0^\infty \left(1+|\lambda+i\sigma|^{\frac{s}{2}}\right)^2 \mathrm{d}\|E_\lambda(\mathcal{F}_t u)(\cdot,\sigma)\|^2\mathrm{d}\sigma\\
&\quad+\sup_{|\xi|>\epsilon}\left|\xi^s K_s(\xi)-2^{s-1}\Gamma(s)\right|^2\int_{\mathbb{R}}\int_0^\infty \chi_{z|\lambda+i\sigma|^{\frac12}>\epsilon}\left(1+|\lambda+i\sigma|^{\frac{s}{2}}\right)^2 \mathrm{d}\|E_\lambda(\mathcal{F}_t u)(\cdot,\sigma)\|^2\mathrm{d}\sigma
\end{aligned}\tag{3.9}$$

Taking $z\to 0+$ in (3.9), we notice that its second term converges to zero. We also use the fact that $\xi^s K_s(\xi)$ is uniformly bounded for large ξ which follows from the fact that for all large ξ

$$|K_s(\xi)|\le Ce^{-\xi}.$$

See for instance [37, (5.11.8)].

After that, we use

$$\lim_{z\to 0} z^s K_s(z)=2^{s-1}\Gamma(s)$$

and let ϵ approach to zero in (3.9) to conclude that the first integral in (3.9) goes to 0. This establishes i).

We now turn our attention to ii), i.e. we show that $\lim_{z\to 0+}\frac{2^{-a}\Gamma(s)}{\Gamma(1-s)}z^a\partial_z U(\cdot,\cdot,z)=\mathcal{H}^s u$ in $\mathbb{H}^{-s}(\mathbb{R}^{n+1})$. We first assume that $u\in\mathcal{S}(\mathbb{R}^{n+1})$. In order to prove ii), we will make use of the following identity holds which was observed in [13, (3.14)]

$$\frac{2^{-a}\Gamma(s)}{\Gamma(1-s)}\,z^a\partial_z\mathcal{F}_t U(\cdot,\sigma,y)=-\frac{1}{\Gamma(1-s)}\int_0^\infty(\lambda+i\sigma)^s\int_0^\infty\frac{e^{-\theta}e^{-(\lambda+i\sigma)\frac{z^2}{4\theta}}}{\theta^s}\mathrm{d}\theta\;\mathrm{d}E_\lambda(\mathcal{F}_t u)(\cdot,\sigma)\tag{3.10}$$

Also we have

$$\mathcal{F}_t\left(\mathcal{H}^s u\right)(\cdot,\sigma)=\int_0^\infty(\lambda+i\sigma)^s\,\mathrm{d}E_\lambda(\mathcal{F}_t u)(\cdot,\sigma).$$

We also make use of the following identity which follows from (3.6) by taking $\beta=\frac{(\lambda+i\sigma)z^2}{4},\gamma=1,\nu=1-s$.

$$\int_0^\infty\frac{e^{-\theta}e^{-(\lambda+i\sigma)\frac{z^2}{4\theta}}}{\theta^s}\mathrm{d}\theta=2\left(\frac{z\sqrt{\lambda+i\sigma}}{2}\right)^{1-s}K_{1-s}\left(z\sqrt{\lambda+i\sigma}\right).\tag{3.11}$$

We thus find

$$\begin{aligned}
&\frac{2^{-a}\Gamma(s)}{\Gamma(1-s)}\,z^a\partial_z\mathcal{F}_t U(\cdot,\sigma,z)-\mathcal{F}_t\left(\mathcal{H}^s u\right)(\cdot,\sigma) \qquad (3.12)\\
&=\frac{2^s}{\Gamma(1-s)}\int_0^\infty(\lambda+i\sigma)^s\left(\left(z\sqrt{\lambda+i\sigma}\right)^{1-s}K_{1-s}\left(z\sqrt{\lambda+i\sigma}\right)-2^{-s}\Gamma(1-s)\right)\mathrm{d}E_\lambda(\mathcal{F}_t u)(\cdot,\sigma).
\end{aligned}$$

Now for any test function $\phi \in \mathbb{H}^s(\mathbb{R}^{n+1})$, using Cauchy-Schwarz inequality we find

$$\begin{aligned}
&\left\langle \frac{2^{-a}\Gamma(s)}{\Gamma(1-s)} z^a \partial_z U(\cdot,\cdot,z) - \mathcal{H}^s u, \phi \right\rangle_{\mathbb{H}^{-s}(\mathbb{R}^{n+1}),\mathbb{H}^s(\mathbb{R}^{n+1})} \qquad (3.13)\\
&= \int_{\mathbb{R}}\int_0^\infty \mathrm{d}\left\langle E_\lambda \left(\frac{2^{-a}\Gamma(s)}{\Gamma(1-s)}\, y^a \partial_y \mathcal{F}_t U(\cdot,\sigma,z) - \mathcal{F}_t\left(\mathcal{H}^s u\right)(\cdot,\sigma)\right), E_\lambda \phi \right\rangle \mathrm{d}\sigma\\
&\preceq \left(\int_{\mathbb{R}}\int_0^\infty (1+|\lambda+i\sigma|)^{-s}\, \mathrm{d}\left\| E_\lambda \left(\frac{2^{-a}\Gamma(s)}{\Gamma(1-s)}\, z^a \partial_z \mathcal{F}_t U(\cdot,\sigma,z) - \mathcal{F}_t\left(\mathcal{H}^s u\right)(\cdot,\sigma)\right)\right\|^2 \mathrm{d}\sigma \right)^{1/2}\\
&\qquad \times \left(\int_{\mathbb{R}}\int_0^\infty (1+|\lambda+i\sigma|)^{s}\, \mathrm{d}\|E_\lambda\phi\|^2 \mathrm{d}\sigma \right)^{1/2}\\
&\preceq \|\phi\|_{\mathbb{H}^s(\mathbb{R}^{n+1})} \left(\int_{\mathbb{R}}\int_0^\infty (1+|\lambda+i\sigma|)^{-s} \mathrm{d}\left\| E_\lambda \left(\frac{2^{-a}\Gamma(s)}{\Gamma(1-s)}\, z^a \partial_z \mathcal{F}_t U(\cdot,\sigma,z) - \mathcal{F}_t\left(\mathcal{H}^s u\right)(\cdot,\sigma)\right)\right\|^2 \mathrm{d}\sigma \right)^{1/2}.
\end{aligned}$$

Then using (3.12) and also by using properties of the projection operators $\{E_\lambda\}$ we infer

$$\begin{aligned}
&\left\| \frac{2^{-a}\Gamma(s)}{\Gamma(1-s)} z^a \partial_z U(\cdot,\cdot,z) - \mathcal{H}^s u \right\|_{\mathbb{H}^{-s}(\mathbb{R}^{n+1})}\\
&\preceq \sup_{|w|\le\epsilon} \left| w^{1-s}K_{1-s}(w) - 2^{-s}\Gamma(1-s)\right| \left(\int_{\mathbb{R}}\int_0^\infty |\lambda+i\sigma|^s \mathrm{d}\|E_\lambda(\mathcal{F}_t u)(\cdot,\sigma)\|^2 \right)^{1/2}\\
&\qquad + \sup_{|w|>\epsilon} \left| w^{1-s}K_{1-s}(w) - 2^{-s}\Gamma(1-s)\right| \left(\int_{\mathbb{R}}\int_0^\infty \chi_{z|\lambda+i\sigma|^{\frac12}>\epsilon} |\lambda+i\sigma|^s \mathrm{d}\|E_\lambda(\mathcal{F}_t u)(\cdot,\sigma)\|^2 \right)^{1/2}.
\end{aligned}$$

Similarly as before, we first take $z \to 0+$ and then let $\epsilon \to 0$ to assert that

$$\left\| \frac{2^{-a}\Gamma(s)}{\Gamma(1-s)} z^a \partial_z U(\cdot,\cdot,z) - \mathcal{H}^s u \right\|_{\mathbb{H}^{-s}(\mathbb{R}^{n+1})} \to 0, \quad \text{as } z \to 0+, \tag{3.14}$$

for $u \in \mathcal{S}(\mathbb{R}^{n+1})$.

Now let $u_k \to u$ in $\mathbb{H}^s$ where u_k's are in $\mathcal{S}(\mathbb{R}^{n+1})$. We denote by U_k and U_l the solutions to the extension problem (3.1) corresponding to Dirichlet data u_k and u_l respectively. Then using (3.10) and by an analogous argument as in (3.13) we find as $k \to \infty$

$$\begin{aligned}
&\left\| \frac{2^{-a}\Gamma(s)}{\Gamma(1-s)} z^a \partial_z (U_k - U)(\cdot,\cdot,z) \right\|_{\mathbb{H}^{-s}(\mathbb{R}^{n+1})} \qquad (3.15)\\
&\preceq \sup_{\mathbb{R}^+} \left| w^{1-s}K_{1-s}(w)\right| \left(\int_{\mathbb{R}}\int_0^\infty |\lambda+i\sigma|^s \mathrm{d}\|E_\lambda(\mathcal{F}_t(u_k-u)(\cdot,\sigma))\|^2 \right)^{1/2} \to 0,
\end{aligned}$$

(since $u_k \to u$ in $\mathbb{H}^s$).

Thus $\{z^a\partial_z U_k\}$ is uniformly Cauchy in z as $z \to 0^+$. This fact combined with (3.14) implies ii) in a standard way.

Now we plan to demonstrate the energy estimate

$$\|U\|_{\mathcal{L}^{1,2}(\mathbb{R}^{n+1}\times(0,M);z^a\mathrm{d}x\mathrm{d}t\mathrm{d}y)} \preceq_M \|u\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}. \tag{3.16}$$

We closely follow the arguments from [13] to establish (3.16). We will not be concerned with proving

$$\|U\|_{L^2(\mathbb{R}^{n+1}\times(0,M);z^a\mathrm{d}x\mathrm{d}t\mathrm{d}y)} \preceq_M \|u\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}.$$

wer as this is already covered in [13, (3.15)]. We first estimate the term

$$\left\|z^{\frac{a}{2}}\partial_z U\right\|_{L^2(\mathbb{R}^{n+1}\times(0,M))} = \int_0^M \int_{\mathbb{R}} z^a \|\partial_z \mathcal{F}_t U(\cdot,\sigma,y)\|^2_{L^2(\mathbb{R}^n)}\mathrm{d}\sigma\mathrm{d}z.$$

By using (3.10) and (3.11), we find that such a term equals

$$\int_0^M \int_{\mathbb{R}}\int_0^\infty z^{-a}|\lambda+i\sigma|^{2s}\left|\int_0^\infty \frac{e^{-\theta}e^{-(\lambda+i\sigma)\frac{z^2}{4\theta}}}{\theta^s}\mathrm{d}\theta\right|^2 \mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2\mathrm{d}\sigma\mathrm{d}z$$
$$\simeq \int_0^M \int_{\mathbb{R}}\int_0^\infty z^{-a}|\lambda+i\sigma|^{2s}\left|\left(z\sqrt{\lambda+i\sigma}\right)^{1-s}K_{1-s}\left(z\sqrt{\lambda+i\sigma}\right)\right|^2 \mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2\mathrm{d}\sigma\mathrm{d}z \quad = I_1+I_2,$$

where I_1 is the integral over the region where $|z\sqrt{\lambda+i\sigma}| \le 1$ and I_2 is the integral over the complement. We first estimate I_1 as follows.

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}}\int_0^\infty \int_0^{|\lambda+i\sigma|^{-\frac{1}{2}}} z^{-a}|\lambda+i\sigma|^{2s}\left|\left(z\sqrt{\lambda+i\sigma}\right)^{1-s}K_{1-s}\left(z\sqrt{\lambda+i\sigma}\right)\right|^2 \mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2\mathrm{d}\sigma\mathrm{d}z \\
&\preceq \sup_{|z|\le 1}\left|z^{1-s}K_{1-s}(z)\right|^2 \int_{\mathbb{R}}\int_0^\infty \left(\int_0^{|\lambda+i\sigma|^{-\frac{1}{2}}} z^{2s-1}\mathrm{d}z\right)|\lambda+i\sigma|^{2s}\mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2\mathrm{d}\sigma \\
&\preceq \int_{\mathbb{R}}\int_0^\infty |\lambda+i\sigma|^s\ \mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2\mathrm{d}\sigma \preceq \|u\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}.
\end{aligned} \tag{3.17}$$

Likewise, I_2 can be estimated as

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}}\int_0^\infty \int_{|\lambda+i\sigma|^{-\frac{1}{2}}}^\infty z^{-a}|\lambda+i\sigma|^{2s}\left|\left(z\sqrt{\lambda+i\sigma}\right)^{1-s}K_{1-s}\left(z\sqrt{\lambda+i\sigma}\right)\right|^2 \mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2\mathrm{d}\sigma\mathrm{d}z \\
&\preceq \int_{\mathbb{R}}\int_0^\infty \left(\int_{|\lambda+i\sigma|^{-\frac{1}{2}}}^\infty z\left|K_{1-s}\left(z\sqrt{\lambda+i\sigma}\right)\right|^2\ \mathrm{d}z\right)|\lambda+i\sigma|^{1+s}\ \mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2\mathrm{d}\sigma \\
&\preceq \int_{\mathbb{R}}\int_0^\infty \left(\int_{|\lambda+i\sigma|^{-\frac{1}{2}}}^\infty e^{-z|\lambda+i\sigma|^{\frac{1}{2}}}\frac{\mathrm{d}z}{|\lambda+i\sigma|^{\frac{1}{2}}}\right)|\lambda+i\sigma|^{1+s}\ \mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2\mathrm{d}\sigma \\
&\preceq \int_{\mathbb{R}}\int_0^\infty \left(\int_1^\infty e^{-m}\ \mathrm{d}m\right)|\lambda+i\sigma|^{s}\ \mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2\mathrm{d}\sigma \preceq \|u\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}.
\end{aligned} \tag{3.18}$$

where we have used the asymptotic $|K_{1-s}(z)|^2 = O\left(\frac{e^{-|z|}}{|z|}\right)$ for $|z|\ge 1$. See for instance [37, 5.11.10]. Combining (3.17) and (3.18), we obtain $\left\|z^{\frac{a}{2}}\partial_z U\right\|_{L^2(\mathbb{R}^{n+1}\times(0,M))} \preceq \|u\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}$. Now we estimate the term $\|\nabla_x U\|_{L^2(\Sigma;z^a\mathrm{d}x\mathrm{d}t\mathrm{d}z)}$. For that, we note that from the resolution of the famous Kato square root problem as in [5] and [6], we have for a smooth function f decaying rapidly at infinity in $\mathbb{R}^n$ that the following holds

$$||\nabla_x f||_{L^2(\mathbb{R}^n)} \approx ||(-\mathcal{L})^{1/2}f||_{L^2(\mathbb{R}^n)}. \tag{3.19}$$

Combining this with Plancherel theorem in the t variable we find

$$\|\nabla_x U\|_{L^2(\Sigma;z^a\mathrm{d}x\mathrm{d}t\mathrm{d}z)} \approx \|z^{\frac{a}{2}}(-\mathcal{L})^{\frac{1}{2}}\mathcal{F}_t U\|_{L^2(\mathbb{R}^{n+1}\times(0,M))} \text{ (using Plancherel theorem in the } t \text{ variable)}$$
$$= \int_0^M \int_{\mathbb{R}} \int_0^\infty z^a \lambda \, \mathrm{d}\|E_\lambda \mathcal{F}_t U(\cdot,\sigma)\|^2 \mathrm{d}\sigma \mathrm{d}z$$
$$= \int_0^M \int_{\mathbb{R}} \int_0^\infty z^a \lambda z^{4s} \left| \int_0^\infty e^{-\frac{z^2}{4\tau}} e^{-(\lambda+i\sigma)\tau} \frac{\mathrm{d}\tau}{\tau^{1+s}} \right|^2 \mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2 \mathrm{d}\sigma \mathrm{d}z$$
$$= \int_0^M \int_{\mathbb{R}} \int_0^\infty \lambda z^{1+2s} |\lambda + i\sigma|^{2s} \left| \int_0^\infty e^{-\theta} e^{-(\lambda+i\sigma)\frac{z^2}{4\theta}} \frac{\mathrm{d}\theta}{\theta^{1+s}} \right|^2 \mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2 \mathrm{d}\sigma \mathrm{d}z$$
$$\approx \int_0^M \int_{\mathbb{R}} \int_0^\infty \lambda z^{1-2s} \left| \left(z\sqrt{\lambda+i\sigma}\right)^s K_s\left(z\sqrt{\lambda+i\sigma}\right)\right|^2 \mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2 \mathrm{d}\sigma \mathrm{d}z \text{ (using (3.6) with } \nu=-s)$$
$$= J_1 + J_2,$$

where J_1 is integration over the region where $|z\sqrt{\lambda+i\sigma}| \le 1$ and J_2 is the integral over the complement. We find

$$\begin{aligned} J_1 &= \int_{\mathbb{R}} \int_0^\infty \int_0^{|\lambda+i\sigma|^{-\frac{1}{2}}} \lambda z^{1-2s} \left| \left(z\sqrt{\lambda+i\sigma}\right)^s K_s\left(z\sqrt{\lambda+i\sigma}\right)\right|^2 \mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2 \mathrm{d}\sigma \mathrm{d}z \\ &\preceq \sup_{|z|\le 1} |z^s K_s(z)|^2 \int_{\mathbb{R}} \int_0^\infty \left(\int_0^{|\lambda+i\sigma|^{-\frac{1}{2}}} z^{1-2s} \mathrm{d}z \right) \lambda \, \mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2 \mathrm{d}\sigma \\ &\preceq \int_{\mathbb{R}} \int_0^\infty |\lambda+i\sigma|^{s-1} \lambda \, \mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2 \mathrm{d}\sigma \preceq \|u\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}. \end{aligned} \tag{3.20}$$

Likewise, J_2 is estimated in the following way.

$$\begin{aligned} J_2 &= \int_{\mathbb{R}} \int_0^\infty \int_{|\lambda+i\sigma|^{-\frac{1}{2}}}^\infty \lambda z^{1-2s} \left| \left(z\sqrt{\lambda+i\sigma}\right)^s K_s\left(z\sqrt{\lambda+i\sigma}\right)\right|^2 \mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2 \mathrm{d}\sigma \mathrm{d}z \\ &\preceq \int_{\mathbb{R}} \int_0^\infty \left(\int_{|\lambda+i\sigma|^{-\frac{1}{2}}}^\infty z \left| K_s\left(z\sqrt{\lambda+i\sigma}\right)\right|^2 \mathrm{d}z \right) \lambda |\lambda+i\sigma|^s \mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2 \mathrm{d}\sigma \mathrm{d}z \\ &\preceq \int_{\mathbb{R}} \int_0^\infty \left(\int_{|\lambda+i\sigma|^{-\frac{1}{2}}}^\infty e^{-z|\lambda+i\sigma|^{\frac{1}{2}}} \frac{\mathrm{d}z}{|\lambda+i\sigma|^{\frac{1}{2}}} \right) |\lambda+i\sigma|^{1+s} \, \mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2 \mathrm{d}\sigma \\ &\preceq \int_{\mathbb{R}} \int_0^\infty \left(\int_1^\infty e^{-m} \, \mathrm{d}m \right) |\lambda+i\sigma|^s \, \mathrm{d}\|E_\lambda \mathcal{F}_t u(\cdot,\sigma)\|^2 \mathrm{d}\sigma \preceq \|u\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}. \end{aligned} \tag{3.21}$$

where we have again used the asymptotic $|K_s(z)|^2 = O\left(\frac{e^{-|z|}}{|z|}\right)$ for $|z| \ge 1$. From the inequalities (3.20) and (3.21), we conclude that $\|\nabla_x U\|_{L^2(\mathbb{R}^{n+1}\times(0,M);z^a\mathrm{d}x\mathrm{d}t\mathrm{d}z)} \preceq \|u\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}$. This finishes the proof of the theorem.

□

3.2. **Fundamental solution of the extension problem.** We now introduce the fundamental solution $\mathcal{G}(Y,X,t)$ associated to the extended operator

$$\mathcal{L}_a := x_{n+1}^a \partial_t - \mathrm{div}\left(x_{n+1}^a A(x,t)\nabla\right)$$

$X=(x,x_{n+1}), Y=(y,y_{n+1})$ will denote generic points in $\mathbb{R}^n\times\mathbb{R}$. It is to be noted that x_{n+1} will play the role of extension variable z that was introduced in Subsection 3.1.

For a function f, we let

$$\partial^a_{x_{n+1}} f = \lim_{x_{n+1}\to 0^+} x^a_{n+1}\partial_{n+1} f(x,x_{n+1}). \tag{3.22}$$

This limit is interpreted in $\mathbb{H}^{-s}$ sense in Theorem 3.1 (where z plays the role of x_{n+1}) but would be eventually interpreted in the strong point wise sense in Section 4 once we have the regularity result in Lemma 4.7.

We now recall that it was shown in [25] that given $\phi\in C_0^\infty(\mathbb{R}^{n+1}_+)$ the solution of the Cauchy problem with Neumann condition

$$\begin{cases}\mathcal{L}_a U = 0 & \text{in } \mathbb{R}^{n+1}_+\times(0,\infty)\\ U(X,0)=\phi(X), & X\in\mathbb{R}^{n+1}_+,\\ \partial^a_{x_{n+1}}U(x,0,t)=0 & x\in\mathbb{R}^n,\ t\in(0,\infty)\end{cases} \tag{3.23}$$

is given by the formula

$$\mathscr{P}^{(a)}_t\phi(Y) \overset{def}{=} U(Y,t) = \int_{\mathbb{R}^{n+1}_+}\phi(X)\mathcal{G}(Y,X,t)x^a_{n+1}dX, \tag{3.24}$$

where

$$\mathcal{G}(Y,X,t) = p(y,x,t)\ p_a(x_{n+1},y_{n+1};t), \tag{3.25}$$

and where $p(y,x,t)$ is the heat-kernel associated to $(\partial_t - \operatorname{div}(A(x,t)\nabla_x))$ as in (2.1) and p_a is the fundamental solution of the Bessel operator $\partial^2_{x_{n+1}} + \frac{a}{x_{n+1}}\partial_{x_{n+1}}$. Such a function p_a is given by the formula

$$p_a(x_{n+1},y_{n+1};t) = (2t)^{-\frac{1+a}{2}}e^{-\frac{x^2_{n+1}+y^2_{n+1}}{4t}}\left(\frac{x_{n+1}y_{n+1}}{2t}\right)^{\frac{1-a}{2}} I_{\frac{a-1}{2}}\left(\frac{x_{n+1}y_{n+1}}{2t}\right), \tag{3.26}$$

where $I_\nu(z)$ the modified Bessel function of the first kind defined by the series

$$I_\nu(w) = \sum_{k=0}^\infty \frac{(w/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+1+\nu)}, \quad |w|<\infty,\ |\arg w|<\pi. \tag{3.27}$$

It follows from (2.1) that

$$\int_{\mathbb{R}^{n+1}_+} x^a_{n+1}\mathcal{G}(Y,X,t)dX = 1, \tag{3.28}$$

and also

$$\mathscr{P}^{(a)}_t\phi(X) \underset{t\to 0^+}{\longrightarrow} \phi(X). \tag{3.29}$$

We finally record the following Gaussian bounds for $p(y,x,t)$ as in [1] which will be needed in Section 4

$$\frac{1}{N_0 t^{n/2}}e^{-\frac{N_0|x-y|^2}{t}} \le p(y,x;t) \le \frac{N_0}{t^{n/2}}e^{-\frac{|x-y|^2}{N_0 t}}. \tag{3.30}$$

3.3. **The initial-exterior problem for $\mathcal{H}^s + q(x,t)$.** In this subsection, we discuss some well-posedness results for the forward problem (1.1). More generally, we consider the presence of a non-trivial source term in the PDE, i.e. we look for existence and uniqueness results of the problem

$$\begin{cases} (\mathcal{H}^s + q(x,t))\, u = F, & \text{in } Q := \Omega \times (-T,T) \\ u(x,t) = f(x,t), & \text{in } Q_e := \Omega_e \times (-T,T) \\ u(x,t) = 0, & \text{for } t \le -T, \end{cases} \tag{3.31}$$

where Ω_e denotes the complement of Ω.

We consider the bilinear map $\mathcal{B}_q(\cdot,\cdot)$ on $\mathbb{H}^s(\mathbb{R}^{n+1}) \times \mathbb{H}^s(\mathbb{R}^{n+1})$ defined by

$$\mathcal{B}_q\,(f,g) = \langle \mathcal{H}^{\frac{s}{2}} f, \mathcal{H}_*^{\frac{s}{2}} g\rangle + \int_Q q f \overline{g}.$$

It follows by an application of Cauchy-Schwartz inequality that

$$|\mathcal{B}_q\,(f,g)| \preceq \|f\|_{\mathbb{H}^s(\mathbb{R}^{n+1})} \|g\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}. \tag{3.32}$$

Akin to that in [18], it turns out that we need to study a time localized problem. Therefore to this end, we introduce the notation

$$u_T(x,t) = u(x,t)\chi_{[-T,T]}(t) \tag{3.33}$$

and note that, $u_T \in \mathbb{H}^s(\mathbb{R}^{n+1})$ whenever $u \in \mathbb{H}^s(\mathbb{R}^{n+1})$. This follows from the fact that $\chi_{[-T,T]}$ is a multiplier in the Sobolev space $H^\gamma(\mathbb{R})$ for $|\gamma| \le \frac{1}{2}$. See for instance [40, Theorem 11.4 in Chapter 1]. Thus we cast all the upcoming analysis for u_T as we can only guarantee the uniqueness up to $t = T$. Also it is to be noted that from the representation of $\mathcal{H}^s$ as in (2.6) it follows that $\mathcal{H}^s u(x,t) = \mathcal{H}^s(\chi_{(-\infty,T]} u)(x,t)$ for all $t \le T$.

Below we simply denote the distributional pairing $\langle \cdot,\cdot\rangle_{(\mathbb{H}^s_{\overline{Q}})^*, \mathbb{H}^s_{\overline{Q}}}$ by $\langle \cdot,\cdot\rangle$ where $(\mathbb{H}^s_{\overline{Q}})^*$ denotes the dual space.

Definition 3.2. *(Weak solutions) Consider Ω to be an open bounded set in $\mathbb{R}^n$ and $T > 0$. For $F \in (\mathbb{H}^s_{\overline{Q}})^*$ and $f \in \mathcal{H}^s\,(Q_e)$, we say $u \in \mathbb{H}^s(\mathbb{R}^{n+1})$ to be a weak solution of (3.31) if $v := (u-f)_T \in \mathbb{H}^s_{\overline{Q}}$ and*

$$\mathcal{B}_q\,(u,\phi) = \langle F, \phi\rangle, \quad \forall \phi \in \mathbb{H}^s_{\overline{Q}}$$

or equivalently,

$$\mathcal{B}_q\,(v,\phi) = \langle F - (\mathcal{H}^s + q) f, \phi\rangle, \quad \forall \phi \in \mathbb{H}^s_{\overline{Q}}.$$

Now, we state the well-posedness results of the initial-exterior problem (1.1).

Theorem 3.3. *Let Ω be an open bounded set in $\mathbb{R}^n$ and $T > 0$. Consider $q \in L^\infty(Q), f \in \mathcal{H}^s(Q_e)$ and $F \in (\mathbb{H}^s_{\overline{Q}})^*$. Then there exists a countable set of real numbers $\Sigma := \{\lambda_i\}_{1 \le i \le \infty}$ such that $\lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots \to \infty$ such that given $\lambda \in \mathbb{R} \setminus \Sigma$, there exists a unique $u_T \in \mathbb{H}^s(\mathbb{R}^{n+1})$ with $(u-f)_T \in \mathbb{H}^s_{\overline{Q}}$ for which*

$$(\mathcal{H}^s + q(x,t) - \lambda)\, u_T = F, \quad \text{in } Q.$$

Moreover u_T satisfies $\|u_T\|_{\mathbb{H}^s(\mathbb{R}^{n+1})} \preceq \Big(\|F\|_{(\mathbb{H}^s_{\overline{Q}})^*} + \|f\|_{\mathcal{H}^s(Q_e)}\Big)$.

Proof. We argue as in Theorem 3.2 in [18]. Let $v := (u-f)_T$ and $\widetilde{F} := F - (\mathcal{H}^s + q)\,f$. We first show the coercivity of the bilinear map $\mathcal{B}_q(v,w) + \mu \int_Q vw$, for μ large enough. For $w \in \mathbb{H}^s_{\overline{Q}}$ and $\mu \geq \|\min\{q,0\}\|_{L^\infty(Q)}$, we notice that

$$\begin{aligned}\mathcal{B}_q(w,w) + \mu \int_Q |w|^2(x,t)\mathrm{d}x\mathrm{d}t &= \langle \mathcal{H}^{\frac{s}{2}} w, \mathcal{H}_*^{\frac{s}{2}} w\rangle + \int_Q (\mu + q(x,t))\,|w|^2(x,t)\;\mathrm{d}x\mathrm{d}t \\ &= \int_{\mathbb{R}}\int_0^\infty (\lambda + i\sigma)^s\,\mathrm{d}\|E_\lambda(\mathcal{F}_t w)(\cdot,\sigma)\|^2 \mathrm{d}\sigma + + \int_Q (\mu + q(x,t))\,|w|^2(x,t)\;\mathrm{d}x\mathrm{d}t \\ &= \int_{\mathbb{R}}\int_0^\infty |\lambda + i\sigma|^s\,(\cos(s\theta) + i\sin(s\theta))\mathrm{d}\|E_\lambda(\mathcal{F}_t w)(\cdot,\sigma)\|^2 \mathrm{d}\sigma + \int_Q (\mu + q(x,t))\,|w|^2(x,t)\;\mathrm{d}x\mathrm{d}t \\ &= \int_{\mathbb{R}}\int_0^\infty |\lambda + i\sigma|^s \cos(s\theta)\mathrm{d}\|E_\lambda(\mathcal{F}_t w)(\cdot,\sigma)\|^2 \mathrm{d}\sigma + \int_Q (\mu + q(x,t))\,|w|^2(x,t)\;\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{3.34}$$

where $\tan(\theta) = \frac{\sigma}{\lambda}$ and where we utilized that $\sin(s\theta)$ is an odd function in the last identity above. Since $\lambda \geq 0$, it is seen that $\theta \in (-\pi/2, \pi/2)$ and thus for a fixed $0 < s < 1$

$$\cos(s\theta) \geq \cos(s\pi/2) \overset{def}{=} c_s > 0. \tag{3.35}$$

Using (3.35) along with (2.11) in (3.34) we obtain

$$\begin{aligned}&\mathcal{B}_q(w,w) + \mu \int_Q |w|^2(x,t)\mathrm{d}x\mathrm{d}t \\ &\succeq \int_{\mathbb{R}^{n+1}} \left||\xi|^2 + i\rho\right|^s |\hat{w}(\xi,\rho)|^2\,\mathrm{d}\xi\mathrm{d}\rho \succeq \int_{\mathbb{R}} \left\|(-\Delta_x)^{\frac{s}{2}}\,\mathcal{F}_t w(\cdot,\sigma)\right\|^2_{L^2(\mathbb{R}^n)} \mathrm{d}\sigma \succeq \|w\|_{L^2(\mathbb{R}^{n+1})}\end{aligned} \tag{3.36}$$

where in the last inequality, we used Hardy-Littlewood-Sobolev inequality in x variable combined with the fact w is compactly supported. Thus from (3.32) and (3.36) we conclude that the bilinear form $\mathcal{B}_q(v,w) + \mu \int_Q vw$ is coercive and bounded. Thus by Lax-Milgram theorem, there is a unique solution $v = G_\mu \widetilde{F} \in \mathbb{H}^s_{\overline{Q}}$ which satisfies

$$\mathcal{B}_q(v,w) + \mu \int_Q vw = \langle \widetilde{F}, w\rangle, \quad \forall w \in \mathbb{H}^s_{\overline{Q}}.$$

alongwith the bound

$$\|v\|_{\mathbb{H}^s_{\overline{Q}}} \preceq \|\widetilde{F}\|_{(\mathbb{H}^s_{\overline{Q}})^*} \tag{3.37}$$

From (3.37), we find $\|u_T\|_{\mathbb{H}^s(\mathbb{R}^{n+1})} \preceq \Big(\|F\|_{(\mathbb{H}^s_{\overline{Q}})^*} + \|f\|_{\mathcal{H}^s(Q_e)}\Big)$. In particular, (3.37) implies that the source to solution map i.e $G_\mu : (\mathbb{H}^s_{\overline{Q}})^* \to \mathbb{H}^s_{\overline{Q}}$ is continuous. Thus using (2.11), by an application of the compact Sobolev embedding we deduce that

$$G_\mu : L^2(Q) \to L^2(Q)$$

is a compact operator and therefore by the spectral theorem, there exists a countable set of eigenvalues of G_μ which are $\frac{1}{\lambda_i + \mu}$ with $\lambda_i \to \infty$. This is evident from the following observation

$$\mathcal{B}_q(v,w) - \lambda \int_Q vw = \langle \widetilde{F} + (\mu + \lambda)v, w\rangle$$

Also it is not hard to see $\Sigma := \{\lambda_i\}_{1\le i\le\infty} \subseteq \big(-\|\min\{q,0\}\|_{L^\infty(Q)}, \infty\big)$. Finally, the Fredholm alternative ensures the existence and uniqueness of the problem under consideration. □

Remark: In view of Theorem 3.3, we could rephrase the eigenvalue condition (1.2) by saying $0 \notin \Sigma$. It follows from the inequality (3.36) that for non-negative potentials i.e. when $q \ge 0$ a.e. in Q, we have $\Sigma \subset \mathbb{R}_+$. The same assertion holds for small enough potentials i.e. when $\|q\|_{L^\infty(Q)}$ is small.

Similarly, one can prove the well-posedness results for the adjoint equation to (1.1) which is the future-exterior problem

$$\begin{cases} (\mathcal{H}^s_* + q(x,t))\, u^* = 0, & \text{in } Q := \Omega\times(-T,T) \\ u^*(x,t) = f(x,t), & \text{in } Q_e := \Omega_e\times(-T,T) \\ u^*(x,t) = 0, & \text{for } t \ge T. \end{cases} \tag{3.38}$$

The analysis here would be identical to that of initial-exterior problem (1.1) and we could have similar well-posedness result here also. Moreover we observe that if we let

$$\tilde{u}(x,t) = u(x,-t)$$

then

$$(\mathcal{H}^s_* u)(x,t) = (\mathcal{H}^s \tilde{u})(x,-t). \tag{3.39}$$

Moreover from (1.2) and Fredholm alternative it follows that

$$0 \text{ is not a Dirichlet eigenvalue for the adjoint problem (3.38).} \tag{3.40}$$

3.4. **The initial-exterior problem for $\mathcal{H}^s + \langle b(x,t), \nabla_x\rangle + q(x,t)$.** We will introduce the notion of weak solutions for the problem (1.4). For the weak formulation, we define the corresponding bilinear form as follows

$$\mathcal{B}_{b,q}(f,g) = \langle \mathcal{H}^{\frac{s}{2}} f, \mathcal{H}^{\frac{s}{2}}_* g\rangle + \int_Q \langle b(x,t), \nabla_x f\rangle\; g + \int_Q q f \overline{g},$$

where $b \in L^\infty\left((-T,T); W^{1-s,\infty}(\Omega)\right)$ and $q \in L^\infty(Q)$. Similar to that in [18], a Kato-Ponce type inequality will be used to obtain the boundedness of the term $\int_Q \langle b(x,t), \nabla_x f\rangle g$ (see (3.42) below). We now define the weak formulation of (3.31).

Definition 3.4. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $s > \frac{1}{2}$ and $T > 0$. For $F \in (\mathbb{H}^s_{\overline{Q}})^*$ and $f \in \mathcal{H}^s(Q_e)$, we say that $u \in \mathbb{H}^s(\mathbb{R}^{n+1})$ is a weak solution of (3.31) if $v := (u-f)_T \in \mathbb{H}^s_{\overline{Q}}$ and*

$$\mathcal{B}_{b,q}(u,\phi) = \langle F, \phi\rangle, \quad \forall \phi \in \mathbb{H}^s_{\overline{Q}}$$

or equivalently,

$$\mathcal{B}_{b,q}(v,\phi) = \langle F - (\mathcal{H}^s + \langle b, \nabla_x\rangle + q) f, \phi\rangle, \quad \forall\phi \in \mathbb{H}^s_{\overline{Q}}.$$

Now, we state and prove the well-posedness result for (1.4).

Theorem 3.5. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $s > \frac{1}{2}$ and $T > 0$. Assume $b \in L^\infty\left((-T,T); W^{1-s,\infty}(\Omega)\right)$, $q \in L^\infty(Q)$, $f \in \mathcal{H}^s(Q_e)$ and $F \in (\mathbb{H}^s_{\overline{Q}})^*$. Then there exists a countable set of real numbers*

$\Sigma := \{\lambda_i\}_{1\leq i\leq\infty}$ with $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \to \infty$ such that whenever $\lambda \in \mathbb{R}\setminus\Sigma$, there exists a unique solution solution $u_T \in \mathbb{H}^s(\mathbb{R}^{n+1})$ satisfying

$$\begin{cases}(\mathcal{H}^s + \langle b(x,t), \nabla_x\rangle + q(x,t) - \lambda)\, u_T = F, & \text{in } Q,\\ u_T(x,t) = f(x,t), & \text{for } (x,t)\in Q_e.\end{cases}$$

such that $\|u_T\|_{\mathbb{H}^s(\mathbb{R}^{n+1})} \preceq \Big(\|F\|_{(\mathbb{H}^s_{\overline{Q}})^*} + \|f\|_{\mathcal{H}^s(Q_e)}\Big)$.

Proof. It suffices to show the boundedness of $\mathcal{B}_{b,q}$ and the coercivity of $\mathcal{B}_{b,q}(w,w) + \mu\int_Q |w|^2$ for large enough μ and then one can argue similarly as in the proof of Theorem 3.3. For boundedness, we need to show

$$|\mathcal{B}_{b,q}(u,v)| \preceq \|u\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}\|v\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}.$$

Now as before, we have

$$\left|\langle \mathcal{H}^s u, \mathcal{H}^s_* v\rangle + \int_Q quv\right| \preceq \|u\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}\|v\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}.$$

Thus it suffices to show that

$$\left|\int_Q u(x,t)\langle b(x,t), \nabla_x v\rangle \,\mathrm{d}x\mathrm{d}t\right| \preceq \|u\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}\|v\|_{\mathbb{H}^s(\mathbb{R}^{n+1})} \tag{3.41}$$

In order to establish (3.41), we argue as in [18]. We first choose $B \in L^\infty\left((-T,T); W^{1-s,\infty}(\mathbb{R}^n)\right)$ with $B = b$ a.e in Q, such that

$$\|B\|_{L^\infty((-T,T);W^{1-s,\infty}(\mathbb{R}^{n+1}))} \leq C\|b\|_{L^\infty((-T,T);W^{1-s,\infty}(\Omega))}$$

and notice the following estimate for all $t \in (-T,T)$

$$\begin{aligned}\left|\int_\Omega u(x,t)\langle b(x,t), \nabla_x v(x,t)\rangle \,\mathrm{d}x\right| & \\ &\preceq \|B(\cdot,t)u(\cdot,t)\|_{\mathbb{H}^{1-s}(\mathbb{R}^n)}\|\nabla_x v\|_{\mathbb{H}^{s-1}(\mathbb{R}^n)}\\ &\preceq \|B\|_{L^\infty((-T,T);W^{1-s,\infty}(\mathbb{R}^n))}\|u(\cdot,t)\|_{\mathbb{H}^{1-s}(\mathbb{R}^n)}\|v(\cdot,t)\|_{\mathbb{H}^s(\mathbb{R}^n)}\\ &\preceq \|u(\cdot,t)\|_{\mathbb{H}^s(\mathbb{R}^n)}\|v(\cdot,t)\|_{\mathbb{H}^s(\mathbb{R}^n)},\end{aligned} \tag{3.42}$$

where we crucially used the assumption $s > \frac{1}{2}$ and also employed the Kato-Ponce inequality in [31, Theorem 1] to obtain

$$\begin{aligned}\|B(\cdot,t)u(\cdot,t)\|_{\mathbb{H}^{1-s}(\mathbb{R}^n)} &\approx \left\|\mathcal{J}^{1-s}(B(\cdot,t)u(\cdot,t))\right\|_{L^2(\mathbb{R}^n)}\\ &\preceq \|B(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\|\mathcal{J}^{1-s}u(\cdot,t)\|_{L^2(\mathbb{R}^n)} + \|\mathcal{J}^{1-s}B(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\|u\|_{L^2(\mathbb{R}^n)}\\ &\preceq \|B\|_{L^\infty((-T,T);W^{1-s,\infty}(\mathbb{R}^n))}\|u(\cdot,t)\|_{\mathbb{H}^s(\mathbb{R}^n)} \preceq \|u(\cdot,t)\|_{\mathbb{H}^s(\mathbb{R}^n)}.\end{aligned}$$

Here $\mathcal{J} := (\mathrm{Id} - \Delta_x)^{\frac{1}{2}}$. Now we use Cauchy-Schwarz inequality and the Plancherel theorem in the t variable to find

$$\begin{aligned}
&\left|\int_Q u(x,t)\langle b(x,t), \nabla_x v\rangle \, \mathrm{d}x\mathrm{d}t\right| \preceq \int_{\mathbb{R}} \|u(\cdot,t)\|_{H^s(\mathbb{R}^n)}\|v(\cdot,t)\|_{H^s(\mathbb{R}^n)}\mathrm{d}t \\
&\preceq \left(\int_{\mathbb{R}^{n+1}} \left(1+|\xi|^{2s}\right)|\mathcal{F}_x u|^2(\xi,t)\mathrm{d}\xi\mathrm{d}t\right)^{1/2} \left(\int_{\mathbb{R}^{n+1}} \left(1+|\xi|^{2s}\right)|\mathcal{F}_x v|^2(\xi,t)\mathrm{d}\xi\mathrm{d}t\right)^{1/2} \\
&\preceq \|u\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}\|v\|_{\mathbb{H}^s(\mathbb{R}^{n+1})}.
\end{aligned}$$

Next we head towards proving the coercivity of $\mathcal{B}_{b,q}(w,w) + \mu\int_Q |w|^2$. In this regard we observe that

$$\langle \mathcal{H}^{\frac{s}{2}} w, \mathcal{H}_*^{\frac{s}{2}} w\rangle + \int_Q w\langle b, \nabla_x w\rangle + \int_Q q|w|^2 \geq c_0\|f\|^2_{\mathbb{H}^s(\mathbb{R}^{n+1})} - \left|\int_Q w\langle b, \nabla_{x,t} w\rangle\right| - \|q\|_{L^\infty(Q)}\|w\|^2_{L^2(Q)}.$$

As in (3.42), we observe that

$$\begin{aligned}
\left|\int_Q w\langle b, \nabla_x w\rangle\right| &\preceq \int_{\mathbb{R}} \|w(\cdot,t)\|_{\mathbb{H}^{1-s}(\mathbb{R}^n)}\|w(\cdot,t)\|_{H^s(\mathbb{R}^n)}\mathrm{d}t \\
&\preceq \left(\int_{\mathbb{R}^{n+1}} \langle\xi\rangle^{2(1-s)}|\mathcal{F}_x w|^2(\xi,t)\mathrm{d}\xi\mathrm{d}t\right)^{1/2} \left(\int_{\mathbb{R}^{n+1}} \left(1+|\xi|^{2s}\right)|\mathcal{F}_x w|^2(\xi,t)\mathrm{d}\xi\mathrm{d}t\right)^{1/2} \\
&\text{(where } \langle\xi\rangle \overset{def}{=} \sqrt{1+|\xi|^2}\text{)} \\
&\preceq \left(\int_{\mathbb{R}^{n+1}} \langle\xi\rangle^{2s}|\mathcal{F}_x w|^2(\xi,t)\mathrm{d}\xi\mathrm{d}t\right)^{\frac{1-s}{2s}} \left(\int_{\mathbb{R}^{n+1}} |\mathcal{F}_x w|^2(\xi,t)\mathrm{d}\xi\mathrm{d}t\right)^{\frac{2s-1}{2s}} \|f\|_{\mathbb{H}^s(\mathbb{R}^{n+1})} \\
&\preceq \|w\|^{\frac{2s-1}{s}}_{L^2(\mathbb{R}^{n+1})}\|w\|^{\frac{1}{s}}_{\mathbb{H}^s(\mathbb{R}^{n+1})}.
\end{aligned}$$

Now an application of Young's inequality gives

$$\left|\int_Q w\langle b, \nabla_x w\rangle\right| \leq \epsilon\|w\|^2_{\mathbb{H}^s(\mathbb{R}^{n+1})} + C_\epsilon\|w\|^2_{L^2(Q)}.$$

Thus by choosing ϵ small enough, we can conclude

$$\mathcal{B}_{b,q}(w,w) \geq \frac{c_0}{2}\|w\|^2_{\mathbb{H}^s(\mathbb{R}^{n+1})} - \left(C_\epsilon + \|q\|_{L^\infty(Q)}\right)\|w\|^2_{L^2(Q)}$$

Finally by choosing $\mu \geq \left(C_\epsilon + \|q\|_{L^\infty(Q)}\right)$, we find that the coercivity of the corresponding bilinear form follows. As previously said, the rest of the proof remains the same as that for Theorem 3.3. □

Likewise, we have the well-posedness result for the adjoint problem defined in the following way

$$\begin{cases}
\mathcal{H}_*^s u^* - \nabla_x\cdot(bu^*) + q(x,t)u^* = 0, & \text{in } Q := \Omega\times(-T,T) \\
u^*(x,t) = f(x,t), & \text{in } Q_e := \Omega_e\times(-T,T) \\
u^*(x,t) = 0, & \text{for } t \geq T.
\end{cases}$$

4. Global unique continuation property

4.1. **Carleman estimate.** In this section, we prove the global weak unique continuation result stated in Theorem 1.3. We follow the strategy in [2] by first establishing a conditional doubling estimate for solutions to the extension problem and then we use a blowup argument to reduce the problem to a weak unique continuation property for the homogeneous extension problem with constant coefficients as in Theorem 1.4. Therefore, as a first step we prove a Carleman estimate for the extended operator. Keeping in mind the possibility of other applications, we allow the matrix coefficients to depend on both space and time variables in the Carleman estimate as stated in Lemma 4.6 below.

Similar to that in Subsection 3.2, in this section, we remind the reader of the following notations. $(X,t) = (x, x_{n+1}, t), (Y,s) = (y, y_{n+1}, s)$ will denote generic points in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$. For a given $r > 0$, we will denote by $\mathbb{B}_r(Y)$, the Euclidean ball in $\mathbb{R}^{n+1}$ of radius r centered at Y and $B_r(y) \overset{def}{=} \{x : (x,0) \in \mathbb{B}_r(Y)\}$. Likewise we let $\mathbb{B}_r^+(Y) \overset{def}{=} \mathbb{B}_r(Y) \cap \{X : x_{n+1} > 0\}$. When the center Y of $\mathbb{B}_r(Y)$ is not explicitly indicated, then we are taking $Y = 0$. Similar agreement for the thick half-balls $\mathbb{B}_r^+(x_0, 0)$.

For notational ease, ∇U and $\operatorname{div} U$ will respectively refer to the quantities $\nabla_X U$ and $\operatorname{div}_X U$. The partial derivative in t will be denoted by $\partial_t U$ and also at times by U_t. The partial derivative $\partial_{x_i} U$ will be denoted by U_i and also by $\partial_i U$.

We will assume that $A(x,t) := (a_{ij}(x,t))_{ij}$ be a $(n+1)\times(n+1)$ is a positive definite block matrix valued function satisfying (2.2) with

$$A(0,0) = \mathbb{I}_{n+1}, \ a_{(n+1)i}(x,t) = \delta_{(n+1)i}, \quad \forall i \in \{1, 2, ..., n+1\}, \tag{4.1}$$

such that the following Lipschitz growth condition holds

$$|A(x,t) - A(y,s)| \le K(|x-y| + |t-s|). \tag{4.2}$$

It follows from (4.2) that if we let $B(X,t) \equiv \{b_{ij}(x,t)\}_{1\le i,j\le n+1} := A(x,t) - \mathbb{I}_{n+1}$, then

$$b_{ij}(x,t) = O(|x| + t) \ \forall i,j \in \{1,2,...,n+1\} \text{ and } b_{(n+1)i}(x,t) = 0, \ \forall i \in \{1,2,...,n+1\}.$$

Corresponding to A as in (4.1) above, we consider the following extended backward parabolic operator

$$\widetilde{\mathcal{H}} := x_{n+1}^a \partial_t + \operatorname{div}\left(x_{n+1}^a A(x,t)\nabla\right). \tag{4.3}$$

For notational convenience, it will be easier to work with this backward extension operator in (4.3) above. Similar to that in Subsection 3.2, it should be clear to the reader that x_{n+1} plays the role of the extension variable z in Subsection 3.1.

In the proof of our Carleman estimate, we adapt the approach in the fundamental works [23] and [22] to our setting of degenerate operators as in (4.3) and this has required some delicate adaptations. It is to be mentioned that although our method is inspired by ideas in [23, 22], nevertheless at a technical level, our proof of the Carleman estimate somewhat differs from that in [23] even in the case when $a = 0$. The proof of such an estimate in [23] relies on first establishing a generic Rellich type identity with respect to appropriate Carleman weights in the Gaussian space (see [23, Lemma 1]). This is then combined with a clever use of some logarithmic inequalities as stated in Lemma 4.2 below which is needed to absorb certain error terms that arises due to the perturbation of the variable coefficient principal part. Differently from that in [23], in our proof we instead analyse the positivity property of the associated conjugate operator directly. Our method however also uses the ODE satisfied by the Carleman weight given in Lemma 4.1 below in the same spirit as in [23, 22]. We are of the opinion that our proof revisits the approach in [23, 22] with a somewhat different viewpoint which can possibly be of independent interest.

Before we state our main Carleman estimate, we first gather some relevant results from [23] that are crucially needed in our context. The following result which is Lemma 4 in [23] is regarding the existence of a suitable weight function σ which has the appropriate pseudo-convexity property needed for the Carleman estimate.

Lemma 4.1. *Let*

$$\theta(t) = t^{\frac{1}{2}} \left(\log \frac{1}{t}\right)^{\frac{3}{2}}. \tag{4.4}$$

Then the solution to the ordinary differential equation

$$\frac{d}{dt} \log\left(\frac{\sigma}{t\sigma'}\right) = \frac{\theta(\lambda t)}{t},\ \sigma(0) = 0,\ \sigma'(0) = 1,$$

where $\lambda > 0$, has the following properties when $0 \leq \lambda t \leq 1$:

(1) $te^{-N} \leq \sigma(t) \leq t$,
(2) $e^{-N} \leq \sigma'(t) \leq 1$,
(3) $|\partial_t[\sigma \log \frac{\sigma}{\sigma' t}]| + |\partial_t[\sigma \log \frac{\sigma}{\sigma'}]| \leq 3N$,
(4) $\left|\sigma\partial_t \left(\frac{1}{\sigma'}\partial_t[\log \frac{\sigma}{\sigma'(t)t}]\right)\right| \leq 3Ne^N \frac{\theta(\gamma t)}{t}$,

where N is some universal constant.

We also need the following real analysis lemma from [23]. See lemma 3.3 in [23].

Lemma 4.2. *Given $m > 0$, $\exists C_m$ such that for all $y \geq 0$ and $0 < \epsilon < 1$,*

$$y^m e^{-y} \leq C_m \left[\epsilon + \left(\log(\frac{1}{\epsilon})\right)^m e^{-y}\right]. \tag{4.5}$$

Finally, we also need the following Hardy type inequality in the Gaussian space which can be found in Lemma 2.2 in [2]. This can be regarded as the weighted analogue of Lemma 3 in [22].

Lemma 4.3 (Hardy type inequality). *For all $h \in C_0^\infty(\overline{\mathbb{R}^{n+1}_+})$ and $b > 0$ the following inequality holds*

$$\int_{\mathbb{R}^{n+1}_+} x_{n+1}^a h^2 \frac{|X|^2}{8b} e^{-|X|^2/4b} dX \leq 2b \int_{\mathbb{R}^{n+1}_+} x_{n+1}^a |\nabla h|^2 e^{-|X|^2/4b} dX$$
$$+ \frac{n+1+a}{2} \int_{\mathbb{R}^{n+1}_+} x_{n+1}^a h^2 e^{-|X|^2/4b} dX.$$

We now state and prove our main Carleman estimate which constitutes the generalization of the Carleman estimate in [22, Lemma 6] to degenerate operators of the type (4.3).

Theorem 4.4. *Let $\widetilde{\mathcal{H}}$ be the backward in time extension operator in (4.3) where $A(x,t)$ is a matrix valued function satisfying (4.1) and (4.2). Let $w \in C_0^\infty\left(\overline{\mathbb{B}_4^+} \times [0, \frac{1}{3\lambda})\right)$ such that $\partial^a_{x_{n+1}} w \equiv 0$ on*

$\{x_{n+1}=0\}$ where $\lambda=\frac{\alpha}{\delta^2}$ for some $\delta\in(0,1)$. Then the following estimate holds for all large α and δ sufficiently small

$$\alpha^2\int_{\mathbb{R}^{n+1}_+\times[c,\infty)} x_{n+1}^a\sigma^{-2\alpha}(t)\ w^2\ G+\alpha\int_{\mathbb{R}^{n+1}_+\times[c,\infty)} x_{n+1}^a\sigma^{1-2\alpha}(t)\ |\nabla w|^2\ G \tag{4.6}$$
$$\preceq \int_{\mathbb{R}^{n+1}_+\times[c,\infty)}\sigma^{1-2\alpha}(t)x_{n+1}^{-a}\ |\widetilde{\mathcal{H}}w|^2\ G+\alpha^{c'\alpha}\ \sup_{t\geq c}\int_{\mathbb{R}^{n+1}_+} x_{n+1}^a\left(w^2+t|\nabla w|^2\right)\ \mathrm{d}X$$
$$+\sigma^{-2\alpha}(c)\left\{-c\int_{t=c} x_{n+1}^a\ |\nabla w(X,c)|^2\ G(X,c)\ \mathrm{d}X+\alpha\int_{t=c} x_{n+1}^a\ |w(X,c)|^2\ G(X,c)\ \mathrm{d}X\right\}.$$

Here σ is as in Lemma 4.1, $G(X,t)=\frac{1}{t^{\frac{n+1+a}{2}}}e^{-\frac{|X|^2}{4t}}$ and $0<c\leq\frac{1}{5\lambda}$.

Proof. Let θ be as in Lemma 4.1. For $t\in[0,\frac{1}{3\lambda})$, we first make the preliminary observation that

$$\frac{\theta(\lambda t)}{t}\geq\lambda^{1/2}t^{-\frac12}\left(\log 3\right)^{\frac32}\geq\sqrt{3}\lambda\left(\log 3\right)^{\frac32}\geq\lambda. \tag{4.7}$$

Also, with a slight abuse of notation, we treat the quantity $-\left(\frac{t\sigma'}{\sigma}\right)'$ as $\frac{\theta(\lambda t)}{t}$ since the term $\frac{t\sigma'}{\sigma}$ is positively bounded from both sides in view of Lemma 4.1. The solid integrals below will be taken in $\mathbb{R}^n\times[c,\infty)$ where $0<c\leq\frac{1}{\lambda}$ and we refrain from mentioning explicit limits in the rest of our discussion.

Note that we have the following equivalent expression for $\widetilde{\mathcal{H}}$

$$x_{n+1}^{-\frac{a}{2}}\widetilde{\mathcal{H}}=x_{n+1}^{\frac{a}{2}}\left(\partial_t+\operatorname{div}(A(x,t)\nabla)+\frac{a}{x_{n+1}}\partial_{n+1}\right).$$

Consider the conjugation

$$w(X,t)=\sigma^\alpha(t)e^{\frac{|X|^2}{8t}}v(X,t).$$

We then note that

$$w_t=e^{\frac{|X|^2}{8t}}\left(\sigma^\alpha(t)v_t+\alpha\sigma^{\alpha-1}(t)\sigma'(t)v-\frac{|X|^2}{8t^2}\sigma^\alpha(t)v\right),\ \nabla w=e^{\frac{|X|^2}{8t}}\sigma^\alpha(t)\left(\nabla v+\frac{X}{4t}v\right). \tag{4.8}$$

From (4.8) we find

$$\operatorname{div}(A(x,t)\nabla w)=\operatorname{div}\left(\sigma^\alpha(t)e^{\frac{|X|^2}{8t}}A(x,t)\left(\nabla v+\frac{X}{4t}v\right)\right)$$
$$=\sigma^\alpha(t)e^{\frac{|X|^2}{8t}}\left[\operatorname{div}(A(x,t)\nabla v)+\frac{\langle X,A(x,t)\nabla v\rangle}{2t}+\left(\frac{\langle X,A(x,t)X\rangle}{16t^2}+\frac{\operatorname{div}(A(x,t)\cdot X)}{4t}\right)v\right]$$

Now we define the vector field

$$\mathcal{Z}:=2t\partial_t+X\cdot A(x,t)\nabla \tag{4.9}$$

and combine the preceding observations to deduce

$$x_{n+1}^{-\frac{a}{2}}\sigma^{-\alpha}(t)e^{-\frac{|X|^2}{8t}}\widetilde{\mathcal{H}}w=x_{n+1}^{\frac{a}{2}}\left[\operatorname{div}\left(A(x,t)\nabla v\right)+\frac{1}{2t}\mathcal{Z}v+\left(\frac{\operatorname{div}(A(x,t)X)+a}{4t}+\frac{\alpha\sigma'}{\sigma}\right)v\right.$$
$$\left.+\left(\frac{\langle X,A(x,t)X\rangle}{16t^2}-\frac{|X|^2}{8t^2}\right)v+\frac{a}{x_{n+1}}\partial_{n+1}v\right].$$

Using (4.1) and (4.2), we further note that the following relations hold for (X,t) varying in a compact set containing the origin

$$\operatorname{div}(A(x,t)X) = n+1+O(|X|+t), \qquad \langle X, A(x,t)X\rangle = |X|^2 + |X|^2 O(|X|+t), \tag{4.10}$$

$$\operatorname{div}(x_{n+1}^a A(x,t)X) = x_{n+1}^a(\operatorname{div}(A(x,t)X)+a) = x_{n+1}^a\left(n+1+a+O(|X|+t)\right). \tag{4.11}$$

Next we consider the expression

$$\begin{aligned}
&\int \sigma^{-2\alpha}(t) t^{-\mu} x_{n+1}^{-a} e^{-\frac{|X|^2}{4t}} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} |\widetilde{\mathcal{H}} w|^2 \qquad (4.12)\\
&= \int x_{n+1}^a t^{-\mu} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \Bigg[\operatorname{div}\left(A(x,t)\nabla v\right) + \frac{1}{2t}\mathcal{Z}v + \frac{a}{x_{n+1}}\partial_{n+1} v\\
&+ \left(\frac{\operatorname{div}(A(x,t)X)+a}{4t} + \frac{\alpha\sigma'}{\sigma}\right) v + \left(\frac{\langle X, A(x,t)X\rangle}{16t^2} - \frac{|X|^2}{8t^2}\right) v\Bigg]^2,
\end{aligned}$$

where μ is to be chosen later. Then we estimate the integral (4.12) from below with an application of the algebraic inequality

$$\int P^2 + 2\int PQ \le \int (P+Q)^2$$

where P and Q are chosen as

$$\begin{aligned}
P &= \frac{x_{n+1}^{\frac{a}{2}} t^{-\frac{\mu+2}{2}}}{2} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{4}} \mathcal{Z}v,\\
Q &= x_{n+1}^{\frac{a}{2}} t^{-\frac{\mu}{2}} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{4}} \Bigg[\left(\frac{\operatorname{div}(A(x,t)X)+a}{4t} + \frac{\alpha\sigma'}{\sigma}\right) v + \operatorname{div}\left(A(x,t)\nabla v\right) + \frac{a\partial_{n+1} v}{x_{n+1}}\\
&\qquad + \frac{\langle X, A(x,t)X\rangle - 2|X|^2}{16t^2} v\Bigg].
\end{aligned}$$

To establish the Carleman estimate, we calculate all the terms coming from the cross product, i.e. from $\int PQ$. We write

$$\int PQ := \sum_{k=1}^{4} \mathcal{I}_k,$$

where

$$\begin{aligned}
\mathcal{I}_1 &= \int x_{n+1}^a t^{-\mu} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \frac{1}{2t}\mathcal{Z}v \left(\frac{\operatorname{div}(A(x,t)X)+a}{4t} + \frac{\alpha\sigma'}{\sigma}\right) v,\\
\mathcal{I}_2 &= \int x_{n+1}^a t^{-\mu} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \frac{\mathcal{Z}v}{2t} \operatorname{div}\left(A(x,t)\nabla v\right),\\
\mathcal{I}_3 &= \int x_{n+1}^a t^{-\mu} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \frac{\mathcal{Z}v}{2t} \left(\frac{\langle X, A(x,t)X\rangle}{16t^2} - \frac{|X|^2}{8t^2}\right) v,\\
\mathcal{I}_4 &= \int x_{n+1}^a t^{-\mu} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \frac{\mathcal{Z}v}{2t} \frac{a\partial_{n+1} v}{x_{n+1}}.
\end{aligned}$$

We start with the term $\mathcal{I}_1$. We have

$$\begin{aligned}\mathcal{I}_1 &= \int x_{n+1}^a t^{-\mu} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \frac{1}{2t}\mathcal{Z}v \left(\frac{\operatorname{div}(A(x,t)X)+a}{4t} + \frac{\alpha\sigma'}{\sigma}\right) v \\ &= \int x_{n+1}^a t^{-\mu} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \frac{\mathcal{Z}v}{2t} \left(\frac{n+1+a+O(|X|+t)}{4t} + \frac{\alpha\sigma'}{\sigma}\right) v \\ &= \frac{n+1+a}{8} \int x_{n+1}^a t^{-\mu-2} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \mathcal{Z}\left(\frac{v^2}{2}\right) + \frac{\alpha}{2} \int x_{n+1}^a t^{-\mu-2} \left(\frac{t\sigma'}{\sigma}\right)^{\frac{1}{2}} \mathcal{Z}\left(\frac{v^2}{2}\right) \\ &\quad + \int x_{n+1}^a t^{-\mu-2} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \mathcal{Z}v \; O(|X|+t)v,\end{aligned}$$

which after employing the AM-GM inequality to the term $\int x_{n+1}^a t^{-\mu-2} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \mathcal{Z}v \; O(|X|+t)v$ can be bounded from below in the following way

$$\begin{aligned}\mathcal{I}_1 \geq \frac{n+1+a}{8} \int x_{n+1}^a t^{-\mu-2} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \mathcal{Z}\left(\frac{v^2}{2}\right) + \frac{\alpha}{2} \int x_{n+1}^a t^{-\mu-2} \left(\frac{t\sigma'}{\sigma}\right)^{\frac{1}{2}} \mathcal{Z}\left(\frac{v^2}{2}\right) \\ - \epsilon \int x_{n+1}^a t^{-\mu-2} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} |\mathcal{Z}v|^2 + \frac{O(1)}{\epsilon} \int x_{n+1}^a t^{-\mu} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \left(\frac{|X|^2 v^2}{t^2} + v^2\right).\end{aligned} \tag{4.13}$$

We remark here that $\epsilon > 0$ will be chosen in a way so that the term

$$-\epsilon \int x_{n+1}^a t^{-\mu-2} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} |\mathcal{Z}v|^2$$

gets absorbed in $\frac{1}{2}\int P^2$. We would also like to mention that the term $\frac{O(1)}{\epsilon} \int x_{n+1}^a t^{-\mu} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \left(\frac{|X|^2 v^2}{t^2} + v^2\right)$ will be eventually estimated favourably by using the log-inequality stated in Lemma 4.2. See (4.37)-(4.39) below. Therefore, we first engage our attention on the terms

$$\frac{n+1+a}{8} \int x_{n+1}^a t^{-\mu-2} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \mathcal{Z}\left(\frac{v^2}{2}\right) \text{ and } \frac{\alpha}{2} \int x_{n+1}^a t^{-\mu-2} \left(\frac{t\sigma'}{\sigma}\right)^{\frac{1}{2}} \mathcal{Z}\left(\frac{v^2}{2}\right).$$

We choose μ such that

$$\operatorname{div}_{X,t}(x_{n+1}^a t^{-\mu-2} \mathcal{Z}(0,0)) = 0.$$

Note that $\mathcal{Z}(0,0) = X\cdot\nabla_X + 2t\partial_t$. This implies that

$$\mu = \frac{n-1+a}{2}. \tag{4.14}$$

With such a choice of μ, by integrating by parts and also by using (4.10) we then observe

$$\begin{aligned}
&\frac{n+1+a}{8}\int x_{n+1}^a t^{-\mu-2}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}}\mathcal{Z}\left(\frac{v^2}{2}\right) \\
&=\frac{(n+1+a)}{16}\int x_{n+1}^a t^{-\mu-1}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{3}{2}}\left(\frac{t\sigma'}{\sigma}\right)' v^2+O(1)\int x_{n+1}^a t^{-\mu-2}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}}(|X|+t)\,v^2 \\
&-\left(\frac{n+1+a}{8}\right)\ c^{-\mu-1}\left(\frac{c\sigma'(c)}{\sigma(c)}\right)^{-\frac{1}{2}}\int_{t=c} x_{n+1}^a v^2(X,c)\ \mathrm{d}X.
\end{aligned}\tag{4.15}$$

Similarly we find

$$\begin{aligned}
&\frac{\alpha}{2}\int x_{n+1}^a t^{-\mu-2}\left(\frac{t\sigma'}{\sigma}\right)^{\frac{1}{2}}\mathcal{Z}\left(\frac{v^2}{2}\right) \\
&=-\frac{\alpha}{4}\int x_{n+1}^a t^{-\mu-1}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}}\left(\frac{t\sigma'}{\sigma}\right)' v^2-\frac{\alpha}{2}c^{-\mu-1}\left(\frac{c\sigma'(c)}{\sigma(c)}\right)^{-\frac{1}{2}}\int_{t=c} x_{n+1}^a v^2(X,c)\ \mathrm{d}X \\
&+O(1)\ \alpha\int x_{n+1}^a t^{-\mu-2}\left(\frac{t\sigma'}{\sigma}\right)^{\frac{1}{2}}(|X|+t)\,v^2.
\end{aligned}\tag{4.16}$$

From (4.13), (4.15) and (4.16) it follows using Lemma 4.1 that the following inequality holds for all α large

$$\begin{aligned}
&\mathcal{I}_1\succeq\alpha\int x_{n+1}^a t^{-\mu-1}\frac{\theta(\lambda t)}{t}v^2-\alpha\int x_{n+1}^a t^{-\mu-2}|X|v^2-\alpha c^{-\mu-1}\int_{t=c} x_{n+1}^a v^2(X,c)\ \mathrm{d}X \\
&-\epsilon\int x_{n+1}^a t^{-\mu-2}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}}|\mathcal{Z}v|^2 \\
&\succeq\alpha\int x_{n+1}^a\sigma^{-2\alpha}(t)\frac{\theta(\lambda t)}{t}\ w^2G-\alpha\int x_{n+1}^a t^{-\mu-2}\sigma^{-2\alpha}(t)|X|\ w^2e^{-\frac{|X|^2}{4t}}-\alpha\sigma^{-2\alpha}(c)\int_{t=c} x_{n+1}^a w^2G \\
&-\epsilon\int x_{n+1}^a t^{-\mu-2}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}}|\mathcal{Z}v|^2,
\end{aligned}\tag{4.17}$$

where all terms with sub-critical power in t can be absorbed in $\alpha\int x_{n+1}^a t^{-\mu-1}\frac{\theta(\lambda t)}{t}v^2$ by using the largeness of $\frac{\theta(\lambda t)}{t}$ as observed in (4.7) above.

Next we consider the term $\mathcal{I}_2$ which finally contributes the positive gradient terms in our Carleman estimate. This is accomplished by a Rellich type argument. By integrating by parts and also by using $\partial^a_{x_{n+1}} v = 0$, we find

$$\mathcal{I}_2 = \int x_{n+1}^a t^{-\mu} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \frac{\mathcal{Z}v}{2t} \operatorname{div}(A(x,t)\nabla v) = \int x_{n+1}^a t^{-\mu} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \sum_{i,j=1}^{n+1} (a_{ij}(x,t)v_j)_i \, \partial_t v \tag{4.18}$$

$$+ \frac{1}{2}\int x_{n+1}^a t^{-\mu-1} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \langle X, A(x,t)\nabla v\rangle \operatorname{div}(A(x,t)\nabla v)$$

$$= -\int x_{n+1}^a t^{-\mu} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \sum_{i,j=1}^{n+1} a_{ij} v_j \partial_t v_i - a\int x_{n+1}^{a-1} t^{-\mu} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} v_{n+1}\partial_t v$$

$$+ \frac{1}{2}\int x_{n+1}^a t^{-\mu-1} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \sum_{i,j,p,q=1}^{n+1} X_i a_{ij} v_j \left(a_{pq} v_q\right)_p$$

$$= -\frac{1}{2}\int x_{n+1}^a t^{-\mu} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \partial_t \langle \nabla v, A\nabla v\rangle + \frac{1}{2}\int x_{n+1}^a t^{-\mu} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \sum_{i,j=1}^{n+1} v_j v_i \partial_t a_{ij}$$

$$- \int x_{n+1}^a t^{-\mu} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \frac{\mathcal{Z}v}{2t}\frac{a\partial_{n+1}v}{x_{n+1}} - \frac{1}{2}\int x_{n+1}^a t^{-\mu-1} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \sum_{i,j,p,q} (X_i a_{ij} v_j)_p a_{pq} v_q$$

$$= -\frac{1}{4}\int x_{n+1}^a t^{-\mu} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{3}{2}} \left(\frac{t\sigma'}{\sigma}\right)' \langle \nabla v, A\nabla v\rangle - \frac{\mu}{2}\int x_{n+1}^a t^{-\mu-1} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \langle \nabla v, A\nabla v\rangle$$

$$+ \frac{1}{2} c^{-\mu} \left(\frac{c\sigma'(c)}{\sigma(c)}\right)^{-\frac{1}{2}} \int_{t=c} x_{n+1}^a \langle \nabla v, A\nabla v\rangle(X,c)\, \mathrm{d}X + O(1)\int x_{n+1}^a t^{-\mu} |\nabla v|^2 - \mathcal{I}_4 + \mathcal{K},$$

where

$$\mathcal{K} = -\frac{1}{2}\int x_{n+1}^a t^{-\mu-1} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \sum_{i,j,p,q} (X_i a_{ij} v_j)_p \; a_{pq} v_q.$$

Using (4.10) and (4.11) we then obtain

$$\mathcal{K} = -\frac{1}{2}\int x_{n+1}^a t^{-\mu-1} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \sum_{i,j,p,q} (X_i a_{ij} v_j)_p \; a_{pq} v_q \tag{4.19}$$

$$= -\frac{1}{2}\int x_{n+1}^a t^{-\mu-1} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \sum_{i,j,p,q} \left(X_i a_{ij} v_{jp} + \delta_{ip} a_{ij} v_j + X_i a_{ij,p} v_j\right) \; a_{pq} v_q$$

$$= -\frac{1}{2}\int x_{n+1}^a t^{-\mu-1} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \left(|A\nabla v|^2 + \sum_{i,p,q} X_i v_{ip} a_{pq} v_q + \sum_{i,j,p,q} X_i b_{ij} v_{jp} a_{pq} v_q + \sum_{i,j,p,q} X_i a_{ij,p} v_j a_{pq} v_q \right)$$

$$= -\frac{1}{2}\int x_{n+1}^a t^{-\mu-1} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \Bigg(|A\nabla v|^2 + \frac{1}{2} X\cdot\nabla\langle A\nabla v, \nabla v\rangle + \frac{1}{2}\langle X, B(X,t)\nabla\langle A\nabla v, \nabla v\rangle\rangle$$

$$+ O(|X|)\langle A\nabla v, \nabla v\rangle \Bigg).$$

Now by integrating by parts the integral $-\frac{1}{4}\int x_{n+1}^a t^{-\mu-1} X\cdot\nabla\langle A\nabla v,\nabla v\rangle$, we obtain from above that the following holds,

$$\mathcal{K}=\frac{n-1+a}{4}\int x_{n+1}^a t^{-\mu-1}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}}\langle A\nabla v,\nabla v\rangle+O(1)\int x_{n+1}^a t^{-\mu-1}(|X|+t)\,|\nabla v|^2. \tag{4.20}$$

where we also used that $a_{ij}(X,t)=\delta_{ij}+b_{ij}(X,t)$. Since $\mu=\frac{n-1+a}{2}$, from (4.18) and (4.20) we obtain

$$\mathcal{I}_2+\mathcal{I}_4=-\frac{1}{4}\int x_{n+1}^a t^{-\mu}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{3}{2}}\left(\frac{t\sigma'}{\sigma}\right)'\langle\nabla v,A\nabla v\rangle+O(1)\int x_{n+1}^a t^{-\mu-1}\left(|X|+t\right)|\nabla v|^2 \tag{4.21}$$
$$+\frac{1}{2}c^{-\mu}\left(\frac{c\sigma'(c)}{\sigma(c)}\right)^{-\frac{1}{2}}\int_{t=c}x_{n+1}^a\langle\nabla v,A\nabla v\rangle(X,c)\,\mathrm{d}X.$$

To express the above relation in terms of u, we first recall

$$\nabla v=\sigma^{-\alpha}(t)e^{-\frac{|X|^2}{8t}}\left(\nabla w-\frac{X}{4t}w\right). \tag{4.22}$$

We now consider the term $-\frac{1}{4}\int x_{n+1}^a t^{-\mu}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{3}{2}}\left(\frac{t\sigma'}{\sigma}\right)'\langle\nabla v,A\nabla v\rangle$. Using (4.22) and also (4.10) and (4.11) we have

$$-\frac{1}{4}\int x_{n+1}^a t^{-\mu}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{3}{2}}\left(\frac{t\sigma'}{\sigma}\right)'\langle\nabla v,A\nabla v\rangle \tag{4.23}$$
$$=-\frac{1}{4}\int x_{n+1}^a t^{-\mu}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{3}{2}}\left(\frac{t\sigma'}{\sigma}\right)'\sigma^{-2\alpha}(t)\left\langle\nabla w-\frac{X}{4t}w,A\left(\nabla w-\frac{X}{4t}w\right)\right\rangle e^{-\frac{|X|^2}{4t}}$$
$$=-\frac{1}{4}\int x_{n+1}^a t^{-\mu}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{3}{2}}\left(\frac{t\sigma'}{\sigma}\right)'\sigma^{-2\alpha}(t)\left(\langle\nabla w,A\nabla w\rangle+\frac{\langle X,AX\rangle}{16t^2}w^2-\frac{1}{4t}\left\langle AX\cdot\nabla(w^2)\right\rangle\right)\,e^{-\frac{|X|^2}{4t}}$$
$$=-\frac{1}{4}\int x_{n+1}^a t^{-\mu}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{3}{2}}\left(\frac{t\sigma'}{\sigma}\right)'\sigma^{-2\alpha}(t)\left(\langle\nabla w,A\nabla w\rangle-\frac{\langle X,AX\rangle}{16t^2}w^2\right)\,e^{-\frac{|X|^2}{4t}}$$
$$-\frac{1}{16}\int t^{-\mu-1}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{3}{2}}\left(\frac{t\sigma'}{\sigma}\right)'\operatorname{div}\left(x_{n+1}^a AX\right)w^2\,e^{-\frac{|X|^2}{4t}}$$
$$=-\frac{1}{4}\int x_{n+1}^a t^{-\mu}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{3}{2}}\left(\frac{t\sigma'}{\sigma}\right)'\sigma^{-2\alpha}(t)\left(\langle\nabla w,A\nabla w\rangle-\frac{\langle X,AX\rangle}{16t^2}w^2\right)\,e^{-\frac{|X|^2}{4t}}$$
$$-\frac{n+1+a}{16}\int x_{n+1}^a t^{-\mu-1}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{3}{2}}\left(\frac{t\sigma'}{\sigma}\right)'\sigma^{-2\alpha}(t)w^2e^{-\frac{|X|^2}{4t}}$$
$$+O(1)\int x_{n+1}^a t^{-\mu-1}\left(\frac{t\sigma'}{\sigma}\right)'\sigma^{-2\alpha}(t)\left(|X|+t\right)w^2e^{-\frac{|X|^2}{4t}}.$$

A purely negative term in (4.23) above is

$$\mathcal{I}_2^*=\frac{1}{64}\int x_{n+1}^a t^{-\mu-2}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{3}{2}}\left(\frac{t\sigma'}{\sigma}\right)'\sigma^{-2\alpha}(t)\langle X,AX\rangle\;w^2\;e^{-\frac{|X|^2}{4t}}. \tag{4.24}$$

This will be handled eventually after combining $\mathcal{I}_2$ with $\mathcal{I}_3$ due to the presence of a similar term in $\mathcal{I}_3$. See (4.29) and (4.32) below. The boundary integral in (4.18) above, i.e. the term

$$\frac{1}{2}c^{-\mu}\left(\frac{c\sigma'(c)}{\sigma(c)}\right)^{-\frac{1}{2}}\int_{t=c} x_{n+1}^a\langle\nabla v, A\nabla v\rangle(X,c)$$

can be treated in a similar fashion which finally results in

$$\begin{aligned}
&\frac{1}{2}c^{-\mu}\left(\frac{c\sigma'(c)}{\sigma(c)}\right)^{-\frac{1}{2}}\int_{t=c} x_{n+1}^a\langle\nabla v, A\nabla v\rangle(X,c)\,\mathrm{d}X\\
&=\frac{1}{2}c^{-\mu}\sigma^{-2\alpha}(c)\left(\frac{c\sigma'(c)}{\sigma(c)}\right)^{-\frac{1}{2}}\int_{t=c} x_{n+1}^a\left(\langle\nabla w, A\nabla w\rangle-\frac{\langle X, A(X)\rangle}{16c^2}w^2+\frac{n+1+a}{4c}w^2\right)\,e^{-\frac{|X|^2}{4c}}\,\mathrm{d}X\\
&\quad+O(1)\;c^{-\mu-1}\sigma^{-2\alpha}(c)\int_{t=c} x_{n+1}^a\,(|X|+c)\,w^2\;e^{-\frac{|X|^2}{4c}}.
\end{aligned}$$

Now a purely negative term in the above expression is

$$\mathcal{I}_2^{**}=-\frac{1}{32}c^{-\mu-2}\sigma^{-2\alpha}(c)\left(\frac{c\sigma'(c)}{\sigma(c)}\right)^{-\frac{1}{2}}\int_{t=c} x_{n+1}^a\langle X, A(X,c)X\rangle\;w^2(X,c)\;e^{-\frac{|X|^2}{4c}}\mathrm{d}X \tag{4.25}$$

which will be taken care of by a similar term in (4.29). See also (4.33) below. Using (4.22) and also by making use of the inequality $(a+b)^2\leq 2(a^2+b^2)$, we obtain

$$\begin{aligned}
\int x_{n+1}^a t^{-\mu-1}|X|\,|\nabla v|^2 \leq 2\int x_{n+1}^a t^{-\mu-1}\sigma^{-2\alpha}(t)|X||\nabla w|^2 e^{-\frac{|X|^2}{4t}}&\\
+\frac{1}{8}\int x_{n+1}^a t^{-\mu-3}\sigma^{-2\alpha}(t)|X|^3|w|^2e^{-\frac{|X|^2}{4t}}&
\end{aligned}\tag{4.26}$$

and

$$\begin{aligned}
\int x_{n+1}^a t^{-\mu-1}t\,|\nabla v|^2 \leq 2\int x_{n+1}^a t^{-\mu}\sigma^{-2\alpha}(t)|\nabla w|^2 e^{-\frac{|X|^2}{4t}}&\\
+\frac{1}{8}\int x_{n+1}^a t^{-\mu-2}\sigma^{-2\alpha}(t)|X|^2|w|^2e^{-\frac{|X|^2}{4t}}.&
\end{aligned}\tag{4.27}$$

Thus from (4.21)-(4.27) and also by using Lemma 4.1 we deduce the following estimate

$$\begin{aligned}
&\mathcal{I}_2+\mathcal{I}_4-\mathcal{I}_2^{*}-\mathcal{I}_2^{**}\succeq\int x_{n+1}^a\sigma^{1-2\alpha}(t)\frac{\theta(\lambda t)}{t}\,|\nabla w|^2\,G+c\sigma^{-2\alpha}(c)\int_{t=c} x_{n+1}^a\,|\nabla w|^2\,G \qquad (4.28)\\
&-\int x_{n+1}^a t^{-\mu-1}\sigma^{-2\alpha}(t)\left(|X||\nabla w|^2+\frac{|X|^2}{t}w^2+\frac{|X|^3}{t^2}w^2\right)e^{-\frac{|X|^2}{4t}}-\;c^{-\mu-1}\sigma^{-2\alpha}(c)\int_{t=c} x_{n+1}^a|X|w^2e^{-\frac{|X|^2}{4c}}\\
&-O(1)\int x_{n+1}^a t^{-\mu-2}\sigma^{-2\alpha}(t)\,(|X|+t)\,w^2e^{-\frac{|X|^2}{4t}}.
\end{aligned}$$

The only cross-product term from $\int PQ$ which remains to be addressed is $\mathcal{I}_3$. We have

$$
\begin{aligned}
\mathcal{I}_3 &= \int x_{n+1}^a t^{-\mu} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \frac{\mathcal{Z}v}{2t} \left(\frac{\langle X, AX\rangle}{16t^2} - \frac{|X|^2}{8t^2}\right) v \qquad (4.29)\\
&= \frac{1}{32} \int x_{n+1}^a t^{-\mu-2} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \left(\langle X, AX\rangle - 2|X|^2\right)\ \partial_t(v^2)\\
&+ \frac{1}{64} \int x_{n+1}^a t^{-\mu-3} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \left(\langle X, AX\rangle - 2|X|^2\right)\ \langle X, A\nabla(v^2)\rangle\\
&= \frac{n+3+a}{64} \int x_{n+1}^a t^{-\mu-3} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \left(\langle X, AX\rangle - 2|X|^2\right)\ v^2 \text{ (using } \mu = \frac{n-1+a}{2})\\
&+ \frac{1}{64} \int x_{n+1}^a t^{-\mu-2} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{3}{2}} \left(\frac{t\sigma'}{\sigma}\right)' \left(\langle X, A(x,t)X\rangle - 2|X|^2\right) v^2 + \frac{1}{32} \int x_{n+1}^a t^{-\mu-2} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \langle X, A_t X\rangle v^2\\
&- \frac{1}{64} \int t^{-\mu-3} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \left(\langle X, AX\rangle - 2|X|^2\right) \operatorname{div}\left(x_{n+1}^a A(x,t)X\right) v^2\\
&- \frac{1}{64} \int x_{n+1}^a t^{-\mu-3} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \left\langle X, A\nabla\left(\langle X, AX\rangle - 2|X|^2\right)\right\rangle v^2\\
&- \frac{1}{32} c^{-\mu-2} \left(\frac{c\sigma'(c)}{\sigma(c)}\right)^{-\frac{1}{2}} \int_{t=c} x_{n+1}^a \left(\langle X, AX\rangle - 2|X|^2\right) v^2.
\end{aligned}
$$

Now using

$$
\begin{aligned}
&\operatorname{div}\left(x_{n+1}^a A(x,t)X\right) = n+1+a+O(|X|+t) \qquad (4.30)\\
&\text{and also that } \left(\langle X, AX\rangle - 2|X|^2\right) O(|X|+t) = O(1)\left(|X|^3 + t|X|^2\right),
\end{aligned}
$$

we find

$$
\begin{aligned}
&\frac{n+3+a}{64} \int x_{n+1}^a t^{-\mu-3} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \left(\langle X, AX\rangle - 2|X|^2\right)\ v^2 \qquad (4.31)\\
&- \frac{1}{64} \int t^{-\mu-3} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \left(\langle X, AX\rangle - 2|X|^2\right) \operatorname{div}\left(x_{n+1}^a A(x,t)X\right) v^2\\
&= \frac{1}{32} \int x_{n+1}^a t^{-\mu-3} \left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}} \left(\langle X, AX\rangle - 2|X|^2\right)\\
&+ O(1) \int x_{n+1}^a t^{-\mu-3} |X|^3 v^2 + O(1) \int x_{n+1}^a t^{-\mu-2} |X|^2 v^2.
\end{aligned}
$$

Similarly by using

$$
\langle X, A(x,t)X\rangle - 2|X|^2 = -\langle X, A(x,t)X\rangle + O(1)(|X|^3 + t|X|^2)
$$

we have

$$\frac{1}{64}\int x_{n+1}^a t^{-\mu-2}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{3}{2}}\left(\frac{t\sigma'}{\sigma}\right)'\left(\langle X, A(x,t)X\rangle - 2|X|^2\right)v^2 = -\mathcal{I}_2^* \tag{4.32}$$
$$+ O(1)\int x_{n+1}^a t^{-\mu-2}\left(\frac{t\sigma'}{\sigma}\right)'|X|^3 v^2 + O(1)\int x_{n+1}^a t^{-\mu-1}\left(\frac{t\sigma'}{\sigma}\right)' v^2,$$

and

$$-\frac{1}{32}c^{-\mu-2}\left(\frac{c\sigma'(c)}{\sigma(c)}\right)^{-\frac{1}{2}}\int_{t=c} x_{n+1}^a\left(\langle X, AX\rangle - 2|X|^2\right)v^2 \tag{4.33}$$
$$= -\mathcal{I}_2^{**} + O(1)\ c^{-\mu-2}\int_{t=c} x_{n+1}^a|X|^3v^2 + O(1)\ c^{-\mu-1}\int_{t=c} x_{n+1}^a|X|^2v^2,$$

where $\mathcal{I}_2^*$ and $\mathcal{I}_2^{**}$ are as in (4.24) and (4.25) respectively. Thus using (4.31)-(4.33) in (4.29) combined with the fact that $\langle X, A_tX\rangle = O(|X|^2)$, we obtain

$$\mathcal{I}_3 = \frac{1}{32}\int x_{n+1}^a t^{-\mu-3}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}}\left(\langle X, AX\rangle - 2|X|^2 - \frac{1}{2}\langle X, A\nabla(\langle X, AX\rangle - 2|X|^2)\rangle\right)v^2 \tag{4.34}$$
$$+ O(1)\int x_{n+1}^a t^{-\mu-2}|X|^2v^2 + O(1)\int x_{n+1}^a t^{-\mu-2}\left(\frac{t\sigma'}{\sigma}\right)'|X|^3v^2 + O(1)\int x_{n+1}^a t^{-\mu-1}\left(\frac{t\sigma'}{\sigma}\right)'v^2$$
$$-\mathcal{I}_2^* - \mathcal{I}_2^{**} + O(1)\int x_{n+1}^a t^{-\mu-3}|X|^3v^2 + O(1)\ c^{-\mu-2}\int_{t=c} x_{n+1}^a|X|^3v^2 + O(1)\ c^{-\mu-1}\int_{t=c} x_{n+1}^a|X|^2v^2.$$

Now using the largeness of $\frac{\theta(\lambda t)}{t}$ and also the fact that

$$\langle X, AX\rangle - 2|X|^2 = -|X|^2 + O(1)\left(|X|^3 + t|X|^2\right),$$
$$\langle X, A\nabla(\langle X, AX\rangle - 2|X|^2)\rangle = -2|X|^2 + O(1)\left(|X|^3 + t|X|^2\right),$$

we find

$$\frac{1}{32}\int x_{n+1}^a t^{-\mu-3}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}}\left(\langle X, AX\rangle - 2|X|^2 - \frac{1}{2}\langle X, A\nabla(\langle X, AX\rangle - 2|X|^2)\rangle\right)v^2 \tag{4.35}$$
$$= O(1)\int x_{n+1}^a t^{-\mu-2}\sigma^{-2\alpha}|X|^2w^2e^{-\frac{|X|^2}{4t}} + O(1)\int x_{n+1}^a t^{-\mu-3}\sigma^{-2\alpha}|X|^3w^2e^{-\frac{|X|^2}{4t}}.$$

Using (4.35) in (4.34) we thus deduce the following estimate

$$\mathcal{I}_3 + \mathcal{I}_2^* + \mathcal{I}_2^{**} = O(1)\int x_{n+1}^a\sigma^{-2\alpha}\frac{\theta(\lambda t)}{t}w^2G + O(1)\int x_{n+1}^a t^{-\mu-2}\sigma^{-2\alpha}|X|^2w^2e^{-\frac{|X|^2}{4t}} \tag{4.36}$$
$$+ O(1)\int x_{n+1}^a t^{-\mu-3}\sigma^{-2\alpha}|X|^3w^2e^{-\frac{|X|^2}{4t}} + O(1)\sigma^{-2\alpha}(c)\ c^{-\mu-2}\int_{t=c} x_{n+1}^a|X|^3w^2e^{-\frac{|X|^2}{4c}}$$
$$+ O(1)\sigma^{-2\alpha}(c)\int_{t=c} x_{n+1}^a w^2G.$$

We now estimate all the error terms in (4.17), (4.28) and (4.36) above using the log inequality in Lemma 4.2.

Taking $\epsilon = (\lambda t)^{2\alpha+\mu+\frac{3}{2}}$ and $m = \frac{1}{2}$ in (4.5), we observe

$$
\begin{aligned}
t^{-\mu-2}|X|e^{-\frac{|X|^2}{4t}} = 2t^{-\mu-\frac{3}{2}}\left(\frac{|X|^2}{4t}\right)^{1/2} e^{-\frac{|X|^2}{4t}} &\leq Ct^{-\mu-\frac{3}{2}}\left((\lambda t)^{2\alpha+\mu+\frac{3}{2}} + \left(2\alpha+\mu+\frac{3}{2}\right)^{\frac{1}{2}}\left(\log\frac{1}{\lambda t}\right)^{\frac{1}{2}} e^{-\frac{|X|^2}{4t}}\right)\\
&\leq C\left(\lambda^{2\alpha+N}t^{2\alpha} + \sqrt{\alpha}t^{-\mu-\frac{3}{2}}\left(\log\frac{1}{\lambda t}\right)^{\frac{1}{2}} e^{-\frac{|X|^2}{4t}}\right)\\
&\leq C\left(\lambda^{2\alpha+N}t^{2\alpha} + \delta\left(\lambda t\log\frac{1}{\lambda t}\right)^{\frac{1}{2}} t^{-\mu-2}e^{-\frac{|X|^2}{4t}}\right)\\
&\leq C\left(\lambda^{2\alpha+N}t^{2\alpha} + \delta\,\frac{\theta(\lambda t)}{t}t^{-\mu-1}e^{-\frac{|X|^2}{4t}}\right).
\end{aligned}
$$

Since $\sigma \sim t$, it follows from the above inequality that the following estimate holds

$$
\alpha\int x_{n+1}^a t^{-\mu-2}\sigma^{-2\alpha}|X|e^{-\frac{|X|^2}{4t}}w^2 \leq C\left(\lambda^{2\alpha+N}\int x_{n+1}^a u^2 + \delta\alpha\int x_{n+1}^a\frac{\theta(\lambda t)}{t}\sigma^{-2\alpha}w^2G\right). \tag{4.37}
$$

Again by applying (4.5) with $m = \frac{3}{2}$ and $\epsilon = (\lambda t)^{2\alpha+\mu+\frac{3}{2}}$ we obtain

$$
t^{-\mu-3}|X|^3e^{-\frac{|X|^2}{4t}} = 8t^{-\mu-\frac{3}{2}}\left(\frac{|X|^2}{4t}\right)^{\frac{3}{2}} e^{-\frac{|X|^2}{4t}} \leq Ct^{-\mu-\frac{3}{2}}\left((\lambda t)^{2\alpha+\mu+\frac{3}{2}} + \left(\alpha+\mu+\frac{3}{2}\right)^{\frac{3}{2}}\left(\log\frac{1}{\lambda t}\right)^{\frac{3}{2}} e^{-\frac{|X|^2}{4t}}\right) \tag{4.38}
$$

$$
\leq C\left(\lambda^{2\alpha+N}t^{2\alpha} + \delta\alpha\,\frac{\theta(\lambda t)}{t}t^{-\mu-1}e^{-\frac{|X|^2}{4t}}\right)
$$

and thus similarly as for (4.37), using (4.38) we deduce the following inequality

$$
\int x_{n+1}^a t^{-\mu-3}\sigma^{-2\alpha}|X|^3e^{-\frac{|X|^2}{4t}}w^2 \leq C\left(\lambda^{2\alpha+N}\int x_{n+1}^a w^2 + \delta\alpha\int x_{n+1}^a\frac{\theta(\lambda t)}{t}\sigma^{-2\alpha}w^2G\right). \tag{4.39}
$$

Also in an essentially similar way, we get

$$
\int x_{n+1}^a t^{-\mu-1}\sigma^{-2\alpha}(t)e^{-\frac{|X|^2}{4t}}|X|\,|\nabla w|^2 \leq C\left(\lambda^{2\alpha+N}\int x_{n+1}^a t|\nabla w|^2 + \delta\int x_{n+1}^a\frac{\theta(\lambda t)}{t}\sigma^{1-2\alpha}|\nabla w|^2G\right). \tag{4.40}
$$

Now we note that the other error terms such as $O(1)\int x_{n+1}^a\sigma^{-2\alpha}\frac{\theta(\lambda t)}{t}w^2G$ that shows up in (4.28) and (4.36), $\delta\alpha\int x_{n+1}^a\frac{\theta(\lambda t)}{t}\sigma^{-2\alpha}w^2G$ that shows up in (4.37) and (4.39) can be absorbed in the integral $\alpha\int x_{n+1}^a\sigma^{-2\alpha}\frac{\theta(\lambda t)}{t}w^2G$ in (4.17) provided δ is sufficiently small.

Likewise for small enough δ, the term $\delta\int x_{n+1}^a\frac{\theta(\lambda t)}{t}\sigma^{1-2\alpha}|\nabla w|^2G$ in (4.40) can be absorbed in the integral $\int x_{n+1}^a\frac{\theta(\lambda t)}{t}\sigma^{1-2\alpha}|\nabla w|^2G$ which appears in (4.28).

We finally control the error term $\sigma^{-2\alpha}(c)\int_{t=c} x_{n+1}^a \frac{|X|^3}{c}\,|w|^2\, G$ that appears in (4.36) in the following way. We have

$$\begin{aligned}
&\sigma^{-2\alpha}(c)\int_{t=c} x_{n+1}^a \frac{|X|^3}{c}\,|w|^2\, G \qquad (4.41)\\
&= \sigma^{-2\alpha}(c)\int_{\mathbb{B}_\rho,\ t=c} x_{n+1}^a \frac{|X|^3}{c}\,|w|^2\, G + \sigma^{-2\alpha}(c)\int_{\mathbb{B}_\rho^c,\ t=c} x_{n+1}^a \frac{|X|^3}{c}\,|w|^2\, G\\
&\le \rho\sigma^{-2\alpha}(c)\int_{t=c} x_{n+1}^a \frac{|X|^2}{c}\,|w|^2\, G + N^\alpha\lambda^{2\alpha+N}\int_{t=c} x_{n+1}^a w^2(X,c),
\end{aligned}$$

where we used the estimate $\frac{|X|^3}{c}G(X,c)\sigma(c)^{-2\alpha} \le N^\alpha\lambda^{2\alpha+N}$ for $x\in\mathbb{B}_\rho^c$ in the last line in the above inequality. Finally the term $\rho\sigma^{-2\alpha}(c)\int_{t=c} x_{n+1}^a \frac{|X|^2}{c}\,|w|^2\, G$ is estimated by using the Hardy inequality in Lemma 4.3 as follows

$$\rho\sigma^{-2\alpha}(c)\int_{t=c} x_{n+1}^a \frac{|X|^2}{c}\,|w|^2\, G \le C\rho\sigma^{-2\alpha}(c)\left(\int x_{n+1}^a w^2 G + c\int x_{n+1}^a |\nabla w|^2 G\right). \qquad (4.42)$$

Now the term $C\rho\sigma^{-2\alpha}(c)c\int x_{n+1}^a|\nabla w|^2 G$ can be absorbed in the integral $c\sigma^{-2\alpha}(c)\int_{t=c} x_{n+1}^a\,|\nabla w|^2\, G$ in (4.28) provided ρ is small enough. Thus from (4.12), (4.17), (4.28), (4.36) and (4.37)-(4.42), we finally deduce the following estimate

$$\begin{aligned}
&\alpha\int_{\mathbb{R}_+^{n+1}\times[c,\infty)} x_{n+1}^a \frac{\theta(\lambda t)}{t}\sigma^{-2\alpha}(t)\ w^2\ G + \int_{\mathbb{R}_+^{n+1}\times[c,\infty)} x_{n+1}^a\sigma^{1-2\alpha}(t)\ \frac{\theta(\lambda t)}{t}|\nabla w|^2\ G \qquad (4.43)\\
&\preceq \int_{\mathbb{R}_+^{n+1}\times[c,\infty)}\int \sigma^{-2\alpha}(t)t^{-\mu}x_{n+1}^{-a}e^{-\frac{|X|^2}{4t}}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}}|\widetilde{\mathcal{H}}w|^2 + \alpha^{c'\alpha}\,\sup_{t\ge c}\int_{\mathbb{R}_+^{n+1}} x_{n+1}^a\left(w^2+t|\nabla w|^2\right)\,\mathrm{d}X\\
&+\sigma^{-2\alpha}(c)\left\{-c\int_{t=c} x_{n+1}^a\ |\nabla w(X,c)|^2\ G(X,c)\,\mathrm{d}X + \alpha\int_{t=c} x_{n+1}^a\ |w(X,c)|^2\ G(X,c)\,\mathrm{d}X\right\}.
\end{aligned}$$

The desired estimate as claimed in (4.6) follows from (4.43) above by using (4.7), the fact that $\lambda\sim\alpha$ and also that

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}\times[c,\infty)}\int \sigma^{-2\alpha}(t)t^{-\mu}x_{n+1}^{-a}e^{-\frac{|X|^2}{4t}}\left(\frac{t\sigma'}{\sigma}\right)^{-\frac{1}{2}}|\widetilde{\mathcal{H}}w|^2\\
&\approx \int_{\mathbb{R}_+^{n+1}\times[c,\infty)} \sigma^{1-2\alpha}(t)x_{n+1}^{-a}\ |\widetilde{\mathcal{H}}w|^2\ G.
\end{aligned}$$

□

Now by a translation in time, we find from Lemma 4.4 that the following estimate holds.

Lemma 4.5. *Let* $\widetilde{\mathcal{H}}$ *be as in* (4.3) *where* $A(x,t)$ *satisfies* (4.1). *Let* $w\in C_0^\infty\left(\overline{\mathbb{B}_4^+}\times[0,\frac{1}{3\lambda})\right)$ *be such that* $\partial_{x_{n+1}}^a w\equiv 0$ *on* $\{x_{n+1}=0\}$ *where* $\lambda=\frac{\alpha}{\delta^2}$ *for* $\delta\in(0,1)$ *sufficiently small. Then the following* *estimate holds for all large* α

$$\alpha^2 \int_{\mathbb{R}^{n+1}_+\times[0,\infty)} x_{n+1}^a \sigma_c^{-2\alpha}(t)\ w^2\ G_c + \alpha \int_{\mathbb{R}^{n+1}_+\times[0,\infty)} x_{n+1}^a \sigma_c^{1-2\alpha}(t)\ |\nabla w|^2\ G_c \tag{4.44}$$
$$\preceq \int_{\mathbb{R}^{n+1}_+\times[0,\infty)} \sigma_c^{1-2\alpha}(t) x_{n+1}^{-a}\ |\widetilde{\mathcal{H}} w|^2\ G_c + \alpha^{c'\alpha} \sup_{t\geq 0} \int_{\mathbb{R}^{n+1}_+} x_{n+1}^a \left(w^2 + t|\nabla w|^2\right)\ \mathrm{d}X$$
$$+ \sigma^{-2\alpha}(c) \left\{ -c \int_{t=0} x_{n+1}^a\ |\nabla w(X,0)|^2\ G(X,c)\ \mathrm{d}X + \alpha \int_{t=0} x_{n+1}^a\ |w(X,0)|^2\ G(X,c)\ \mathrm{d}X \right\}.$$

Here $\sigma_c(t) = \sigma(c+t)$, $G_c(X,t) = G(X,t+c)$ *and* $0 < c \leq \frac{1}{5\lambda}$.

4.2. **Some basic regularity estimates for the extension problem.** We now gather some important qualitative properties of the solution to the extension problem (3.1). We first note that it follows from Theorem 3.1 that the following result holds.

Lemma 4.6. *Let* U *be the solution to the extension problem* (3.1) *corresponding to* $u \in \mathbb{H}^s$. *Assume that* $\mathcal{H}^s u = 0$ *in* $B_1 \times (-1,0)$ *in the sense of Defintion 3.2. Then* U *is a weak solution to*

$$\begin{cases} \operatorname{div}(x_{n+1}^a A(x) \nabla U) = x_{n+1}^a U_t \text{ in } \mathbb{B}_1^+ \times (-1,0), \\ \partial_{x_{n+1}}^a U = 0 \text{ at } \{x_{n+1} = 0\}. \end{cases} \tag{4.45}$$

We refer to Section 4 in [11] for the precise notion of weak solutions. See also [13].

Proof. Step 1: Let $\tilde{U}$ denote the even reflection of U across $\{x_{n+1} = 0\}$. See (4.51) below. We claim that

$$\int_{\mathbb{B}_1\times(-1,0)} |x_{n+1}|^a \langle \nabla\tilde{U}, \nabla\phi\rangle dX dt \int_{\mathbb{B}_1\times(-1,0)} |x_{n+1}|^a \tilde{U}\phi_t dX dt, \tag{4.46}$$

for all $\phi \in C_0^\infty(\mathbb{B}_1^+ \times (-1,0))$. We first let

$$I_\epsilon = \int_{\mathbb{B}_1\times(-1,0)\cap\{|x_{n+1}|>\epsilon\}} |x_{n+1}|^a \langle \nabla\tilde{U}, \nabla\phi\rangle dX dt.$$

Then by using the equation satisfied by $\tilde{U}$ in $\{|x_{n+1}| > \epsilon\}$ and divergence theorem, we find

$$I_\epsilon = \int_{\mathbb{B}_1\times(-1,0)\cap\{|x_{n+1}|>\epsilon\}} |x_{n+1}|^a \tilde{U}\phi_t dX dt + A_\epsilon + B_\epsilon, \tag{4.47}$$

where

$$\begin{cases} A_\epsilon = -\int_{x_{n+1}=\epsilon} x_{n+1}^a \partial_{x_{n+1}} \tilde{U}(x,\epsilon,t)\phi(x,\epsilon,t) dX dt \\ B_\epsilon = \int_{x_{n+1}=-\epsilon} |x_{n+1}|^a \partial_{x_{n+1}} \tilde{U}(x,-\epsilon,t)\phi(x,-\epsilon,t) dX dt. \end{cases} \tag{4.48}$$

Using (ii) in Theorem 3.1, we now show that $A_\epsilon, B_\epsilon \to 0$ as $\epsilon \to 0$. We only show it for A_ϵ as the arguments for B_ϵ is analogous. Now A_ϵ can be rewritten as

$$A_\epsilon = -\int_{x_{n+1}=\epsilon} x_{n+1}^a \partial_{x_{n+1}} \tilde{U}(x,\epsilon,t)\phi(x,0,t) dX dt - \int_{x_{n+1}=\epsilon} x_{n+1}^a \partial_{x_{n+1}} \tilde{U}(x,\epsilon,t)(\phi(x,\epsilon,t) - \phi(x,0,t)) dX dt.$$

Using ii) in Theorem 3.1, the fact that $\mathcal{H}^s u = 0$ in $B_1 \times (-1,0)$ and also that $\phi(\cdot,0,\cdot)$ is smooth and compactly supported, it follows that as $\epsilon \to 0$

$$\int_{x_{n+1}=\epsilon} x_{n+1}^a \partial_{x_{n+1}} \tilde{U}(x,\epsilon,t)\phi(x,0,t) dX dt \to 0.$$

Now by using Fundamental theorem of calculus in x_{n+1}, we can write

$$(\phi(x,\epsilon,t) - \phi(x,0,t) = \epsilon\psi(x,\epsilon,t).$$

where ψ is smooth and compactly supported. Thus using inequality (3.15), Plancherel theorem and Cauchy-Schwartz inequality, we find that the term $\int_{x_{n+1}=\epsilon} x_{n+1}^a \partial_{x_{n+1}} \tilde{U}(x,\epsilon,t)(\phi(x,\epsilon,t) - \phi(x,0,t))dXdt$ can be estimated in the following way

$$\left|\int_{x_{n+1}=\epsilon} x_{n+1}^a \partial_{x_{n+1}} \tilde{U}(x,\epsilon,t)(\phi(x,\epsilon,t) - \phi(x,0,t))dXdt\right| \tag{4.49}$$

$$\leq C\epsilon||u||_{\mathbb{H}^s(\mathbb{R}^{n+1})} \times \sup_{0<b<1}||\psi(\cdot,b,\cdot)||_{\mathbb{H}^s(\mathbb{R}^{n+1})} \to 0, \tag{4.50}$$

as $\epsilon \to 0$. Thus we find also by using iii) in Theorem 3.1 that A_ϵ and likewise $B_\epsilon \to 0$ as $\epsilon \to 0$ which establishes the claim in (4.46).

Step 2(Conclusion): Now given that (4.46) holds, by a density argument as in [35, Corollary 1.7] and also by using iii) in Theorem 3.1, it is seen that (4.46) holds for all ϕ such that $\nabla\phi, \phi_t \in L^2(\mathbb{B}_1, x_{n+1}^a dXdt)$.

□

We now state the relevant regularity result for the extension problem which is Lemma 2.2 in [12]. We refer to [39, Chapter 4] for the relevant notion of parabolic Hölder spaces $H^{k+\alpha}$ that appears below.

Lemma 4.7. *Let U be a weak solution to (4.45) in $\mathbb{B}_1^+ \times (-1,0]$ where A satisfies (2.2) and (2.3) or equivalently (4.1). Then the extended function $\tilde{U}$ which is defined as*

$$\begin{cases} \tilde{U}(x,x_{n+1}) = U(x,x_{n+1}) \text{ for } x_{n+1} > 0 \\ \tilde{U}(x,x_{n+1}) = U(x,-x_{n+1}) \text{ for } x_{n+1} < 0 \end{cases} \tag{4.51}$$

solves

$$\operatorname{div}(|x_{n+1}|^a A(x)\nabla\tilde{U}) - |x_{n+1}|^a \partial_t \tilde{U} = 0 \tag{4.52}$$

in $\mathbb{B}_1 \times (-1,0]$, and moreover $\tilde{U} \in H^{1+\alpha}(\mathbb{B}_{1/2} \times (-1/4,0])$ for all $\alpha > 0$. Moreover, the $H^{1+\alpha}$ norm in $\mathbb{B}_{1/2} \times (-1/4,0])$ can be estimated by $C\int_{\mathbb{B}_1\times(-1,0]} |x_{n+1}|^a \tilde{U}^2 dXdt$ where C depends on the dimension, the ellipticity and the Lipschitz character of A.

Moreover by arguing as in the proof of Lemma 5.5 in [11], we have the following result regarding the integrability of the second derivatives.

Lemma 4.8. *Let U be as in Lemma 4.7 above. Then we have that the following estimate holds,*

$$\int_{\mathbb{B}_{1/2}^+\times(-1/4,0])} x_{n+1}^a(|\nabla U|^2 + |\nabla_x\nabla U|^2 + U_t^2) + x_{n+1}^{-a}|\nabla(x_{n+1}^a U_{x_{n+1}})|^2 \leq C\int_{\mathbb{B}_1\times(-1,0]} |x_{n+1}|^a\tilde{U}^2 dXdt, \tag{4.53}$$

where C has a similar dependence as in Lemma 4.7 above.

As previously said, for notational purposes it will be convenient to work with the following backward version of problem (3.2) in the cylinder $\mathbb{B}_4^+ \times (0,16]$

$$\begin{cases} x_{n+1}^a \partial_t U + \operatorname{div}(x_{n+1}^a A(x) \nabla U) = 0 & \text{in } \mathbb{B}_4^+ \times [0,16), \\ U(x,0,t) = u(x,t) \\ \partial_{x_{n+1}}^a U(x,0,t) = 0 & \text{in } B_4 \times [0,16). \end{cases} \tag{4.54}$$

We note that the former can be transformed into the latter by changing $t \to -t$.

We now introduce an assumption that will remain in force for the rest of the section up to the proof of Theorem 1.3. When we work with a solution U of the problem (4.45) in $\mathbb{B}_4^+ \times (-16,0]$, we will always assume that

$$\int_{\mathbb{B}_1^+} x_{n+1}^a U(X,0)^2 dX > 0. \tag{4.55}$$

As a consequence of such hypothesis the number

$$\theta \overset{def}{=} \frac{\int_{\mathbb{B}_4^+ \times (-16,0]} x_{n+1}^a U(X,t)^2 dX dt}{\int_{\mathbb{B}_1^+} x_{n+1}^a U(X,0)^2 dX} \tag{4.56}$$

will be well-defined. In the remainder of this work the symbol θ will always mean the number defined by (4.56).

We now state and prove the relevant monotonicity in time result which is analogous to Lemma 3.1 in [2].

Lemma 4.9. *Let U be a solution of (4.54). Then there exists a constant $N = N(n,a,A) > 2$ such that $N \log(N\theta) \geq 1$, and for which the following inequality holds for $0 \leq t \leq 1/N \log(N\theta)$*

$$N \int_{\mathbb{B}_2^+} x_{n+1}^a U(X,t)^2 dX \geq \int_{\mathbb{B}_1^+} x_{n+1}^a U(X,0)^2 dX.$$

Proof. Let $f = \phi U$, where $\phi \in C_0^\infty(\mathbb{B}_2)$ is a spherically symmetric cutoff such that $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on $\mathbb{B}_{3/2}$. Since U solves (4.54) and ϕ is independent of t and symmetric in x_{n+1}, it is easily seen that the function f solves the problem

$$\begin{cases} x_{n+1}^a f_t + \operatorname{div}(x_{n+1}^a \nabla f) = 2 x_{n+1}^a \langle \nabla U, \nabla \phi \rangle + \operatorname{div}(x_{n+1}^a \nabla \phi) U & \text{in } \mathbb{B}_4^+ \times (-16,0], \\ f(x,0,t) = u(x,t)\phi(x,0) \\ \partial_{x_{n+1}}^a f(x,0,t) = 0 & \text{in } B_4 \times [0,16). \end{cases} \tag{4.57}$$

Again since ϕ is symmetric in x_{n+1}, we have $\partial_{n+1}\phi \equiv 0$ on the thin set $\{x_{n+1} = 0\}$. This fact and the smoothness of ϕ imply that $\frac{\phi_y}{y}$ be bounded up to $\{y = 0\}$. Therefore we observe that the following is true

$$\begin{cases} \operatorname{supp}(\nabla \phi) \cap \{x_{n+1} > 0\} \subset \mathbb{B}_2^+ \setminus \mathbb{B}_{3/2}^+ \\ |\operatorname{div}(x_{n+1}^a \nabla \phi)| \leq C x_{n+1}^a \, \mathbf{1}_{\mathbb{B}_2^+ \setminus \mathbb{B}_{3/2}^+}, \end{cases} \tag{4.58}$$

where for a set E we have denoted by $\mathbf{1}_E$ its indicator function.

We now fix a point $Y \in \mathbb{R}_+^{n+1}$ and introduce the quantity

$$H(t) = \int_{\mathbb{R}_+^{n+1}} x_{n+1}^a f(X,t)^2 \mathcal{G}(Y,X,t) dX,$$

where $\mathcal{G}$ is as in (3.25). We note that for $t > 0$, $\mathcal{G} = \mathcal{G}(Y, \cdot)$ solves

$$\operatorname{div}(x_{n+1}^a \nabla \mathcal{G}) = x_{n+1}^a \partial_t \mathcal{G}. \tag{4.59}$$

Before proceeding further, we remark that in the ensuing computations below, the formal differentiation under the integral sign and the integration by parts can be justified by an approximation argument by first considering the integrals in the region $\{x_{n+1} > \epsilon\}$ and then by letting $\epsilon \to 0$ using the regularity estimates in Lemma 4.7 and Lemma 4.8. Thus in view of this, By differentiating H', we observe using (4.59) that the following holds

$$\begin{aligned} H'(t) &= 2\int_{\mathbb{R}^{n+1}_+} x_{n+1}^a f f_t \mathcal{G} + \int x_{n+1}^a f^2 \partial_t \mathcal{G} \\ &= 2\int_{\mathbb{R}^{n+1}_+} x_{n+1}^a f f_t \mathcal{G} + \int_{\mathbb{R}^{n+1}} f^2 \operatorname{div}\left(x_{n+1}^a A(x,t)\nabla \mathcal{G}\right) \\ &= 2\int f\mathcal{G}\left(x_{n+1}^a f_t + \operatorname{div}\left(x_{n+1}^a A(x,t)\cdot \nabla f\right)\right) + 2\int x_{n+1}^a \mathcal{G}\langle \nabla f, A(x,t)\nabla f\rangle. \end{aligned} \tag{4.60}$$

For $Y \in \mathbb{B}_1^+$, we now claim that the following estimate holds

$$I_1 := 2\int f\mathcal{G}\left(x_{n+1}^a f_t + \operatorname{div}\left(x_{n+1}^a A(x,t)\cdot \nabla f\right)\right) \geq -Ne^{-1/Nt}\int_{\mathbb{B}_4^+\times(-16,0]} x_{n+1}^a U^2 dX dt, \tag{4.61}$$

for some universal N. We argue as in [2]. In order to establish (4.61), we need the following asymptotics of $I_{\frac{a-1}{2}}$ which asserts that there exists $C(a), c(a) > 0$ such that

$$I_{\frac{a-1}{2}}(z) \leq C(a) z^{\frac{a-1}{2}} \quad \text{if } 0 < z \leq c(a), \qquad I_{\frac{a-1}{2}}(z) \leq C(a) z^{-1/2} e^z \quad \text{if } z \geq c(a). \tag{4.62}$$

See for instance [37, formulas (5.7.1) and (5.11.8)]. We then write the integral on the left hand side in (4.61) as $I_1^1 + I_1^2$, where I_1^1 is integral on the set $\mathcal{A} = \{X \in \mathbb{R}^{n+1}_+ \mid x_{n+1}y_{n+1} > 2tc(a)\}$ and I_1^2 is the integral on the complement $\mathcal{A}_e$ of $\mathcal{A}$. We want to bound I_1 by appropriately bounding $\mathcal{G}$ from above in each of the sets $\mathcal{A}$ and $\mathcal{A}_e$. In this respect it is important to note that in view of (4.57) and (4.58), the integral in the definition of I_1 is actually performed in $X \in \mathbb{B}_2^+ \setminus \mathbb{B}_{3/2}^+$ and on such set we have for every $Y \in \mathbb{B}_1^+$

$$\frac{1}{2} \leq |X - Y| \leq 3. \tag{4.63}$$

Our objective is to prove that when $Y \in \mathbb{B}_1^+$, $X \in \mathbb{B}_2^+ \setminus \mathbb{B}_{3/2}^+$ and $0 < t \leq 1$, the following bound holds for some universal $M > 0$

$$\mathcal{G}(Y, X, t) \leq e^{-\frac{1}{Mt}}. \tag{4.64}$$

To prove that (4.64) holds when $X \in \mathcal{A} \cap (\mathbb{B}_2^+ \setminus \mathbb{B}_{3/2}^+)$ we argue as follows. Since for $X \in \mathcal{A}$ we have $\frac{x_{n+1}y_{n+1}}{2t} > c(a)$, by the second inequality in (4.62) we have

$$I_{\frac{a-1}{2}}\left(\frac{x_{n+1}y_{n+1}}{2t}\right) \leq C(a)\left(\frac{x_{n+1}y_{n+1}}{2t}\right)^{-1/2} e^{\frac{x_{n+1}y_{n+1}}{2t}}. \tag{4.65}$$

Consider first the case $-1 < a \leq 0$. Since for $X \in \mathbb{B}_2^+$ and $Y \in \mathbb{B}_1^+$ we trivially have $\frac{x_{n+1}y_{n+1}}{2t} \leq \frac{4}{t}$, in such case we have $\left(\frac{x_{n+1}y_{n+1}}{2t}\right)^{-a/2} \leq 2^{-a/2} t^{a/2}$. Using this estimate and (4.65) in (3.26), we obtain

$$p_a(y_{n+1}, x_{n+1}, t) \leq C^\star(a) t^{-1/2} e^{-\frac{(y_{n+1}-x_{n+1})^2}{4t}}.$$

Combining this bound with (3.30) we infer that for $Y \in \mathbb{B}_1^+$ and $X \in \mathcal{A}$

$$\mathcal{G}(Y,X,t) \leq C\ t^{-\frac{n+1}{2}} e^{-\frac{|x-y|^2}{N_0 t} - \frac{(y_{n+1}-x_{n+1})^2}{4t}}. \tag{4.66}$$

On the other hand if $a > 0$, then we have $\left(\frac{x_{n+1}y_{n+1}}{2t}\right)^{-a/2} \leq c(a)^{-a/2}$ for $X \in \mathcal{A}$. Using this estimate and (4.65) in (3.26) we find

$$p_a(y_{n+1}, x_{n+1}, t) \leq C^{\star\star}(a) t^{-\frac{a+1}{2}} e^{-\frac{(y_{n+1}-x_{n+1})^2}{4t}}.$$

Combining this bound with (3.30) we infer that for $Y \in \mathbb{B}_1^+$ and $X \in \mathcal{A}$

$$\mathcal{G}(Y,X,t) \leq C\ t^{-\frac{n+1+a}{2}} e^{-\frac{|x-y|^2}{N_0 t} - \frac{(y_{n+1}-x_{n+1})^2}{4t}}. \tag{4.67}$$

From (4.66) and (4.67) and (4.63) we conclude that when $Y \in \mathbb{B}_1^+$, $X \in \mathcal{A} \cap (\mathbb{B}_2^+ \setminus \mathbb{B}_{3/2}^-)$ and $0 < t \leq 1$, the following bound holds for some universal $C > 0$ and for $l = \max\{\frac{n+1}{2}, \frac{n+1+a}{2}\}$

$$\mathcal{G}(Y,X,t) \leq C\ t^{-l} e^{-\frac{1}{Ct}}.$$

From this inequality above, (4.64) immediately follows when $X \in \mathcal{A} \cap (\mathbb{B}_2^+ \setminus \mathbb{B}_{3/2}^+)$. If instead $X \in \mathcal{A}_e \cap (\mathbb{B}_2^+ \setminus \mathbb{B}_{3/2}^+)$, keeping in mind that on the set $\mathcal{A}_e$ we have $\frac{x_{n+1}y_{n+1}}{2t} \leq c(a)$, by the first inequality in (4.62) we obtain that for all $a \in (-1,1)$

$$I_{\frac{a-1}{2}}\left(\frac{x_{n+1}y_{n+1}}{2t}\right) \leq C(a)\left(\frac{x_{n+1}y_{n+1}}{2t}\right)^{\frac{a-1}{2}}.$$

Using this in (3.26) we find

$$p_a(y_{n+1}, x_{n+1}, t) \leq C(a)(2t)^{-\frac{a+1}{2}} e^{-\frac{y_{n+1}^2 + x_{n+1}^2}{4t}} \leq C^{\star}(a) t^{-\frac{a+1}{2}} e^{-\frac{(y_{n+1}-x_{n+1})^2}{8t}}.$$

Combining this bound with (3.30) we again conclude that for $Y \in \mathbb{B}_1^+$, $0 < t \leq 1$ and $X \in \mathcal{A}_e \cap (\mathbb{B}_2^+ \setminus \mathbb{B}_{3/2}^+)$

$$\mathcal{G}(Y,X,t) \leq C t^{-\frac{n+1+a}{2}} e^{-\frac{1}{Ct}}.$$

Thus we find that (4.64) holds. Now using (4.64) in the definition of I_1 and also by using (4.57) and (4.58) we finally obtain

$$|I_1| \leq C e^{-\frac{1}{Mt}} \int_{\mathbb{B}_2^+} x_{n+1}^a \left(|\nabla U| + |U|\right)|U|.$$

We can now appeal to the L^∞ bounds for $U, \nabla U, U_t$ as in Lemma 4.7 to finally conclude that for every $Y \in \mathbb{B}_1^+$ and $0 < t \leq 1$ the inequality (4.61) holds.

Using (4.61) in (4.60), we obtain

$$H'(t) \geq -N e^{-1/Nt} \int_{\mathbb{B}_4^+ \times (-16,0]} x_{n+1}^a U^2 dX dt. \tag{4.68}$$

Now from the approximation to identity property (3.29) it follows that

$$\lim_{t \to 0^+} H(t) = U(Y,0)^2. \tag{4.69}$$

Using (4.69) in (4.68) we obtain

$$H(t) \geq U(Y,0)^2 - N e^{-1/Nt} \int_{\mathbb{B}_4^+ \times (-16,0]} x_{n+1}^a U^2 dX dt. \tag{4.70}$$

Now by integrating (4.70) with respect to Y in $\mathbb{B}_1^+$, exchanging the order of integration and using (3.28) we obtain

$$\int_{\mathbb{B}_2^+} x_{n+1}^a U(X,t)^2 dX \geq \int_{\mathbb{B}_1^+} x_{n+1}^a U(X,0)^2 dX - Ne^{-1/Nt} \int_{\mathbb{B}_4^+ \times (-16,0]} x_{n+1}^a U^2 dX dt. \tag{4.71}$$

Note that in (4.71) above, we have renamed the variable Y as X. Now from the L^∞ bound on U as in Lemma 4.7 that the following estimate holds

$$\int_{\mathbb{B}_1^+} x_{n+1}^a U(X,0)^2 dX \leq C \int_{\mathbb{B}_4^+ \times (-16,0]} x_{n+1}^a U^2 dX dt. \tag{4.72}$$

Note that (4.72) in particular implies that θ as defined in (4.56) is bounded from below away from zero. Now if we let

$$t \leq \frac{1}{N \log(2N\theta)}, \tag{4.73}$$

then we find from the definition of θ in (4.56) that

$$Ne^{-1/Nt} \int_{\mathbb{B}_4^+ \times (-16,0]} x_{n+1}^a U^2 dX dt < \frac{1}{2} \int_{\mathbb{B}_1^+} x_{n+1}^a U(X,0)^2 dX. \tag{4.74}$$

Using (4.74) in (4.71) we have

$$2 \int_{\mathbb{B}_2^+} x_{n+1}^a U(X,t)^2 dX \geq \int_{\mathbb{B}_1^+} x_{n+1}^a U(X,0)^2 dX, \tag{4.75}$$

for all t satisfying (4.73). Thus by letting $2N$ as our new N, we find that the conclusion of the lemma follows. □

Now given the Carleman estimate in Lemma 4.5 and the monotonicity result in Lemma 4.9, using Lemma 4.7 and the integrability of the second derivatives as in Lemma 4.8, one can now repeat the arguments as in [22, pages 11- 13] (see also [2]) to assert that the following conditional doubling inequality holds under the assumption (4.55).

Theorem 4.10. *Let U be a solution of* (4.54) *in* $\mathbb{B}_4^+ \times [0, 16)$. *There exists $N > 2$, depending on n, a for which $N \log(N\theta) \geq 1$ and such that:*

(i) *For $r \leq 1/2$, we have*

$$\int_{\mathbb{B}_{2r}^+} x_{n+1}^a U(X,0)^2 dX \leq (N\theta)^N \int_{\mathbb{B}_r} x_{n+1}^a U(X,0)^2 dX.$$

Moreover for $r \leq 1/\sqrt{N \log(N\theta)}$ the following two inequalities hold:

(ii)

$$\int_{\mathbb{B}_{2r}^+ \times [0,4r^2)} x_{n+1}^a U(X,t)^2 dX dt \leq \exp(N \log(N\theta) \log(N \log(N\theta)))r^2 \int_{\mathbb{B}_r^+} x_{n+1}^a U^2(X,0) dX.$$

(iii)

$$\int_{\mathbb{B}_{2r}^+ \times [0,4r^2)} x_{n+1}^a U(X,t)^2 dX dt \leq \exp(N \log(N\theta) \log(N \log(N\theta))) \int_{\mathbb{B}_r^+ \times [0,r^2)} x_{n+1}^a U(X,t)^2 dX dt.$$

4.3. **Proof of Theorem 1.3.** With Theorem 4.10 in hand, we now proceed with the proof of Theorem 1.3 by means of blowup argument inspired by that in [2] and [11].

Proof of Theorem 1.3. Without loss of generality, we assume that $A(0) = \mathbb{I}$ and also that $u \in \mathbb{H}^s$ solves $\mathcal{H}^s u = 0$ in $B_4 \times (-16, 0]$ and vanishes in $B_4 \times (-16, 0]$. It suffices to show that for the solution U to the extension problem (4.54) (by changing $t \to -t$), we must have

$$U(X, 0) \equiv 0, \qquad \text{for every } X \in \mathbb{B}_1^+. \tag{4.76}$$

Once (4.76) is proven, we then note that, away from the thin set $\{x_{n+1} = 0\}$, U solves a uniformly parabolic PDE with Lipschitz coefficients and vanishes identically in the half-ball $\mathbb{B}_1^+$. We can thus appeal to [9, Theorem 1] to assert that U vanishes to infinite order both in space and time at every $(X, 0)$ for $X \in \mathbb{B}_1^+$. At this point, we can use the strong unique continuation result in [23, Theorem 1] to finally conclude that $U(X, 0) \equiv 0$ for $X \in \mathbb{R}^{n+1}_+$. Letting $x_{n+1} = 0$, this implies $u(x, 0) = U(x, 0, 0) \equiv 0$ for $x \in \mathbb{R}^n$. Similarly, we can show that $u(\cdot, t) \equiv 0$ for all $t \in (-16, 0)$ and thus Theorem 1.3 would follow.

Therefore we are left with establishing the claim in (4.76). We argue by contradiction and assume that (4.76) is not true. Consequently, (4.55) does hold and therefore we can use the results in Theorem 4.10. In particular from (i) in Theorem 4.10 it follows that $\int_{\mathbb{B}_r^+} x_{n+1}^a U(X, 0)^2 dX > 0$ for all $0 < r \leq \frac{1}{2}$. From this fact and the continuity of U up to the thin set $\{x_{n+1} = 0\}$ we deduce that

$$\int_{\mathbb{B}_r^+ \times [0, r^2)} x_{n+1}^a U^2 dX dt > 0, \tag{4.77}$$

for all $0 < r \leq 1/2$. Moreover, the inequality (iii) in Theorem 4.10 holds, i.e. there exist r_0 and C depending on θ in (4.56) such that for all $r \leq r_0$ one has

$$\int_{\mathbb{B}_r^+ \times [0, r^2)} x_{n+1}^a U^2 dX dt \leq C \int_{\mathbb{B}_{r/2}^+ \times [0, r^2/4)} x_{n+1}^a U^2 dX dt. \tag{4.78}$$

From this doubling estimate we can derive in a standard manner the following inequality for all $r \leq \frac{r_0}{2}$

$$\int_{\mathbb{B}_r^+ \times [0, r^2)} x_{n+1}^a U^2 dX dt \geq \frac{r^M}{C} \int_{\mathbb{B}_{r_0}^+ \times [0, r_0^2)} x_{n+1}^a U^2 dX dt,$$

where $M = \log_2 C$. Letting $C_0 = \frac{1}{C} \int_{\mathbb{B}_{r_0}^+ \times [0, r_0^2)} U^2 y^a dX dt$, and noting that $C_0 > 0$ in view of (4.77), we can rewrite the latter inequality as

$$\int_{\mathbb{B}_r^+ \times [0, r^2)} U^2 y^a dX dt \geq C_0 r^M. \tag{4.79}$$

Let now $r_j \searrow 0$ be a sequence such that $r_j \leq r_0$ for every $j \in \mathbb{N}$, and define

$$U_j(X, t) = \frac{U(r_j X, r_j^2 t)}{\left(\frac{1}{r_j^{n+3+a}} \int_{\mathbb{B}_{r_j}^+ \times [0, r_j^2)} x_{n+1}^a U^2 dX dt \right)^{1/2}}.$$

Note that thanks to (4.77) the functions U_j's are well defined. Furthermore, by a change of variable, we note

$$\int_{\mathbb{B}_1^+ \times [0, 1)} x_{n+1}^a U_j^2 dX dt = 1. \tag{4.80}$$

Again by a change of variable and by using the doubling inequality (4.78), we have for all j

$$\int_{\mathbb{B}^+_{1/2}\times[0,1/4)} x_{n+1}^a U_j^2 dXdt \geq C^{-1}. \tag{4.81}$$

Moreover U_j solves the following problem in $\mathbb{B}_1^+ \times [0,1)$

$$\begin{cases} \operatorname{div}(x_{n+1}^a A(r_j x)\nabla U_j) + x_{n+1}^a \partial_t U_j = 0, \\ \partial^a_{x_{n+1}} U_j(x,0,t) = 0. \end{cases} \tag{4.82}$$

From (4.80) and the regularity estimates in Lemma 4.7 and Lemma 4.8 we infer that, possibly passing to a subsequence which we continue to indicate with U_j, we have $U_j \to U_0$ in $H^{1+\alpha}(\mathbb{B}^+_{3/4} \times [0,9/16))$ up to $\{x_{n+1}=0\}$. We infer in a standard way by a weak type argument that the blowup limit U_0 solves in $\mathbb{B}^+_{3/4}\times[0,9/16)$

$$\begin{cases} \operatorname{div}(y^a \nabla U_0) + y^a \partial_t U_0 = 0, \\ \partial^a_{x_{n+1}} U_0(x,0,t) = 0. \end{cases} \tag{4.83}$$

Since $U(x,0,t)$ vanishes identically in $B_4 \times [0,16)$, it follows on account of uniform convergence of U_j's to U_0 that $U_0(x,0,t) \equiv 0$ in $B_{1/2}\times[0,1/4)$. On the other hand, from the uniform convergence of U_j's in $\mathbb{B}^+_{1/2}\times[0,1/4)$ and the non-degeneracy estimate (4.81) we also have

$$\int_{\mathbb{B}^+_{1/2}\times[0,1/4)} x_{n+1}^a U_0^2 dXdt \geq C^{-1}, \tag{4.84}$$

and thus $U_0 \not\equiv 0$ in $\mathbb{B}^+_{1/2}\times[0,1/4)$. This violates the weak unique continuation property in Theorem 1.4. Therefore (4.76) must be true. Now in view of our discussion after (4.76), we find that the conclusion of the theorem thus follows.

□

5. Applications to Calderón inverse problems

In this section, we obtain the unique recovery result of the potential q as in the initial-exterior problem (1.1) from the nonlocal DN map (1.3). We rigorously define the DN map introduced in (1.3) and then derive an Alessandrini type identity in this context. This will be followed by the Runge approximation result which will be a byproduct of unique continuation result Theorem 1.3. This is similar to that in [29] and [18]. As previously mentioned in the introduction, such a Runge type approximation argument allows to bypass the method of CGO solutions and this aspect is quite specific to nonlocal problems. In this section, we closely follow the approach in [18]. We start by defining the abstract trace space as follows

$$\mathbb{X} := \mathcal{H}^s\left(\mathbb{R}^n \times [-T,T]\right) \setminus \mathbb{H}^s_{\overline{Q}}.$$

The norm in $\mathbb{X}$ is defined in an analogous way as (2.10). Before moving on to the definition of the DN map, we would like to stress the fact that the solution u in (1.1) corresponding to $f \in \mathbb{H}^s(\mathbb{R}^{n+1})$ depends only on $f|_{Q_e}$ where $Q_e = \Omega_e \times (-T,T)$ which can be seen as a consequence of uniqueness and the weak formulation in Definition 3.2. To emphasize the dependence of the solution on the data, we declare u_f and u_f^* to be the solutions of (1.1) and (3.38) respectively for the exterior value

f. The following proposition below constitutes the rigorous definition of the DN map analogous to Proposition 3.5 in [18]. This crucially relies on the well-posedness which is accounted by (1.2).

Proposition 5.1 (The DN map for $\mathcal{H}^s + q$). *Let $s \in (0,1)$, $T > 0$ and Ω be a bounded open set in $\mathbb{R}^n, n \geq 1$ and let $Q = \Omega \times (-T,T)$. Further assume $q \in L^\infty(Q)$ satisfies the eigenvalue condition (1.2). For $f, g \in \mathcal{H}^s(\mathbb{R}^n \times [-T,T])$, we define*

$$\langle \Lambda_q[f], [g] \rangle_{\mathbb{X}^* \times \mathbb{X}} = \mathcal{B}_q(u_f, g).$$

Then $\Lambda_q : \mathbb{X} \to \mathbb{X}^*$ is a bounded operator.

Proof. First we need to justify well-definedness of Λ_q. For that, we consider $f' \in [f]$ and $g' \in [g]$ or, in other words $f' = f + \phi$ and $g' = g + \psi$ for some $\phi, \psi \in \mathbb{H}^s_Q$ and wish to show $\langle \Lambda_q[f], [g] \rangle_{\mathbb{X}^* \times \mathbb{X}} = \langle \Lambda_q[f'], [g'] \rangle_{\mathbb{X}^* \times \mathbb{X}}$. In this regard, we notice

$$\mathcal{B}_q(u_{f+\phi}, g + \psi) = \mathcal{B}_q(u_{f+\phi}, g) = \mathcal{B}_q(u_f, g)$$

where the above implications follow directly from the weak formulation of (1.1) and the fact that u_f depends only on $f|_{Q_e}$. Moreover from (3.32) we find

$$|\langle \Lambda_q[f], [g] \rangle_{\mathbb{X}^* \times \mathbb{X}}| \leq C \, \|f + \phi\|_{\mathcal{H}^s(\mathbb{R}^n \times [-T,T])} \, \|g + \psi\|_{\mathcal{H}^s(\mathbb{R}^n \times [-T,T])}$$

where the constant $C > 0$ is independent of the choices $\phi, \psi \in \mathbb{H}^s_Q$. This implies $\Lambda_q[f] \in \mathbb{X}^*$ with $\|\Lambda_q\|_{\mathbb{X} \to \mathbb{X}^*} \leq C$. □

Following the natural pairing, we define the DN map for the adjoint problem (3.38) as

$$\langle [f], \Lambda^*_q[g] \rangle_{\mathbb{X} \times \mathbb{X}^*} = \langle \Lambda_q[f], [g] \rangle_{\mathbb{X}^* \times \mathbb{X}}. \tag{5.1}$$

We also note that if $u^*_g \in H^s(\mathbb{R}^{n+1})$ solves (3.38) corresponding to the exterior data $g \in \mathcal{H}^s(Q_e)$ then we have

$$\langle [f], \Lambda^*_q[g] \rangle_{\mathbb{X} \times \mathbb{X}^*} = \mathcal{B}_q(f, u^*_g) \tag{5.2}$$

which follows from the variational formulation of the problems (1.1) and (3.38).

We now state and prove an Alessandrini type identity in our context which plays an essential role in proving the uniqueness result.

Lemma 5.2 (Integral identity for $\mathcal{H}^s + q$). *Let $s \in (0,1), T > 0$ and Ω be a bounded open set in $\mathbb{R}^n$. Furthermore, let $q_1, q_2 \in L^\infty(Q)$ be such that the eigenvalue condition (1.2) holds. Then for $f, g \in \mathcal{H}^s(\mathbb{R}^n \times [-T,T])$, we have*

$$\langle (\Lambda_{q_1} - \Lambda_{q_2}) [f], [g] \rangle_{\mathbb{X}^* \times \mathbb{X}} = \int_Q (q_1 - q_2) u_f u^*_g.$$

where u_f solves the problem (1.1) for $q = q_1$ associated to the exterior data f and u^*_g is a solution to (3.38) when $q = q_2$ corresponding to exterior data g.

Proof. From the adjoint property (5.1), its characterization in (5.2) and also by using the fact that $u_f \in [f], u^*_g \in [g]$, we find

$$\begin{aligned}
\langle (\Lambda_{q_1} - \Lambda_{q_2}) [f], [g] \rangle_{\mathbb{X}^* \times \mathbb{X}} &= \langle \Lambda_{q_1}[f], [g] \rangle_{\mathbb{X}^* \times \mathbb{X}} - \langle [f], \Lambda^*_q[g] \rangle_{\mathbb{X} \times \mathbb{X}^*} \\
&= \mathcal{B}_{q_1}(u_f, u^*_g) - \mathcal{B}_{q_2}(u_f, u^*_g) \\
&= \int_Q (q_1 - q_2) u_f u^*_g.
\end{aligned}$$

□

The final step in the uniqueness proof is the following density result where we crucially use the weak unique continuation result Theorem 1.3. We remark here that the unique determination result (1.1) only requires the density with respect to L^2 norm. But we eventually need the approximation result in $\mathbb{H}^s_{\overline{Q}}$ for recovering both the first and zeroth order perturbations in Theorem 1.2. With an intent to omit a similar discussion for the (b, q) case, we present a general approximation result (in $\mathbb{H}^s_{\overline{Q}}$ topology) for the case when $b = 0$.

Theorem 5.3 (Runge approximation for $\mathcal{H}^s + q$)**.** *Let $s \in (0,1), T > 0$ and Ω be a bounded open set in $\mathbb{R}^n$. Consider W to be a bounded open set in $\mathbb{R}^n$, such that $\overline{\Omega} \cap \overline{W} = \emptyset$. Then the set*

$$\mathcal{D}_q(W) = \{u_f - f; \quad f \in C_0^\infty\left(W \times (-T,T)\right)\}$$

is dense in $\mathbb{H}^s_{\overline{Q}}$ where $Q = \Omega \times (-T,T)$ and u_f is the solution to (1.1) corresponding to f.

Proof. The proof is similar to that in [29] given the validity of Theorem 1.3. Invoking the Hahn-Banach theorem, it suffices to show that there is no non-trivial $F \in (\mathbb{H}^s_{\overline{Q}})^*$ which satisfies

$$\langle F, u_f - f\rangle = 0, \quad \forall f \in C_0^\infty\left(W \times (-T,T)\right). \tag{5.3}$$

In order to establish (5.3), we first construct $\phi \in \mathbb{H}^s_{\overline{Q}}$ solving the adjoint problem

$$\begin{cases} \left(\mathcal{H}^s_* + q(x,t)\right)\phi = F, & \text{in } Q \\ \phi(x,t) = 0, & \text{in } Q_e, \text{ and for } t \geq T. \end{cases}$$

Then the weak formulation (3.38) together with (5.3) implies

$$0 = \langle F, u_f - f\rangle = \mathcal{B}_q\left(u_f - f, \phi\right). \tag{5.4}$$

Now using the weak formulation for u_f, we find that

$$\mathcal{B}_q\left(u_f, \phi\right) = 0, \tag{5.5}$$

since $\phi \equiv 0$ in Q_e. Therefore it follows that

$$\mathcal{B}_q\left(f, \phi\right) = 0 \tag{5.6}$$

for all $f \in C_0^\infty(W \times (-T,T))$. Since f is supported in $W \times (-T,T)$, thus from (5.6) we deduce that

$$\mathcal{H}^s_*\phi = 0, \ \phi = 0, \quad \text{in } W \times (-T,T). \tag{5.7}$$

Now in view of the change of variable in (3.39), we can invoke Theorem 1.3 to conclude that $\phi = 0$ in $\mathbb{R}^{n+1}$ which then implies that $F = 0$. This finishes the proof of the Theorem. □

With the Runge type approximation result as in Theorem 5.3 in hand, we now proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1. Let us fix some $\phi \in C_0^\infty(Q)$ and also let W_1 and W_2 be as in Theorem 1.1. We also let $\psi \in C_0^\infty(Q)$ such that $\psi \equiv 1$ on $\mathrm{supp}(\phi)$. By virtue of Theorem 5.3, for $k = 1, 2$ and $j \in \mathbb{N}$, there exists exterior values (for both the forward and adjoint problems) $f_{j,k} \in C_0^\infty(W_k \times (-T,T))$ for which

$$\begin{aligned} &(\mathcal{H}^s + q_1)\, u_{j,1} = (\mathcal{H}^s_* + q_2)\, u^*_{j,2} = 0, \quad \text{ in } Q,\\ &u_{j,1}(x,t) = f_{j,1}(x,t), \text{ and } u^*_{j,2}(x,t) = f_{j,2}(x,t), \text{ in } Q_e,\\ &u_{j,1}|_{t\le -T} = 0, \text{ and } u^*_{j,2}|_{t\ge T} = 0,\\ &u_{j,1} - f_{j,1} = \phi + r_{j,1}, \; u^*_{j,2} - f_{j,2} = \psi + r_{j,2}, \end{aligned}$$

such that

$$\|r_{j,k}\|_{\mathbb{H}^s_{\overline{Q}}} \to 0 \text{ as } j \to \infty \text{ for } k = 1,2. \tag{5.8}$$

Plugging these solutions into the Alessandrini type identity in Lemma 5.2 and by using $\Lambda_{q_1}([f_{j,1}])|_{W_2\times(-T,T)} = \Lambda_{q_2}([f_{j,1}])|_{W_2\times(-T,T)}$ and also that $f_{j,2} \in C_0^\infty(W_2 \times (-T,T))$, we obtain

$$\int_Q (q_1 - q_2) u_{j,1} u^*_{j,2} \,\mathrm{d}x\mathrm{d}t = 0, \quad \text{ for } j \in \mathbb{N}. \tag{5.9}$$

Now since $f_{j,k} \equiv 0$ in Q, by letting $j \to \infty$ and by using (5.8), we find that (5.9) reduces to

$$\int_Q (q_1 - q_2)\phi \,\mathrm{d}x\mathrm{d}t = 0.$$

Since this is valid for any $\phi \in C_0^\infty(Q)$, we deduce that $q_1 = q_2$ in Q. This finishes the proof of the theorem. □

Now we prove Theorem 1.2. The rigorous definition of the DN map and derivation of related integral identity along with Runge approximation result are exactly similar to the ones discussed for the q case. For this reason, we choose to skip the details and merely mention the statements in this setting. The only part different from the previous discussion is the determination of b and q simultaneously. To accomplish that, we follow the strategy in [18] by determining q first and then use it to recover the drift term b. Throughout we assume that (1.5) holds.

Proposition 5.4 (The DN map for $\mathcal{H}^s + \langle b, \nabla_x\rangle + q$). *Let $s \in \left(\frac{1}{2}, 1\right), T > 0$ and Ω be a bounded Lipschitz open set in $\mathbb{R}^n$. Further assume that $b \in L^\infty\left((-T,T); W^{1-s,\infty}(\Omega)\right)$, $q \in L^\infty(Q)$. For $f, g \in \mathbb{H}^s(\mathbb{R}^n \times [-T,T])$, we define*

$$\langle \Lambda_{b,q}[f], [g]\rangle_{\mathbb{X}^*\times\mathbb{X}} = \mathcal{B}_{b,q}(u_f, g).$$

Then $\Lambda_{b,q} : \mathbb{X} \to \mathbb{X}^*$ is well defined and is a bounded operator.

Lemma 5.5 (Integral identity for $\mathcal{H}^s + \langle b, \nabla_x\rangle + q$). *Let $s \in \left(\frac{1}{2}, 1\right), T > 0$ and Ω be a bounded Lipschitz open set in $\mathbb{R}^n$. For $q_1, q_2 \in L^\infty(Q)$, $b_1, b_2 \in L^\infty\left((-T,T); W^{1-s,\infty}(\Omega)\right)$ and $f, g \in \mathbb{H}^s(\mathbb{R}^n \times [-T,T])$, we have*

$$\langle \left(\Lambda_{b_1,q_1} - \Lambda_{b_2,q_2}\right)[f], [g]\rangle_{\mathbb{X}^*\times\mathbb{X}} = \int_Q \langle (b_1 - b_2), \nabla_x u_f\rangle\, u^*_g + \int_Q (q_1 - q_2) u_f u^*_g.$$

Theorem 5.6 (Runge approximation for $\mathcal{H}^s + \langle b, \nabla_x\rangle + q$). *Let $s \in \left(\frac{1}{2}, 1\right), T > 0$ and Ω be a bounded Lipschitz open set in $\mathbb{R}^n$ and W be a bounded open set in $\mathbb{R}^n$, such that $\overline{\Omega} \cap \overline{W} = \emptyset$. Then the set*

$$\mathcal{D}_{b,q}(W) = \{u_f - f;\; f \in C_0^\infty\left(W \times (-T,T)\right)\}$$

is dense in $\mathbb{H}^s_{\widetilde{Q}}$.

Proof of Theorem 1.2. Let $\phi, \psi \in C_0^\infty(Q)$ be such that $\psi \equiv 1$ in $\mathrm{supp}(\phi)$. Thanks to Theorem 5.6, for $k = 1, 2$ and $j \in \mathbb{N}$, we can choose the exterior values $f_{j,k} \in C_0^\infty(W_k \times (-T, T))$ for the forward and adjoint problems in a way so that

$$
\begin{aligned}
&(\mathcal{H}^s + \langle b_1, \nabla_x \rangle + q_1)\, u_{j,1} = (\mathcal{H}^s_* + \langle b_2, \nabla_x \rangle + q_2)\, u^*_{j,2} = 0, \qquad \text{in } Q,\\
&u_{j,1}(x,t) = f_{j,1}(x,t), \text{ and } u^*_{j,2}(x,t) = f_{j,2}(x,t), \text{ in } Q_e,\\
&u_{j,1}|_{t \le -T} = 0 \text{ and } u^*_{j,2}|_{t \ge T} = 0,
\end{aligned}
$$

where

$$
u_{j,1} - f_{j,1} = \psi + r_{j,1} \text{ and } u^*_{j,2} - f_{j,2} = \phi + r_{j,2}, \tag{5.10}
$$

with $r^j_k \in \mathbb{H}^s_{\widetilde{Q}}$ and

$$
\|r_{j,k}\|_{\mathbb{H}^s_{\widetilde{Q}}} \to 0 \text{ as } j \to \infty \text{ for } k = 1, 2. \tag{5.11}
$$

Proceeding as before, using that $\Lambda_{b_1,q_1}[f_{ji}]|_{W_2 \times (-T,T)} = \Lambda_{b_2,q_2}[f_{j1}]|_{W_2 \times (-T,T)}$, we find from the integral identity Lemma 5.5 that the following holds

$$
\int_Q \langle (b_1 - b_2), \nabla_x u_{j,1} \rangle \; u^*_{j,2} \; \mathrm{d}x\mathrm{d}t + \int_Q (q_1 - q_2) u_{j,1} u^*_{j,2} \; \mathrm{d}x\mathrm{d}t = 0. \tag{5.12}
$$

Now we analyze the term $\int_Q \langle (b_1 - b_2), \nabla_x u_{j,1} \rangle \; u^*_{j,2} \; \mathrm{d}x\mathrm{d}t$. Using (5.10) and the fact that $\nabla \psi \equiv 0$ on the support of ϕ, we find

$$
\int_Q \langle (b_1 - b_2), \nabla_x u_{j,1} \rangle \; u^*_{j,2} = \int_Q \langle (b_1 - b_2), \nabla_x r_{j,1} \rangle \; r_{j,2} + \int_Q \langle (b_1 - b_2), \nabla_x r_{j,1} \rangle \; \phi. \tag{5.13}
$$

We show that both the integrals in (5.13) converges to zero as $j \to \infty$. Arguing as in (3.42), we observe

$$
\left| \int_Q \langle (b_1 - b_2), \nabla_x r_{j,1} \rangle \; r_{j,2} \right| \preceq \|r_{j,1}\|_{\mathbb{H}^s_{\widetilde{Q}}} \|r_{j,2}\|_{\mathbb{H}^s_{\widetilde{Q}}}
$$

and similarly

$$
\left| \int_Q \langle (b_1 - b_2), \nabla_x r_{j,1} \rangle \; \phi \right| \preceq \|r_{j,2}\|_{\mathbb{H}^s_{\widetilde{Q}}} \|\phi\|_{\mathbb{H}^s_{\widetilde{Q}}}.
$$

Therefore using (5.11) we can conclude that

$$
\lim_{j \to \infty} \int_Q \langle (b_1 - b_2), \nabla_x u_{j,1} \rangle \; u^*_{j,2} = 0. \tag{5.14}
$$

Using (5.14) in (5.12) we deduce

$$
\lim_{j \to \infty} \int_Q (q_1 - q_2) u_{j,1} u^*_{j,2} \; \mathrm{d}x\mathrm{d}t = 0. \tag{5.15}
$$

By following the proof of Theorem 1.1, we get that (5.15) reduces to

$$
\int_Q (q_1 - q_2) \phi \; \mathrm{d}x\mathrm{d}t = 0.
$$

Since $\phi \in C_0^\infty(Q)$ is arbitrary, we thus have $q_1 = q_2$ in Q.

We now uniquely determine the drift term. We first choose some $\phi \in C_0^\infty(Q)$ and then for $i = 1, 2$ and $k \in \{1, 2, ..., n\}$, we choose $\phi_{i,k} \in C_0^\infty(Q)$ in such a way that

$$\phi_{2,k} = \phi \text{ and } \phi_{1,k} = x_k \text{ on } \mathrm{supp}(\phi_{2,k}). \tag{5.16}$$

Thanks to Theorem 5.6, we can find $f_{j,l}^k \in C_0^\infty(W_l \times (-T, T))$ for $k \in \{1, 2, .., n\}, j \in \mathbb{N}, l = 1, 2$ and solutions $\{u_{j,1}^k, u_{j,2}^{*,k}\}$ to the forward and adjoint problems associated to the exterior data $f_{j,l}^k$ such that

$$u_{j,1}^k - f_{j,1}^k = \phi_{1,k} + r_{j,1}^k, \qquad \text{and } u_{j,2}^{*,k} - f_{j,2}^k = \phi_{2,k} + r_{j,2}^k,$$

where $\{r_{jl}^k\}$ satisfies the limiting condition in (5.11) as $j \to \infty$. Now given $k \in \{1, 2, ..., n\}$, using the identity in Lemma 5.5 along with the fact that $q_1 = q_2$ in Q, we have

$$\begin{aligned} 0 = \int_Q \langle (b_1 - b_2), \nabla_x u_{1,j}^k \rangle \; u_{2,j}^{*,k} &= \int_Q \langle (b_1 - b_2), \nabla_x \phi_{1,k} \rangle \; \phi_{2,k} + \int_Q \langle (b_1 - b_2), \nabla_x \phi_{1,k} \rangle \; r_{j,2}^k \\ &+ \int_Q \langle (b_1 - b_2), \nabla_x r_{j,1}^k \rangle \; \phi_{2,k} + \int_Q \langle (b_1 - b_2), \nabla_x r_{j,1}^k \rangle \; r_{j,2}^k. \end{aligned}$$

Using $\lim_{j\to\infty} \|r_{j,l}^k\|_{\mathbb{H}_Q^s} = 0$ for $l = 1, 2$ and $k \in \{1, 2, .., n\}$, we notice as before that

$$\lim_{j\to\infty} \int_Q \langle (b_1 - b_2), \nabla_x \phi_{1,k} \rangle \; r_{j,2}^k = \lim_{j\to\infty} \int_Q \langle (b_1 - b_2), \nabla_x r_{j,1}^k \rangle \; \phi_{2,k} = \lim_{j\to\infty} \int_Q \langle (b_1 - b_2), \nabla_x r_{j,1}^k \rangle \; r_{j,2}^k = 0$$

which then implies

$$\int_Q \langle (b_1 - b_2), \nabla_x \phi_{1,k} \rangle \; \phi_{2,k} \; \mathrm{d}x\mathrm{d}t = 0. \tag{5.17}$$

Now using (5.16) in (5.17) we find

$$\int_Q (b_1 - b_2)_k(x, t) \phi(x, t) \; \mathrm{d}x\mathrm{d}t = 0, \quad \text{for } k \in \{1, 2, .., n\}.$$

Since $\phi \in C_0^\infty(Q)$ is arbitrary, we can thus infer that $b_1 = b_2$ in Q. □

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